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RESEARCH ARTICLE

Admissibility and uniform dichotomy for evolution families

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Abstract The present paper studies the problem of uniform dichotomy for evolution families in general. We obtain a result identical with the one for differential systems in infinite dimensional spaces (Massera and Schäffer in Linear Diferential Equations and Function Spaces, Academic Press, New York, 1966, Preda in An Univ Timisoara Ser Stiint Mat 17:65–71, 1979), as well as the existence of a family of projectors compatible with the evolution family. This family of projectors has similar properties as the one obtained by van Minh et al. (Integr Equ Oper Theory 32:332–353, 1998) and by van Minh and Thieu Huy (J Math Anal Appl 261:28–44, 2001), but for evolution families with uniform exponential growth.

Keywords Admissibility · Evolution family · Uniform dichotomy

1 Introduction

The present paper belongs to the field of evolution equations theory (differential equations and asymptotic properties of their solutions) which started to develop in the

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1930s. As a starting point for a vast literature concerning this subject, we mention the pioneering work of Perron [12], who in 1930 was the first to establish the connection between the asymptotic behavior of the solution of the differential equation

$$\dot{x}(t) = A(t)x(t) \qquad (A)$$

and the associated non-homogeneous equation

$$\dot{x}(t) = A(t)x(t) + f(t) \qquad (A, f)$$

in finite dimensional spaces, where A is a $n \times n$ dimensional, continuous and bounded matrix and f is a continuous and bounded function on $\mathbb{R}_+$.

In a paper from 1958 ([8]), Massera and Schäffer study the same problem as Perron for differential systems in infinite dimensional spaces. This time they used the pair of spaces (L^p, L^∞), $1 < p < \infty$, where

$$L^p = \{f : [0, \infty) \to X : \int_0^\infty ||f(t)||^p dt < \infty\},$$
$$L^\infty = \{f : [0, \infty) \to X : ess \sup_{t \geq 0} ||f(t)|| < \infty\}.$$

In the monograph [9], the same authors prove that the differential system (A) has uniform dichotomy if and only if the pair of spaces (L^1, L^∞) is admissible to it.

The case of differential systems in finite dimensional spaces was later studied by Coppel in the monographs [2,3] and by Hartman [6].

Further developments for differential systems in infinite dimensional spaces can be found in the monographs of Daleckij and Krein [4] and Massera and Schäffer [9]. The case of dynamical systems described by evolution processes was studied by Chicone and Latushkin [1], by Engel and Nagel [5] and by Levitan and Zhikov [7].

Another important result is the one presented by van Minh et al. in [10]. In this paper the authors give a characterization for uniform exponential dichotomy of evolution families with uniform exponential growth, i.e. $||\Phi(t, t_0)|| \leq Me^{\omega(t-t_0)}$ for all $t \geq t_0 \geq 0$, using the pair of spaces $(\mathcal{C}_{00}, \mathcal{C})$, where

$$\mathcal{C} = \{f : [0, \infty) \to X : f \text{ is continuous and bounded}\},$$
$$\mathcal{C}_{00} = \{f \in \mathcal{C} : \lim_{t \to \infty} f(t) = f(0) = 0\}.$$

The admissibility of the above mentioned pair of spaces also implies the existence of a family of projectors compatible with the evolution families used. In order to prove our main result, we use similar techniques as the ones in this paper.

The next important step in studying asymptotic properties of evolution families with uniform exponential growth is the paper [11] of van Minh and Thieu Huy. The authors use the pair of spaces $(L^p, L^p \cap \mathcal{C})$ and the input-output technique. The admissibility of this pair of spaces implies uniform dichotomy for the evolution families, as well as the existence of a family of projectors compatible with them. Inspired by this paper, we also use this technique, i.e. we choose carefully selected input functions that allow us to prove our main result.

In the present paper, we study uniform dichotomy of evolution families without any exponential growth, by using the input-output technique. We generalize the results found in the papers [8–11,13].

2 Preliminaries

Let X be a Banach space and $\mathbb{B}(X)$ the space of all linear and bounded operators acting on X. The norms on X and on $\mathbb{B}(X)$ will be denoted by $||\cdot||$.

Definition 2.1 A family of linear operators $\Phi : \Delta = \{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0 \geq 0\} \to \mathbb{B}(X)$ on a Banach space X is an evolution family if:

(i) $\Phi(t, t) = I$, for all $t \in \mathbb{R}_+$, where I is the identity on X;
(ii) $\Phi(t, t_0) = \Phi(t, s)\Phi(s, t_0)$, for all $t \geq s \geq t_0 \geq 0$;
(iii) the map $\Phi(\cdot, t_0)x$ is continuous on $[t_0, \infty)$ for all $x \in X$ and $\Phi(t, \cdot)x$ is continuous on $[0, t]$ for all $x \in X$.

Let $X_1(0) = \{x \in X : \Phi(\cdot, 0)x \in L^\infty\}$ be a complemented space in X and $X_2(0)$ one of its complements, i.e. $X = X_1(0) \oplus X_2(0)$.

We use the spaces

$$L^1(X) = \{f : \mathbb{R}_+ \to X : \ f \text{ is Bochner measurable and } \int_0^\infty ||f(\tau)||d\tau < \infty\},$$
$$L^\infty(X) = \{f : \mathbb{R}_+ \to X : \ f \text{ is Bochner measurable and } ess \sup_{t\geq 0} ||f(t)|| < \infty\}.$$

The norms on these spaces are

$$||f||_1 = \int_0^\infty ||f(\tau)||d\tau,$$
$$||f||_\infty = ess \sup_{t\geq 0} ||f(t)||.$$

The spaces $(L^1(X), ||\cdot||_1)$ and $(L^\infty(X), ||\cdot||_\infty)$ are Banach spaces.

Definition 2.2 Let $\{\Phi(t, t_0)\}_{t\geq t_0\geq 0}$ be an evolution family. The pair of spaces $(L^1(X), L^\infty(X))$ is admissible to $\{\Phi(t, t_0)\}_{t\geq t_0\geq 0}$ if and only if for every f in $L^1(X)$ there exists an element x in X such that the function $x_f : \mathbb{R}_+ \to X$,

$$x_f(t) = \Phi(t, 0)x + \int_0^t \Phi(t, \tau)f(\tau)d\tau \text{ is in } L^\infty(X).$$

Remark 2.1 If the pair $(L^1(X), L^\infty(X))$ is admissible to the evolution family $\{\Phi(t, t_0)\}_{t\geq t_0\geq 0}$, then for every f in $L^1(X)$ there exists an unique element x in $X_2(0)$ such that the function

$$x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)d\tau \text{ is in } L^\infty(X).$$

Indeed, if f is in $L^1(X)$, then there exists x in X such that the function

$$x_f : \mathbb{R}_+ \to X, x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)d\tau, \text{ is in } L^\infty(X).$$

Since $X = X_1(0) \oplus X_2(0)$, then there exist two unique elements $x_1 \in X_1(0)$ and $x_2 \in X_2(0)$ such that $x = x_1 + x_2$.

Then the function $x_f - \Phi(\cdot,0)x_1$ is in $L^\infty(X)$, which means that

$$t \mapsto \Phi(t,0)x_2 + \int_0^t \Phi(t,\tau)f(\tau)d\tau : \mathbb{R}_+ \to X \text{ is in } L^\infty(X).$$

So we have found an element $x_2 \in X_2(0)$ such that the above function is in $L^\infty(X)$. We show that this element is unique.

We assume that there exist $z_1, z_2 \in X_2(0)$ such that the functions

$$\begin{aligned} &x_f^1, x_f^2 : \mathbb{R}_+ \to X, \\ &x_f^1(t) = \Phi(t,0)z_1 + \int_0^t \Phi(t,\tau)f(\tau)d\tau, \\ &x_f^2(t) = \Phi(t,0)z_2 + \int_0^t \Phi(t,\tau)f(\tau)d\tau, \end{aligned}$$

are in $L^\infty(X)$. Then $x_f^1 - x_f^2$ is in $L^\infty(X)$ and

$$x_f^1(t) - x_f^2(t) = \Phi(t,0)(z_1 - z_2),$$

for all $t \geq 0$, so $z_1 - z_2 \in X_1(0)$. Since $z_1 - z_2 \in X_2(0)$, the conclusion is that $z_1 = z_2$. Throughout the following we denote by

$$x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)d\tau$$

the function for which $x_f(0) = x \in X_2(0)$.

Definition 2.3 A family of operators $P : \mathbb{R}_+ \to \mathbb{B}(X)$ is called a family of projectors if and only if

$$P^2(t) = P(t), \text{ for all } t \geq 0.$$

We denote $Q(t) = I - P(t)$, for all $t \geq 0$. Therefore the subspaces $X_1(t) = P(t)X$ and $X_2(t) = Q(t)X$, for all $t \geq 0$, are closed and complemented, i.e. $X = X_1(t) \oplus X_2(t)$, for all $t \geq 0$.

Definition 2.4 Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an evolution family. The family of projectors $P:\mathbb{R}_+\to\mathbb{B}(X)$ is called compatible to the evolution family if and only if

(i) $P(t)\Phi(t,t_0)=\Phi(t,t_0)P(t_0)$, for all $t\geq t_0\geq 0$;
(ii) $\Phi(t_1,t_0):X_2(t_0)\to X_2(t_1)$ is invertible, for all $t_1\geq t_0\geq 0$;
(iii) the function $t\mapsto P(t)x:\mathbb{R}_+\to X$ is continuous and bounded, for all $t\geq 0$, $x\in X$.

Definition 2.5 The evolution family $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is uniform dichotomic if and only if there exist a family of projectors $P:\mathbb{R}_+\to\mathbb{B}(X)$, compatible to $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$, and $N>0$ such that

(i) $||\Phi(t,t_0)P(t_0)x||\leq N||P(t_0)x||$, for all $x\in X$ and $t\geq t_0\geq 0$;
(ii) $N||\Phi(t,t_0)Q(t_0)x||\geq ||Q(t_0)x||$, for all $x\in X$ and $t\geq t_0\geq 0$.

Throughout the following we assume that only $X_1(0)$ is complemented in X and that $X_2(0)$ is one of its complements.

Remark 2.2 If $x\in X_2(0)\setminus\{0\}$, then $\Phi(t,0)x\neq 0$, for all $t\geq 0$.

Indeed, if we assume that there exists $t_0>0$ such that

$$\Phi(t_0,0)x=0,$$

where $x\in X_2(0)\setminus\{0\}$, then

$$\Phi(t,0)x=0,$$

for all $t\geq t_0\geq 0$.

Therefore $\Phi(\cdot,0)x$ is in $L^\infty(X)$, since it is continuous on the compact interval $[0,t_0]$, so $x\in X_1(0)$, which is absurd, because $X_1(0)\cap X_1(0)=\{0\}$.

3 Main results

Lemma 3.1 *Let $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ be an evolution family. If $\lim\limits_{n\to\infty} f_n=f$ in $L^1(X)$, then*

$$\lim_{n\to\infty}\int_0^t \Phi(t,\tau)f_n(\tau)d\tau=\int_0^t \Phi(t,\tau)f(\tau)d\tau,\ \textit{for all } t\geq 0.$$

Proof Since $\Phi(t,\cdot)x$ is continuous on $[0,t]$, for all $x\in X$, then there exists $M_{t,x}>0$ such that

$$||\Phi(t,\tau)x||\leq M_{t,x},\ \text{ for all } \tau\in[0,t] \text{ and } x\in X.$$

By the uniform boundedness principle there exists $M_t>0$ such that

$$||\Phi(t,\tau)x||\leq M_t||x||,\ \text{ for all } \tau\in[0,t] \text{ and } x\in X.$$

This implies $||\Phi(t, \tau)|| \leq M_t$ for all $\tau \in [0, t]$.

We have that

$$||\int_0^t \Phi(t,\tau) f_n(\tau)d\tau - \int_0^t \Phi(t,\tau) f(\tau)d\tau||$$
$$\leq M_t \int_0^t ||f_n(\tau) - f(\tau)||d\tau \leq M_t||f_n - f||_1 \xrightarrow[n\to\infty]{} 0, \text{ for all } t \geq 0.$$

□

The following theorem is one of the most important tools in proving our main result.

Theorem 3.1 *If the pair* $(L^1(X), L^\infty(X))$ *is admissible to the evolution family* $\{\Phi(t, t_0)\}_{t\geq t_0\geq 0}$*, then there exists* $k > 0$ *such that*

$$||x_f(t)|| \leq k||f||_1, \text{ a.e. } t \geq 0,$$

for all $f \in L^1(X)$.

Proof Let $\mathcal{U} : L^1(X) \to X_2(0) \oplus L^\infty(X) = \{(x, g)|x \in X_2(0), g \in L^\infty(X)\}$, $\mathcal{U}f = (x_f(0), x_f)$. The norm on $X_2(0) \oplus L^\infty(X)$ is $||(x, g)|| = ||x|| + ||g||_\infty$. We will show that it is a closed linear operator.

Let

$$f_n \xrightarrow[n\to\infty]{L^1(X)} f \text{ and } \mathcal{U}f_n \xrightarrow[n\to\infty]{X_2(0)\oplus L^\infty(X)} (y, g).$$

We prove that $\mathcal{U}f = (y, g)$.

By Lemma 3.1

$$\lim_{n\to\infty} x_{f_n}(t) = \Phi(t, 0)y + \int_0^t \Phi(t,\tau) f(\tau)d\tau$$

for all $t \geq 0$, but we also have that

$$\lim_{n\to\infty} x_{f_n}(t) = g(t) \text{ a.e. } t \geq 0.$$

This proves that $\mathcal{U}f = (y, g)$ in $X_2(0) \oplus L^\infty(X)$, which means that $\mathcal{U}$ is closed. By the closed graph principle we have that there exists $k > 0$ such that

$$||\mathcal{U}f|| \leq k||f||_1, \text{ for all } f \in L^1(X).$$

This shows that

$$||x_f(t)|| \leq k||f||_1 \text{ a.e. } t \geq 0,$$

for all $f \in L^1(X)$. □

The following theorem is our main result.

Theorem 3.2 *Let $X_1(0)$ be a closed and complemented subspace in X and $X_2(0)$ one of its complemets.*

The pair $(L^1(X), L^\infty(X))$ is admissible to the evolution family $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ if and only if $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is uniform dichotomic.

Proof Necessity. Let $t_0 \geq 0$, $X_1(t_0) = \{x \in X : \Phi(\cdot, t_0)x \in L^\infty(X)\}$, for all $t_0 \geq 0$, $x \in X_1(t_0)$ such that $\Phi(t,t_0)x \neq 0$, for all $t \geq t_0 \geq 0$, $\Delta > 0$ and $f : \mathbb{R}_+ \to X$,

$$f(t) = \varphi_{[t_0,t_0+\Delta]}(t)\frac{\Phi(t,t_0)x}{||\Phi(t,t_0)x||},$$

where $\varphi_{[t_0,t_0+\Delta]}$ is the characteristic function of the interval $[t_0, t_0+\Delta]$.

Then $||f||_1 = \Delta$ and let $y : \mathbb{R}_+ \to X$,

$$y(t) = \begin{cases} 0, & 0 \leq t \leq t_0 \\ \int_{t_0}^t \varphi_{[t_0,t_0+\Delta]}(\tau)\frac{d\tau}{||\Phi(\tau,t_0)x||}\Phi(t,t_0)x, & t \geq t_0 \end{cases}.$$

We have that

$$y(t) = \int_0^t \Phi(t,\tau)f(\tau)d\tau, \text{ for all } t \geq 0.$$

Obviously $y \in L^\infty(X)$ and since $y(0) = 0$ is an element of $X_2(0)$, then $y = x_f$.

The function y is continuous and by Theorem 3.1 and we have that

$$||y(t)|| \leq k\Delta, \text{ for all } t \geq 0.$$

Therefore, if $t \geq t_0 + \Delta$,

$$\frac{1}{\Delta}\int_{t_0}^{t_0+\Delta}\frac{d\tau}{||\Phi(\tau,t_0)x||}||\Phi(t,t_0)x|| \leq k.$$

If $\Delta \to 0$, then the above inequality becomes

$$||\Phi(t,t_0)x|| \leq k||x||, \text{ for all } t \geq t_0 \geq 0.$$

Let $x \in X_1(t_0)$ such that there exists $t' > t_0$ with $\Phi(t',t_0)x = 0$. Then $\Phi(t,t_0)x = 0$, for all $t \geq t'$.

We denote $\sigma = \inf\limits_{t\geq t_0}\{t : \Phi(t,t_0)x = 0\}$, so $\Phi(\sigma,t_0)x = 0$ and $\Phi(t,t_0)x \neq 0$, for all $t \in [t_0,\sigma)$.

We have shown that $||\Phi(t,t_0)x|| \leq k||x||$, for all $t \in [t_0,\sigma)$, so

$$||\Phi(t,t_0)x|| \leq k||x||, \text{ for all } t \geq t_0 \geq 0, \ x \in X_1(t_0). \tag{1}$$

Next we show that $X_1(t_0)$ is a closed subspace, for all $t_0 \geq 0$.

Let $t_0 \geq 0$ and $x \in \overline{X_1(t_0)}$. Then there exists $(x_n)_n$ in $X_1(t_0)$ such that $\lim_{n\to\infty} x_n = x$. Since $x_n \in X_1(t_0)$, then

$$||\Phi(t, t_0)x_n|| \leq k||x_n||, \text{ for all } n \in \mathbb{N},\ t \geq t_0 \geq 0.$$

Therefore if $n \to \infty$, $||\Phi(t, t_0)x|| \leq k||x||$, for all $t \geq t_0 \geq 0$.

So $\Phi(\cdot, t_0)x \in L^\infty(X)$ and $x \in X_1(t_0)$. This shows that $X_1(t_0)$ is a closed subspace.

Let $\Delta > 0,\ t_0 \geq 0,\ x \in X_2(0) \setminus \{0\}$ and $g : \mathbb{R}_+ \to X$,

$$g(t) = \varphi_{[t_0, t_0+\Delta]}(t)\frac{\Phi(t, 0)x}{||\Phi(t, 0)x||}.$$

Then $g \in L^1(X)$ and $||g||_1 = \Delta$. Let $z : \mathbb{R}_+ \to X$,

$$\begin{aligned} z(t) &= -\int_t^\infty \varphi_{[t_0,t_0+\Delta]}(\tau)\frac{d\tau}{||\Phi(\tau, 0)x||}\Phi(t, 0)x \\ &= \Phi(t, 0)\left(-\int_0^\infty \varphi_{[t_0,t_0+\Delta]}(\tau)\frac{d\tau}{||\Phi(\tau, 0)x||}x\right) + \int_0^t \Phi(t, \tau)g(\tau)d\tau. \end{aligned}$$

Obviously $z(t) = 0$ for all $t \geq t_0 + \Delta$, so $z \in L^\infty(X)$. Since $z(0) \in X_2(0)$, then $z = x_g$. The function z is continuous and by Theorem 3.1 we have that

$$||z(t)|| \leq k\Delta, \text{ for all } t \geq 0.$$

If $t \leq t_0$, the above inequality becomes

$$\frac{1}{\Delta}\int_{t_0}^{t_0+\Delta} \frac{d\tau}{||\Phi(\tau, 0)x||}||\Phi(t, 0)x|| \leq k.$$

If $\Delta \to 0$, then

$$||\Phi(t, 0)x|| \leq k||\Phi(t_0, 0)x||, \text{ for all } t_0 \geq t \geq 0,\ x \in X_2(0).$$

Therefore

$$||\Phi(t, 0)x|| \geq \frac{1}{k}||\Phi(t_0, 0)x||, \text{ for all } t \geq t_0 \geq 0,\ x \in X_2(0).$$

We denote $X_2(t_0) = \Phi(t_0, 0)X_2(0)$, so

$$||\Phi(t, t_0)x'|| \geq \frac{1}{k}||x'||, \text{ for all } t \geq t_0 \geq 0,\ x' \in X_2(t_0). \tag{2}$$

Next we show that $X_2(t_0)$ is a closed subspace, for all $t_0 \geq 0$.

Let $x \in \overline{X_2(t_0)}$, then there exists $(x_n)_n$ in $X_2(t_0)$ such that $\lim_{n\to\infty} x_n = x$.

Since $x_n \in X_2(t_0)$, then there exists $u_n \in X_2(0)$ such that $x_n = \Phi(t_0, 0)u_n$, for all $n \in \mathbb{N}$.

We have that

$$||x_n - x_m|| = ||\Phi(t_0, 0)(u_n - u_m)|| \geq \frac{1}{k}||u_n - u_m||, \text{ for all } n, m \in \mathbb{N},$$

so $(u_n)_n$ is a fundamental sequence in $X_2(0)$. Therefore there exists $u \in X_2(0)$ such that $\lim_{n\to\infty} u_n = u$.

We have that $\lim_{n\to\infty} x_n = \Phi(t_0, 0)u$. But $\lim_{n\to\infty} x_n = x$, so $x = \Phi(t_0, 0)u$, which implies $x \in X_2(t_0)$. This shows that $X_2(t_0)$ is closed, for all $t_0 \geq 0$.

Next we show that $X = X_1(t_0) \oplus X_2(t_0)$, for all $t_0 \geq 0$.

Let $z \in X$ and $f : \mathbb{R}_+ \to X$,

$$f(t) = \begin{cases} -\Phi(t, t_0)z, \ t \in [t_0, t_0 + 1] \\ 0, \qquad\qquad \text{otherwise} \end{cases}.$$

Obviously $f \in L^1(X)$, so there exists an unique $x' \in X_2(0)$ such that the function $x_f : \mathbb{R}_+ \to X$,

$$x_f(t) = \Phi(t, 0)x' + \int_0^t \Phi(t, \tau) f(\tau) d\tau, \text{ is in } L^\infty(X).$$

But

$$x_f(t) = \Phi(t, t_0)(\Phi(t_0, 0)x' - z), \text{ for all } t \geq t_0 + 1.$$

Therefore $\Phi(t_0, 0)x' - z \in X_1(t_0)$.

Clearly

$$z = z - \Phi(t_0, 0)x' + \Phi(t_0, 0)x' \in X_1(t_0) + X_2(t_0), \text{ for all } t_0 \geq 0.$$

So

$$X = X_1(t_0) + X_2(t_0), \text{ for all } t_0 \geq 0. \tag{3}$$

Next we show that $X_1(t_0) \cap X_2(t_0) = \{0\}$, for all $t_0 \geq 0$.

Let $t_0 \geq 0$ and $y \in X_1(t_0) \cap X_2(t_0)$. So $\Phi(\cdot, t_0)y \in L^\infty(X)$ and there exists $v \in X_2(0)$ such that $y = \Phi(t_0, 0)v$.

Therefore

$$\Phi(\cdot, t_0)y = \Phi(\cdot, t_0)\Phi(t_0, 0)v = \Phi(\cdot, 0)v \in L^\infty(X),$$

which shows that $v \in X_1(0) \cap X_2(0) = \{0\}$. So $y = 0$. By (3) this proves that $X = X_1(t_0) \oplus X_2(t_0)$, for all $t \geq t_0 \geq 0$.

Thus we have shown that $X_1(t_0)$ and $X_2(t_0)$ are closed complements in X.

We show next that the function $\Phi(t_1, t_0) : X_2(t_0) \to X_2(t_1)$ is invertible.

Let $t_1 \geq t_0 \geq 0,\ z \in X_2(t_1)$ and $f : \mathbb{R}_+ \to X$,

$$f(t) = \begin{cases} -\Phi(t, t_1)z, & t \in [t_1, t_1 + 1] \\ 0, & \text{otherwise} \end{cases}.$$

Since $f \in L^1(X)$, then there exists an unique $u \in X_2(0)$ such that the function $x_f : \mathbb{R}_+ \to X$,

$$x_f(t) = \Phi(t, 0)u + \int_0^t \Phi(t, \tau) f(\tau) d\tau,$$

is in $L^\infty(X)$.

But

$$x_f(t) = \Phi(t, t_1)(\Phi(t_1, 0)u - z) = \Phi(t, t_1)(\Phi(t_1, t_0)\Phi(t_0, 0)u - z),$$

for all $t \geq t_1 + 1$.

We know that $\Phi(t_1, t_0)\Phi(t_0, 0)u - z$ is an element of $X_1(t_1) \cap X_2(t_1)$, so

$$z = \Phi(t_1, t_0)\Phi(t_0, 0)u.$$

Obviously $\Phi(t_0, 0)u \in X_2(t_0)$ and u is unique, so $\Phi(t_1, t_0)$ is invertible, for all $t_1 \geq t_0 \geq 0$.

Let $t_0 \geq 0,\ \Delta > 0,\ x_1 \in X_1(t_0) \setminus \{0\},\ x_2 \in X_2(t_0) \setminus \{0\}$ and the functions $y, z, w : \mathbb{R}_+ \to X$,

$$y(t) = \begin{cases} \Phi(t, t_0)\frac{x_1}{||x_1||}, & t \geq t_0 \\ \frac{x_1}{||x_1||}, & t < t_0 \end{cases},\quad z(t) = \begin{cases} \Phi(t, t_0)\frac{x_2}{||x_2||}, & t \geq t_0 \\ \frac{x_2}{||x_2||}, & t < t_0 \end{cases},$$
$$w(t) = y(t) + z(t).$$

It can easily be seen that if $x \in X_1(t_0)$, then $\Phi(t_1, t_0)x \in X_1(t_1)$, for all $t_1 \geq t_0 \geq 0$.

Therefore $y(t) \in X_1(t)$ and $z(t) \in X_2(t)$, so $w(t) \neq 0$, for all $t \geq 0$.

We also consider the functions $f, v : \mathbb{R}_+ \to X$,

$$f(t) = \varphi_{[t_0, t_0+\Delta]}(t)\frac{w(t)}{||w(t)||} \text{ and}$$
$$v(t) = \int_0^t \varphi_{[t_0, t_0+\Delta](\tau)} \tfrac{d\tau}{||w(\tau)||} y(t) - \int_t^\infty \varphi_{[t_0, t_0+\Delta](\tau)} \tfrac{d\tau}{||w(\tau)||} z(t).$$

If $x_2 \in X_2(t_0) \setminus \{0\}$, then there exists $u_2 \in X_2(0) \setminus \{0\}$ such that $x_2 = \Phi(t_0, 0)u_2$.

If $t \geq t_0$, we have

$$v(t) = \int_0^t \varphi_{[t_0, t_0+\Delta]}(\tau)\frac{1}{||w(\tau)||} d\tau \frac{\Phi(t, t_0)x_1}{||x_1||}$$

$$-\Phi(t,0)\left[\int_0^\infty \varphi_{[t_0,t_0+\Delta]}(\tau)\frac{1}{||w(\tau)||}d\tau\frac{u_2}{||\Phi(t_0,0)u_2||}\right]$$
$$+\int_0^t \varphi_{[t_0,t_0+\Delta]}(\tau)\frac{1}{||w(\tau)||}d\tau\frac{\Phi(t,t_0)x_2}{||x_2||}$$
$$=\Phi(t,0)\left[-\int_0^\infty \varphi_{[t_0,t_0+\Delta]}(\tau)\frac{1}{||w(\tau)||}d\tau\frac{u_2}{||\Phi(t_0,0)u_2||}\right]$$
$$+\int_0^t \varphi_{[t_0,t_0+\Delta]}(\tau)\frac{1}{||w(\tau)||}d\tau\, w(t).$$

Therefore $v(t)=\Phi(t,0)v(0)+\int_0^t \Phi(t,\tau)f(\tau)d\tau$ for all $t\geq t_0$.

But $v(0)\in X_2(0)$ and $v\in L^\infty(X)$, so $v(t)=x_f(t)$, for all $t\geq t_0$. The function v is continuous, so by Theorem 3.1

$$||v(t)||\leq k||f||_1,\ \text{for all } t\geq t_0\geq 0.$$

If $t=t_0$ and since $||f||_1=\Delta$, the above inequality becomes

$$\frac{1}{\Delta}\int_{t_0}^{t_0+\Delta}\frac{d\tau}{||w(\tau)||}\leq k.$$

If $\Delta\to 0$, then

$$||w(t_0)||\geq\frac{1}{k},\ \text{for all } t_0\geq 0.$$

But $w(t_0)=\frac{x_1}{||x_1||}+\frac{x_2}{||x_2||}$, so $\gamma[X_1(t_0),X_2(t_0)]\geq\frac{1}{k}$, for all $t_0\geq 0$, where

$$\gamma[X_1(t_0),X_2(t_0)]=\inf_{x_1\in X_1(t_0),x_2\in X_2(t_0)}\left|\left|\frac{x_1}{||x_1||}+\frac{x_2}{||x_2||}\right|\right|.$$

We denote $P_1(t)=P(t)$ and $P_2(t)=I-P(t)$, for all $t\geq 0$.

We have that $\frac{2}{||P_i(t_0)||}\geq\gamma[X_1(t_0),X_2(t_0)]$, for all $t_0\geq 0,\ i=1,2$ ([9], Theorem 11.D, p. 8), so

$$||P_i(t_0)||\leq 2k,\ \text{for all } t_0\geq 0, i=1,2,$$

which proves that the functions $t\mapsto||P_i(t)||:\mathbb{R}_+\to\mathbb{R}_+$ are bounded, $i=1,2$.

Let $x\in X$ and $t\geq t_0\geq 0$. We have that

$$\begin{aligned}||P_i(t)x-P_i(t_0)x|| &= ||P_i(t)x-\Phi(t,t_0)P_i(t_0)x+\Phi(t,t_0)P_i(t_0)x-P_i(t_0)x||\\ &\leq ||P_i(t)||||x-\Phi(t,t_0)x||+||\Phi(t,t_0)P_i(t_0)x-P_i(t_0)x||,\end{aligned}$$

for all $t\geq t_0\geq 0$ and $x\in X,\ i=1,2$.

If $t \to t_0$, then

$$\lim_{\substack{t \to t_0 \\ t > t_0}} ||P_i(t)x - P_i(t_0)x|| = 0,$$

for all $x \in X$, so the functions $t \mapsto P_i(t)x : \mathbb{R}_+ \to X$ are righthand-side continuous, $i = 1, 2$.

If $t < t_0$, it follows that $\Phi(t_0, t) : X_2(t) \to X_2(t_0)$ is invertible.

We show that $t \mapsto \Phi^{-1}(t_0, t)x : [0, t_0] \to X$ is continuous on $[0, t_0]$, for all $x \in X_2(t_0)$.

Let $s \in [0, t_0]$, $(s_n)_n$ such that $\lim\limits_{n\to\infty} s_n = s$ and $x \in X_2(t_0)$. Then there exists $y \in X_2(0)$ such that $x = \Phi(t_0, 0)y$.

We have

$$\begin{aligned}\Phi^{-1}(t_0, s_n)x &= \Phi^{-1}(t_0, s_n)\Phi(t_0, 0)y \\ &= \Phi^{-1}(t_0, s_n)\Phi(t_0, s_n)\Phi(s_n, 0)y = \Phi(s_n, 0)y \xrightarrow[n\to\infty]{} \Phi(s, 0)y \\ &= \Phi(s, 0)\Phi^{-1}(t_0, 0)x = \Phi(s, 0)(\Phi(t_0, s)\Phi(s, 0))^{-1}x = \Phi^{-1}(t_0, s)x.\end{aligned}$$

This proves that $t \mapsto \Phi^{-1}(t_0, t)x : [0, t_0] \to X$ is continuous on $[0, t_0]$, for all $x \in X_2(t_0)$.

We show that

$$\Phi(t_0, t)P_i(t) = P_i(t_0)\Phi(t_0, t), \text{ for all } t_0 \geq t \geq 0, i = 1, 2.$$

It is sufficient to show that the relation above is true for $i = 2$. Let $x \in X$, since $P_2(t_0)x \in X_2(t_0)$, there exists $u \in X_2(0)$ such that $P_2(t_0)x = \Phi(t_0, 0)u$, so we have that

$$\begin{aligned}\Phi(t, t_0)P_2(t_0)x &= \Phi(t, t_0)\Phi(t_0, 0)u = \Phi(t, 0)u = P_2(t)\Phi(t, t_0)P_2(t_0)x \\ &= P_2(t)\Phi(t, t_0)x - P_2(t)\Phi(t, t_0)P_1(t_0)x,\end{aligned}$$

but $P_2(t)\Phi(t, t_0)P_1(t_0)x = 0$, for all $t \geq t_0 \geq 0$, so

$$\Phi(t, t_0)P_2(t_0)x = P_2(t)\Phi(t, t_0)x, \text{ for all } t \geq t_0 \geq 0, x \in X.$$

Therefore

$$P_i(t)x = \Phi^{-1}(t_0, t)P_i(t_0)\Phi(t_0, t)x \xrightarrow[\substack{t \to t_0 \\ t < t_0}]{} P_i(t_0)x,$$

for all $x \in X,\ i = 1, 2$.

This implies that $t \mapsto P_i(t)x : \mathbb{R}_+ \to X$ are lefthand-side continuous, $i = 1, 2$.

Thus the functions $t \mapsto P_i(t)x : \mathbb{R}_+ \to X$ are continuous, for all $x \in X,\ i = 1, 2$.

We can conclude now that the evolution family $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ is uniform dichotomic. Sufficiency.

Let $f:\mathbb{R}_+\to X,\ f\in L^1(X)$, and the function $y:\mathbb{R}_+\to X$,

$$y(t)=\int_0^t \Phi(t,\tau)P(\tau)f(\tau)d\tau-\int_t^\infty \Phi^{-1}(\tau,t)Q(\tau)f(\tau)d\tau,$$

where $P(t):\mathbb{R}_+\to\mathbb{B}(X)$ is the family of projectors compatible to the evolution family $\{\Phi(t,t_0)\}_{t\geq t_0\geq 0}$ and $Q(t)=I-P(t)$, for all $t\geq 0$. Obviously both integrals from the relation above are convergent. We denote $x=y(0)$ and so $x\in X_2(0)$.

We have

$$\begin{aligned}
x_f(t)&=\Phi(t,0)x+\int_0^t \Phi(t,\tau)f(\tau)d\tau\\
&=-\int_0^t \Phi(t,0)\Phi^{-1}(\tau,0)Q(\tau)f(\tau)d\tau-\int_t^\infty \Phi(t,0)\Phi^{-1}(\tau,0)Q(\tau)f(\tau)d\tau\\
&\quad+\int_0^t \Phi(t,\tau)f(\tau)d\tau\\
&=-\int_0^t \Phi(t,\tau)Q(\tau)f(\tau)d\tau-\int_t^\infty \Phi^{-1}(\tau,t)Q(\tau)f(\tau)d\tau+\int_0^t \Phi(t,\tau)f(\tau)d\tau\\
&=\int_0^t \Phi(t,\tau)P(\tau)f(\tau)d\tau-\int_t^\infty \Phi^{-1}(\tau,t)Q(\tau)f(\tau)d\tau=y(t),
\end{aligned}$$

for all $t\geq 0$.

So $x_f=y$. But

$$||y(t)||\leq N(\sup_{\tau\geq 0}||P(\tau)||+\sup_{\tau\geq 0}||Q(\tau)||)||f||_1<\infty,$$

for all $t\geq 0$. Therefore y is an element of $L^\infty(X)$. □

Remark 3.1 In the hypothesis of the theorem above it can easily be seen that the subspace $X_1(t_0)=\{x\in X:\Phi(\cdot,t_0)x\in L^\infty\}$ is actually $P(t_0)X$, for all $t_0\geq 0$, i.e. all the bounded solutions verify condition (i) in Definition 2.5.

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