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RESEARCH ARTICLE

A characterization of arithmetical invariants by the monoid of relations II: the monotone catenary degree and applications to semigroup rings

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Abstract The investigation and classification of nonunique factorization phenomena has attracted some interest in recent literature. For finitely generated monoids, S.T. Chapman and P.A. García-Sánchez, together with several co-authors, derived a method to calculate the catenary and tame degree from the monoid of relations. Then, in Philipp (Semigroup Forum 81:424–434, 2010), the algebraic structure of this approach was investigated and the restriction to finitely generated monoids was removed. We now extend these ideas further to the monotone catenary degree and then apply all these results to the explicit computation of arithmetical invariants of semigroup rings.

Keywords Nonunique factorizations · Monoid of relations · Monotone catenary degree

1 Introduction

An integral domain and, more generally, a commutative, cancellative monoid is called *atomic* if every nonzero nonunit has a factorization into irreducible elements, and it is called *factorial* if this factorization is unique up to ordering and associates. Nonunique factorization theory is concerned with the description and classification of nonunique factorization phenomena arising in atomic domains. It has its origin in algebraic number theory—the ring of integers of an algebraic number field is atomic but generally

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not factorial—but in recent decades it became an autonomous theory with many connections to zero-sum theory, commutative ring theory, module theory, additive combinatorics, and representations of monoids. We refer to the monograph [14] for a recent presentation of the various aspects of the theory.

To describe these phenomena, various invariants have been studied in the literature. Among these, the tame degree, the catenary degree, and—a variant thereof—the monotone catenary degree received some attention in recent research; for some new results, see, e.g. [3,12], and [13]; for an overview of known results and additional references, see, e.g., the monograph [14]; for a statement of the formal definitions, see Sect. 2 and the beginning of Sect. 4. Additionally, monotone and near monotone chains of factorizations have been studied in [10,11], and [12, Sect. 7].

There are natural connections between arithmetical invariants on the one side and presentations of the semigroup on the other side. It is quite surprising that investigations of this type were started only fairly recently by Chapman et al. in [4]. First results were restricted to finitely generated monoids (with a focus on numerical monoids) and to the catenary and tame degrees. But then the study was extended to more general settings and to a wider range of arithmetical invariants (see [4,29], and especially [3,5], and [6]). In the present paper, we establish results of this flavor. More precisely, in Sects. 3 and 4 we extend the tools from [24] to study the monotone catenary degree by submonoids of the monoid of relations.

Apart from being of interest in their own right, all these results, which characterize arithmetical invariants by certain monoids of relations, can often be used successfully for further arithmetical investigations. So we do in Sect. 5, where we apply the abstract semigroup theoretical results of the previous sections to semigroup rings and to generalized power series rings. This is part of a more general strategy in factorization theory. Indeed, domains of arithmetical interest (such as orders in algebraic number fields) are rarely studied in a direct way but mainly by transfer homomorphisms to auxiliary monoids which have a simpler structure but which still carry much arithmetical information of the starting domains (in case of orders in algebraic number fields, the auxiliary monoids are T-block monoids). The auxiliary monoids have such simple constituents for which monoids of relations can actually be determined and then they give the required information on the arithmetical invariants. For a more detailed description of such an approach we refer to [26] where semigroup theoretical results of the present paper—in particular, Lemma 3.4—are used to study the arithmetic of non-principal orders in algebraic number fields.

2 Preliminaries

In this note, our notation and terminology will be consistent with [14]. Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \uplus \{0\}$. For integers $n, m \in \mathbb{Z}$, we set $[n, m] = \{x \in \mathbb{Z} \mid n \leq x \leq m\}$. By convention, the supremum of the empty set is zero and we set $\frac{0}{0} = 1$. The term "monoid" always means a commutative, cancellative semigroup with unit element. When not mentioned otherwise, we will write all monoids multiplicatively. For a monoid H, we denote by $H^\times$ the set of invertible elements of H and by $\mathsf{q}(H)$ the quotient group of H, i.e., $H^{-1}H$. We call

H reduced if $H^\times = \{1\}$ and call $H_{\mathrm{red}} = H/H^\times$ the reduced monoid associated with H. Of course, H_{red} is always reduced. Note that the arithmetic of H is determined by H_{red} and therefore we can restrict our attention to reduced monoids whenever convenient. We denote by $\mathcal{A}(H)$ the *set of atoms* of H, by $\mathcal{A}(H_{\mathrm{red}})$ the set of atoms of the associated reduced monoid H_{red}, by $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$ the free (abelian) monoid with basis $\mathcal{A}(H_{\mathrm{red}})$, and by $\pi_H : \mathsf{Z}(H) \to H_{\mathrm{red}}$ the unique homomorphism such that $\pi_H | \mathcal{A}(H_{\mathrm{red}}) = \mathrm{id}$. We call $\mathsf{Z}(H)$ the *factorization monoid* and call π_H the *factorization homomorphism* of H. For $a \in H$, we denote by $\mathsf{Z}(a) = \pi_H^{-1}(aH^\times)$ the *set of factorizations* of a and denote by $\mathsf{L}(a) = \{|z| \mid z \in \mathsf{Z}(a)\}$ the *set of lengths* of a, where $|\cdot|$ is the ordinary length function in the free monoid $\mathsf{Z}(H)$. In this terminology, a monoid H is called *half-factorial* if $|\mathsf{L}(a)| = 1$ for all $a \in H \setminus H^\times$ and *factorial* if $|\mathsf{Z}(a)| = 1$ for all $a \in H \setminus H^\times$. This definition of being half-factorial coincides with the classical definition, since then every two factorizations of an element have the same length.

With all these notions at hand, for $a \in H$, we call

$$\rho(a) = \frac{\sup \mathsf{L}(a)}{\min \mathsf{L}(a)} \quad \text{the } \textit{elasticity} \text{ of } a \text{ and}$$
$$\rho(H) = \sup\{\rho(a) \mid a \in H\} \quad \text{the } \textit{elasticity} \text{ of } H.$$

Note that H is half-factorial if and only if $\rho(H) = 1$.

For two factorizations $z, z' \in \mathsf{Z}(H)$, we call

$$\mathsf{d}(z, z') = \max\left\{\left|\frac{z}{\gcd(z, z')}\right|, \left|\frac{z'}{\gcd(z, z')}\right|\right\} \quad \text{the } \textit{distance} \text{ between } z \text{ and } z'$$

and, for two subset $X, Y \subset \mathsf{Z}(H)$, we call

$$\mathsf{d}(X, Y) = \min\{\mathsf{d}(x, y) \mid x \in X, \ y \in Y\} \quad \text{the } \textit{distance} \text{ between } X \text{ and } Y.$$

If one of the sets is a singleton, say $X = \{x\}$, we write $\mathsf{d}(\{x\}, Y) = \mathsf{d}(x, Y)$.

Let $a \in H$. We call two lengths $k, l \in \mathsf{L}(a)$ with $k < l$ *adjacent* if $[k, l] \cap \mathsf{L}(a) = \{k, l\}$ and, for $M \subset \mathbb{N}$, we set $\mathsf{Z}_M(a) = \{x \in \mathsf{Z}(a) \mid |x| \in M\}$. If the set is a singleton, say $M = \{k\}$, then we write $\mathsf{Z}_{\{k\}}(a) = \mathsf{Z}_k(a)$. Additionally, we call $\triangle H = \{|l - k| \mid k, l \in \mathsf{L}(a) \text{ are adjacent for some } a \in H\}$ the *set of distances* of H.

Definition 2.1 Let H be an atomic monoid and let $a \in H$.

1. Factorizations $z_0, \ldots, z_n \in \mathsf{Z}(a)$ with $n \in \mathbb{N}$ and $\mathsf{d}(z_{i-1}, z_i) \le N$ for some $N \in \mathbb{N}$ and $i \in [1, n]$ are called
   - an *N-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$).
   - a *monotone N-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$) if $|z_{i-1}| \le |z_i|$ for all $i \in [1, n]$.
   - an *equal-length N-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$) if $|z_{i-1}| = |z_i|$ for all $i \in [1, n]$.
2. The
   - *catenary degree* $\mathsf{c}(a)$

- *monotone catenary degree* $\mathsf{c}_{\mathrm{mon}}(a)$
- *equal catenary degree* $\mathsf{c}_{\mathrm{eq}}(a)$

denotes, respectively, the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that, for all $z, z' \in \mathsf{Z}(a)$ with $|z| \le |z'|$, there is
- an N-chain concatenating z and z'.
- a monotone N-chain concatenating z to z' or concatenating z' to z.
- an equal-length N-chain concatenating z and z'.

Then we call
- $\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\}$ the *catenary degree* of H.
- $\mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{mon}}(a) \mid a \in H\}$ the *monotone catenary degree* of H.
- $\mathsf{c}_{\mathrm{eq}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(a) \mid a \in H\}$ the *equal-length catenary degree* of H.

Note that $\sup\{\mathsf{c}(H), \mathsf{c}_{\mathrm{eq}}(H)\} \le \mathsf{c}_{\mathrm{mon}}(H)$.

For the description and computation of the monotone catenary degree, we follow the same two step procedure as in [3]. In order to formulate this precisely, we need to define another variant of the catenary degree.

Definition 2.2 Let H be an atomic monoid. For $a \in H$, we define

$$\mathsf{c}_{\mathrm{ad}}(a) = \sup\{\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) \mid k, l \in \mathsf{L}(a) \text{ are adjacent}\}$$

as the *adjacent catenary degree* of a.

Also, $\mathsf{c}_{\mathrm{ad}}(H) = \sup\{\mathsf{c}_{\mathrm{ad}}(a) \mid a \in H\}$ is called the *adjacent catenary degree* of H.

By [3, (4.1)], we find

$$\mathsf{c}(H) \le \mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(H), \mathsf{c}_{\mathrm{ad}}(H)\}.$$

Here we follow the same strategy as in [24, Sect. 3] for the definition of the $\mathcal{R}$-relation and the μ-invariant.

Definition 2.3 Let H be an atomic monoid and let $a \in H$.

1. Factorizations $z_0, \ldots, z_n \in \mathsf{Z}(a)$ with $n \in \mathbb{N}$ and $\gcd(z_{i-1}, z_i) \ne 1$ for all $i \in [1, n]$ are called
   - an $\mathcal{R}$*-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$).
   - a *monotone* $\mathcal{R}$*-chain* concatenating z_0 to z_n (in $\mathsf{Z}(H)$) if $|z_{i-1}| \le |z_i|$ for all $i \in [1, n]$.
   - an *equal-length* $\mathcal{R}$*-chain* concatenating z_0 and z_n (in $\mathsf{Z}(H)$) if $|z_{i-1}| = |z_i|$ for all $i \in [1, n]$.
2. Two elements $z, z' \in \mathsf{Z}(H)$ are
   - $\mathcal{R}$*-related*
   - $\mathcal{R}_{\mathrm{eq}}$*-related*

   respectively, if there is an
   - $\mathcal{R}$-chain
   - equal-length $\mathcal{R}$-chain

   concatenating z and z'. We then write $z \approx z'$ respectively $z \approx_{\mathrm{eq}} z'$.

Note that, with the above definitions, $\approx$ and $\approx_{\text{eq}}$ are congruences on $\mathsf{Z}(H) \times \mathsf{Z}(H)$.

Based on these definitions, we can now recall the definition of the μ-invariant (for reference see [24, Sect. 3]) and give the definition of the μ_{eq}-invariant and the μ_{ad}-invariant. Note that the definition of the last one differs significantly from the other two since there is no appropriate equivalence relation we can make use of.

Definition 2.4 Let H be an atomic monoid and let $a \in H$.

1. Let $\mathcal{R}_a$ denote the set of $\mathcal{R}$-*equivalence classes* of $\mathsf{Z}(a)$ and, for $\eta \in \mathcal{R}_a$, let $|\eta| = \min\{|z| \mid z \in \eta\}$. We set
$$\mu(a) = \sup\{|\eta| \mid \eta \in \mathcal{R}_a\} \le \sup \mathsf{L}(a)$$
and define $\mu(H) = \sup\{\mu(a) \mid a \in H\}$.
2. For $k \in \mathsf{L}(a)$, let $\mathcal{R}_{a,k}$ denote the set of $\mathcal{R}_{\text{eq}}$-*equivalence classes* of $\mathsf{Z}_k(a)$. We set
$$\mu_{\text{eq}}(a) = \sup\{k \in \mathsf{L}(a) \mid |\mathcal{R}_{a,k}| > 1\} \le \sup \mathsf{L}(a)$$
and define $\mu_{\text{eq}}(H) = \sup\{\mu_{\text{eq}}(a) \mid a \in H\}$.
3. We set
$$\mu_{\text{ad}}(a) = \sup\{k \in \mathsf{L}(a) \mid \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = k \text{ for } l \in \mathsf{L}(a),\ l < k,\ l \text{ adjacent to } k\}.$$
Then we set $\mu_{\text{ad}}(H) = \sup\{\mu_{\text{ad}}(a) \mid a \in H\}$.

Then $\mu(H) = 0$ if and only if $|\mathcal{R}_a| \le 1$ for all $a \in H$ and $\mu_{\text{eq}}(H) = 0$ if and only if $|\mathcal{R}_{a,k}| \le 1$ for all $a \in H$ and $k \in \mathsf{L}(a)$.

Definition 2.5 Let $H \subset D$ be monoids.

1. We call $H \subset D$ *saturated* or, equivalently, *a saturated submonoid* if, for all $a, b \in H$, $a \mid b$ in D already implies that $a \mid b$ in H; that is, for all $a, b \in H$ and $c \in D$, $a = bc$ implies $c \in H$.
2. If $H \subset D$ is a saturated submonoid, then we set $D/H = \{a\mathsf{q}(H) \mid a \in D\}$ and $[a]_{D/H} = a\mathsf{q}(H)$ and we call $\mathsf{q}(D)/\mathsf{q}(H) = \mathsf{q}(D/H)$ the *class group* of H in D.

Definition 2.6 Let H be an atomic monoid. We call

$$\sim_H = \{(x, y) \in \mathsf{Z}(H) \times \mathsf{Z}(H) \mid \pi(x) = \pi(y)\} \quad \text{the \textit{monoid of relations} of } H,$$
$$\sim_{H,\text{eq}} = \{(x, y) \in \sim_H \mid |x| = |y|\} \quad \text{the \textit{monoid of equal-length relations} of } H,$$
$$\sim_{H,\text{mon}} = \{(x, y) \in \sim_H \mid |x| \le |y|\} \quad \text{the \textit{monoid of monotone relations} of } H,$$

and, for $a \in H$, we set

$$\mathcal{A}_a(\sim_H) = \mathcal{A}(\sim_H) \cap (\mathsf{Z}(a) \times \mathsf{Z}(a)),$$
$$\mathcal{A}_a(\sim_{H,\text{eq}}) = \mathcal{A}(\sim_{H,\text{eq}}) \cap (\mathsf{Z}(a) \times \mathsf{Z}(a)),$$
$$\mathcal{A}_a(\sim_{H,\text{mon}}) = \mathcal{A}(\sim_{H,\text{mon}}) \cap (\mathsf{Z}(a) \times \mathsf{Z}(a)).$$

By [24, Lemma 11], $\sim_H \subset \mathsf{Z}(H) \times \mathsf{Z}(H)$ is a saturated submonoid of a free monoid and thus a Krull monoid by [14, Theorem 2.4.8.1]. By [3, Proposition 4.4.1], $\sim_{H,\mathrm{eq}} \subset \sim_H$ is a saturated submonoid and hence a Krull monoid, and, by [3, Proposition 4.4.2], $\sim_{H,\mathrm{eq}}$ is finitely generated if H_{red} is finitely generated. Unfortunately, $\sim_{H,\mathrm{mon}} \subset \sim_H$ is not saturated, but, by Lemma 5.8, we find that $\sim_{H,\mathrm{mon}}$ is a finitely generated Krull monoid if H is finitely generated.

We briefly recall the main result on the catenary degree from [24] and offer a corrected proof for monoids fulfilling the ascending chain condition on principal ideals here.

Lemma 2.7 (cf. [24, Proposition 8, Corollary 9, Proposition 16]). *Let H be an atomic monoid which fulfills the ascending chain condition on principal ideals. Then*

1. $\mathsf{c}(a) \ge \mu(a)$ *for all* $a \in H$, *and* $\mathsf{c}(H) = \mu(H)$.
2. $\mathsf{c}(H) = \sup\{\mu(a) \mid a \in H,\ \mathcal{A}_a(\sim_H) \neq \emptyset,\ |\mathcal{R}_a| > 1\}$.

Proof 1. Let $a \in H$ be such that $|\mathcal{R}_a| > 1$. We may assume that $\mathsf{c}(a) < \infty$. Let $N \in \mathbb{N}_0$ be such that $\mu(a) \ge N$. Let $\rho \in \mathcal{R}_a$ be such that $|\rho| \ge N$ and $z \in \rho$ such that $|z| = |\rho|$. Let $z' \in \mathsf{Z}(a)$ be such that $z \not\approx z'$ and let $z = z_0, z_1, \ldots, z_k = z'$ be a $\mathsf{c}(a)$-chain concatenating z and z'. Let $i \in [1, k]$ be minimal such that $z \not\approx z_i$. Then $z_{i-1} \not\approx z_i$, and therefore

$$N \le |z_0| \le |z_{i-1}| \le \mathsf{d}(z_i, z_{i-1}) \le \mathsf{c}(a).$$

The preceding argument has shown $\mathsf{c}(H) \ge \mu(H)$. Next we show

$$\mu(H) \ge \mathsf{c}(H).$$

We show that, for all $a \in H$, we have $\mathsf{c}(a) \le \mu(H)$. Since H fullfills the ascending chain condition on principal ideals we can proceed by induction on a. For $a = 1$, this is trivial. Now suppose $a \neq 1$ and that, for all $b \in H$ with $b \mid a$ and b not an associate of a, we have $\mathsf{c}(b) \le \mu(H)$. Now let $z, z' \in \mathsf{Z}(a)$. If $z \not\approx z'$, then there are $z'', z''' \in \mathsf{Z}(a)$ such that $z'' \approx z$, $z''' \approx z'$, and z'' and z''' are minimal in their $\mathcal{R}$-classes with respect to their lengths. Since $\gcd(z'', z''') = 1$, we find $\mathsf{d}(z'', z''') = \max\{|z''|, |z'''|\} \le \mu(a) \le \mu(H)$. Now it remains to show that, for any two factorizations $z, z' \in \mathsf{Z}(a)$ with $z \approx z'$, there is a $\mu(H)$-chain concatenating them. By definition, there is an $\mathcal{R}$-chain $z_0, \ldots, z_k$ with $z = z_0$ and $z' = z_k$, and $g_i = \gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$. Since $\pi_H(g_i^{-1} z_{i-1}) \mid a$, we find a $\mu(H)$-chain concatenating $g_i^{-1} z_{i-1}$ and $g_i^{-1} z_i$ for all $i \in [1, k]$ by induction hypothesis, and thus there is a $\mu(H)$-chain concatenating z_{i-1} and z_i for all $i \in [1, k]$; thus there is a $\mu(H)$-chain concatenating z and z'. So $\mathsf{c}(a) \le \mu(H)$.

2. When we compare the definitions, we see that the only thing remaining is

$$\{a \in H \mid \mathcal{A}_a(\sim_H) \neq \emptyset,\ |\mathcal{R}_a| > 1\} = \{a \in H \mid |\mathcal{R}_a| > 1\}.$$

One inclusion is trivial and, for the other one, let $a \in H$ be such that $|\mathcal{R}_a| > 1$, and let $z, z' \in \mathsf{Z}(a)$ be two factorizations of a such that $z \not\approx z'$ and such that both are minimal in their $\mathcal{R}$-equivalence classes with respect to their lengths. Now assume $(z, z') \notin \mathcal{A}(\sim_H)$. Then there is $k \geq 2$ and $(x_1, y_1), \ldots, (x_k, y_k) \in \mathcal{A}(\sim_H)$ such that $(z, z') = (x_1, y_1) \cdot \ldots \cdot (x_k, y_k)$. But now we find the following $\mathcal{R}$-chain from z to z': $z_0 = z$ and $z_i = z_{i-1} x_i^{-1} y_i$ for $i \in [1, k]$. Then $z_k = z'$ and $\gcd(z_{i-1}, z_i) \neq 1$. Since this is a contradiction, we have $(z, z') \in \mathcal{A}(\sim_H)$, and thus $(z, z') \in \mathcal{A}_a(\sim_H) \neq \emptyset$. □

3 A characterization of the monotone catenary degree by monoids of relations

Lemma 3.1 *Let H be an atomic monoid, $a \in H$ and $x, y \in \mathsf{Z}(a)$.*

1. *If $x \not\approx_{\mathrm{eq}} y$ with $|x| = |y|$, then $(x, y) \in \mathcal{A}_a(\sim_{H,\mathrm{eq}})$.*
2. *Let $k, l \in \mathsf{L}(a)$ be adjacent with $k < l$. If $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = l$, then $(x, y) \in \mathcal{A}_a(\sim_{H,\mathrm{mon}})$ for all $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_l(a)$.*

Proof Since the arithmetic of H is determined solely by H_{red}, we may assume that H is reduced.

1. Let $a \in H$ and $x, y \in \mathsf{Z}(a)$ be such that $(x, y) \notin \mathcal{A}_a(\sim_{H,\mathrm{eq}})$. Then, trivially, $(x, y) \notin \mathcal{A}(\sim_{H,\mathrm{eq}})$ and thus there are $(x_1, y_1), \ldots, (x_k, y_k) \in \mathcal{A}(\sim_{H,\mathrm{eq}})$ with $k \geq 2$ such that $(x, y) = (x_1, y_1) \cdot \ldots \cdot (x_k, y_k)$. Then $x = x_1 \cdot \ldots \cdot x_k,\ y_1 x_2 \cdot \ldots \cdot x_k,\ y_1 \cdot \ldots \cdot y_k = y$ is an $\mathcal{R}_{\mathrm{eq}}$-chain concatenating x and y, and therefore $x \approx_{\mathrm{eq}} y$.
2. Let $a \in H$, let $k, l \in \mathsf{L}(a)$ be adjacent with $k < l$ and $\mathsf{d}(Z_k(a), Z_l(a)) = l$, and let $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_l(a)$. Now suppose $(x, y) \notin \mathcal{A}_a(\sim_{H,\mathrm{mon}})$. Then, trivially, $(x, y) \notin \mathcal{A}(\sim_{H,\mathrm{mon}})$ and there are $(x_1, y_1), \ldots, (x_k, y_k) \in \mathcal{A}(\sim_{H,\mathrm{mon}})$ with $k \geq 2$ and $|y_1| - |x_1| \leq \cdots \leq |y_k| - |x_k|$. Then we set $x' = x_1^{-1} y_1 x$. If $|y_1| - |x_1| = 0$, we find $|x'| = k$ and $\gcd(x', y) \neq 1$, a contradiction to $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = l$. Otherwise, if $|y_1| - |x_1| > 0$, then $k = |x| < |x'| < |y| = l$, a contradiction to k and l being adjacent. □

In principal, we follow the same strategy as in [24, Sect. 3] for the μ-invariant when studying the μ_{eq}-invariant. For the μ_{ad}-invariant, we cannot construct an equivalence relation like the $\mathcal{R}$-relation or the $\mathcal{R}_{\mathrm{eq}}$-relation. Thus we follow a slightly modified strategy in the proofs of parts 3 and 4 from Theorem 3.2

Theorem 3.2 *Let H be an atomic monoid. Then*

1. *$\mathsf{c}_{\mathrm{eq}}(a) \geq \mu_{\mathrm{eq}}(a)$ for all $a \in H$, and $\mathsf{c}_{\mathrm{eq}}(H) = \mu_{\mathrm{eq}}(H)$.*
2. *$\mathsf{c}_{\mathrm{eq}}(H) = \sup\{\mu_{\mathrm{eq}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \neq \emptyset,\ |\mathcal{R}_{a,k}| > 1$ for some $k \in \mathsf{L}(a)\}$ $= \sup\{k \in \mathbb{N} \mid a \in H,\ \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \neq \emptyset,\ k \in \mathsf{L}(a),\ |\mathcal{R}_{a,k}| > 1\}$.*
3. *$\mathsf{c}_{\mathrm{ad}}(a) \geq \mu_{\mathrm{ad}}(a)$ for all $a \in H$, and $\mathsf{c}_{\mathrm{ad}}(H) = \mu_{\mathrm{ad}}(H)$.*
4. *$\mathsf{c}_{\mathrm{ad}}(H) = \sup\{\mu_{\mathrm{ad}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\mathrm{mon}}) \neq \emptyset\}$.*

In particular,

$$\mathsf{c}_{\mathrm{mon}}(H) = \sup(\{\mu_{\mathrm{eq}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \neq \emptyset,\ |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathsf{L}(a)\} \cup \{\mu_{\mathrm{ad}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\mathrm{mon}}) \neq \emptyset\}).$$

Proof Since the arithmetic of H is determined solely by H_{red} we may assume that H is reduced.

1. First we prove $\mathsf{c}_{\mathrm{eq}}(a) \ge \mu_{\mathrm{eq}}(a)$ for all $a \in H$. We may assume that $\mathsf{c}_{\mathrm{eq}}(a) < \infty$ and $\mu_{\mathrm{eq}}(a) \ge 1$. Let $N \in \mathbb{N}$ be such that $N \le \mu_{\mathrm{eq}}(a)$. Then there exists $k \in \mathsf{L}(a)$ such that $|\mathcal{R}_{a,k}| > 1$ and $k \ge N$. Let $z,\ z' \in \mathsf{Z}_k(a)$ be such that $z \not\approx_{\mathrm{eq}} z'$, and let $z = z_0, z_1, \ldots, z_n = z'$ be an equal-length $\mathsf{c}_{\mathrm{eq}}(a)$-chain concatenating z and z'. Now we choose $i \in [1, n]$ minimal such that $z \not\approx_{\mathrm{eq}} z_i$. Then $z_{i-1} \not\approx_{\mathrm{eq}} z_i$, and we find
$$\mathsf{c}_{\mathrm{eq}}(a) \ge \mathsf{d}(z_{i-1}, z_i) = k \ge N.$$
Now we prove $\mu_{\mathrm{eq}}(H) \ge \mathsf{c}_{\mathrm{eq}}(H)$. We show that, for all $N \in \mathbb{N}_0$, all $a \in H$, and all factorizations $z,\ z' \in \mathsf{Z}(a)$ with $|z| = |z'| \le N$, there is an equal-length $\mu_{\mathrm{eq}}(H)$-chain from z to z'. We proceed by induction on N. If $N = 0$, then $z = z' = 1$ and $\mathsf{d}(z, z') = 0 \le \mu_{\mathrm{eq}}(H)$. Suppose $N \ge 1$ and that, for all $a \in H$ and all $z,\ z' \in \mathsf{Z}(a)$ with $|z| = |z'| < N$, there is an equal-length $\mu_{\mathrm{eq}}(H)$-chain from z to z'. Now let $a \in H$ and let $z,\ z' \in \mathsf{Z}(a)$ with $|z| = |z'| \le N$. If $z \not\approx_{\mathrm{eq}} z'$, then $\mu_{\mathrm{eq}}(H) \ge \mu_{\mathrm{eq}}(a) \ge |z| = \mathsf{d}(z, z')$. Now it remains to show that, for any two factorizations $z,\ z' \in \mathsf{Z}(a)$ with $|z| = |z'| \le N$ and $z \approx_{\mathrm{eq}} z'$, there is an equal-length $\mu_{\mathrm{eq}}(H)$-chain concatenating them. By definition, there is an $\mathcal{R}_{\mathrm{eq}}$-chain $z_0, \ldots, z_k$ with $z_0 = z$, $z' = z_k$, $g_i = \gcd(z_{i-1}, z_i) \ne 1$, and $|z_i| = |z|$ for all $i \in [1, k]$. By induction hypothesis, there is an equal-length $\mu_{\mathrm{eq}}(H)$-chain from $g_i^{-1} z_{i-1}$ to $g_i^{-1} z_i$ for all $i \in [1, k]$, and hence there is an equal-length $\mu_{\mathrm{eq}}(H)$-chain from z_{i-1} to z_i for $i \in [1, k]$; thus there is an equal-length μ_{eq} (H)-chain from z to z'.
2. By part 1, we have $\mathsf{c}_{\mathrm{eq}}(H) = \mu_{\mathrm{eq}}(H)$ and, by Definition 2.4.2, the third equality is obvious. Thus it suffices to show that
$$\begin{aligned}&\{\mu_{\mathrm{eq}}(a) \mid a \in H,\ |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathsf{L}(a)\}\\ &\quad = \{\mu_{\mathrm{eq}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \ne \emptyset,\ |\mathcal{R}_{a,k}| > 1 \text{ for some } k \in \mathsf{L}(a)\}.\end{aligned}$$
The inclusion from right to left is clear. Now let $a \in H$ and $k \in \mathsf{L}(a)$ be such that $|\mathcal{R}_{a,k}| > 1$. Then there exist $z,\ z' \in \mathsf{Z}_k(a)$ such that $z \not\approx_{\mathrm{eq}} z'$. By Lemma 3.1.1, we find $(z, z') \in \mathcal{A}_a(\sim_{H,\mathrm{eq}}) \ne \emptyset$.
3. First let $a \in H$. We show that $\mathsf{c}_{\mathrm{ad}}(a) \ge \mu_{\mathrm{ad}}(a)$, and then $\mathsf{c}_{\mathrm{ad}}(H) \ge \mu_{\mathrm{ad}}(H)$ follows by passing to the supremum on both sides. If $\mu_{\mathrm{ad}}(a) = 0$ or $\mu_{\mathrm{ad}}(a) = \infty$, this is trivial. Now let $\mu_{\mathrm{ad}}(a) = l \in \mathbb{N}$. Then there is $k \in \mathsf{L}(a)$ and $k < l$ with l adjacent to k. Then, by Definition 2.2, $\mathsf{c}_{\mathrm{ad}}(a) \ge \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = \max\{k, l\} = l = \mu_{\mathrm{ad}}(a)$.
Now we prove $\mu_{\mathrm{ad}}(H) \ge \mathsf{c}_{\mathrm{ad}}(H)$. We must prove that $\mathsf{c}_{\mathrm{ad}}(a) \le \mu_{\mathrm{ad}}(H)$ for all $a \in H$. Assume to the contrary that there is some $a \in H$ such that $\mathsf{c}_{\mathrm{ad}}(a) > \mu_{\mathrm{ad}}(H)$. Let $l \in \mathbb{N}$ be minimal such that there is some $k < l$ and $a \in H$ with k and l adjacent lengths of a and $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) > \mu_{\mathrm{ad}}(H)$. If $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = l$, then $l \le \mu_{\mathrm{ad}}(H) < \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) \le l$ a contradiction. Thus $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) < l$ and there are some $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_l(a)$ such that $g = \gcd(x, y) \ne 1$. We set

$b = \pi_H(g^{-1}x)$ and find

$$\mu_{\text{ad}}(H) < \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = \mathsf{d}(x, y) = \mathsf{d}(g^{-1}x, g^{-1}y) = \mathsf{d}(\mathsf{Z}_{k-|g|}(b), \mathsf{Z}_{l-|g|}).$$

Since $k - |g|$ and $l - |g|$ are adjacent lengths of b and $l - |g| < l$, this contradicts the minimal choice of l. Thus we infer that $\mathsf{c}_{\text{ad}}(a) \leq \mu_{\text{ad}}(H)$.

4. By part 3 and Definition 2.4.3, we find

$$\mathsf{c}_{\text{ad}}(H) = \mu_{\text{ad}}(H) = \sup\{\mu_{\text{ad}}(a) \mid a \in H\}.$$

If $\mu_{\text{ad}}(a) = \infty$ for some $a \in H$ then $\mathsf{c}_{\text{ad}}(H) = \infty$ by part 3. Thus we may assume that $\mu_{\text{ad}}(a) < \infty$ for all $a \in H$ and it suffices to show that

$$\sup\{\mu_{\text{ad}}(a) \mid a \in H\} = \sup\{\mu_{\text{ad}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\text{mon}}) \neq \emptyset\}.$$

In fact, we only have to show that $\sup\{\mu_{\text{ad}}(a) \mid a \in H\} \leq \sup\{\mu_{\text{ad}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\text{mon}}) \neq \emptyset\}$. Now let $a \in H$ and $\mu_{\text{ad}}(a) = k \in \mathbb{N}$. Then there is $l \in \mathsf{L}(a)$ with $l < k$, l adjacent to k, and $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = k$. Now let $x \in \mathsf{Z}_l(a)$ and $y \in \mathsf{Z}_k(a)$. Then we have $\gcd(x, y) = 1$. By Lemma 3.1.2, we have $(x, y) \in \mathcal{A}_a(\sim_{H,\text{mon}}) \neq \emptyset$.

The additional statement now follows easily by parts 2 and 4. □

Lemma 3.3 *Let H be an atomic monoid and let $a \in H$.*

1. *Let $x, y \in \mathsf{Z}(a)$ with $\min\{|x|, |y|\} > \mathsf{c}_{\text{mon}}(a)$. Then there is a monotone $\mathcal{R}$-chain concatenating x and y, and thus $x \approx y$; in particular, if $|x| = |y|$, then $x \approx_{\text{eq}} y$.*
2. *Let $k, l \in \mathsf{L}(a)$. Then*

$$\begin{aligned}\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) = \max\{k, l\} \quad &\textit{if and only if} \quad \gcd(x, y)\\ &= 1 \textit{ for all } x \in \mathsf{Z}_k(a) \textit{ and } y \in \mathsf{Z}_l(a).\end{aligned}$$

3. *Let $k, l \in \mathsf{L}(a)$ be adjacent with $k < l$ such that there are $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_l(a)$ such that there is a monotone $\mathcal{R}$-chain concatenating x and y. Then $\mu_{\text{ad}}(a) \neq l$.*

Proof Since the arithmetic of H is determined solely by H_{red}, we may assume that H is reduced.

1. Let $a \in H$ and $x, y \in \mathsf{Z}(a)$ be such that $\min\{|x|, |y|\} > \mathsf{c}_{\text{mon}}(a)$. We may assume that $|x| \leq |y|$. Then there is a monotone $\mathsf{c}_{\text{mon}}(a)$-chain concatenating x and y, say $z_0 = x, z_1, \ldots, z_k = y$. Since, for all $i \in [1, k]$, we have $\mathsf{d}(z_{i-1}, z_i) \leq \mathsf{c}_{\text{mon}}(a) < |x| = |z_0|$, we have $\gcd(z_{i-1}, z_i) \neq 1$ for all $i \in [1, k]$. Thus $z_0, \ldots, z_k$ is a monotone $\mathcal{R}$-chain concatenating x and y, and therefore $x \approx y$. If $|x| = |y|$, then $z_0, \ldots, z_k$ is an equal-length chain, and therefore $x \approx_{\text{eq}} y$.
2. Follows immediately by the definition of the distance of factorizations in $\mathsf{Z}(H)$.
3. Let $a \in H$, let $k, l \in \mathsf{L}(a)$ be adjacent with $k < l$, let $x \in \mathsf{Z}_k(a)$, and $y \in \mathsf{Z}_l(a)$ be such that there is a monotone $\mathcal{R}$-chain from x to y, say $z_0 = x, z_1, \ldots, z_n = y$ for some $n \in \mathbb{N}$. Now choose $i \in [1, n]$ minimal such that $|z_i| = l$. Due to

the minimality of i, we find $z_{i-1} \in \mathsf{Z}_k(a)$. Since $\gcd(z_{i-1}, z_i) \neq 1$, we find $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) < l$, and therefore $\mu_{\mathrm{ad}}(a) \neq l$. □

Lemma 3.4 *Let H be an atomic monoid. Then*

1. $\mathsf{c}_{\mathrm{eq}}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{H,\mathrm{eq}}),\ x \not\approx_{\mathrm{eq}} y\}$.
2. $\mathsf{c}_{\mathrm{ad}}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{H,\mathrm{mon}}),\ |x| < |y|,\ |x|, |y| \in \mathsf{L}(\pi_H(x))$ *adjacent, and there is no monotone $\mathcal{R}$-chain from x to y*$\}$.

In particular,

$$\mathsf{c}_{\mathrm{mon}}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{H,\mathrm{mon}}),\ \textit{there is no monotone } \mathcal{R}\textit{-chain from } x \textit{ to } y,\ \textit{and either } |x| = |y| \textit{ or } |x|,\ |y| \in \mathsf{L}(\pi_H(x)) \textit{ are adjacent}\}.$$

Proof Since the arithmetic of H is determined solely by H_{red}, we may assume that H is reduced.

1. The inequality $\mathsf{c}_{\mathrm{eq}}(H) \leq \sup\{|y| \mid (x, y) \in \mathcal{A}(\sim_{H,\mathrm{eq}})\}$ has been proven in [3, Proposition 4.4.3]. The slightly stronger statement here follows immediately by the definition of $\mu_{\mathrm{eq}}(\cdot)$; see Definition 2.4.2.
2. By Theorem 3.2.4, we have $\mathsf{c}_{\mathrm{ad}}(H) = \sup\{\mu_{\mathrm{ad}}(a) \mid a \in H,\ \mathcal{A}_a(\sim_{H,\mathrm{mon}}) \neq \emptyset\}$. Now the assertion follows from Lemma 3.3.3, Lemma 3.1.2, and the definition of $\mu_{\mathrm{ad}}(\cdot)$; see Definition 2.4.3.

The additional statement follows from

$$\mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(H), \mathsf{c}_{\mathrm{ad}}(H)\} \quad \text{and} \quad \mathcal{A}(\sim_{H,\mathrm{eq}}) \subset \mathcal{A}(\sim_{H,\mathrm{mon}}).$$

□

4 Tameness and monotone chains

Definition 4.1 Let H be an atomic monoid.

1. For $a \in H$ and $x \in \mathsf{Z}(H)$, let $\mathsf{t}(a, x)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:
   If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \neq \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists some $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \leq N$.
   For subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$, we define
   $$\mathsf{t}(H', X) = \sup\{\mathsf{t}(a, x) \mid a \in H',\ x \in X\},$$
   and we define $\mathsf{t}(H) = \mathsf{t}(H, \mathcal{A}(H_{\mathrm{red}}))$. This is called the *tame degree* of H.
2. If $\mathsf{t}(H) < \infty$, then we call H *tame*.

Being tame is a very strong finiteness condition within nonunique factorization theory, in particular, the finiteness of the tame degree implies the finiteness of the elasticity and the catenary degree among other invariants. Next, we give a list of examples where tameness is characterized in various classes of monoids and domains; for a similar list, the reader is referred to [16, Examples 3.2].

1. *Finitely generated monoids*. If $H_{\rm red}$ is finitely generated, then H is tame (see [14, Theorem 3.1.4]).
2. *Finitely primary monoids*. Let H be finitely primary of rank $s \in \mathbb{N}$. Then H is tame if and only if $s = 1$ (see [14, Theorem 3.1.5]).
3. *Weakly Krull domains*. Let R be a v-noetherian weakly Krull domain with nonzero conductor $\mathfrak{f} = (R : \widehat{R})$ and finite v-class group $\mathcal{C}_v(R)$. Note that, in particular, orders in algebraic number fields fulfill all these properties.
   Then R is tame if and only if, for every nonzero prime ideal $\mathfrak{p} \in \mathfrak{X}(R)$ with $\mathfrak{p} \supset \mathfrak{f}$, there is precisely one $\mathfrak{P} \in \mathfrak{X}(\widehat{R})$ such that $\mathfrak{P} \cap R = \mathfrak{p}$ (see [14, Theorem 3.7.1]).
4. *Krull monoids* and therefore *Krull domains*. Let H be a Krull monoid, $F = \mathcal{F}(P)$ a monoid of divisors and $G_P = \{[p] \mid p \in P\} \subset F/H_{\rm red} = G$ the set of classes containing prime divisors. Suppose that one of the following conditions hold:
   (a) H has the approximation property.
   (b) Every $g \in G_P$ contains at least two prime divisors.
   (c) There is an $m \in \mathbb{N}$ such that $-G_p \subset m(G_P \cup \{0\})$.
   (d) The torsion free rank of G is finite.
   Then H is tame if and only if $\mathsf{D}(G_P) < \infty$ (see [16, Theorem 4.2]). In particular, all principal orders in algebraic number fields are tame.
5. *C-like monoids*. Let H be a C-like monoid. Then H is tame if and only if the natural map $s\text{-spec}(\widehat{H}) \to s\text{-spec}(H)$ is bijective (see [19, Theorem 8.3] and [19, Definition 5.6] for a precise definition of C-like monoids).
   Next we give two examples of C-like monoids. $R^\bullet = (R\setminus\{0\}, \cdot)$, the multiplicative monoid of the domain, is a C-like monoid if
   - (see [19, Proposition 6.1]) R is an integral domain and $R^\bullet$ is finitely primary.
   - (see [19, Proposition 6.5]) R is a Mori domain with complete integral closure $\widehat{R}$, $\mathcal{C}_v(\widehat{R})$ is finite, $(R : \widehat{R}) \neq 0$, and either
     – R is semilocal, and $\widehat{R}/(R : \widehat{R})$ is quasi artinian or
     – $\mathcal{C}_v(R)$ is finite and $S^{-1}\widehat{R}/S^{-1}(R : \widehat{R})$ is quasi artinian, where $S \subset R^\bullet$ is the submonoid of regular elements.

While the tameness of a monoid implies the finiteness of the catenary degree, it does not imply the finiteness of the equal catenary degree and therefore not the finiteness of the monotone catenary degree. In order to point this out, we discuss a monoid originally introduced in [9, Example 4.5].

Recall that a monoid H is called *finitely primary* if there exist $s, k \in \mathbb{N}$ and a factorial monoid $F = [p_1, \ldots, p_s] \times F^\times$ with the following properties:

- $H \setminus H^\times \subset p_1 \cdot \ldots \cdot p_s F$ and
- $(p_1 \cdot \ldots \cdot p_s)^k F \subset H$.

If this is the case, then we call H a *finitely primary monoid* of *rank* s and *exponent* k.

Example 4.2 (cf. [9, Example 4.5]). There exists a tame monoid H such that $\mathsf{c}_{\rm eq}(H) = \infty$, and thus $\mathsf{c}_{\rm mon}(H) = \infty$ but $\mathsf{c}_{\rm ad}(H) < \infty$.

Proof We proceed in four steps.

1. We start with the construction of a finitely primary monoid. Let G be an additively written abelian group and $f : G \to N_0$ a map with $f(0) = 0$ and finite image $f(G)$ such that, for all $g, g' \in G$, the following two conditions are satisfied:

(a) $f(g+g') \le f(g) + f(g')$ and
(b) if $f(g) = 0$, then $f(-g) = 0$.
Then, by construction,

$$H(G, f) = \{(g, k) \mid g \in G,\ k \in \mathbb{N}_0 \text{ with } k \ge f(g)\} \subset (G \times \mathbb{N}_0, +)$$

is a finitely primary monoid of rank one and exponent $\max f(G)$.

2. We consider a group G with basis $E = \{e_m, e'_m \mid m \in \mathbb{N}\}$, where $\operatorname{ord}(e_m) = \operatorname{ord}(e'_m) = m$ for all $m \in \mathbb{N}$, whence

$$G = \bigoplus_{m \in \mathbb{N}} (\langle e_m \rangle \oplus \langle e'_m \rangle) = \bigoplus_{m \in \mathbb{N}} (\mathbb{Z}/m\mathbb{Z})^2.$$

Let $f : G \to \mathbb{N}_0$ be defined by $f(0) = 0$, $f(e) = 1$ for all $e \in E$, and $f(g) = 2$ for all $g \in G \setminus (E \cup \{0\})$. Then f satisfies all properties required in part 1, and we study $H(G, f) = H$.
In this case H is reduced because $H^\times \subset \{(x, 0) \mid x \in f^{-1}(0)\} = \{(0, 0)\}$ since $f^{-1}(0) = \{0\}$ by definition.

3. Let $n \in \mathbb{N}$ and $a_n = (0, n) \in H$. Then $z_n = (e_n, 1) + \cdots + (e_n, 1) \in \mathsf{Z}_n(a_n)$, $z'_n = (e'_n, 1) + \cdots + (e'_n, 1) \in \mathsf{Z}_n(a_n)$, and we assert that, for every $z \in \mathsf{Z}_n(a_n) \setminus \{z_n\}$, we have $\mathsf{d}(z_n, z) = n$. Then we find

$$\mathsf{c}_{\text{mon}}(H) \ge \mathsf{c}_{\text{eq}}(H) \ge \mathsf{c}_{\text{eq}}(a_n) \ge n,$$

whence $\mathsf{c}_{\text{mon}}(H) = \mathsf{c}_{\text{eq}}(H) = \infty$. Let $z \in \mathsf{Z}_n(a_n) \setminus \{z_n\}$. Then z has the form $z = (g_1, 1) + \cdots + (g_n, 1)$ with $g_1, \ldots, g_n \in G$. Since $1 \ge f(g_i)$ for every $i \in [1, n]$, it follows that $\{g_1, \ldots, g_n\} \subset E \cup \{0\}$. If $e_n \in \{g_1, \ldots, g_n\}$, then $g_1 = \cdots = g_n = e_n$, because E is a basis. Since $z \ne z_n$, we infer that $e_n \notin \{g_1, \ldots, g_n\}$, whence $\mathsf{d}(z_n, z) = n$.

4. Let $(g, n) \in H(G, f)$. Since $(0, 1) \notin H(G, f)^\times$ and $(g, n) = (g, 2) + (n-2)(0, 1)$, we conclude that $\max \mathsf{L}((g, n)) \in \{n-1, n\}$. Now we prove that either $\max \mathsf{L}((g, n)) = n - 1$ or $\mathsf{d}(\mathsf{Z}_{n-1}((g, n)), \mathsf{Z}_n((g, n))) = 3$. If $\max \mathsf{L}((g, n)) = n - 1$, then the assertion is trivial. Thus assume $\max \mathsf{L}((g, n)) = n$. Then there is $z \in \mathsf{Z}_n((g, n))$ of the form $z = (e_{m_1}, 1) + \cdots + (e_{m_n}, 1)$ with $e_{m_1}, \ldots, e_{m_n} \in E$ and $e_{m_1} + \cdots + e_{m_n} = g$. Now we find $z' = (e_{m_1} + e_{m_2} + e_2, 2) + (e_2, 1) + (e_{m_3}, 1) + \cdots + (e_{m_n}, 1) \in \mathsf{Z}_{n-1}((g, n))$ and $\mathsf{d}(\mathsf{Z}_{n-1}((g, n)), \mathsf{Z}_n((g, n))) \le \mathsf{d}(z, z') = 3$. This proves the assertion in the second case.
If $n \le 6$, then $\max \mathsf{L}((g, n)) \le 6$, and thus $\mathsf{c}_{\text{ad}}((g, n)) \le 6$. Let now $n \ge 7$. Then there are $n' \in [3, 6]$ and $n'' \in \mathbb{N}$ such that $n = n' + 4n''$ and $(g, n) = (g, n') + n''(0, 4)$. Since we have $z_1 = 4(e_4, 1)$, $z_2 = (e_2, 1) + (e_4, 1) + (e_2 + 3e_4, 2)$, $z_3 = 2(e_2 + 2e_4, 2) \in \mathsf{Z}((0, 4))$, $|z_1| = 4$, $|z_2| = 3$, $|z_3| = 2$, and $\mathsf{d}(z_1, z_2) = \mathsf{d}(z_2, z_3) = 3$, we find that $\mathsf{c}_{\text{ad}}((g, n)) \le \max\{3, n'\} = n' \le 6 < \infty$.

5. By [14, Theorem 3.1.5.2.a], each finitely primary monoid of rank one is tame. □

Note that, for the monoid H in Example 4.2, we have $\mathsf{c}_{\text{ad}}(H) < \infty$, and therefore the question of whether the finiteness of the tame degree implies the finiteness of the adjacent catenary degree remains open. Nevertheless, the following result from [16] might be interpreted as a strong sign that the tame degree can dominate the adjacent catenary degree.

Lemma 4.3 *(cf. [16, Theorem 5.1.b],). Let H be a tame monoid. Then there exists a constant $M \in \mathbb{N}_0$ such that, for all $a \in H$ and for each two adjacent lengths $k, l \in \mathsf{L}(a) \cap [\min \mathsf{L}(a) + M, \max \mathsf{L}(a) - M]$, we have $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) \le M$.*

Lemma 4.4 *Let H be an atomic monoid.*

1. *If H is half-factorial, then $\mathsf{c}_{\mathrm{ad}}(H) = 0$ and $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{eq}}(H) = \mathsf{c}(H)$.*
2. *If $a \in H$ satisfies $|\mathsf{L}(a)| \le 2$, then $\mu_{\mathrm{ad}}(a) \le \mathsf{t}(H)$.*

Proof Since the arithmetic of H is determined solely by H_{red}, we may assume that H is reduced.

1. Since, for all $a \in H$ with $|\mathsf{L}(a)| = 1$, we have no adjacent lengths, it follows that $\mathsf{c}_{\mathrm{ad}}(H) = 0$, and thus $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{eq}}(H)$. As—in this special situation—every chain of factorizations is an equal-length chain of factorizations, we get $\mathsf{c}_{\mathrm{eq}}(H) = \mathsf{c}(H)$.
2. Choose $a \in H$ such that $|\mathsf{L}(a)| \le 2$. If $|\mathsf{L}(a)| = 1$, then $\mu_{\mathrm{ad}}(a) = 0$. Now suppose $|\mathsf{L}(a)| = 2$. If $\mu_{\mathrm{ad}}(a) = 0$, then there is nothing to show. Now suppose $\mu_{\mathrm{ad}}(a) > 0$. Then $\mu_{\mathrm{ad}}(a) = \max \mathsf{L}(a)$, and thus $\gcd(x, y) = 1$ for all $x, y \in \mathsf{Z}(a)$ with $|x| = \min \mathsf{L}(a)$ and $|y| = \max \mathsf{L}(a)$. Let $x, y \in \mathsf{Z}(a)$ with $|x| = \min \mathsf{L}(a)$ and $|y| = \max \mathsf{L}(a)$ and choose $u \in \mathcal{A}(H)$ such that $x \in \mathsf{Z}(a) \cap uH^\times \mathsf{Z}(H)$. Then there is no $y' \in \mathsf{Z}(a) \cap uH^\times \mathsf{Z}(H)$ with $|y'| = |y|$. Now we find
$$\mathsf{t}(H) \ge \mathsf{t}(a, uH^\times) \ge \mathsf{d}(y, \mathsf{Z}(a) \cap uH^\times \mathsf{Z}(H)) = |y| = \max \mathsf{L}(a) = \mu_{\mathrm{ad}}(a). \square$$

Next we formulate another variant of the catenary degree, which is somewhat similar to the adjacent catenary degree and equals it in a special situation. The main difference is that we can prove that the m-adjacent catenary degree is finite for tame monoids when m is sufficiently large.

Definition 4.5 Let H be an atomic monoid, let $a \in H$ and let $m \in \mathbb{N}$.

1. We set
$$\mu_{\mathrm{ad},m}(a) = \sup\{k \in \mathsf{L}(a) \mid \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) = k\} \quad \text{and} \quad \mu_{\mathrm{ad},m}(H) = \sup\{\mu_{\mathrm{ad},m}(a) \mid a \in H\}.$$
2. We define
$$\mathsf{c}_{\mathrm{ad},m}(a) = \sup\{\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) \mid k \in \mathsf{L}(a)\}$$
as the *m-adjacent catenary degree* of a.
Also, $\mathsf{c}_{\mathrm{ad},m}(H) = \sup\{\mathsf{c}_{\mathrm{ad},m}(a) \mid a \in H\}$ is called the *m-adjacent catenary degree* of H.

Obviously, we find

$$\mathsf{c}_{\mathrm{ad},m}(H) \begin{cases} = 0 & m < \min \Delta(H) \\ \le \mathsf{c}_{\mathrm{ad}}(H) & \\ = \mathsf{c}_{\mathrm{ad}}(H) & \Delta(H) = \{n\} \text{ and } n \le m < 2n. \end{cases}$$

Since the definitions of the m-adjacent catenary degree and $\mu_{\mathrm{ad},m}(H)$ are similar to those of the adjacent catenary degree and $\mu_{\mathrm{ad}}(H)$, we can now prove the analog of Theorem 3.2.3 for the two newly defined invariants.

Theorem 4.6 *Let H be an atomic monoid and let $m \in \mathbb{N}$. Then*

1. $\mathsf{c}_{\mathrm{ad},m}(a) \ge \mu_{\mathrm{ad},m}(a)$ *for all* $a \in H$, *and* $\mathsf{c}_{\mathrm{ad},m}(H) = \mu_{\mathrm{ad},m}(H)$.
2. $\mathsf{c}_{\mathrm{ad},m}(H) \le \mathsf{t}(H)$ *for all* $m \ge \mathsf{t}(H)$.

Proof 1. For $m < \min \Delta(H)$, we have $\mathsf{c}_{\mathrm{ad},m}(H) = 0 = \mu_{\mathrm{ad},m}(H)$ by definition. Now let $m \in \mathbb{N}$ and $m \ge \min \Delta(H)$.

First we let $a \in H$ and show that $\mathsf{c}_{\mathrm{ad},m}(a) \ge \mu_{\mathrm{ad},m}(a)$, after which $\mathsf{c}_{\mathrm{ad},m}(H) \ge \mu_{\mathrm{ad},m}(H)$ follows by passing to the supremum on both sides. If $\mu_{\mathrm{ad},m}(a) = 0$ or $\mu_{\mathrm{ad},m}(a) = \infty$, this is trivial. Now let $\mu_{\mathrm{ad},m}(a) = k \in \mathbb{N}$ and $[k - m, k + m] \setminus \{k\} \cap \mathsf{L}(a) = \{l_1, \dots, l_n\}$. Then, by Definition 4.5.2, $\mathsf{c}_{\mathrm{ad},m}(a) \ge \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) = k = \mu_{\mathrm{ad},m}(a)$.

Now we prove $\mu_{\mathrm{ad},m}(H) \ge \mathsf{c}_{\mathrm{ad},m}(H)$. We must prove that $\mathsf{c}_{\mathrm{ad},m}(a) \le \mu_{\mathrm{ad},m}(H)$ for all $a \in H$. Assume to the contrary that there is some $a \in H$ such that $\mathsf{c}_{\mathrm{ad},m}(a) > \mu_{\mathrm{ad},m}(H)$. Let $k \in \mathbb{N}$ be minimal such that there is $a \in H$ with $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) > \mu_{\mathrm{ad},m}(H)$. If $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) = k$, then $k \le \mu_{\mathrm{ad},m}(H) < \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) = k$, a contradiction. Thus $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) < k$ and then there are some $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)$ such that $g = \gcd(x, y) \ne 1$ and $\mathsf{d}(x, y) = \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a))$. If $b = \pi_H(g^{-1}x)$, then $|x| - |g| = k - |g|$, $|y| - |g| \in \mathsf{L}(b) \cap [k - m - |g|, k + m - |g|]$ and

$$\begin{aligned} \mu_{\mathrm{ad},m}(H) &< \mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) = \mathsf{d}(x, y) \\ &\le \mathsf{d}(\mathsf{Z}_{k-|g|}(b), \mathsf{Z}_{[k-m-|g|,k+m-|g|]\setminus\{k-|g|\}}(b)) \le k - |g| < k, \end{aligned}$$

a contradiction to the minimal choice of k. Thus we infer that $\mathsf{c}_{\mathrm{ad},m}(a) \le \mu_{\mathrm{ad},m}(H)$.

2. If H is not tame, then there is no $m \in \mathbb{N}$ with $m \ge \mathsf{t}(H)$. Thus we may assume that $\mathsf{t}(H) < \infty$. Let $m \ge \mathsf{t}(H)$. If $\mu_{\mathrm{ad},m}(H) = \infty$, then $\mathsf{t}(H) > m$. Thus we may assume that $\mu_{\mathrm{ad},m}(H) < \infty$. By part 1, it suffices to show that $\mu_{\mathrm{ad},m}(a) \le \mathsf{t}(H)$ for all $a \in H$. Let $a \in H$. If $\mu_{\mathrm{ad},m}(a) = 0$, then there is nothing to show. Now suppose $\mu_{\mathrm{ad},m}(a) = k > 0$ and $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{[k-m,k+m]\setminus\{k\}}(a)) = k$. Then we have $\mathsf{L}(a) \cap [k - m, k + m] \setminus \{k\} = \{l_1, \dots, l_n\}$ and $\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_{l_i}(a)) = k$ for all $i \in [1, n]$. Then $\gcd(x, y) = 1$ for all $x \in \mathsf{Z}_k(a)$ and $y \in \mathsf{Z}_{l_1}(a)$. Now let $x \in \mathsf{Z}_k(a)$, $y \in \mathsf{Z}_{l_1}(a)$, and choose $u \in \mathcal{A}(H)$ such that $y \in \mathsf{Z}(a) \cap uH^{\times}\mathsf{Z}(H)$.

We find

$$\mathsf{t}(H) \geq \mathsf{t}(a, uH^\times) \geq \mathsf{d}(x, \mathsf{Z}(a) \cap uH^\times \mathsf{Z}(H)) = \min\{\mathsf{d}(x, \mathsf{Z}_l(a) \\ \cap uH^\times \mathsf{Z}(H)) \mid l \in \mathsf{L}(a),\ l \neq k\} \geq \min\{k, m+1\} = k = \mu_{\mathrm{ad},m}(a). \tag{4.1}$$

For $l \in [k-m, k+m]$ the inequality is clear by the choice of k. and for $|l-k| \geq m$ the inequality holds trivially since $m+1 > \mathsf{t}(H)$. □

Another interesting observation arising from the proof of Theorem 4.6.2 is the fact that the crucial inequality (4.1) might fail for $m < \mathsf{t}(H)$ for some $a \in H$ (of course, with $\mu_{\mathrm{ad},m}(a) > 0$). Additionally, Theorem 4.6.2 can never be used to bound $\mathsf{c}_{\mathrm{ad}}(H)$, since $\mathsf{c}_{\mathrm{ad}}(H) = \mathsf{c}_{\mathrm{ad},m}(H)$ for $m = \min \triangle(H)$ if $|\triangle(H)| = 1$, but then $\mathsf{t}(H) \geq m+2 > m$, and therefore Theorem 4.6.2 does not hold for $\mathsf{c}_{\mathrm{ad}}(H)$.

5 Applications to semigroup rings and generalized power series rings

The arithmetic of semigroup rings and generalized power series rings has attracted a lot of interest; for an overview, we refer to [1] and [2]; and for some recent results, we refer to [20] and [21]. Nevertheless, there are nearly no precise results on their arithmetic. In order to apply our monoid theoretic tools from Sect. 3 and [24] to the explicit computation of various arithmetical invariants of semigroup rings and generalized power series rings, we follow a 2-step strategy. In the first step, we apply transfer principles as described in much detail in [14, Sect. 3.2], and in the second step, we make use of the monoid theoretic tools.

Definition 5.1 A monoid homomorphism $\theta : H \to B$ is called a *transfer homomorphism* if it has the following properties:

T1 $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.
T2 If $a \in H$, $r, s \in B$ and $\theta(a) = rs$, then there exist $b, c \in H$ such that $\theta(b) \sim r$, $\theta(c) \sim s$, and $a = bc$.

Definition 5.2 Let $\theta : H \to B$ be a transfer homomorphism of atomic monoids and $\bar{\theta} : \mathsf{Z}(H) \to \mathsf{Z}(B)$ the unique homomorphism satisfying $\bar{\theta}(uH^\times) = \theta(u)B^\times$ for all $u \in \mathcal{A}(H)$. We call $\bar{\theta}$ the extension of θ to the factorization monoids.

For $a \in H$, the *catenary degree in the fibers* $\mathsf{c}(a, \theta)$ denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For any two factorizations $z, z' \in \mathsf{Z}(a)$ with $\bar{\theta}(z) = \bar{\theta}(z')$, there exists a finite sequence of factorizations $(z_0, z_1, \ldots, z_k)$ in $\mathsf{Z}(a)$ such that $z_0 = z$, $z_k = z'$, $\bar{\theta}(z_i) = \bar{\theta}(z)$, and $\mathsf{d}(z_{i-1}, z_i) \leq N$ for all $i \in [1, k]$; that is, z and z' can be concatenated by an N-chain in the fiber $\mathsf{Z}(a) \cap \bar{\theta}^{-1}((\bar{\theta}(z)))$.

Also, $\mathsf{c}(H, \theta) = \sup\{\mathsf{c}(a, \theta) \mid a \in H\}$ is called the *catenary degree in the fibers* of H.

We briefly fix the notation concerning sequences over finite abelian groups. Let G be an additively written, finite abelian group. For a subset $A \subset G$ and an element $g \in G$, we set $-A = \{-a \mid a \in A\}$ and $A - g = \{a - g \mid a \in A\}$. Let $\mathcal{F}(G)$ be the free abelian monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$. For a sequence $S = g_1 \cdot \ldots \cdot g_l$, we call

- $|S| = l$ the *length* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i \in G$ the *sum* of S,
- $\mathrm{supp}(S) = \{g_1, \ldots, g_l\} \subset G$ the *support* of S,
- $\Sigma(S) = \{\sum_{i \in I} g_i \mid \emptyset \neq I \subset [1, l]\} \subset G$ the *set of subsums* of S, and
- $-\Sigma(S) = \{\sum_{i \in I}(-g_i) \mid \emptyset \neq I \subset [1, l]\} = \{-g \mid g \in \Sigma(S)\} \subset G$ the *set of negative subsums* of S.

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- *zero-sum free* if there is no non-trivial zero-sum subsequence, i.e. $0 \notin \Sigma(S)$, and
- a *minimal zero-sum sequence* if S is nontrivial, $\sigma(S) = 0$, and every subsequence $S' \mid S$ with $1 \le |S'| < |S|$ is zero-sum free.

For a subset $G_0 \subset G$, we set

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\} \text{ for the } \textit{block monoid} \text{ over } G_0 \text{ and}$$
$$\mathcal{A}(G_0) = \{S \in \mathcal{F}(G_0) \mid S \text{ minimal zero-sum sequence}\} \subset \mathcal{B}(G_0).$$

Then, in fact, $\mathcal{B}(G_0)$ is an atomic monoid and $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ is its set of atoms.

The *Davenport constant* $\mathsf{D}(G_0) \in \mathbb{N}$ is defined to be the supremum of all lengths of sequences in $\mathcal{A}(G_0)$.

Definition 5.3 Let G be an additive abelian group, $G_0 \subset G$ a subset, T a monoid, $\iota : T \to G$ a homomorphism, and $\sigma : \mathcal{F}(G_0) \to G$ the unique homomorphism such that $\sigma(g) = g$ for all $g \in G_0$. Then we call

$$\mathcal{B}(G_0, T, \iota) = \{St \in \mathcal{F}(G_0) \times T \mid \sigma(S) + \iota(t) = \mathbf{0}\}$$

the *T-block monoid* over G_0 defined by ι.

If $T = \{1\}$, then $\mathcal{B}(G_0, T, \iota) = \mathcal{B}(G_0)$ is the block monoid of all zero-sum sequences over G_0 and if $G_0 = \{\mathbf{0}\}$ then $\mathcal{B}(G_0, T, \iota) = [\mathbf{0}] \times T$. Since $\mathbf{0} \in \mathcal{B}(G_0, T, \iota)$ is prime, the arithmetic of T and $\mathcal{B}(G_0, T, \iota)$ coincide in this situation.

Lemma 5.4 *Let D be an atomic monoid, $P \subset D$ a set of prime elements, and $T \subset D$ an atomic submonoid such that $D = \mathcal{F}(P) \times T$. Let $H \subset D$ be a saturated atomic submonoid, let $G = \mathsf{q}(D/H)$ be its class group, let $\iota : T \to G$ be a homomorphism defined by $\iota(t) = [t]_{D/H}$, and suppose each class in G contains some prime element from P.*

1. *The map $\beta : H \to \mathcal{B}(G, T, \iota)$, given by $\beta(pt) = [p]_{D/H} + \iota(t) = [p]_{D/H} + [t]_{D/H}$, is a transfer homomorphism onto the T-block monoid over G defined by ι and $\mathsf{c}(H, \beta) \le 2$*

2. *The following inequalities hold:*

$$\begin{aligned} \mathsf{c}(\mathcal{B}(G,T,\iota)) \le \quad \mathsf{c}(H) \quad &\le \max\{\mathsf{c}(\mathcal{B}(G,T,\iota)), \mathsf{c}(H,\beta)\},\\ \mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)) \le \mathsf{c}_{\mathrm{mon}}(H) &\le \max\{\mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)), \mathsf{c}(H,\beta)\}, \text{ and}\\ \mathsf{t}(\mathcal{B}(G,T,\iota)) \le \quad \mathsf{t}(H) \quad &\le \mathsf{t}(\mathcal{B}(G,T,\iota)) + \mathsf{D}(G) + 1. \end{aligned}$$

   In particular, the equality $\mathsf{c}(H) = \mathsf{c}(\mathcal{B}(G,T,\iota))$ *holds if* $\mathsf{c}(\mathcal{B}(G,T,\iota)) \ge 2$, *and the equality* $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota))$ *holds if* $\mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G,T,\iota)) \ge 2$.
3. $\mathcal{L}(H) = \mathcal{L}(\mathcal{B}(G,T,\iota))$, $\triangle(H) = \triangle(\mathcal{B}(G,T,\iota))$, $\min\triangle(H) = \min\triangle(\mathcal{B}(G,T,\iota))$, *and* $\rho(H) = \rho(\mathcal{B}(G,t,\iota))$.
4. *We set* $\mathcal{B} = \{S \in \mathcal{B}(G,T,\iota) \mid \mathbf{0} \nmid S\}$. *Then* $\mathcal{B}$ *and* $\mathcal{B}(G,T,\iota)$ *have the same arithmetical properties, and*

$$\begin{aligned} \mathsf{c}(\mathcal{B}) \le \quad \mathsf{c}(H) \quad &\le \max\{\mathsf{c}(\mathcal{B}), \mathsf{c}(H,\beta)\},\\ \mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \le \mathsf{c}_{\mathrm{mon}}(H) &\le \max\{\mathsf{c}_{\mathrm{mon}}(\mathcal{B}), \mathsf{c}(H,\beta)\}, \text{ and}\\ \mathsf{t}(\mathcal{B}) \le \quad \mathsf{t}(H) \quad &\le \mathsf{t}(\mathcal{B}) + \mathsf{D}(G) + 1. \end{aligned}$$

   In particular, the equality $\mathsf{c}(H) = \mathsf{c}(\mathcal{B})$ *holds if* $\mathsf{c}(\mathcal{B}) \ge 2$, *and the equality* $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B})$ *holds if* $\mathsf{c}_{\mathrm{mon}}(\mathcal{B}) \ge 2$.
   Additionally, $\mathcal{L}(H) = \mathcal{L}(\mathcal{B})$, $\triangle(H) = \triangle(\mathcal{B})$, $\min\triangle(H) = \min\triangle(\mathcal{B})$, *and* $\rho(H) = \rho(\mathcal{B})$.

Proof 1. Follows by [14, Proposition 3.2.3.3 and Proposition 3.4.8.2].
2. The assertion on the catenary degree follows by [14, Theorem 3.2.5.5], the assertion on the monotone catenary degree by [14, Lemma 3.2.6], and the assertion on the tame degree by [14, Theorem 3.2.5.1].
3. Follows by [14, Proposition 3.2.3.5].
4. Since $\mathbf{0} \in \mathcal{B}(G,T,\iota)$ is a prime element, it defines a partition $\mathcal{B}(G,T,\iota) = [\mathbf{0}] \times \mathcal{B}$ with $\mathcal{B} = \{S \in \mathcal{B}(G,T,\iota) \mid \mathbf{0} \nmid S\}$. Thus all studied arithmetical invariants coincide for $\mathcal{B}$ and $\mathcal{B}(G,T,\iota)$. Now the assertions follow from part 2 and part 3. □

From now on, we write monoids additively. Then, for a reduced monoid H, $H^\times = \{0\}$.

Definition 5.5 Let K be a field and H a reduced atomic monoid. Then we call

- $K[\![H]\!] = K[\![X^s \mid s \in \mathcal{A}(H)]\!]$ the *generalized power series ring*
- $K[H] = K[X^s \mid s \in \mathcal{A}(H)]$ the *semigroup ring*

defined by H over K.

We restrict ourselves to the simplest monoids possible, i.e., to numerical monoids. Recall that a submonoid $H \subset (\mathbb{N}_0, +)$ is called a numerical monoid if the set of gaps $\mathsf{G}(H) = \mathbb{N}_0 \setminus H$ is finite. Special cases of generalized power series rings and semigroup rings are studied in [14, Example 3.7.3]).

Theorem 5.6 *Let $H \subset (\mathbb{N}_0, +)$ be a numerical monoid.*

1. *$K[\![H]\!]$ has finite catenary degree and finite elasticity.*
2. *Let*

$$\phi : \begin{cases} K[\![H]\!]^\bullet & \to & H \\ f = \sum_{h \in H} f_h X^h & \mapsto & \min\{h \in H \mid f_h \neq 0\}\,. \end{cases}$$

Then ϕ is a transfer homomorphism if and only if there is some $m \in \mathbb{N}$ such that $H = [m, m+1, \ldots, 2m-1]$ (equivalently, $H = \mathbb{N}_{\geq m} \cup \{0\}$).

Proof 1. See [15, Proposition 6.10 and Theorem 6.7].

2. Suppose first that $m \in \mathbb{N}$ and $H = \mathbb{N}_{\geq m} \cup \{0\}$. Since $\phi^{-1}(0) = K^\times = K[\![H]\!]^\times$, condition (T1) holds. For the proof of (T2), let $u \in K[\![H]\!]^\bullet$ and $\phi(u) = d = a + b$ for some $a,\ b \in H$. We may assume that $a,\ b \in H \setminus \{0\}$. Then $a + b \geq 2m$, and if $h \in H$ and $h > a + b$, then $h - a > b \geq m$ and thus $h - a \in H$. If

$$u = u_d X^d + \sum_{h > d} u_h X^h\,, \quad \text{where} \quad u_d \in K^\times \ \text{and} \quad u_h \in K \ \text{ for all } h > d\,,$$

then

$$u = X^a \Big(u_d X^b + \sum_{h > d} u_h X^{h-a} \Big) = vw\,, \quad \text{where} \quad \phi(v) = a \ \text{ and } \ \phi(w) = b\,.$$

To prove the converse, assume that there is some $b \in H \setminus \{0\}$ such that $b + 1 \notin H$, and yet (T2) holds. Let $b \in H$ be maximal such that $b + 1 \notin H$. Then $2b,\ 2b+1 \in H$, hence $u = X^{2b} + X^{2b+1} \in K[\![H]\!]$, and $\phi(u) = b + b$. By (T2), there exist $v,\ w \in K[\![H]\!]$ such that $\phi(v) = \phi(w) = b$ and $vw = u$. But as $b + 1 \notin H$, it follows that

$$v = v_b X^b + \sum_{\substack{h \in H \\ h \geq b+2}} v_h X^h\,, \qquad w = w_b X^b + \sum_{\substack{h \in H \\ h \geq b+2}} w_h X^h\,,$$

and hence $vw \neq u$, a contradiction. □

For the study of semigroup rings, the situation is even more difficult, since there is then no transfer homomorphism $R = F[H] \to H$; see [14, Example 3.7.3, Special Case 3.3]. Thus—even after applying the transfer principles in order to be in an easier situation—it is necessary to compute all the invariants of nonunique factorization for more general T-block monoids, $\mathcal{B}(G, T, \iota)$, where neither T nor G are trivial. In the upcoming subsections, we exploit the results from [5,6], [24, Proposition 16] (repeated as Lemma 2.7), [24, Theorem 19.2], and Sect. 3 (mainly Theorem 3.2) together with recent programming techniques (see [17] and [23, Sect. 8]) and parallelization to explicitly compute various arithmetical invariants, namely, the elasticity, the catenary degree, the monotone catenary degree, and a bound for the tame degree of the T-block monoids associated with the studied domains, and therefore for the domains themselves.

5.1 Preliminaries about zero-sum sequences and T-block monoids

In order to be able to describe the set of atoms of a T-block monoid precisely, we use the terminology of sequences over finite abelian groups.

For our algorithmic considerations in the forthcoming sections, it will be very useful to have some sort of order defined on the elements of a finite abelian group G. By the structure theorem for finitely generated abelian groups, there are uniquely determined $r \in \mathbb{N}_0$ and $n_1, \ldots, n_r \in \mathbb{N}$ such that there is a group isomorphism $\varphi : G \to \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$ and $1 < n_1 \mid \cdots \mid n_r$. For $i \in [1, r)$, we choose $[0, n_i)$ as a system of representatives for $\mathbb{Z}/n_i\mathbb{Z}$. Now we can compare two elements $g_1, g_2 \in G$ by comparing $\varphi(g_1)$ and $\varphi(g_2)$ with respect to the lexicographic order. For short, we simply write $g_1 \leq g_2$ respectively $g_1 \geq g_2$.

In particular, in Sect. 5.4, we will need some kind of coordinate representation for the elements of a T-block monoid, i.e., a monoid isomorphism mapping a T-block monoid onto a submonoid of $\mathbb{Z}^m \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}$ for some $m, r \in \mathbb{N}_0$ and $n_1, \ldots, n_r \in \mathbb{N}$. Let G be a finite abelian group, T a finitely generated monoid, and $\iota : T \to G$ a homomorphism. Let $T = D_1 \times \cdots \times D_r$ be a product of finitely primary monoids $D_i \subset [p_1^{(i)}, \ldots, p_{r_i}^{(i)}] \times \widehat{D_i}^{\times} = \widehat{D_i}$ where $r_i \in \mathbb{N}$, and the $\widehat{D_i}^{\times}$ are finitely generated abelian groups for $i \in [1, r]$. Then there are uniquely determined $l_i, k_i \in \mathbb{N}_0$ such that there is an isomorphism $\phi_i : \widehat{D_i}^{\times} \to \mathbb{Z}^{l_i} \times \mathbb{Z}/n_1^{(i)}\mathbb{Z} \times \cdots \times \mathbb{Z}/n_{k_i}^{(i)}$ with $1 < n_1^{(i)} \mid \cdots \mid n_{k_i}^{(i)}$ for $i \in [1, r]$. This isomorphism can be extended to an isomorphism $\bar{\phi}_i : \widehat{D_i} \to N_0^{r_i} \times \phi_i(\widehat{D_i}^{\times})$ for $i \in [1, r]$. Now there is an isomorphism $\phi = \bar{\phi}_1 \times \cdots \times \bar{\phi}_r : \widehat{T} \to \bar{\phi}_1(\widehat{D_1}) \times \cdots \times \bar{\phi}_r(\widehat{D_r})$. This again can be extended to an isomorphism $\bar{\varphi} : \mathcal{F}(G) \times \widehat{T} \to N_0^{|G|} \times \phi(\widehat{T})$. Now we can define the desired isomorphism by restriction of $\bar{\varphi}$ to the T-block monoid $\mathcal{B}(G, T, \iota)$ as follows:

$$\varphi = \bar{\varphi}|\mathcal{B}(G, T, \iota) : \mathcal{B}(G, T, \iota) \to \bar{\varphi}(\mathcal{B}(G, T, \iota)) \subset \mathbb{N}_0^{|G|} \times \prod_{i=1}^{r} \left(\mathbb{N}_0^{r_i} \times \mathbb{Z}^{l_i} \times \prod_{j=1}^{k_i} \mathbb{Z}/n_j^{(i)}\mathbb{Z} \right). \tag{5.1}$$

5.2 The set of atoms $\mathcal{A}(G)$ of a block monoid

Based on ideas from [17], we give an algorithm for the computation of the set of atoms $\mathcal{A}(G)$ for a finite additive abelian group G. The problem of computing $\mathcal{A}(G)$ grows exponentially in terms of $|G|$, but, for very small groups as the ones involved in Sect. 5.5, it can be easily performed—sometimes even by hand. Unfortunately, we have to do some sort of brute force search in the set of all $S \in \mathcal{F}(G)$ with $|S| \leq \mathsf{D}(G)$. But with the algorithm presented below, we can avoid most of the redundant checks and therefore speed up the computation dramatically.

Since modular arithmetic on vectors with multiple coordinates is quite inefficient, it is necessary for a fast execution of the RAS, Algorithm 1, to pre-compute the sums

Algorithm 1 Recursive Atom Search: $A \leftarrow \text{RAS}(A, S, \Sigma, B)$

```
for all g ∈ B do
  S′ ← Sg
  if g ≤ −σ(S′) then
    A ← A ∪ {S′(−σ(S′))}
  end if
  Σ′ ← Σ
  B′ ← ∅
  for all g′ ∈ B do
    if g + g′ ∈ Σ then
      Σ′ ← Σ′ ∪ {g′}
    else
      B′ ← B′ ∪ {g′}
    end if
  end for
  if |B′| > 0 then
    A ← RAS(A, S′, Σ′, B′)
  end if
end for
return A
```

Algorithm 2 Atoms Computation Algorithm 1: $\mathcal{A}(G) \leftarrow \text{ACA1}(G)$

```
A ← {0}
for all g ∈ G \ {0} do
  if g ≤ −g then
    A ← A ∪ {g(−g)}
  end if
  Σ ← {0, g}
  B ← G \ {0, −g}
  S ← g
  if |B| > 0 then
    A ← RAS(A, S, Σ, B)
  end if
end for
return A
```

$g + g'$. This can be done once in the ACA1, Algorithm 2, before the main loop. For additional details on speeding up these types of algorithms by special alignment of the pre-computed data and on the parallelization aspects, the reader is referred to [17, Sect. 3].

5.3 The set of atoms of a T-block monoid

Lemma 5.7 *Let G be a finite additive abelian group, T a reduced atomic monoid, $\iota : T \to G$ a homomorphism, and $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$ the T-block monoid over G defined by ι. Furthermore, suppose each class in G contains some $p \in P$, and let $\bar{\iota} : \mathsf{Z}(T) \to \mathcal{F}(G)$ be the homomorphism generated by the extension of ι onto $\mathsf{Z}(T)$ such that, for a factorization $z = a_1 \cdot \dots \cdot a_n \in \mathsf{Z}(T)$ with $a_i \in \mathcal{A}(T)$ for $i \in [1, n]$, we have $\bar{\iota}(z) = \iota(a_1) \cdot \dots \cdot \iota(a_n)$.*

Then we have

$$\begin{aligned}&\mathcal{A}(\mathcal{B}(G,T,\iota)) =\\ &\quad \{S\pi(z) \mid S \in \mathcal{F}(G),\ z \in \mathsf{Z}(T),\ S\bar{\iota}(z) \in \mathcal{A}(G), \nexists n \geq 2 :\ \exists S_i \in \mathcal{F}(G),\ z_i \in \mathsf{Z}(T)\\ &\quad \textit{with } S_i\bar{\iota}(z_i) \in \mathcal{A}(G) \textit{ for } i \in [1,n] :\ S_1\pi(z_1)\cdot\dots\cdot S_n\pi(z_n) = S\pi(z)\} \qquad (5.2)\end{aligned}$$

Proof Clearly, every atom $a \in \mathcal{A}(\mathcal{B}(G,T,\iota))$ is of the form $a = S\pi(z)$ with $S \in \mathcal{F}(G)$, $z \in \mathsf{Z}(T)$, and $S\bar{\iota}(z) \in \mathcal{A}(G)$. Now suppose we have $n \in [2, \mathsf{D}(G)]$, $S_i \in \mathcal{F}(G)$, $z_i \in \mathsf{Z}(T)$, $S_i\bar{\iota}(z_i) \in \mathcal{A}(G)$ for $i \in [1,n]$ and $S\pi(z) = S_1\pi(z_1)\cdot\dots\cdot S_n\pi(z_n)$. Obviously then, $a \notin \mathcal{A}(\mathcal{B}(G,T,\iota))$. Now the other inclusion is obvious. □

In general, it is very hard to calculate $\mathcal{A}(\mathcal{B}(G,T,\iota))$ explicitly by the characterization in (5.2). But if we restrict ourselves to a finite group G and a finitely generated reduced monoid T such that $\mathcal{A}(G)$, $\mathcal{A}(T)$, and $\iota(a)$ for $a \in \mathcal{A}(T)$ are all known explicitly, we can formulate the ACA2, Algorithm 3, for the computation of the set of atoms of a T-block monoid.

Algorithm 3 Atoms Computation Algorithm 2: $\mathcal{A}(\mathcal{B}(G,T,\iota)) \leftarrow \mathrm{ACA2}(G,T,\mathcal{A}(G),\mathcal{A}(T),\iota)$

```
A ← ∅
D ← 0
for all S ∈ A(G) do
  if |S| > D then
    D ← |S|
  end if
  A ← A ∪ {(S, 1)}
end for
F_0 ← ∅
for all a ∈ A(T) do
  for all (S, 1) ∈ A do
    if ι(a) | S then
      F_0 ← F_0 ∪ {(ι(a)^{-1} S, a)}
    end if
  end for
end for
E ← ∅
n ← 1
while n < D and F_{n-1} ≠ ∅ do
  E ← E ∪ F_{n-1}
  E ← E F_0
  F_n ← ∅
  for all a ∈ A(T) do
    for all (S, b) ∈ A do
      if ι(a) | S then
        F_n ← F_n ∪ {(ι(a)^{-1} S, ab)}
      end if
    end for
  end for
  n ← n + 1
end while
return A ∪ F_0 ∪ ... ∪ F_{n-1}
```

5.4 Computing arithmetical invariants of a T-block monoid

Throughout this section, we implicitly use the isomorphism defined in (5.1). Thus we only have to work with submonoids

$$S \subset \mathbb{Z}^m \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z} \quad \text{with } m,\ r \in \mathbb{N}_0 \text{ and } n_1, \ldots, n_r \in \mathbb{N}$$

such that $S \cong \varphi(\mathcal{B}(G, T, \iota))$, where G is an additively written finite abelian group, T is a product of finitely many reduced finitely primary monoids of rank 1, $\iota : G \to T$ is a homomorphism, and φ is the isomorphism defined in (5.1). If T is not the product of finitely many reduced finitely primary monoids of rank 1, then T would not be finitely generated. Now we know $\mathcal{A}(S)$ explicitly, since, obviously, $\mathcal{A}(S) = \varphi(\mathcal{A}(\mathcal{B}(G, T, \iota)))$ and $\mathcal{A}(\mathcal{B}(G, T, \iota))$ can be computed explicitly by the ACA2; see Algorithm 3.

For the computation of the tame degree, we use the definition of the distance of factorizations and [24, Theorem 19.2]; for additional reference on this computation, see [6, Sect. 4].

Now we are ready to describe the computation step by step.

5.4.1 Finding the elements of $\mathcal{A}(\sim_S)$

The first step is finding the elements of $\mathcal{A}(\sim_S)$ explicitly. Unfortunately, this is a very hard task. Probably, the most efficient way is the following one as described in [5, Sects. 1 and 2].

1. Since we know $\mathcal{A}(S)$ explicitly, we can write the atoms of S in their coordinates as vectors:

$$\mathcal{A}(S) = \left\{(a_1^{(1)}, \ldots, a_m^{(1)}, a_{m+1}^{(1)} \mod n_1, \ldots, a_{m+r}^{(1)} \mod n_r), \ldots\right\}.$$

2. By [5, Sect. 2], finding the elements of $\mathcal{A}(\sim_S)$ is equivalent to determining the minimal positive solutions of the following system of linear diophantine equations:

$$\begin{array}{lllll} x_1 a_1^{(1)} + \ldots + x_k a_1^{(k)} & - y_1 a_1^{(1)} & - \ldots - y_k a_1^{(k)} & = 0 & \\ \vdots & \vdots & \vdots & \vdots & \\ x_1 a_m^{(1)} + \ldots + x_k a_m^{(k)} & - y_1 a_m^{(1)} & - \ldots - y_k a_m^{(k)} & = 0 & \\ x_1 a_{m+1}^{(1)} + \ldots + x_k a_{m+1}^{(k)} & - y_1 a_{m+1}^{(1)} & - \ldots - y_k a_{m+1}^{(k)} & \equiv 0 & \mod n_1 \\ \vdots & \vdots & \vdots & \vdots & \\ x_1 a_{m+r}^{(1)} + \ldots + x_k a_{m+r}^{(k)} & - y_1 a_{m+r}^{(1)} & - \ldots - y_k a_{m+r}^{(k)} & \equiv 0 & \mod n_r \end{array} \tag{5.3}$$

   We write a solution $(x_1, \ldots, x_k, y_1, \ldots, y_k)$ as $((x_1, \ldots, x_k), (y_1, \ldots, y_k))$.

3. Again, by [5, Sect. 2] and [27, Sect. 2], finding the set of minimal positive solutions is equivalent to finding the set of minimal positive solutions for the following enlarged system and then projecting back by the map and removing the zero

element (if appearing after the projection) from the set of solutions:

$$\begin{array}{lllllll} x_1 a_1^{(1)} & + \ldots - y_1 a_1^{(1)} & - \ldots & & & = 0 \\ \vdots & \vdots & & & & \vdots \\ x_1 a_m^{(1)} & + \ldots - y_1 a_m^{(1)} & - \ldots & & & = 0 \\ x_1 a_{m+1}^{(1)} & + \ldots - y_1 a_{m+1}^{(1)} & - \ldots & + x_{k+1} n_1 & - y_{k+1} n_1 & = 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ x_1 a_{m+r}^{(1)} & + \ldots - y_1 a_{m+r}^{(1)} & - \ldots & + x_{k+r} n_r & - y_{k+r} n_r & = 0 \end{array} \tag{5.4}$$

$$\Phi : \begin{cases} \mathbb{N}_0^{k+r} \times \mathbb{N}_0^{k+r} \rightarrow \mathbb{N}_0^{k} \times \mathbb{N}_0^{k} \\ ((x_1, \ldots, x_{k+r}), (y_1, \ldots, y_{k+r})) \mapsto ((x_1, \ldots, x_k), (y_1, \ldots, y_k)). \end{cases}$$

One of the most efficient algorithms for finding these solutions is due to Contejean and Devie; see [7]. Nevertheless, this might take a very long time since the problem of determining the set of all minimal non-negative solutions of a system of linear diophantine equations is well known to be NP-complete.

5.4.2 Removing unnecessary elements

Clearly, elements of the form $((0, \ldots, 0, 1, 0, \ldots, 0), (0, \ldots, 0, 1, 0, \ldots, 0))$ are minimal solutions. But as elements of $\mathcal{A}(\sim_S)$, these elements do not carry any information about the arithmetic of S. Therefore we may simply drop them. Since, for any two factorizations, $(x, y) \in \mathsf{Z}(S)$ is equivalent to $(y, x) \in \mathsf{Z}(S)$, we may also reduce the number of pairs by a factor of two. This smaller set will be denoted by $\mathcal{A}(\sim_S)^* = \{((x_1, \ldots, x_k), (y_1, \ldots, y_k)), \ldots\}$.

5.4.3 Computing the elasticity

By our finiteness assumptions on T, i.e., since T is finitely generated, we know the set $\mathcal{A}(\sim_S)^*$ is finite. Thus we can simply compute the elasticity using [24, Proposition 14.2] as follows:

$$\rho(S) = \max \left\{ \frac{x_1 + \cdots + x_k}{y_1 + \cdots + y_k}, \frac{y_1 + \cdots + y_k}{x_1 + \cdots + x_k} \,\middle|\, ((x_1, \ldots, x_k), (y_1, \ldots, y_k)) \in \mathcal{A}(\sim_S)^* \right\}.$$

5.4.4 Computing the catenary degree

By Lemma 2.7.2, we need only consider elements $a \in S$ such that their factorizations appear as part of an element of $\mathcal{A}(\sim_S)$ and such that their sets of factorizations consist of more than one $\mathcal{R}$-equivalence class. Then we get the catenary degree by taking the maximum over $\mu(a)$ for all those $a \in S$.

Algorithm 4 Recursive $\mathcal{R}$-Class Finder: $\mathcal{R} \leftarrow \text{RCF}(\mathcal{R}, \mathsf{Z} = \{z_1, \ldots, z_n\})$

```
r ← {z_1}
Z ← Z \ {z_1}
n ← n − 1
Z = {z_1, ..., z_n} {renumber}
i ← 1
while i < n do
  for i = 1 to n do
    if gcd(z_i, x) ≠ 1 for some x ∈ r then
      r ← r ∪ {z_i}
      Z ← Z \ {z_i}
      n ← n − 1
      Z = {z_1, ..., z_n} {renumber}
      break
    end if
  end for
end while
R ∪ {r}
if Z ≠ ∅ then
  R ← RCF(R, Z)
end if
return R
```

Algorithm 5 Catenary degree Computation Algorithm: $\mathsf{c}(S) \leftarrow \text{CCA}(\mathcal{A}(S), \mathcal{A}(\sim_S)^*)$

```
A ← ∅
for all (x, y) ∈ A(∼_S)* do
  A ← A ∪ {π(x)}
end for
c ← 0
for all a ∈ A do
  R_a ← RCF(Z(a))
  if |R_a| > 1 then
    μ ← min{|x| | R_a}
    if c < μ then
      c ← μ
    end if
  end if
end for
return c
```

5.4.5 Computing the tame degree

After having computed $\mathsf{Z}(a)$ for all $a \in S$ such that $\mathcal{A}_a(\sim_S) \neq \emptyset$, we can apply [24, Theorem 19.1] for every $u \in \mathcal{A}(S)$. Since there are only finitely many, we get the tame degree as the maximum of these values.

5.4.6 Computing the monotone catenary degree

For computing the monotone catenary degree, we compute the equal catenary degree $\mathsf{c}_{\text{eq}}(S)$ and the adjacent catenary degree $\mathsf{c}_{\text{ad}}(S)$. We start with the adjacent catenary degree and proceed like in 5.4.1. We use the fact that $\sim_{S,\text{mon}} = \{(x, y) \in \sim_S | \ |x| \leq$

$|y|\}$ and again [5, Sect. 2]. Now finding the elements of $\mathcal{A}(\sim_{S,\mathrm{mon}})$ is equivalent to determining the minimal positive solutions of a system of linear diophantine equations.

Before we construct this finite system of linear diophantine equations explicitly, we formulate a short lemma.

Lemma 5.8 *Let H be a finitely generated monoid.*
Then $\sim_{H,\mathrm{mon}}$ is a finitely generated Krull monoid.

Proof Let H be a finitely generated monoid. Since $\sim_H \subset \mathsf{Z}(H) \times \mathsf{Z}(H)$ is then a saturated submonoid of a finitely generated monoid, $\sim_H$ is finitely generated by [14, Proposition 2.7.5]. Now assume $\sim_H$ has $n \in \mathbb{N}$ generators. Then the atoms of $\sim_H$ can be described as the minimal solutions of a system of finitely many, say k, linear diophantine equations in $2n$ variables as in step 5.4.1 above. Then the atoms of $\sim_{H,\mathrm{mon}}$ can be described as the minimal solutions of a system of $k+1$ linear diophantine equations in $2n+1$ variables—see below for the explicit description of this system of linear diophantine equations. Thus H is a finitely generated Krull monoid by [14, Theorem 2.7.14]. □

The system is (5.3), with one additional variable z and one equation, namely,

$$x_1 + \cdots + x_k - y_1 - \cdots - y_k + z = 0.$$

The coefficients at z are zero in all other equations. Now we have two possibilities.

- Either we proceed by the same steps as in 5.4.1 and solve this directly
- or we use the incremental version of the algorithm of Devie and Contejoud (see [7, Sect. 9]) and the set $\mathcal{A}(\sim_S)$, which we already computed in 5.4.1.

Next we can reduce the set of relations which we must consider, as in 5.4.2. By Theorem 3.2.4, we have to consider only elements $a \in S$ such that $\mathcal{A}_a(\sim_{S,\mathrm{mon}}) \neq \emptyset$. Then we get the adjacent catenary degree by taking the maximum over $\mu_{\mathrm{ad}}(a)$ for all those a. For the computation of the equal catenary degree, we must know the elements of $\mathcal{A}(\sim_{S,\mathrm{eq}})$. But these are already known, since $\mathcal{A}(\sim_{S,\mathrm{eq}}) \subset \mathcal{A}(\sim_{S,\mathrm{mon}})$. Here we can again reduce the set of relations which we must consider, as in 5.4.2. By Theorem 3.2.4, we have to consider only elements $a \in S$ such that $\mathcal{A}_a(\sim_{S,\mathrm{eq}}) \neq \emptyset$ and $|\mathcal{R}_{a,k}| > 1$ for some $k \in \mathsf{L}(a)$. Now this can be done by applying the RCF, Algorithm 4, to $\mathsf{Z}_k(a)$ instead of $\mathsf{Z}(a)$. Then we get the equal catenary degree by taking the maximum over $\mu_{\mathrm{eq}}(a)$ for all those a.

Now we find the monotone catenary degree by $\mathsf{c}_{\mathrm{mon}}(S) = \max\{\mathsf{c}_{\mathrm{ad}}(S), \mathsf{c}_{\mathrm{eq}}(S)\}$.

5.4.7 Reducing the computation time for the catenary degree

If we are only interested in the computation of the catenary degree, we can speed up the very time consuming computations in Step 5.4.1 in the following way. In favor of Lemma 2.7.2, we may restrict our search for minimal solutions of the system of linear diophantine equations (5.4) to solutions $(x_1, \ldots, x_{k+r}, y_1, \ldots, y_{k+r})$ such that $\sum_{i=1}^k x_i \le \mathsf{c}(S)$ and $\sum_{i=1}^k y_i \le \mathsf{c}(S)$. Of course, we do not know $\mathsf{c}(S)$ a priori, but we may replace it with any upper bound—the better the bound, the faster the computation.

In our special situation of T-block monoids, we can find a reasonably good bound by [14, Theorem 3.6.4.1] and by [14, Proposition 3.6.6]. Formulated in our terminology, these results read as follows.

Theorem 5.9 *Let G be an additively written abelian group, T a reduced finitely generated monoid, $\iota : T \to G$ a homomorphism, and $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$ the T-block monoid over G defined by ι. Then*

1. $\rho(\mathcal{B}(G, T, \iota), \mathcal{F}(G) \times T) \le \rho(T)$.
2. $\mathsf{c}(\mathcal{B}(G, T, \iota)) \le \rho(T)\mathsf{D}(G) \max\{\mathsf{c}(T), \mathsf{D}(G)\}$.

Now we set $C = \rho(T)\mathsf{D}(G) \max\{\mathsf{c}(T), \mathsf{D}(G)\}$ for the upper bound. Though this does not speed up the search for minimal solutions itself that much, it is a very efficient (additional) termination criterion in our variant of the algorithm due to Contejean and Devie; for reference on the originally proposed algorithm, see [7].

Unfortunately, this method has one drawback for the computation of the elasticity and the tame degree. As we no longer compute all minimal solutions to our system of linear diophantine equations, we no longer compute all elements in $\mathcal{A}(\sim_S)$, and therefore we cannot compute more than a lower bound for the elasticity in Step 5.4.3 and for the tame degree in Step 5.4.5.

5.4.8 Computing the elasticity from an appropriate subset of $\mathcal{A}(\sim_S)$

In [8], Domenjoud proposed an algorithm for computing the set of minimal solutions of a system of linear diophantine equations, which computes the set of minimal solutions with minimal support in a first step. All other minimal solutions can then be found by "appropriate" linear combinations of them using non-negative rational coefficients. With this interesting fact in mind, we consider the following lemma.

Definition 5.10 Let H be an atomic monoid. For $x \in \mathsf{Z}(H)$, we set

$$\operatorname{supp}(x) = \{u \in \mathcal{A}(H_{\text{red}}) \mid u \mid x\}.$$

Lemma 5.11 *Let H be a finitely generated monoid. Then*

$$\rho(H) = \sup\left\{ \frac{|x|}{|y|} \,\middle|\, (x, y) \in \mathcal{A}'(\sim_H) \right\},$$

where $\mathcal{A}'(\sim_H) = \{(x, y) \in \mathcal{A}(\sim_H) \mid \operatorname{supp}(x) \cup \operatorname{supp}(y)$ is minimal under inclusion$\}$.

Proof Let $(x, y) \in \sim_H$. Then there are $n \in \mathbb{N}$, $(x_i, y_i) \in \mathcal{A}'(\sim_H)$, and $q_i \in \mathbb{Q}$ with $0 \le q_i < 1$ for $i \in [1, n]$ such that

$$(x, y) = \prod_{i=1}^{n} (x_i, y_i)^{q_i}.$$

Such a decomposition exists, since the equivalent one exists for the set of solutions of the associated system of linear diophantine equations, see [8, Theorem 3]. When we pass to the lengths, we find $|x| = \sum_{i=1}^n q_i |x_i|$ and $|y| = \sum_{i=1}^n q_i |y_i|$. This yields

$$\frac{|x|}{|y|} \cdot |y| = |x| = \sum_{i=1}^{n} q_i |x_i| = \sum_{i=1}^{n} q_i \frac{|x_i|}{|y_i|} |y_i| \le \max_{i=1}^{n} \frac{|x_i|}{|y_i|} \sum_{i=1}^{n} q_i |y_i| = \max_{i=1}^{n} \frac{|x_i|}{|y_i|} \cdot |y| \, .$$

Thus we find

$$\frac{|x|}{|y|} \le \max_{i=1}^{n} \frac{|x_i|}{|y_i|}.$$

Since $\mathcal{A}'(\sim_H) \subset \mathcal{A}(\sim_H)$, the assertion now follows by [24, Proposition 14.2]. □

Thus we can restrict ourselves to minimal solutions with minimal support for computing the elasticity.

As far as computational performance is concerned, the most interesting point of this approach is that there are straightforward optimizations of Domenjoud's algorithm for symmetric systems of linear diophantine equations like the one in (5.4).

5.5 Three explicit examples

Let $p \in \mathbb{P}$ be a prime. Let $R = \mathbb{F}_p[X^{e_1}, X^{e_2}]$ with $e_1, e_2 \ge 2$ being coprime. Then R is a one-dimensional noetherian domain with integral closure $\widehat{R} = \mathbb{F}_p[X]$ and conductor $\mathfrak{f} = (R : \widehat{R}) = X^f \widehat{R}$, where $X \in \widehat{R}$ is a prime element and f is the Frobenius number of the numerical monoid generated by e_1 and e_2. Thus R is an order in the Dedekind domain $\widehat{R}$, and $X\widehat{R}$ is the only maximal ideal of $\widehat{R}$ containing $\mathfrak{f}$. Furthermore, $\widehat{R}^\times = R^\times = \mathbb{F}_p^\times$. By the computations in [14, Special case 3.2 in Example 3.7.3], we have $G = \mathrm{Pic}(R) \cong \mathbb{F}_p$.

For a more detailed presentation of the explicit computations, the reader is referred to [25, Sect. 2.3.5].

5.5.1 $\mathbb{F}_3[X^2, X^3]$

Now let $p = 3$. Since $|G| = 3$, we write $G = \{\mathbf{0}, \mathbf{e}, \mathbf{e}'\}$. Clearly—or by applying the ACA1, see Algorithm 2—we have $\mathcal{A}(G) = \{\mathbf{0}, \mathbf{e}\mathbf{e}', \mathbf{e}^3, \mathbf{e}'^3\}$. Now we apply [14, Theorem 3.7.1] and switch to the block monoid, which is a T-block monoid over G, say $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by $\mathcal{A}(T) = \{X^n g \mid n \in \{2, 3\},\ g \in G\}$ and ι is the uniquely determinated homomorphism $\iota : G \to T$ such that $\iota(X^n g) = g$ for all $n \in \{2, 3\}$ and $g \in G$.

Now we apply the ACA2, see Algorithm 3, and find

$$\begin{aligned}\mathcal{A}(\mathcal{B}(G, T, \iota)) = \{&(\mathbf{0}, 1), (\mathbf{e}\mathbf{e}', 1), (\mathbf{e}^3, 1), (\mathbf{e}'^3, 1), (1, X^2\mathbf{0}), (1, X^3\mathbf{0}),\\ &(\mathbf{e}, X^2\mathbf{e}'), (\mathbf{e}, X^3\mathbf{e}'), (\mathbf{e}', X^2\mathbf{e}), (\mathbf{e}', X^3\mathbf{e}), (\mathbf{e}^2, X^2\mathbf{e}),\\ &(\mathbf{e}^2, X^3\mathbf{e}), (\mathbf{e}'^2, X^2\mathbf{e}'), (\mathbf{e}'^2, X^3\mathbf{e}')\}.\end{aligned}$$

Using the construction from the beginning of Sect. 5.4, we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/3\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G, T, \iota) \cong S \subset \mathbb{N}_0^4 \times \mathbb{Z}/3\mathbb{Z}\,.$$

Then, for the set of atoms, we find

$$\begin{aligned}\mathcal{A}(S) = \{&(1,0,0,0,\bar{0}), (0,1,1,0,\bar{0}), (0,3,0,0,\bar{0}), (0,0,3,0,\bar{0}), (0,0,0,2,\bar{0}),\\ &(0,0,0,3,\bar{0}), (0,1,0,2,\bar{2}), (0,1,0,3,\bar{2}), (0,0,1,2,\bar{1}), (0,0,1,3,\bar{1}),\\ &(0,2,0,2,\bar{1}), (0,2,0,3,\bar{1}), (0,0,2,2,\bar{2}), (0,0,2,3,\bar{2})\}\,.\end{aligned}$$

Since the atom $(1, 0, 0, 0, \bar{0})$ is prime, we can restrict to a monoid $\bar{S} \subset \mathbb{N}_0^3 \times Z/3\mathbb{Z}$. Now we can find everything by using the algorithms presented at the end of Sect. 5.4. Even in the modified version of the algorithm in Step 5.4.1—here the bound is 13.5—we find about 7,500 minimal representations to consider after the reduction in Step 5.4.2.

From those, we get $\mathsf{c}(\mathbb{F}_3[X^2, X^3]) = 3$ in Step 5.4.4. Since we did not compute all minimal solutions, we find $\mathsf{t}(\mathbb{F}_3[X^2, X^3]) \geq 4$ in Step 5.4.5.

By using the alternative approach from Step 5.4.8, we find $\rho(\mathbb{F}_3[X^2, X^3]) = \frac{5}{2}$. Note that this particular result on the elasticity can also be obtained by [14, Example 3.7.3, Special Case 3.2].

5.5.2 $\mathbb{F}_2[X^2, X^3]$

Let $p = 2$. Then $|G| = 2$, so write $G = \{\mathbf{0}, \mathbf{e}\}$. Obviously—or by applying the ACA1, see Algorithm 2—we have $\mathcal{A}(G) = \{\mathbf{0}, \mathbf{e}^2\}$. Now we apply [14, Theorem 3.7.1] as in the case $p = 3$ and switch to the block monoid, which is a T-block monoid over G, say $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by $\mathcal{A}(T) = \{X^n g \mid n \in \{2, 3\},\ g \in G\}$ and ι is the uniquely determinated homomorphism $\iota : G \to T$ such that $\iota(X^n g) = g$ for all $n \in \{2, 3\}$ and $g \in G$.

Now we apply the ACA2, see Algorithm 3, as before and find

$$\mathcal{A}(\mathcal{B}(G, T, \iota)) = \{(\mathbf{0}, 1), (\mathbf{e}^2, 1), (1, X^2\mathbf{0}), (1, X^3\mathbf{0}), (\mathbf{e}, X^2\mathbf{e}), (\mathbf{e}, X^3\mathbf{e})\}\,.$$

Using the construction from the beginning of Sect. 5.4, we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G, T, \iota) \cong S \subset \mathbb{N}_0^3 \times \mathbb{Z}/2\mathbb{Z}\,.$$

Then, for the set of atoms, we find

$$\mathcal{A}(S) = \{(1,0,0,\bar{0}), (0,2,0,\bar{0}), (0,0,2,\bar{0}), (0,0,3,\bar{0}), (0,1,2,\bar{1}), (0,1,3,\bar{1})\}\,.$$

Since the atom $(1, 0, 0, \bar{0})$ is prime, we can use the same arguments as in Lemma 5.4.4 and restrict to a monoid $\bar{S} \subset \mathbb{N}_0^2 \times \mathbb{Z}/2\mathbb{Z}$ with a reduced set of atoms.

Since, in this case, Step 5.4.1 can be performed easily without any bound, we compute all atoms. Given this list, we immediately find $\rho(\mathbb{F}_2[X^2, X^3]) = 2$ in Step

5.4.3. Note that this particular result on the elasticity can also be obtained by [14, Example 3.7.3, Special Case 3.2].

Now we proceed with Step 5.4.4 and we deduce $\mathsf{c}(\mathbb{F}_2[X^2, X^3]) = 3$ and $\mathsf{D}(\mathbb{F}_2) + 1 + \mathsf{t}(S) = 6 \geq \mathsf{t}(\mathbb{F}_2[X^2, X^3]) \geq \mathsf{t}(S) = 3$.

Next, we compute the monotone catenary degree. For this, we proceed as in Step 5.4.6 and find $\mathsf{c}_{\mathrm{ad}}(S) = 3$ and $\mathsf{c}_{\mathrm{eq}}(S) = 3$, and thus $\mathsf{c}_{\mathrm{mon}}(\mathbb{F}_2[X^2, X^3]) = \mathsf{c}_{\mathrm{mon}}(S) = 3$.

5.5.3 $\mathbb{F}_2[X^2, X^5]$

The results in this case differ slightly from then ones we obtained above. We have $|G| = 2$, say $G = \{\mathbf{0}, \mathbf{e}\}$. Again, we have $\mathcal{A}(G) = \{\mathbf{0}, \mathbf{e}^2\}$. Now we apply [14, Theorem 3.7.1] as before and switch to the block monoid, which is a T-block monoid over G, say $\mathcal{B}(G, T, \iota) \subset \mathcal{F}(G) \times T$, where T is the reduced finitely primary monoid generated by $\mathcal{A}(T) = \{X^n g \mid n \in \{2, 5\},\ g \in G\}$ and ι is the uniquely determinated homomorphism $\iota : G \to T$ such that $\iota(X^n g) = g$ for all $n \in \{2, 5\}$ and $g \in G$.

Now we apply the ACA2, see Algorithm 3, as before and find

$$\mathcal{A}(\mathcal{B}(G, T, \iota)) = \{(\mathbf{0}, 1), (\mathbf{e}^2, 1), (1, X^2\mathbf{0}), (1, X^5\mathbf{0}), (\mathbf{e}, X^2\mathbf{e}), (\mathbf{e}, X^5\mathbf{e})\}\,.$$

Using the construction from the beginning of Sect. 5.4, we find

$$\widehat{T} \cong \mathbb{N}_0 \times \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathcal{B}(G, T, \iota) \cong S \subset \mathbb{N}_0^3 \times \mathbb{Z}/2\mathbb{Z}\,.$$

Then, for the set of atoms, we find

$$\mathcal{A}(S) = \{(1, 0, 0, \bar{0}), (0, 2, 0, \bar{0}), (0, 0, 2, \bar{1}), (0, 0, 5, \bar{1}), (0, 1, 2, \bar{1}), (0, 1, 5, \bar{1})\}\,.$$

Since the atom $(1, 0, 0, \bar{0})$ is prime, we can use the same arguments as in Lemma 5.4.4 and restrict to a monoid $\bar{S} \subset \mathbb{N}_0^2 \times \mathbb{Z}/2\mathbb{Z}$ with a reduced set of atoms.

Since, in this case, Step 5.4.1 can be performed without any bound, we compute all atoms. Now, we find a list of 25 atoms after Step 5.4.2. Given this list, we immediately find $\rho(\mathbb{F}_2[X^2, X^5]) = 3$ in Step 5.4.3. Now we proceed with Step 5.4.4 and obtain $\mathsf{t}(S) = 4$, $\mathsf{c}(\mathbb{F}_2[X^2, X^5]) = 5$, and $\mathsf{D}(\mathbb{F}_2) + 1 + \mathsf{t}(S) = 7 \geq \mathsf{t}(\mathbb{F}_2[X^2, X^5]) \geq \max\{\mathsf{t}(S), \mathsf{c}(\mathbb{F}_2[X^2, X^5])\} = 5$. Next, we compute the monotone catenary degree. For this, we proceed as in Step 5.4.6 and start with the adjacent catenary degree. We find $\mathsf{c}_{\mathrm{ad}}(S) = 5$. Next we compute the equal catenary degree and find $\mathsf{c}_{\mathrm{eq}}(S) = 6$. Now we find $\mathsf{c}_{\mathrm{mon}}(\mathbb{F}_2[X^2, X^5]) = \mathsf{c}_{\mathrm{mon}}(S) = 6 > 5 = \mathsf{c}(\mathbb{F}_2[X^2, X^5])$.

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References

1. Barucci, V.: Numerical semigroup algebras. In: Multiplicative Ideal Theory in Commutative Algebra, pp. 39–53. Springer, New York (2006)

2. Barucci, V., Dobbs, D.E., Fontana, M.: Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains. Mem. Am. Math. Soc. **125**(598), x+78 (1997)
3. Blanco, V., García-Sánchez, P.A., Geroldinger, A.: Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids. Illinois J. Math. **56** (2012)
4. Chapman, S.T., García-Sánchez, P.A., Llena, D.: The catenary and tame degree of numerical monoids. Forum Math. **21**(1), 117–129 (2009)
5. Chapman, S.T., García-Sánchez, P.A., Llena, D., Rosales, J.C.: Presentations of finitely generated cancellative commutative monoids and nonnegative solutions of systems of linear equations. Discrete Appl. Math. **154**(14), 1947–1959 (2006)
6. Chapman, S.T., García-Sánchez, P.A., Llena, D., Ponomarenko, V., Rosales, J.C.: The catenary and tame degree in finitely generated commutative cancellative monoids. Manuscripta Math. **120**(3), 253–264 (2006)
7. Contejean, E., Devie, H.: An efficient incremental algorithm for solving systems of linear Diophantine equations. Inf. Comput. **113**(1), 143–172 (1994)
8. Domenjoud, E.: Solving systems of linear Diophantine equations: an algebraic approach. In: Mathematical Foundations of Computer Science, 1991, (Kazimierz Dolny, 1991). Lecture Notes in Computer Science, vol. 520, pp. 141–150. Springer, Berlin (1991)
9. Foroutan, A.: Monotone chains of factorizations. In: Focus on Commutative Rings Research, pp. 107–130. Nova Sci. Publ., New York (2006)
10. Foroutan, A., Geroldinger, A.: Monotone chains of factorizations in C-monoids. In: Arithmetical Properties of Commutative Rings and Monoids. Lecture Notes in Pure Applied Mathematics, vol. 241, pp. 99–113. Chapman & Hall/CRC, Boca Raton (2005)
11. Foroutan, A., Hassler, W.: Chains of factorizations and factorizations with successive lengths. Commun. Algebra **34**(3), 939–972 (2006)
12. Geroldinger, A., Grynkiewicz, D.J., Schaeffer, G.J., Schmid, W.A.: On the arithmetic of Krull monoids with infinite cyclic class group. J. Pure Appl. Algebra **214**(12), 2219–2250 (2010)
13. Geroldinger, A., Grynkiewicz, D.J., Schmid, W.: The catenary degree of Krull monoids I. J. Théor. Nombres Bordeaux **23**(1), 137–169 (2011)
14. Geroldinger, A., Halter-Koch, F.: Nonunique factorizations. In: Pure and Applied Mathematics, vol. 278. Algebraic, Combinatorial and Analytic Theory. Chapman & Hall/CRC, Boca Raton (2006)
15. Geroldinger, A., Hassler, W.: Arithmetic of Mori domains and monoids. J. Algebra **319**(8), 3419–3463 (2008)
16. Geroldinger, A., Kainrath, F.: On the arithmetic of tame monoids with applications to Krull monoids and Mori domains. J. Pure Appl. Algebra **214**(12), 2199–2218 (2010)
17. Geroldinger, A., Liebmann, A., Philipp, A.: Long zero-sum free sequences. Periodica Math. Hung. **64**, 213–225 (2012)
18. Hassler, W.: Faktorisierung in eindimensionalen Integritätsbereichen. PhD thesis, Karl-Franzens Universität Graz (Advisor: F. Halter-Koch) (2000)
19. Kainrath, F.: Arithmetic of Mori domains and monoids: the global case (manuscript)
20. Malcolmson, P., Okoh, F.: Factorization in subalgebras of the polynomial algebra. Houston J. Math. **35**(4), 991–1012 (2009)
21. Malcolmson, P., Okoh, F.: Power series extensions of half-factorial domains. J. Pure Appl. Algebra **213**(4), 493–495 (2009)
22. Omidali, M.: The catenary and tame degree of certain numerical semigroups. Forum Mathematicum **24**, 627–640 (2012)
23. Ordaz, O., Philipp, A., Santos, I., Schmid, W.: On the Olson and the strong Davenport constants. Journal de Théorie des Nombres Bordeaux **23**, 715–750 (2011)
24. Philipp, A.: A characterization of arithmetical invariants by the monoid of relations. Semigroup Forum **81**, 424–434 (2010)
25. Philipp, A.: Nonunique factorizations—a semigroup-theoretic algorithmic approach with applications to non-principal orders in algebraic number fields. PhD thesis, Karl-Franzens Universität Graz (Advisors: F. Halter-Koch und A. Geroldinger) (2010)
26. A. Philipp. A precise result on the arithmetic of non-principal orders in algebraic number fields. J. Algebra Appl. **11** (2012)

27. Rosales, J.C., García-Sánchez, P.A.: Nonnegative elements of subgroups of Z^n. Linear Algebra Appl. **270**, 351–357 (1998)
28. Rosales, J.C., García-Sánchez, P.A.: Numerical semigroups. In: Developments in Mathematics, vol. 20. Springer, New York (2009)
29. Rosales, J.C., García-Sánchez, P.A., Urbano-Blanco, J.M.: On presentations of commutative monoids. Int. J. Algebra Comput. **9**(5), 539–553 (1999)