Semigroup Forum (2015) 90:615–631
DOI 10.1007/s00233-014-9633-9

RESEARCH ARTICLE

Generation of analytic semigroups with generalized Wentzell boundary condition

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Received: 10 June 2014 / Accepted: 7 August 2014 / Published online: 25 September 2014

Abstract In this paper we study second order ordinary and partial differential equations with generalized Wentzell boundary condition. We prove, by a perturbation method, that certain second order ordinary differential operators generate an analytic semigroup in $W^{1,p}([0, 1])$ and the same result has been extended for the degenerate operator $Au(x) := x(1 - x)u''(x)$. Finally, we prove that certain linear partial differential operators of the second order generate analytic semigroups in the space of continuous functions.

Keywords Analytic semigroups · Generation of semigroups · Wentzell boundary conditions

1 Introduction

The study of ordinary and partial differential equations with Wentzell boundary conditions was stimulated by the theory of probability in the pioneering paper of Wentzell [17]. He considered a second order elliptic differential operator A in a domain Ω in $\mathbb{R}^n$,

Communicated by Jerome A. Goldstein.

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with sufficiently smooth boundary $\partial\Omega$, and he looked for the most general supplementary conditions which restrict the given operator A to be the infinitesimal generator of a semigroup corresponding to a Markov process in the domain. In [17] it is proved that, when the domain Ω has some special shape, the boundary conditions could involve the operator A. Generalized Wentzell boundary conditions contain, as special cases, Dirichlet, Neumann and Robin boundary conditions. In a paper by Feller [9] we find a diffusion process involving the operator

$$Au(x) := x(1-x)u''(x), \quad x \in (0,1), \tag{1.1}$$

with the boundary conditions

$$\lim_{x\to 0^+,\ x\to 1^-} Au(x) = 0. \tag{1.2}$$

In [3] it is proved that A with the domain

$$\mathcal{D}(A) = \{u \in C[0,1] \cap C^1(0,1) \ : \lim_{x\to 0^+,\ x\to 1^-} Au(x) = 0\} \tag{1.3}$$

is the generator of a C_0-contraction semigroup on $C[0,1]$. For this reason in this paper we will consider the operator $x(1-x)u''(x)$. Several motivations to study elliptic operators with Wentzell boundary conditions can also be found in [11,13–15].

In [7] generation results in suitable L^p spaces were proved. The present paper can be considered an extension of that work to the case $W^{1,p}$. We recall what has been proved in [7], introducing the problem

$$\begin{cases} \lambda u - \nabla\cdot(a\nabla u) = f, \ \text{in} \ \ \Omega, \\ \nabla\cdot(a\nabla u) + \beta\frac{\partial u}{\partial n} + \gamma u = 0, \ \ \text{on} \ \ \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded open set with smooth boundary $\partial\Omega$, $\dfrac{\partial u}{\partial n}$ is the outward normal derivative, $a(x)$ is a strictly positive C^2 function for all $x \in \overline{\Omega}$ and $\beta(x) > 0, \gamma(x) \geq 0$, for all $x \in \partial\Omega$.

We start by pointing out the surprising fact that the presence of the term $\beta\frac{\partial u}{\partial n}$ in the boundary conditions led in [7] to introduce suitable L^p weighted spaces to prove a generation result. More precisely the authors in [7] define $L^p(\overline{\Omega}, d\mu)$, where

$$d\mu = dx\Big|_{\Omega} \oplus \frac{a}{\beta}\ dS\Big|_{\Gamma};$$

here dx denotes the Lebesgue measure on Ω, while $(a/\beta)\ dS$ denotes the measure with weight a/β on Γ: it is assumed that $a > 0$ in $\Omega\cup\Gamma$, where $\emptyset \neq \Gamma \subset \partial\Omega$ and $a \in C^2(\Omega\cup\Gamma)\cap C\left(\overline{\Omega}\right)$.

Let $U = (u, v) \in L^p(\overline{\Omega}, d\mu)$, where $u : \Omega \to \mathbb{C}$ and $v : \partial\Omega \to \mathbb{C}$ are measurable functions such that

$$\int_{\Omega} |u(x)|^p \, dx + \int_{\partial\Omega} |v(x)|^p \, \frac{a}{\beta} \, dS$$

is finite. With the norm

$$\|U\|_{L^p(\overline{\Omega}, d\mu)} := \left[\int_{\Omega} |u(x)|^p \, dx + \int_{\partial\Omega} |v(x)|^p \, \frac{a}{\beta} \, dS \right]^{1/p},$$

$L^p(\overline{\Omega}, d\mu)$ becomes a Banach space. Moreover, the space $L^p(\overline{\Omega}, d\mu)$ can be identified with

$$L^p(\Omega, dx) \times L^p\left(\Gamma, \frac{a}{\beta} \, dS\right).$$

This space can be defined in an equivalent way: consider $u \in C\left(\overline{\Omega}\right)$, set $U = (u|_{\Omega}, u|_{\partial\Omega})$ and define $X_p(\overline{\Omega})$ to be the completion of $C(\overline{\Omega})$ with the norm $\|U\|_{L^p(\overline{\Omega}, d\mu)}$. It can be shown that

$$X_p\left(\overline{\Omega}\right) = L^p(\overline{\Omega}, d\mu).$$

In the spaces $C\left(\overline{\Omega}\right)$ the trace of a function has a clear meaning. We recall, for the sake of completeness, the main generation theorem in the space $X_p\left(\overline{\Omega}\right)$ proved in [7] (see Theorem 3.1).

Theorem 1.1 *Let Ω be an open bounded set in $\mathbb{R}^n$ with boundary of class C^2. Let $a \in C^1(\overline{\Omega})$ with $a > 0$ in $\overline{\Omega}$, $Au := \nabla \cdot (a \nabla u)$, $\Gamma := \{x \in \partial\Omega : a(x) > 0\} \neq \emptyset$. If β and γ are non negative functions in $C^1(\partial\Omega)$ with $\beta > 0$, then $\overline{G}$, the closure of the operator*

$$G = \begin{bmatrix} A & 0 \\ -\beta \frac{\partial}{\partial n} & -\gamma \end{bmatrix} \tag{1.5}$$

with domain

$$D_p(G) := \Big\{ u \in C^2(\Omega \cup \Gamma) \cap C(\overline{\Omega}) : \; Au \in L^p(\Omega, dx), \quad Au + \beta \frac{\partial u}{\partial n} + \gamma u = 0, \quad on \;\; \Gamma \Big\}$$

generates a (C_0) contraction semigroup on X_p, for $p \in [1, \infty)$. The semigroup is analytic if $p \in (1, \infty)$.

This result was significally extended in [6] and [4].

We are able to generalize the above result for ordinary differential operators in the space $W^{1,p}([0, 1])$ with Wentzell boundary conditions also for a degenerate case. The plan of the paper is the following.

- In Sect. 2 we give a generation result in $W^{1,p}([0,1])$ with Wentzell boundary conditions.
- In Sect. 3 we establish a generation result in $C^1([0,1])$.
- In Sect. 4 we prove a generation result for a degenerate operator in $W^{1,p}([0,1])$ with Wentzell boundary conditions.
- In Sect. 5 we consider a linear partial differential operator of the second order and prove a generation result in $C(\overline{\Omega})$ with Wentzell boundary conditions.

We set $I = [0,1]$ and, in the following, we shall always suppose that $a, b, c \in C^\infty(I)$ and, for every $x \in I$, $a(x) \geq a_0 > 0$.

Set

$$\left(A^\sharp u\right)(x) := a(x)u'' + b(x)u' + c(x)u, \qquad x \in [0,1],$$

We make the following hypothesis:

(H1) for every $c_0, c_1 \in \mathbf{C}$, there exists a unique solution u of the boundary value problem

$$A^\sharp u = 0, \quad u(0) = c_0, \quad u(1) = c_1.$$

Then, if B_0, B_1 are boundary operators of order not greater than one, the mapping $(A^\sharp, B_0, B_1)$ is an isomorphism from $W^{3,p}([0,1])$ onto $W^{1,p}([0,1]) \times \mathbb{C} \times \mathbb{C}$, by e.g. [16], Theorem 5.5.2.

2 Generation results in $W^{1,p}([0,1])$ with Wentzell boundary conditions

Theorem 2.1 *Define, for $p > 1$, the operator*

$$\begin{cases} \mathcal{D}(A) = \{u \in W^{1,p}(I) \ : \ au'' + bu' + cu \in W_0^{1,p}(I)\}, \\ Au := au'' + bu' + cu, \quad u \in \mathcal{D}(A), \end{cases} \tag{2.1}$$

and suppose that A satisfies assumption $(H1)$. Then $(A, \mathcal{D}(A))$ generates an analytic semigroup in $W^{1,p}(I)$.

Proof If $\lambda \in \mathbb{C} \setminus \{0\}$ and $f \in W^{1,p}(I)$, consider the boundary problem

$$\begin{cases} \lambda u(x) - Au(x) = f(x), \qquad x \in I, \\ Au(j) = 0, \quad j = 0, 1, \end{cases} \tag{2.2}$$

and the auxiliary problem

$$\begin{cases} AG(x) = 0, \qquad x \in I, \\ G(j) = f(j), \qquad j = 0, 1. \end{cases} \tag{2.3}$$

By hypothesis $(H1)$, problem (2.3) has a unique solution $G \in W^{3,p}(I)$. Then we can rewrite Problem (2.2) as

$$\begin{cases} \lambda\Big(u(x) - \frac{G(x)}{\lambda}\Big) - A\Big(u(x) - \frac{G(x)}{\lambda}\Big) = f(x) - G(x), & x \in I, \\ u(j) - \frac{G(j)}{\lambda} = 0, \quad A\Big(u - \frac{G}{\lambda}\Big)(j) = 0, & j = 0, 1. \end{cases} \tag{2.4}$$

Since $f \in W^{1,p}(I)$ we have $f - G \in W_0^{1,p}(I)$, and the operator

$$\begin{cases} \mathcal{D}(A_0) = \{u \in W_0^{1,p}(I) \ : \ au'' + bu' + cu \in W_0^{1,p}(I)\}, \\ A_0 u := au'' + bu' + cu, \quad u \in \mathcal{D}(A_0), \end{cases} \tag{2.5}$$

generates an analytic semigroup in $W_0^{1,p}(I)$, by [1]. So we have proved that Problem (2.4) has a unique solution $v := u - G/\lambda \in \mathcal{D}(A_0)$ and the estimate

$$\|v\|_{W_0^{1,p}(I)} \le \frac{c_p}{|\lambda|}\|f - G\|_{W_0^{1,p}(I)} \tag{2.6}$$

holds, where c_p is a positive real constant independent of λ. Moreover, for Problem (2.3) we have

$$\|G\|_{W^{1,p}(I)} \le K_p \ \|f\|_{W^{1,p}(I)} \tag{2.7}$$

where K_p is a positive real constant. Then we get the generation estimate

$$\begin{aligned} \|u\|_{W^{1,p}(I)} &\le \left\| u - \frac{G}{\lambda}\right\|_{W^{1,p}(I)} + \left\|\frac{G}{\lambda}\right\|_{W^{1,p}(I)} \le \frac{c_p}{|\lambda|} \|f - G\|_{W^{1,p}(I)} \\ &\quad + \frac{K_p}{|\lambda|} \|f\|_{W^{1,p}(I)} . \end{aligned} \tag{2.8}$$

□

Theorem 2.2 *Define, for* $p > 1$*, the operator*

$$\begin{cases} \mathcal{D}(A_0) = \{u \in W^{1,p}(I) \ : \ au'' \in W^{1,p}(I), \ \ au'' + bu' + cu \in W_0^{1,p}(I)\}, \\ A_0 u := au'', \quad u \in \mathcal{D}(A_0). \end{cases} \tag{2.9}$$

Then $(A_0, \mathcal{D}(A_0))$ *generates an analytic semigroup in* $W^{1,p}(I)$.

Proof The proof is based on a perturbation method and on Theorem 2.1. Consider the operator B defined on $W^{1,p}(I)$ as follows:

$$\begin{cases} \mathcal{D}(B) = \mathcal{D}(A_0), \\ Bu := -bu' - cu, \quad \forall u \in \mathcal{D}(B). \end{cases} \tag{2.10}$$

Consider the chain of inequalities, for $\varepsilon > 0$:

$$\begin{aligned} \|Bu\|_{L^p(I)} &\le K_0\|u'\|_{L^p(I)} + M_0\|u\|_{L^p(I)} \le \frac{K_0\varepsilon}{a_0}\|au''\|_{L^p(I)} + M_\varepsilon\|u\|_{L^p(I)} \\ &\quad + M_0\|u\|_{L^p(I)} \end{aligned}$$

$$\leq \frac{K_0\varepsilon}{a_0}\|au''+bu'+cu\|_{L^p(I)}+\frac{K_0\varepsilon}{a_0}\|bu'+cu\|_{L^p(I)}+M_\varepsilon\|u\|_{L^p(I)}$$
$$+K_0\|u\|_{L^p(I)}$$
$$=\frac{K_0\varepsilon}{a_0}\|Au\|_{L^p(I)}+\frac{K_0\varepsilon}{a_0}\|Bu\|_{L^p(I)}+(M_\varepsilon+K_0)\,\|u\|_{L^p(I)}, \tag{2.11}$$

where, here and in the sequel, K_i, M_i, M_ε, M'_ε and M''_ε denote positive constants. Furthermore,

$$\begin{aligned}
\|(Bu)'\|_{L^p(I)} &\leq K_0\|u''\|_{L^p(I)}+\|(b'+c)u'\|_{L^p(I)}+\|c'u\|_{L^p(I)}\\
&\leq K_0\;\varepsilon\;\|u'''\|_{L^p(I)}+M_\varepsilon\|u'\|_{L^p(I)}+M_1\|u'\|_{L^p(I)}+M_2\|u\|_{L^p(I)}\\
&= K_0\;\varepsilon\;\|u'''\|_{L^p(I)}+M_\varepsilon\|u\|_{W^{1,p}(I)}\\
&\leq \frac{K_0}{a_0}\;\varepsilon\;\|(au'')'-a'u''\|_{L^p(I)}+M'_\varepsilon\|u\|_{W^{1,p}(I)}\\
&\leq \frac{K_0}{a_0}\;\varepsilon\;\|(au''+bu'+cu)'-(bu'+cu)'\|_{L^p(I)}+\frac{K_1\varepsilon}{a_0}\|u''\|_{L^p(I)}\\
&\quad+M'_\varepsilon\|u\|_{W^{1,p}(I)}\\
&\leq \frac{K_0}{a_0}\;\varepsilon\;\left(\|Au\|_{W^{1,p}(I)}+\|(Bu)'\|_{L^p(I)}\right)\\
&\quad+\frac{K_1}{a_0^2}\;\varepsilon\left(\|Au\|_{L^p(I)}+\|Bu\|_{L^p(I)}\right)+M'_\varepsilon\|u\|_{W^{1,p}(I)}.
\end{aligned} \tag{2.12}$$

Then, from (2.11)–(2.12), we get the estimates

$$\left(1-\frac{K_0\varepsilon}{a_0}\right)\|(Bu)'\|_{L^p(I)}\leq M_1\varepsilon\|Au\|_{W^{1,p}(I)}+M'_\varepsilon\|u\|_{W^{1,p}(I)}, \tag{2.13}$$

$$\left(1-\frac{K_0\varepsilon}{a_0}\right)\|Bu\|_{L^p(I)}\leq \frac{K_0\varepsilon}{a_0}\|Au\|_{L^p(I)}+M''_\varepsilon\|u\|_{L^p(I)}. \tag{2.14}$$

By taking ε sufficiently small we get the desired estimate:

$$\|Bu\|_{W^{1,p}(I)}\leq M\varepsilon\|Au\|_{W^{1,p}(I)}+M'''_\varepsilon\|u\|_{W^{1,p}(I)}. \tag{2.15}$$

Since A generates an analytic semigroup in $W^{1,p}(I)$, thanks to Theorem 2.1, and B is A-bounded with A-bound equal to zero, we conclude that $A_0=A+B$ generates an analytic semigroup as well. □

Remark The above result should be compared with Warma [18].

More generally, we have the following result.

Theorem 2.3 *Let A_1 be the operator in $W^{1,p}(I)$, with $p>1$,*

$$\begin{cases} \mathcal{D}(A_1)=\{u\in W^{1,p}(I)\;:\;au''+b_1u'+c_1u\in W_0^{1,p}(I)\},\\ A_1u:=au''+bu'+cu,\quad \forall u\in\mathcal{D}(A_1), \end{cases} \tag{2.16}$$

where $b_1, c_1 \in C^{\infty}(I)$. *Then* $(A_1, D(A_1))$ *generates an analytic semigroup in* $W^{1,p}([0,1])$.

Proof Define the operator C in $W^{1,p}([0,1])$ by

$$\mathcal{D}(C) = \mathcal{D}(A), \qquad Cu = (b_1 - b)\, u' + (c_1 - c)\, u. \tag{2.17}$$

Then $A_1 u = Au + Cu$, where A is the operator defined in Theorem 2.1.

It is well known (see, e.g., [5], Example III.2.2) that the operator B, defined by

$$\mathcal{D}(B) = W^{1,p}(I), \qquad Bu = u',$$

is $\mathcal{A}$-bounded, with $\mathcal{A}$-bound equal to 0, where

$$\mathcal{D}(\mathcal{A}) = W^{2,p}(I), \qquad \mathcal{A}u = u''.$$

Therefore, for every $\varepsilon \in \mathbf{R}^+$, we have

$$\begin{aligned}
\|Cu\|_{L^p(I)} &\le K_1 \left(\|u'\|_{L^p(I)} + \|u\|_{L^p(I)}\right) \\
&\le K_1 \left(\varepsilon \|u''\|_{L^p(I)} + M_\varepsilon \|u\|_{L^p(I)} + \|u\|_{L^p(I)}\right) \\
&\le K_1 a_0^{-1}\, \varepsilon\, \|au''\|_{L^p(I)} + K_2 M_\varepsilon \|u\|_{L^p(I)} \\
&\le K_1 a_0^{-1}\, \varepsilon\, \|au'' + bu' + cu\|_{L^p(I)} + K_1 a_0^{-1}\, \varepsilon\, \|bu'\|_{L^p(I)} \\
&\quad + K_1 a_0^{-1}\, \varepsilon\, \|cu\|_{L^p(I)} + K_2 M_\varepsilon \|u\|_{L^p(I)} \\
&\le K_1 a_0^{-1}\, \varepsilon\, \|Au\|_{L^p(I)} + M'_\varepsilon \|u\|_{W^{1,p}(I)} \le K_1 a_0^{-1}\, \varepsilon \|Au\|_{W^{1,p}(I)} \\
&\quad + M'_\varepsilon \|u\|_{W^{1,p}(I)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|(Cu)'\|_{L^p(I)} &\le \|Cu\|_{W^{1,p}(I)} \le K_3 \left(\|u'\|_{W^{1,p}(I)} + \|u\|_{W^{1,p}(I)}\right) \\
&\le K_3 \varepsilon \|u''\|_{W^{1,p}(I)} + M''_\varepsilon \|u\|_{W^{1,p}(I)} \\
&\le K_3 a_0^{-1} \varepsilon \left(\|au'' + bu' + cu\|_{W^{1,p}(I)} + \|bu' + cu\|_{W^{1,p}(I)}\right) \\
&\qquad + M''_\varepsilon \|u\|_{W^{1,p}(I)} \\
&\le K_3 a_0^{-1} \varepsilon \|Au\|_{W^{1,p}(I)} + K_4 a_0^{-1} \varepsilon \|u'\|_{W^{1,p}(I)} \\
&\qquad + \left(K_5 a_0^{-1} \varepsilon + M''_\varepsilon\right) \|u\|_{W^{1,p}(I)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(K_3 - K_4 a_0^{-1} \varepsilon\right) \|u'\|_{W^{1,p}(I)} + \left(K_3 - K_5 a_0^{-1} \varepsilon\right) \|u\|_{W^{1,p}(I)} \\
&\quad \le K_3 a_0^{-1} \varepsilon \|Au\|_{W^{1,p}(I)} + M'_\varepsilon \|u\|_{W^{1,p}(I)}.
\end{aligned}$$

Taking ε sufficiently small, we get

$$\|u\|_{W^{1,p}(I)} \leq K_6\varepsilon\|Au\|_{W^{1,p}(I)} + M_\varepsilon'''\|u\|_{W^{1,p}(I)},$$
$$\|u'\|_{W^{1,p}(I)} \leq K_6\varepsilon\|Au\|_{W^{1,p}(I)} + M_\varepsilon'''\|u\|_{W^{1,p}(I)}$$

whence

$$\|(Cu)'\|_{L^p(I)} \leq K_7\varepsilon\|Au\|_{W^{1,p}(I)} + M_\varepsilon'''\|u\|_{W^{1,p}(I)}.$$

Then,

$$\|Cu\|_{W^{1,p}(I)} \leq \varepsilon\|Au\|_{W^{1,p}(I)} + K_\varepsilon\|u\|_{W^{1,p}(I)}.$$

This proves that C is A-bounded, with A-bound equal to 0 and hence $A_1 = A + C$ generates an analytic semigroup in $W^{1,p}(I)$. □

3 A generation result in $C^1(I)$

In the monograph [12] it is proved the generation theorem of an analytic semigroup in $C_0^1\left(\overline{\Omega}\right)$, where Ω is a bounded open subset of R^N with smooth boundary. Here

$$C_0^1\left(\overline{\Omega}\right) = \left\{u \in C^1\left(\overline{\Omega}\right); u_{/\partial\Omega} = 0\right\}.$$

Let $\mathcal{L}(\cdot, D)$ denote a linear second order partial differential operator in $\overline{\Omega}$ with smooth coefficients, whose leading part satifies the uniform ellipticity condition

$$\forall x \in \overline{\Omega}, \forall z \in \mathbb{C}^n, \quad \sum_{i,j=1}^{N} a_{ij}(x) z_i \overline{z_j} \geq \alpha_0 |z|^2,$$

and set

$$\begin{cases} \mathcal{D}(A_N) = \left\{u \in \mathcal{D}(A_{00}) : A_N u \in C_0^1\left(\overline{\Omega}\right)\right\}, \\ A_N u := \mathcal{L}(\cdot, D)u. \end{cases} \tag{3.1}$$

Here

$$\mathcal{D}(A_{00}) = \left\{u \in \bigcap_{p=1}^{\infty} W_{loc}^{2,p}(\Omega) : u,\ \mathcal{L}(\cdot, D)u \in C\left(\overline{\Omega}\right)\right\}.$$

Notice that, if $N = 1$, then $\mathcal{D}(A_{00}) = \left\{u \in C^2\left(\overline{\Omega}\right) : u_{/\partial\Omega} = 0\right\}$.

It is known (see [12], Theorem 3.1.25), that A_N is sectorial in $C_0^1\left(\overline{\Omega}\right)$.

We confine to $N = 1$ and define the operator E by

$$\begin{cases} \mathcal{D}(E) = \left\{u \in C_0^1(I) \ : \alpha u'' \in C_0^1(I)\right\}, \\ Eu := \alpha u'', \end{cases} \tag{3.2}$$

where $\alpha \in C^1(I)$, with $\alpha(x) \geq \alpha_0 > 0$ in I.

Theorem 3.1 *Under the previous assumptions, the operator E is sectorial in $C^1(I)$.*

Proof Recall that the operator A_1 defined above is sectorial in $C_0^1(I)$. Then, if $f \in C^1(I)$, for any $\lambda \in \mathbb{C}$, with Re λ large enough, there exists a unique $u \in \mathcal{D}(A_1)$, such that

$$\begin{aligned} &\lambda u(x) - \alpha u''(x) = f(x) - (1-x)f(0) - xf(1), \\ &\|u\|_{C_0^1(I)} \leq C|\lambda|^{-1}\|f(x) - (1-x)f(0) - xf(1)\|_{C_0^1(I)}. \end{aligned} \tag{3.3}$$

Observe that Eq. (3.3) reads

$$\begin{aligned} &\lambda \left(u(x) + \lambda^{-1}((1-x)f(0) + xf(1))\right) \\ &\quad -\alpha(x)\left(u(x) + \lambda^{-1}((1-x)f(0) + xf(1))\right)'' = f(x). \end{aligned}$$

Set $v(x) = u(x) + \lambda^{-1}((1-x)f(0) + xf(1))$; then $v \in C^2(I)$ and $\alpha v''(= \alpha u'') \in C_0^1(I)$. Moreover,

$$\begin{aligned} \|v\|_{C^1(I)} &\leq \|u\|_{C^1(I)} + 2|\lambda|^{-1}\|f\|_{C(I)} \\ &\leq K|\lambda|^{-1}\|f\|_{C(I)} + 2|\lambda|^{-1}\|f\|_{C(I)} = K_1|\lambda|^{-1}\|f\|_{C(I)}. \end{aligned}$$

On the other hand, if $\lambda v_1 - Ev_1 = f = \lambda v_2 - Ev_2$, then $\lambda(v_1 - v_2) - E(v_1 - v_2) = 0$, and hence $\alpha(v_1 - v_2)''_{/\partial[0,1]} = 0$, so that $(v_1 - v_2)(0) = (v_1 - v_2)(1) = 0$.
Then, $v_1 - v_2 \in \mathcal{D}(A_1)$ and

$$\lambda(v_1 - v_2) - A_1(v_1 - v_2) = 0.$$

It follows that $v_1 = v_2$ and uniqueness of solutions follows. This implies that E is sectorial in $C^1([0, 1])$, as required. □

4 A generation result for a degenerate operator in $W^{1,p}([0, 1])$ with Wentzell boundary condition

In this section we show that a second order degenerate operator in I generates an analytic semigroup in $W^{1,p}(I)$, where $1 < p < \infty$. This result must be compared to Theorem 7.9, in the monograph [8]. Here we give an alternative proof of an important case of large interest.

Recall that the norms

$$\left(\int_0^1 |u(x)|^p\,dx\right)^{1/p} + \left(\int_0^1 |u'(x)|^p\,dx\right)^{1/p} \quad \text{and} \quad \left(\int_0^1 |u'(x)|^p\,dx\right)^{1/p}$$

are equivalent in $W_0^{1,p}(I)$, and define the operator:

$$\begin{cases} \mathcal{D}\left(A_{(p)}\right) = \{u \in W_0^{1,p}(I);\ x(1-x)u'' \in W_0^{1,p}(I)\}, \\ A_{(p)}u = x(1-x)u'', \quad u \in \mathcal{D}\left(A_{(p)}\right). \end{cases} \tag{4.1}$$

We prove the following generation result.

Theorem 4.1 *The operator* $\left(A_{(p)}, \mathcal{D}\left(A_{(p)}\right)\right)$ *defined in* (4.1) *generates an analytic semigroup in* $W_0^{1,p}(I)$*, for* $1 < p < \infty$.

Proof Consider the resolvent equation for $\lambda \in \mathbb{C}$, such that $\mathrm{Re}\lambda > 0$, and $f \in W_0^{1,p}(I)$,

$$\lambda u(x) - A_{(p)}u(x) = f(x),$$

so that

$$\frac{\lambda u(x)}{x(1-x)} - u''(x) = \frac{f(x)}{x(1-x)}.$$

Note that $f \in W_0^{1,p}(I)$ implies

$$\int_0^1 \left|\frac{f(x)}{x(1-x)}\right|^p dx \le c\|f\|^p_{W_0^{1,p}(I)},$$

by the Hardy inequality. On the other hand, if $x(1-x)u'' = g \in W_0^{1,p}(I)$, then

$$|g(x)| \le x^{1/p'}\|g\|_{L^p(I)}, \quad |g(x)| \le (1-x)^{1/p'}\|g\|_{L^p(I)}$$

so that

$$u'(x) - u'(y) = \int_y^x u''(t)\,dt = \int_y^x \frac{g(t)}{t(1-t)}\,dt$$

and in a neighborhood of zero for $y < x$ we have

$$|u'(x) - u'(y)| \le \int_y^x \frac{|g(t)|}{t(1-t)}\,dt \le c\int_y^x t^{-(1-1/p')}\,dt \le c(x-y)^{1-1/p},$$

where c, here and in the sequel, denotes a real positive constant. Analogously one has the same bound near 1. Hence, from

$$\lambda \int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx - \int_0^1 u''(x)\,\overline{u(x)}\,|u(x)|^{p-2}\,dx = \int_0^1 \frac{f(x)\,\overline{u(x)}}{x(1-x)}\,|u(x)|^{p-2}\,dx$$

by integration by parts, one obtains, using the boundary conditions, if $p \geq 2$:

$$\begin{aligned}&\lambda \int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx + \int_0^1 |u'(x)|^2\,|u(x)|^{p-2}\,dx\\ &+ (p-2)\int_0^1 u'(x)\overline{u(x)}(\mathrm{Re}(u'(x)\overline{u(x)}))\,|u(x)|^{p-4}\,dx\\ &\leq \int_0^1 \frac{f(x)\,\overline{u(x)}}{x(1-x)}\,|u(x)|^{p-2}\,dx.\end{aligned}$$

We consider the case $p \geq 2$ to show the strategy to obtain suitable estimates for our problem. Attention must be devoted to the case $1 < p < 2$ that follows with a few modifications of the previous case. Taking real and imaginary parts and integrating by parts yields

$$\begin{aligned}&(\mathrm{Re}\,\lambda)\int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx + \int_0^1 |u'(x)|^2|u(x)|^{p-2}\,dx\\ &+ (p-2)\int_0^1 \left(\mathrm{Re}(u'(x)\overline{u(x)})\right)^2\,|u(x)|^{p-4}\,dx\\ &= \mathrm{Re}\int_0^1 \frac{f(x)\,\overline{u(x)}}{x(1-x)}\,|u(x)|^{p-2}\,dx\end{aligned} \tag{4.2}$$

and

$$(\mathrm{Im}\,\lambda)\int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx = \mathrm{Im}\int_0^1 \frac{f(x)\,\overline{u}(x)}{x(1-x)}\,|u(x)|^{p-2}\,dx. \tag{4.3}$$

Then we get, for any $\varepsilon > 0$,

$$\begin{aligned}|\mathrm{Im}\,\lambda|\int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx &\leq (p-2)\int_0^1 \left|\mathrm{Im}(u'(x)\overline{u(x)})\right|\,\left|\mathrm{Re}(u'(x)\overline{u(x)})\right|\,|u(x)|^{p-4}\,dx\\ &\quad+\left|\int_0^1 \frac{f(x)\,\overline{u(x)}}{x(1-x)}\,|u(x)|^{p-2}\,dx\right|\\ &\leq (p-2)\left(\varepsilon\int_0^1 \left|\mathrm{Im}\left(u'(x)\overline{u(x)}\right)\right|^2\,|u(x)|^{p-4}\,dx\right.\\ &\quad\left.+\frac{1}{\varepsilon}\int_0^1 \left|\mathrm{Re}\left(u'(x)\overline{u(x)}\right)\right|^2\,|u(x)|^{p-4}\,dx\right)\\ &\quad+\left|\int_0^1 \frac{f(x)\,\overline{u(x)}}{x(1-x)}\,|u(x)|^{p-2}\,dx\right|.\end{aligned}$$

Let $\eta > 0$ be an arbitrary real number. Multiplying the last inequality by η and adding it to Eq. (4.2), one gets

$$
\begin{aligned}
&(\mathrm{Re}\lambda + \eta|\mathrm{Im}\lambda|) \int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx \\
&\quad + \left(p-1-(p-2)\frac{\eta}{\varepsilon}\right)\int_0^1 \left(\mathrm{Re}\left(u'(x)\overline{u(x)}\right)\right)^2 |u(x)|^{p-4}\,dx \\
&\quad + (1-(p-2)\varepsilon\eta)\int_0^1 \left(\mathrm{Im}\left(u'(x)\overline{u(x)}\right)\right)^2 |u(x)|^{p-4}\,dx \\
&\quad \le \left|\mathrm{Re}\int_0^1 \frac{f(x)\,\overline{u(x)}}{x(1-x)}\,|u(x)|^{p-2}\,dx\right| \\
&\quad + \eta\left|\mathrm{Im}\int_0^1 \frac{f(x)\,\overline{u(x)}}{x(1-x)}\,|u(x)|^{p-2}\,dx\right|.
\end{aligned}
$$

Since η and ε are arbitrary positive constants, we can choose them so that

$$\eta < \varepsilon, \quad \varepsilon^2(p-2) < 1\,;$$

this assures that $p-1-(p-2)\eta/\varepsilon > 0$ and that $1-(p-2)\varepsilon\eta > 0$; then in the sector $\Sigma := \{\lambda \in \mathbb{C} \ : \ \mathrm{Re}\lambda + \eta|\mathrm{Im}\lambda| \ge \delta_0 > 0\,\}$, we have the estimates

$$
\begin{aligned}
&\int_0^1 \left(\mathrm{Re}\left(u'(x)\overline{u(x)}\right)\right)^2 |u(x)|^{p-4}\,dx \le c\|f\|_{W_0^{1,p}(I)}\|u\|_{L^p(I)}^{p-1}, \\
&\int_0^1 \left(\mathrm{Im}\left(u'(x)\overline{u(x)}\right)\right)^2 |u(x)|^{p-4}\,dx \le c\|f\|_{W_0^{1,p}(I)}\|u\|_{L^p(I)}^{p-1}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mathrm{Re}\lambda \int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx &\le c\|f\|_{W_0^{1,p}(I)}\|u\|_{L^p(I)}^{p-1}, \\
|\mathrm{Im}\lambda| \int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx &\le c\|f\|_{W_0^{1,p}(I)}\|u\|_{L^p(I)}^{p-1}.
\end{aligned}
$$

Therefore

$$(\mathrm{Re}\lambda + |\mathrm{Im}\lambda|)\,\|u\|_{L^p(I)}^p \le (\mathrm{Re}\lambda + |\mathrm{Im}\lambda|)\int_0^1 \frac{|u(x)|^p}{x(1-x)}\,dx \le c\|f\|_{W_0^{1,p}(I)}\|u\|_{L^p(I)}^{p-1},$$

which gives

$$(\mathrm{Re}\lambda + |\mathrm{Im}\lambda|)\,\|u\|_{L^p(I)} \le c\|f\|_{W_0^{1,p}(I)}.$$

We need now to estimate $\|u'\|_{L^p(I)}$. Consider

$$\lambda u'(x) - [x(1-x)u''(x)]' = f'(x)$$

and multiply it by $\overline{u'(x)}|u'(x)|^{p-2}$, so that

$$\begin{aligned}&\lambda\int_0^1 |u'(x)|^p\,dx - \left[x(1-x)u''(x)\overline{u'(x)}|u'(x)|^{p-2}\right]_0^1\\&+\int_0^1 x(1-x)|u''(x)|^2\,|u'(x)|^{p-2}\,dx\\&+(p-2)\int_0^1 x(1-x)\overline{u'(x)}\,u''(x)\,\mathrm{Re}\left(\overline{u'(x)}\,u''(x)\right)\,|u'(x)|^{p-4}\,dx\\&=\int_0^1 f(x)\,\overline{u'(x)}\,|u'(x)|^{p-2}\,dx.\end{aligned}$$

As $x(1-x)u''(x)$ vanishes on the boundary and $u'(x)$ has a finite limit as $x \to 0$ or $x \to 1$, we obtain

$$\left[x(1-x)u''(x)\overline{u}'(x)|u'(x)|^{p-2}\right]_0^1 = 0.$$

By repeating for u' the arguments above for u we get the estimate, if $\lambda \in \Sigma$,

$$|\lambda|\|u'\|_{L^p(I)} \le c\|f'\|_{L^p(I)}\,;$$

so we have the bound, if $\lambda \in \Sigma$,

$$|\lambda|\|u\|_{W_0^{1,p}(I)} \le c\|f'\|_{W_0^{1,p}(I)}.$$

Take now $f \in W_0^{1,p}(I)$, so that $f' \in L^p(I)$ and consider the problem

$$\begin{cases}\lambda v(x) - (x(1-x)v'(x))' = f'(x), & x \in I,\\ x(1-x)v'(x) \in W_0^{1,p}(I),\end{cases} \tag{4.4}$$

for $\mathrm{Re}\lambda > 0$. It is known, from the paper [2], that it has a unique solution such that

$$|\lambda|\|v\|_{L^p(I)} \le c\|f'\|_{L^p(I)}.$$

Integrating over $(0, x)$ we obtain

$$\lambda\int_0^x v(t)\,dt - x(1-x)v'(x) = f(x).$$

Set

$$u(x) = \int_0^x v(t)\,dt\,;$$

then

$$\lambda u(x) - x(1-x)u''(x) = f(x), \quad 0 \le x \le 1,$$

so u is the desired solution to our problem. Therefore $A_{(p)}$ generates an analytic semigroup in $W_0^{1,p}(I)$. □

We now define the operator W, for $1 < p < \infty$,

$$\begin{cases} \mathcal{D}(W) = \{u \in W^{1,p}(I) \ : \ x(1-x)u'' \in W_0^{1,p}(I)\ \}, \\ Wu := x(1-x)u'', \quad \forall\, u \in \mathcal{D}(W). \end{cases} \tag{4.5}$$

Theorem 4.2 *The operator* $(W, \mathcal{D}(W))$ *generates an analytic semigroup in* $W^{1,p}(I)$.

Proof We only need to observe, see Theorem 2.1, that

$$x(1-x)u''(x) = 0, \quad x \in I,$$

and

$$u(0) = u_0, \ \ u(1) = u_1$$

has a unique solution in $C^\infty([0,1])$. □

5 A generation result in $C(\overline{\Omega})$ with Wentzell boundary conditions

In this case we have a more general result. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. Let us assume the conditions

- $(K1)$ a_{jk}, b_k, d are real $C^\infty(\overline{\Omega})$ functions, with $a_{jk} = a_{kj}$, $k, j = 1, ..., n$, and $d(x) < 0$;
- (K2) $\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c\sum_{j=1}^k \xi_j^2, \quad \forall x \in \overline{\Omega}$ and $(\xi_1, ..., \xi_n) \in \mathbb{R}^n$; here $c > 0$.

Let us define the elliptic operator :

$$Lu(x) = \sum_{j,k=1}^n \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u(x)}{\partial x_k}\right) + \sum_{k=1}^n b_k(x)\frac{\partial u(x)}{\partial x_k} + d(x)u(x) \tag{5.1}$$

with domain

$$\mathcal{D}(L) = \left\{ u \in \bigcap_{p>1} W^{2,p}_{\mathrm{loc}}(\Omega) \ : \ u,\ Lu \in C(\overline{\Omega}) \right\}. \tag{5.2}$$

We recall the following well known theorem (see [10] Corollary 9.18).

Theorem 5.1 *Let* L *be the operator defined in* (5.1)–(5.2)*, satisfying assumptions* $(K1)$ *and* $(K2)$. *Then, for any* $\phi \in C(\Gamma)$*, the boundary value problem*

$$\begin{cases} LG(x) = 0, \quad x \in \Omega, \\ G(x) = \phi(x), \quad x \in \Gamma, \end{cases} \tag{5.3}$$

has a unique solution $G \in \mathcal{D}(L)$ and there exists a positive constant M such that

$$\|G\|_{C(\overline{\Omega})} \leq M\|\phi\|_{C(\Gamma)}. \tag{5.4}$$

In this section we will use the functional space

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \ : \ u|_\Gamma = 0\} \tag{5.5}$$

with its natural norm. Thanks to Corollary 3.1.21 (ii) in Lunardi [12] we observe that operator

$$\begin{cases} \mathcal{D}(L_0) = \left\{u \in \bigcap_{p>1} W^{2,p}_{\mathrm{loc}}(\Omega) \ : \ u, Lu \in C(\overline{\Omega}), \ \ u|_\Gamma = 0\right\}, \\ L_0 u = Lu, \quad u \in \mathcal{D}(L_0), \end{cases} \tag{5.6}$$

is sectorial in $C(\overline{\Omega})$. This implies that the operator L_1

$$\begin{cases} \mathcal{D}(L_1) = \left\{u \in \bigcap_{p>1} W^{2,p}_{\mathrm{loc}}(\Omega) \ : \ u, Lu \in C_0(\overline{\Omega})\right\} \\ L_1 u = Lu, \quad u \in \mathcal{D}(L_1) \end{cases} \tag{5.7}$$

generates an analytic semigroup in $C_0(\overline{\Omega})$. We are now in position to prove the following generation result.

Theorem 5.2 *Let L be the operator defined in (5.1), satisfying assumptions $(K1)$ and $(K2)$. Then the operator L_W defined by*

$$\begin{cases} \mathcal{D}(L_W) = \left\{u \in \bigcap_{p>1} W^{2,p}_{\mathrm{loc}}(\Omega) \ : \ u \in C(\overline{\Omega}), \ \ Lu \in C_0(\overline{\Omega})\right\}, \\ L_W u = Lu, \quad u \in \mathcal{D}(L_W), \end{cases} \tag{5.8}$$

generates an analytic semigroup in $C(\overline{\Omega})$.

Proof Consider the resolvent equation, for $f \in C(\overline{\Omega})$

$$\begin{cases} \lambda u(x) - Lu(x) = f(x), \quad x \in \overline{\Omega}, \\ Lu(x)|_\Gamma = 0, \end{cases} \tag{5.9}$$

and let G be the solution of the problem

$$\begin{cases} LG(x) = 0, \quad x \in \overline{\Omega}, \\ G|_\Gamma = f|_\Gamma. \end{cases} \tag{5.10}$$

Problem (5.9), taking into account (5.10), can be transformed into

$$\begin{cases} \lambda\Big(u - \frac{G}{\lambda}\Big) - L\Big(u - \frac{G}{\lambda}\Big) = f - G, \quad x \in \overline{\Omega}, \\ \Big(u - \frac{G}{\lambda}\Big)\Big|_\Gamma = 0, \quad L\Big(u - \frac{G}{\lambda}\Big)\Big|_\Gamma = 0. \end{cases} \tag{5.11}$$

Note that $f - G \in C_0(\overline{\Omega})$ so that there is a unique $u - \frac{G}{\lambda} \in C_0(\overline{\Omega})$, in view of the previous considerations, that satisfies:

$$\left\| u - \frac{G}{\lambda} \right\|_{C_0(\overline{\Omega})} \le \frac{c}{|\lambda|} \| f - G \|_{C_0(\overline{\Omega})}. \tag{5.12}$$

On the other hand, if $v := u - G/\lambda$ satisfies $\lambda v - A_W v = f - G$, then $u = v + G/\lambda$ belongs to $\mathcal{D}(A_W)$ and satisfies Problem (5.9). Moreover,

$$\|u\|_{C(\overline{\Omega})} \le \left\| u - \frac{G}{\lambda} \right\|_{C(\overline{\Omega})} + \left\| \frac{G}{\lambda} \right\|_{C(\overline{\Omega})} \le \frac{c}{|\lambda|} \left(\| f - G\|_{C(\overline{\Omega})} + \|f\|_{C(\Gamma)} \right) \tag{5.13}$$

$$\le \frac{c_1}{|\lambda|} \left(\|f\|_{C(\overline{\Omega})} + 2\|f\|_{C(\Gamma)} \right) \le \frac{c_2}{|\lambda|} \|f\|_{C(\overline{\Omega})}, \tag{5.14}$$

where c_i denote positive constants, so that L_W generates an analytic semigroup in $C(\overline{\Omega})$. □

We now apply a perturbation argument to obtain the following generation result. Let $\tilde{L}$ be the operator in $C(\overline{\Omega})$ defined by

$$\begin{cases} \mathcal{D}\left(\tilde{L}\right) = \Big\{ u \in \bigcap_{p>1} W^{2,p}_{\text{loc}}(\Omega) \ : \ \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(a_{jk}(\cdot) \frac{\partial u}{\partial x_k} \right) \\ + \sum_{k=1}^n b_k(\cdot) \frac{\partial u}{\partial x_k} + d(\cdot) u \in C_0\left(\overline{\Omega}\right) \Big\}, \\ \tilde{L}u = \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(a_{jk}(x) \frac{\partial u(x)}{\partial x_k} \right), \quad u \in \mathcal{D}\left(\tilde{L}\right). \end{cases} \tag{5.15}$$

By a perturbation method we get the following generation result.

Theorem 5.3 *The operator $\tilde{L}$, defined in (5.15), generates an analytic semigroup in $C(\overline{\Omega})$.*

Proof From Theorem 5.2 we know that A_W, defined in (5.8) generates an analytic semigroup in $C(\overline{\Omega})$. Let us introduce the operator

$$\begin{cases} \mathcal{D}(\tilde{C}) = \mathcal{D}(\tilde{L}) = \Big\{ u \in \bigcap_{p>1} W^{2,p}_{\text{loc}}(\Omega) \ : \ \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(a_{jk}(\cdot) \frac{\partial u}{\partial x_k} \right) \\ + \sum_{k=1}^n b_k(\cdot) \frac{\partial u}{\partial x_k} + d(\cdot) u \in C_0(\overline{\Omega}) \Big\}, \\ \tilde{C}u = - \sum_{k=1}^n b_k(\cdot) \frac{\partial u}{\partial x_k} - d(\cdot) u, \quad u \in \mathcal{D}(\tilde{C}). \end{cases} \tag{5.16}$$

Then

$$\mathcal{D}(\tilde{C}) \hookrightarrow W^{2,p}(\Omega), \quad \text{for any } p > 1,$$

and by the Rellich's imbedding theorem we have

$$W^{2,p}(\Omega) \hookrightarrow^c C^1(\overline{\Omega}) \hookrightarrow C(\overline{\Omega}), \quad \text{for any } p > n.$$

Applying the Ehrling Lemma (see, e.g. [19], Theorem 7A16) , we have that for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\|u\|_{C^1(\overline{\Omega})} \leq \varepsilon \|u\|_{\mathcal{D}(\tilde{C})} + M_\varepsilon \|u\|_{C(\overline{\Omega})}$$

in other words $\|Cu\|_{C(\overline{\Omega})}$ is estimated by

$$\|Cu\|_{C(\overline{\Omega})} \leq \varepsilon \|A_W u\|_{C(\overline{\Omega})} + +M_\varepsilon \|u\|_{C(\overline{\Omega})}.$$

Since $\tilde{C}$ is A_W-bounded with A_W-bound equal to zero, the statement follows. □

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