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RESEARCH ARTICLE

A groupoid generalisation of Leavitt path algebras

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Abstract Let G be a locally compact, Hausdorff, étale groupoid whose unit space is totally disconnected. We show that the collection $A(G)$ of locally-constant, compactly supported complex-valued functions on G is a dense $*$-subalgebra of $C_c(G)$ and that it is universal for algebraic representations of the collection of compact open bisections of G. We also show that if G is the groupoid associated to a row-finite graph or k-graph with no sources, then $A(G)$ is isomorphic to the associated Leavitt path algebra or Kumjian–Pask algebra. We prove versions of the Cuntz–Krieger and graded uniqueness theorems for $A(G)$.

Keywords Topological groupoids · Leavitt algebra · Groupoid algebra · Graded algebra

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1 Introduction

A ring R is said to have invariant basis number if any two bases (i.e., R-linearly independent spanning sets) of a free left R-module have the same number of elements. Many familiar rings (e.g., fields, commutative rings, left-Noetherian rings) have invariant basis number, but there are many examples of noncommutative rings that do not. A ring R without invariant basis number is said to have module type (m, n) if $m < n$ are natural numbers chosen minimally with $R^m \cong R^n$ as left R-modules. In the 1940's, Leavitt constructed algebras $L_{m,n}$ with module type (m, n) for all pairs of natural numbers with $m < n$ [13,14]. The $L_{m,n}$ are now known as the *Leavitt algebras*, and when $m = 1$, the Leavitt algebra $L_{1,n}$ is the unique nontrivial unital complex algebra generated by elements $x_1 \ldots x_n$ and $y_1, \ldots, y_n$ such that $\sum_{i=1}^n x_i y_i = 1$ and $y_i x_j = \delta_{i,j} 1$ for all $i, j \leq n$. In the 1970's, independent of Leavitt's work and motivated by the search for C^*-algebraic analogues of Type III factors, Cuntz defined a class of C^*-algebras $\mathcal{O}_n$, one for each integer $n \geq 2$, which are generated by elements $s_1, \ldots, s_n$ satisfying $\sum_{i=1}^n s_i s_i^* = 1$ and $s_i^* s_i = 1$ for all i (it follows that $s_i^* s_j = \delta_{i,j} 1$ for all $i, j \leq n$). A consequence of the uniqueness of $L_{1,n}$ is that it is isomorphic to the dense $*$-subalgebra of $\mathcal{O}_n$ generated by $s_1, \ldots, s_n$ via an isomorphism that carries each x_i to s_i and each y_i to s_i^*.

Shortly after Cuntz's work, Cuntz and Krieger [6] generalised Cuntz's results to describe a class of C^*-algebras $\mathcal{O}_A$ associated to binary-valued matrices A. At about the same time, Enomoto and Watatani provided a very elegant description of these Cuntz–Krieger algebras in terms of the directed graphs encoded by the matrices. Nearly twenty years later, Kumjian et al. [12] developed the class of C^*-algebras now known as graph C^*-algebras, as a far-reaching generalisation of the Cuntz–Krieger algebras patterned on Enomoto and Watatani's approach. Each graph C^*-algebra is described in terms of generators associated to the vertices and edges in the graph subject to relations encoded by connectivity in the graph. The Cuntz algebra $\mathcal{O}_n$ corresponds to the graph with one vertex and n edges. A remarkable assortment of important C^*-algebraic properties of a graph C^*-algebra can be characterised in terms of the structure of the graph (see [15] for a good overview). Shortly afterwards, Kumjian and Pask [10] introduced a sort of higher-dimensional graph, now known as a k-graph, and an associated class of C^*-algebras, as a flexible visual model for the higher-rank Cuntz–Krieger algebras discovered by Robertson and Steger [20]. When $k = 1$, a k-graph is essentially a directed graph, and Kumjian and Pask's C^*-algebras coincide with the graph C^*-algebras of [12].

In the early 2000's, the algebraic community became interested in the similarity between the constructions of Leavitt and Cuntz and the potential for the graph C^*-algebra template to provide a broad class of interesting new algebras. Following the lead of [12], Abrams and Aranda Pino associated *Leavitt path algebras* to a broad class of directed graphs. The Leavitt path algebra of a directed graph is the universal algebra whose presentation in terms of generators and relations is essentially the same as that of the graph C^*-algebra. Moreover, the graded uniqueness theorem for Leavitt path algebras implies that the C^*-algebra of a directed graph is a norm completion of its Leavitt path algebra [16,23]. Further generalising Leavitt path algebras, Aranda

Pino et al. [3] recently constructed a class of algebras associated to k-graphs, which they call Kumjian–Pask algebras.

A very powerful framework for constructing C^*-algebras is the notion of a groupoid C^*-algebra. Renault's structure theory for groupoid C^*-algebras [19] is exploited in [12] where structural properties of the graph C^*-algebra are deduced by showing that the graph C^*-algebra is isomorphic to a groupoid C^*-algebra and then tapping into Renault's results [19]. The same approach was taken in [10] to establish important structural properties of k-graph C^*-algebras: the C^*-algebra of a k-graph is defined in terms of generators and relations, but its structure is analysed by identifying it with a groupoid C^*-algebra.

In this paper, from a sufficiently well-behaved groupoid G, we construct a complex algebra $A(G)$ with the following properties:

(1) $A(G)$ has a natural description as a universal algebra (Theorem 3.10);
(2) $A(G)$ is isomorphic to a dense subalgebra of the groupoid C^*-algebra $C^*(G)$ (Proposition 4.2); and
(3) given a k-graph Λ, if $G = G_\Lambda$ is the groupoid corresponding to Λ as in [10] (Proposition 4.3), then $A(G)$ is isomorphic to the Kumjian–Pask algebra $\mathrm{KP}_{\mathbb{C}}(\Lambda)$. In particular, if E is a directed graph and $G = G_E$ is the graph groupoid associated to E, then $A(G)$ is isomorphic to the Leavitt path algebra $L_{\mathbb{C}}(E)$.

In [22], Steinberg defines a groupoid algebra KG for an arbitrary commutative ring K with unit and shows that KG is a quotient of an associated inverse semigroup algebra. We show that the algebra $A(G)$ is identical to KG for $K = \mathbb{C}$ (the complex numbers).[1] Our approach is different from that of [22] and our universal property and uniqueness theorems (see below) provide new tools for studying KG and the inverse semigroup algebras associated to them in the case where $K = \mathbb{C}$; it would be interesting to investigate versions of these theorems for general K.

The Cuntz–Krieger uniqueness theorem and gauge-invariant uniqueness theorem are important tools in the study of graph C^*-algebras. Versions of these theorems have been established for many generalisations of Cuntz–Krieger algebras [5,8,10–12,17,18]. For Leavitt path algebras, the graded uniqueness theorem is the analogue of the gauge-invariant uniqueness theorem. The first version of this graded uniqueness theorem was a corollary to Ara, Moreno, and Pardo's characterisation [2, Theorem 4.3] of the graded ideals in a Leavitt path algebra. It was first stated explicitly by Raeburn who proved both the graded uniqueness theorem and Cuntz–Krieger uniqueness theorem for Leavitt path algebras of row-finite graphs with no sinks and over fields equipped with a positive definite $*$-operation [16, Theorem 1.3.2 and Theorem 1.3.4]. Tomforde extended these results to Leavitt path algebras of arbitrary graphs over arbitrary fields in [23, Theorem 4.8 and Theorem 6.8], and later proved the two uniqueness theorems for Leavitt path algebras of arbitrary graphs over a ring [24, Theorem 5.3 and Theorem 6.5]. Aranda Pino et al. [3] subsequently proved versions of these theorems for Kumjian–Pask algebras. In Sect. 5 we prove versions of the Cuntz–Krieger

[1] We would like to thank Steinberg who brought this to our attention after reading an earlier version of this paper.

uniqueness theorem (Theorem 5.1) and the graded uniqueness theorem (Theorem 5.4) for $A(G)$. We also give an example of a groupoid satisfying our hypothesis that is not necessarily the groupoid of a k-graph.

Our aim in defining and initiating the analysis of $A(G)$ is twofold: (1) to provide a broad framework for future generalisations of Leavitt path algebras from other combinatorial structures; and (2) to make available the powerful toolkit of groupoid analysis to study these algebras. In addition, we hope this will provide a new and useful perspective on the interplay between algebra and analysis at the interface between Leavitt path algebras and graph C^*-algebras.

2 Preliminaries

A groupoid is a small category with inverses. We write $G^{(2)} \subseteq G \times G$ for the set of composable pairs in G; we write $G^{(0)}$ for the unit space of G, and we denote by r and s the range and source maps $r, s : G \to G^{(0)}$. So $(\alpha, \beta) \in G^{(2)}$ if $s(\alpha) = r(\beta)$. For $U, V \subseteq G$, we define

$$UV := \{\alpha\beta : \alpha \in U,\ \beta \in V,\ \text{and } r(\beta) = s(\alpha)\}. \tag{2.1}$$

A topological groupoid is a groupoid endowed with a topology under which r and s are continuous, the inverse map is continuous, and such that composition is continuous with respect to the relative topology on $G^{(2)}$ inherited from $G \times G$.

Recall that if G is a groupoid, then an *open bisection* of G is an open subset $U \subseteq G$ such that $r|_U$ and $s|_U$ are homeomorphisms. We will work exclusively with locally compact, Hausdorff groupoids which are étale in the sense that the source map $s : G \to G^{(0)}$ is a local homeomorphism. The range map is then a local homeomorphism as well. If G is étale then $G^{(0)}$ is open in G and G admits a Haar system consisting of counting measures. The following also appears as [7, Proposition 4.1].

Lemma 2.1 *Let G be a locally compact, Hausdorff, étale groupoid. Suppose that $G^{(0)}$ is totally disconnected. Then the topology on G has a basis of clopen bisections. Moreover, if G is locally compact and Hausdorff, then G has a basis of compact open bisections.*

Proof Proposition 2.8 of [19] implies that G has a basis of open bisections. For each $\gamma \in G$, let U be an open bisection containing γ. Since r is an open map there exists a basic clopen neighbourhood X of $r(\gamma)$ such that $X \subseteq r(U)$. Then $XU = \{h \in U : r(h) \in X\} = U \cap r^{-1}(X)$ is homeomorphic to X by choice of U and in particular is a clopen bisection containing γ. If G is also locally compact, then U may be chosen to be precompact. Hence the clopen subset XU is a compact open bisection. □

Notation 2.2 For the remainder of this paper, Γ will denote a discrete group, G will denote a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and c will denote a continuous cocycle from G to Γ (that is, c carries composition in G to the group operation in Γ).

By Lemma 2.1, with Γ, G and c as above, G has a basis of compact open bisections. Since G is Hausdorff, compact subsets of G are closed. We will use this fact frequently and without further comment.

Remark 2.3 These hypotheses might sound very restrictive, but, for instance, every k-graph groupoid satisfies them (see, for example, [8]).

Remark 2.4 Let U be a compact open subset of a topological space X. Let F be a finite cover of U by compact open subsets of U. For each nonempty $H \subseteq F$, let $V_H := \big(\bigcap H\big) \setminus \big(\bigcup(F \setminus H)\big)$. Since each $V \in F$ is compact and open, so is each V_H. In particular, since F is finite, so is $K := \{H \subseteq F : H \neq \emptyset, V_H \neq \emptyset\}$, and

$$U = \bigsqcup_{H \in K} V_H$$

is an expression for U as a finite disjoint union of nonempty compact open sets such that for each $W \in K$ we have $W \subseteq V$ for at least one $V \in F$, and such that whenever $W \in K$ and $V \in F$ satisfy $W \not\subseteq V$, we have $W \cap V = \emptyset$. We refer to this as the *disjointification* of the cover F of U.

Throughout this paper, unless stated otherwise, all algebras are taken to be complex *-algebras, and all representations are assumed to preserve adjoints.

3 The algebra $A(G)$

Definition 3.1 Let X be a topological space. A function $f : X \to Y$ is *locally constant* if for every $x \in X$ there exists a neighbourhood U of x such that $f|_U$ is constant.

Observe that if $f : X \to \mathbb{C}$ is locally constant then it is automatically continuous, and the support of f is clopen in X.

Definition 3.2 Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. We define $A(G)$ to be the compex vector space

$$A(G) = \{f \in C_c(G) : f \text{ is locally constant}\}$$

with pointwise addition and scalar multiplication.

The following lemma shows that $A(G)$ is precisely the algebra $\mathbb{C}G$ of [22, Definition 4.1]. (In fact, Definition 3.2 agrees precisely with the definition of $\mathbb{C}G$ given in the preprint version of [22]—see [21, Definition 3.1]—and then the following Lemma is [21, Proposition 3.3]).

Lemma 3.3 *Let Γ be a discrete group and G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. If $\mathcal{U}$ is the basis of all compact open subsets of G, we have $A(G) = \operatorname{span}\{1_U : U \in \mathcal{U}\}$.*

Proof For any $U \in \mathcal{U}$, the function 1_U is locally constant, and hence $\operatorname{span}\{1_U : U \in \mathcal{U}\} \subseteq A(G)$. We must show that $A(G) \subseteq \operatorname{span}\{1_U : U \in \mathcal{U}\}$. Fix $f \in A(G)$. Since f is locally constant and $\mathcal{U}$ is a basis, for each $\alpha \in \operatorname{supp}(f)$, there is a neighbourhood $U_\alpha \in \mathcal{U}$ of α such that $f|_{U_\alpha}$ is constant. Since $\operatorname{supp}(f)$ is clopen, we may assume that $U_\alpha \subseteq \operatorname{supp}(f)$. Since $\operatorname{supp}(f)$ is compact there is a finite subset $F \subseteq \{U_\alpha\}_{\alpha \in \operatorname{supp} f}$ such that $\operatorname{supp}(f) = \bigcup F$. Let K be the disjointification of F discussed in Remark 2.4. Since f is constant on each $V \in F$ and each $W \in K$ is a subset of some $V \in F$, the function f is constant on each $W \in K$. Hence, writing $f(W)$ for the unique value taken by f on $W \in K$, we have $f = \sum_{W \in K} f(W) 1_W$. □

Definition 3.4 Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. For $n \in \Gamma$ we write $G_n := c^{-1}(n)$. We write $A_n(G)$ for the subset of $A(G)$ consisting of functions whose support is contained in G_n. We say that a subset S of G is *graded* if the cocycle c is constant on S. If $S \subseteq G_n$, we say that S is *n-graded*. For each $n \in \Gamma$ we write $B_n^{\mathrm{co}}(G)$ for the collection of all n-graded compact open bisections of G. We write $B_*^{\mathrm{co}}(G)$ for $\bigcup_{n \in \Gamma} B_n^{\mathrm{co}}(G)$.

Lemma 3.5 *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. We have $A(G) = \operatorname{span}\{A_n : n \in \Gamma\} \subseteq C_c(G)$. Every $f \in A(G)$ can be expressed as $f = \sum_{U \in F} a_U 1_U$ where F is a finite subset of $B_*^{\mathrm{co}}(G)$ whose elements are mutually disjoint and $a : U \mapsto a_U$ is a function from F to $\mathbb{C}$.*

Proof We have $A(G) \supseteq \operatorname{span}\{A_n : n \in \Gamma\}$ because each A_n consists of locally constant functions. For the reverse inclusion, fix $f \in A(G)$. Since Γ is discrete and c is continuous, each G_n is clopen. Since $\operatorname{supp}(f)$ is compact, there is a finite collection $N \subseteq G$ such that $\operatorname{supp}(f) \subseteq \bigcup_{n \in N} G_n$. For $n \in N$ let f_n denote the pointwise product $f 1_{G_n}$. Then f_n is locally constant and continuous because 1_{G_n} and f are. We then have $f = \sum_{n \in N} f_n \in \operatorname{span}\{A_n : n \in \Gamma\}$.

Let $f \in A(G)$. By Lemma 3.3, there is a finite set K_0 of compact open sets and an assignment $W \mapsto d_W$ of scalars to the elements of K_0 such that $f = \sum_{W \in K_0} d_W 1_W$. Let

$$K := \{W \cap G_n : W \in K_0, n \in \Gamma, W \cap G_n \neq \emptyset\}.$$

Since Γ is discrete and c is continuous, each G_n is open. Since each $W \in K_0$ is compact, K is finite. Each $V \in K$ is graded; we write $c(V)$ for the unique value taken by c on V. For each $V \in K$, let

$$b_V = \sum_{W \in K_0, W \cap G_{c(V)} = V} d_W$$

Then $f = \sum_{V \in K} b_V 1_V$.

Let F be the disjointification of K. Each $U \in F$ is graded because F is a refinement of K. For $U \in F$, define

$$a_U = \sum_{V \in K,\, U \subseteq V} b_V.$$

Then $f = \sum_{U \in F} a_U 1_U$ is the desired expression. □

Recall that given a locally compact, Hausdorff, étale groupoid G such that $s : G \to G^{(0)}$ is a local homeomorphism, and given $f, g \in A(G) \subseteq C_c(G)$, the functions f^* and $f * g$ are given by

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \tag{3.1}$$

$$(f * g)(\gamma) = \sum_{r(\alpha)=r(\gamma)} f(\alpha)g(\alpha^{-1}\gamma). \tag{3.2}$$

Proposition 3.6 *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Under the operations (3.1) and (3.2), $A(G)$ is a Γ-graded $*$-algebra with graded subspaces A_n as described in Definition 3.4.*

Remark 3.7 For us, an involution on a $*$-algebra over $\mathbb{C}$ is always *conjugate* linear.

Remark 3.8 We do not assume that Γ is abelian so we will write the group operation multiplicatively.

Proof That $A(G)$ is a complex algebra follows from [22, Proposition 4.6]. We must verify that $A(G)$ is a $*$-algebra and that $A(G)$ is graded. The A_n are mutually linearly independent because the G_n are disjoint and restriction of functions gives a vector-space isomorphism of each A_n onto the space of locally constant functions on G_n. Observe that the $*$-operation is a conjugate-linear involution on $A(G)$ and takes A_n to $A_{n^{-1}}$. Next we will show that the multiplication defined on $A(G)$ is a graded multiplication. If $f \in A_m$ and $g \in A_n$, then if $(f * g)(\gamma) \neq 0$ we have $f(\alpha) \neq 0$ and $g(\alpha^{-1}\gamma) \neq 0$ for some α with $r(\alpha) = r(\gamma)$. In particular, $c(\alpha) = m$, and $c(\alpha^{-1}\gamma) = n$ forcing $c(\gamma) = mn$ (because $c(\gamma) = c(\alpha\alpha^{-1}\gamma) = c(\alpha)c(\alpha^{-1}\gamma)$). Hence $\operatorname{supp}(f * g) \subseteq G_{mn}$. □

We finish this section by presenting of $A(G)$ as a universal algebra.

Definition 3.9 Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Let B be an algebra over $\mathbb{C}$. A *representation of* $B_*^{\mathrm{co}}(G)$ in B is a family $\{t_U : U \in B_*^{\mathrm{co}}(G)\} \subseteq B$ satisfying

(R1) $t_\emptyset = 0$;
(R2) $t_U t_V = t_{UV}$ for all $U, V \in B_*^{\mathrm{co}}(G)$; and
(R3) $t_U + t_V = t_{U \cup V}$ whenever U and V are disjoint elements of $B_n^{\mathrm{co}}(G)$ for some n such that $U \cup V$ is a bisection.

The following theorem gives an alternative formulation of [21, Theorem 3.11].

Theorem 3.10 *Let* Γ *be a discrete group,* G *a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and* $c : G \to \Gamma$ *a continuous cocycle. Then* $\{1_U : U \in B_*^{co}(G)\} \subseteq A(G)$ *is a representation of* $B_*^{co}(G)$ *which spans* $A(G)$*. Moreover,* $A(G)$ *is universal for representations of* $B_*^{co}(G)$ *in the sense that for every representation* $\{t_U : U \in B_*^{co}(G)\}$ *of* $B_*^{co}(G)$ *in an algebra* B*, there is a unique homomorphism* $\pi : A(G) \to B$ *such that* $\pi(1_U) = t_U$ *for all* $U \in B_*^{co}(G)$.

Proof The collection $\{1_U : U \in B_*^{co}(G)\}$ certainly satisfies (R1) and (R3), and it satisfies (R2) by [22, Proposition 4.5 (3)]. That this family spans $A(G)$ follows from Lemma 3.5.

Let B be a complex algebra and let $\{t_U : U \in B_*^{co}(G)\}$ be a representation of $B_*^{co}(G)$ in B. We must show that there is a homomorphism $\pi : A(G) \to B$ satisfying $\pi(1_U) = t_U$ for all $U \in B_*^{co}(G)$; uniqueness follows from the previous paragraph. We begin by showing that

$$\sum_{U\in F} t_U = t_{\bigcup F} \text{ for } n \in \Gamma \text{ and finite } F \subseteq B_n^{co}(G) \text{ consisting of mutually disjoint bisections such that } \bigcup F \in B_n^{co}(G). \tag{3.3}$$

Let $F \subseteq B_n^{co}(G)$ be a finite collection of mutually disjoint bisections such that $\bigcup F$ is a bisection. We claim that $r(U) \cap r(V) = \emptyset$ for distinct $U, V \in F$. To see this, fix $x \in r(U)$. There exists $\alpha \in U$ such that $r(\alpha) = x$, and this α is the unique element of $\bigcup F$ whose range is x because $\bigcup F$ is a bisection. Since $U \cap V = \emptyset$, we have $\alpha \notin V$ and hence $x \notin r(V)$. So the sets $r(U)$ where $U \in F$ are mutually disjoint as claimed. Thus each $U \in F$ satisfies $U = r(U)(\bigcup F)$. A standard induction extends (R3) to finite collections of mutually disjoint compact open subsets of $G^{(0)}$. Combining this with (R2), we obtain

$$t_{\bigcup F} = t_{r(\bigcup F)} t_{\bigcup F} = \sum_{U\in F} t_{r(U)} t_{\bigcup F} = \sum_{U\in F} t_{r(U)(\bigcup F)} = \sum_{U\in F} t_U.$$

We show next that the formula $\sum_{U\in F} a_U 1_U \mapsto \sum_{U\in F} a_U t_U$ is well-defined on linear combinations of indicator functions where $F \subseteq B_*^{co}(G)$ is a finite collection of mutually disjoint bisections. It will follow from Lemma 3.5 that there is a unique linear map $\pi : A(G) \to B$ such that $\pi(1_U) = t_U$ for each $U \in B_*^{co}(G)$. Fix $f \in A(G)$ and suppose that

$$\sum_{U\in F} a_U 1_U = f = \sum_{V\in H} b_V 1_V$$

where each of F and H is a finite set of mutually disjoint elements of $B_*^{co}(G)$. We must show that

$$\sum_{U\in F} a_U t_U = \sum_{V\in H} b_V t_V.$$

Since the G_n are mutually disjoint, for each $n \in \Gamma$ we have

$$\sum_{U \in F \cap B_n^{\mathrm{co}}(G)} a_U 1_U = f|_{G_n} = \sum_{V \in H \cap B_n^{\mathrm{co}}(G)} b_V 1_V,$$

so we may assume that $F, G \subseteq B_n^{\mathrm{co}}(G)$ for some $n \in \Gamma$.

Let $K = \{U \cap V : U \in F, V \in H, U \cap V \neq \emptyset\}$. Then each $W \in K$ belongs to $B_n^{\mathrm{co}}(G)$. Moreover, for $U \in F$ we have $U = \bigsqcup\{W \in K : W \subseteq U\}$. Hence (3.3) gives $t_U = \sum_{W \in K, W \subseteq U} t_W$ for each $U \in F$; a similar decomposition holds for t_V for each $V \in H$. Therefore

$$\sum_{U \in F} a_U t_U = \sum_{U \in F} \sum_{W \in K,\, W \subseteq U} a_U t_W = \sum_{W \in K} \left(\sum_{U \in F,\, W \subseteq U} a_U \right) t_W,$$

and similarly

$$\sum_{V \in F} b_V t_V = \sum_{W \in K} \left(\sum_{V \in F,\, W \subseteq V} b_V \right) t_W.$$

Fix $W \in K$. It suffices now to show that $\sum_{U \in F,\, W \subseteq U} a_U = \sum_{V \in F,\, W \subseteq V} b_V$. By definition of K, the set W is nonempty, so let $\alpha \in W$. Then for $U \in F$, we have $\alpha \in U \implies W \cap U \neq \emptyset \implies W \subseteq U$. Since $\alpha \in W$, this implies that $\alpha \in U \iff W \subseteq U$. Hence

$$f(\alpha) = \sum_{U \in F} a_U 1_U(\alpha) = \sum_{U \in F,\, \alpha \in U} a_U = \sum_{U \in F,\, W \subseteq U} a_U.$$

a similar calculation shows that $\sum_{V \in F,\, W \subseteq V} b_V = f(\alpha)$ as well. So there is a linear map $\pi : A(G) \to B$ such that $\pi(1_U) = t_U$ for all $U \in B_*^{\mathrm{co}}(G)$.

We must check that π is a homomorphism. To see that π is multiplicative, fix $f, g \in A(G)$. Express $f = \sum_{U \in F} a_U 1_U$ and $g = \sum_{V \in H} b_V 1_V$ where F and H are finite subsets of $B_*^{\mathrm{co}}(G)$, and calculate:

$$\begin{aligned} \pi(fg) &= \pi\left(\Big(\sum_{U \in F} a_U 1_U \Big) \Big(\sum_{V \in H} b_V 1_V \Big) \right) \\ &= \pi\left(\sum_{U \in F} \sum_{V \in H} a_U b_V 1_U 1_V \right). \end{aligned}$$

Since [22, Proposition 4.5 (3)] gives $1_U 1_V = 1_{UV}$ for all U, V, we then have

$$\pi(fg) = \pi\left(\sum_{U \in F} \sum_{V \in H} a_U b_V 1_{UV} \right) = \sum_{U \in F} \sum_{V \in H} a_U b_V t_{UV}.$$

Each $t_{UV} = t_U t_V$ by (R2), so

$$\pi(fg) = \sum_{U\in F}\sum_{V\in H} a_U b_V t_U t_V = \Big(\sum_{U\in F} a_U t_U\Big)\Big(\sum_{V\in H} b_V t_V\Big) = \pi(f)\pi(g)$$

as required. □

4 $A(G)$ is dense in $C^*(G)$

Since our aim is to produce algebras associated to totally disconnected, locally compact, Hausdorff groupoids whose relationship to the groupoid C^*-algebra is analogous to that of Leavitt path algebras to graph C^*-algebras, we show in this section that the subalgebra $A(G)$ of $C_c(G)$ is dense in the full (and hence also the reduced) C^*-algebra of G. We could prove this as in [12, Proposition 4.1] or [22, Proposition 6.7] by using the Stone-Weierstrass theorem, but a direct argument takes about the same amount of effort.

We first prove a technical lemma.

Lemma 4.1 *Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. Fix a compact open bisection U of G and suppose that $f \in C_c(G)$ is supported on U. Fix $\varepsilon > 0$. There exists a finite set $\mathcal{V}$ of nonempty compact open bisections of G such that $U = \bigsqcup \mathcal{V}$ and such that for each $V \in \mathcal{V}$, we have $|f(\alpha) - f(\beta)| \le \varepsilon$ for all $\alpha, \beta \in V$.*

Proof For each $\gamma \in U$ let U_γ be a compact open neighbourhood of γ such that $U_\gamma \subseteq U$ and $|f(\alpha) - f(\gamma)| < \varepsilon/2$ for all $\alpha \in U_\gamma$. Since U is compact, there is a finite subset F of U such that $\{U_\gamma : \gamma \in F\}$ covers U. Let $\mathcal{V}$ be the disjointification of the U_γ as in Remark 2.4. Fix $V \in \mathcal{V}$. Then there exists $\gamma \in F$ such that $V \subseteq U_\gamma$, and then for $\alpha, \beta \in V$, we have $|f(\alpha) - f(\beta)| \le |f(\alpha) - f(\gamma)| + |f(\gamma) - f(\beta)| < \varepsilon$. □

To state the next proposition, we recall from [19] that for a locally compact, Hausdorff, étale groupoid G, the I-norm on $C_c(G)$ is defined as follows. For $f \in C_c(G)$, let

$$\|f\|_{I,r} := \sup_{u\in G^{(0)}} \Big\{\sum_{r(\alpha)=u} |f(\alpha)|\Big\} \quad \text{and} \quad \|f\|_{I,s} := \sup_{u\in G^{(0)}} \Big\{\sum_{s(\alpha)=u} |f(\alpha)|\Big\}.$$

Then the I-norm of f is $\|f\|_I := \max\{\|f\|_{I,r}, \|f\|_{I,s}\}$. The I-norm dominates each of the universal norm, the reduced norm, and the uniform norm on $C_c(G)$. (See [19] for further details.)

Proposition 4.2 *Let Γ be a discrete group, G a locally compact, Hausdorff étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. With notation as above, $A(G)$ is dense in $C_c(G)$ under each of the reduced norm, the universal norm, the I-norm, and the uniform norm.*

Proof Since the I-norm dominates the other three norms, it suffices to prove the result for the I-norm. Fix $f \in C_c(G)$ and $\varepsilon > 0$. Since f has compact support, $\mathrm{supp}(f)$ can be written as a finite union of elements of $B_*^{\mathrm{co}}(G)$. So we can write $f = \sum_{i=1}^n f_i$ where each f_i is supported on an element of $B_*^{\mathrm{co}}(G)$. For each i, apply Lemma 4.1 to $\mathrm{supp}(f_i)$ to obtain a cover $\mathcal{U}_i$ of the support of f_i by disjoint compact open bisections such that for $U \in \mathcal{U}_i$, we have $|f_i(\alpha) - f_i(\beta)| \le \varepsilon/n$ for all $\alpha, \beta \in U_i$. For each $i \le n$ and each $U \in \mathcal{U}_i$, fix $z_{i,U} \in f(U)$, so $|f_i(\alpha) - z_{i,U}| \le \varepsilon/n$ for all $\alpha \in U$. Then let $g_i := \sum_{U \in \mathcal{U}_i} z_{i,U} 1_U$ for all $i \le n$ and define $g := \sum_{i=1}^n g_i \in A(G)$. We have

$$\|f - g\|_I \le \sum_{i=1}^n \|f_i - g_i\|_I.$$

Fix $i \le n$. It suffices to show that $\|f_i - g_i\|_I \le \varepsilon/n$. Fix $u \in G^{(0)}$. Since f_i is supported on a bisection, there is at most one $\alpha \in s^{-1}(u) \cap \mathrm{supp}(f_i)$. If there is no such α, then $\sum_{s(\alpha)=u} |(f_i - g_i)(\alpha)| = 0$ and we are done. So suppose that $\alpha \in s^{-1}(u) \cap \mathrm{supp}(f_i)$. Then there is a unique $U_0 \in \mathcal{U}_i$ such that $\alpha \in U_0$. Therefore $\sum_{s(\alpha)=u} |(f_i - g_i)(\alpha)| = |f_i(\alpha) - z_{i,U_0}| \le \varepsilon/n$. Since $u \in G^{(0)}$ was arbitrary, we conclude that $\|f_i - g_i\|_{I,s} \le \varepsilon/n$. A symmetric argument gives $\|(f_i - g_i)(\alpha)\|_{I,r} \le \varepsilon/n$. Hence $\|f_i - g_i\|_I \le \varepsilon/n$ as required. □

Proposition 4.3 *Suppose that Λ is a row-finite, k-graph with no sources and that G_Λ is the corresponding k-graph groupoid. Then $A(G_\Lambda)$ as constructed above is isomorphic to the Kumjian–Pask algebra* $\mathrm{KP}(\Lambda, \mathbb{C})$.

Proof By [10, Corollary 3.5], $t_\lambda := 1_{Z(\lambda,s(\lambda))}$ determines a Cuntz–Krieger Λ-family in $C^*(G)$. In particular, there is a Kumjian–Pask family ([3, Definition 3.1]) for Λ determined by $t_\lambda = 1_{Z(\lambda,s(\lambda))}$ and $t_{\lambda^*} = 1_{Z(s(\lambda),\lambda)}$ for all $\lambda \in \Lambda$. It follows from the universal property of $\mathrm{KP}(\Lambda, \mathbb{C})$ that there is a homomorphism $\phi : \mathrm{KP}(\Lambda, \mathbb{C}) \to A(G_\Lambda)$ which carries each s_λ to t_λ and each s_{λ^*} to t_{λ^*}.

By [3, Theorem 3.4] the algebra $\mathrm{KP}(\Lambda, \mathbb{C})$ is spanned by the elements $t_\mu t_{\nu^*}$ where $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, and the $\mathbb{Z}$-grading of $\mathrm{KP}(\Lambda, \mathbb{C})$ carries each $s_\mu s_{\nu^*}$ to $d(\mu) - d(\nu)$. So to see that ϕ is graded, it suffices to show that it preserves the grading of each $s_\mu s_{\nu^*}$, which it does since

$$\phi(s_\mu s_{\nu^*}) = 1_{Z(\mu,\nu)} = 1_{\{(\mu x, d(\mu)-d(\nu), \nu x) : x \in \Lambda^\infty, r(x)=s(\mu)\}} \in A_{d(\mu)-d(\nu)}.$$

Since each $Z(v)$ is nonempty, $\phi(p_v) \neq 0$ for each $v \in E^0$. Thus the graded uniqueness theorem for Kumjian–Pask algebras [3, Theorem 4.1] implies that ϕ is injective.

It remains to show that ϕ is surjective. By Lemma 3.3, $A(G_\Lambda)$ is spanned by the functions 1_U where U ranges over all compact open bisections in G_Λ. Let U be a compact open bisection. Since the grading is continuous and U is compact, we can write 1_U as the finite sum $\sum_{U \cap G_n \neq \emptyset} 1_{U \cap G_n}$ where each $U \cap G_n$ is a graded compact open bisection. So fix $n \in \mathbb{N}^k$ and a compact open n-graded bisection V. It suffices to show that $1_V \in \mathrm{span}\{1_{Z(\mu,\nu)} : s(\mu) = s(\nu)\}$. Because V is compact and the sets $Z(\mu, \nu)$ form a basis for the topology on G_Λ [10, Proposition 2.8], we can write $V =$

$\bigcup_{(\mu,\nu)\in F} Z(\mu,\nu)$ for some finite set $F \subseteq \{(\mu,\nu) \in \Lambda\times\Lambda : s(\mu) = s(\nu)\}$. Since V is n-graded, we have $d(\mu)-d(\nu) = n$ for all $(\mu,\nu) \in F$. Let $p := \bigvee_{(\mu,\nu)\in F} d(\mu)$. Then for each $(\mu,\nu) \in F$ we have $Z(\mu,\nu) = \bigcup\{Z(\mu\alpha,\nu\alpha) : \alpha \in s(\mu)\Lambda^{p-d(\mu)}\}$. Let $H := \{(\mu\alpha,\nu\alpha) : (\mu,\nu) \in F, \alpha \in s(\mu)\Lambda^{p-d(\mu)}\}$. Then $Z(\eta,\zeta)\cap Z(\eta',\zeta') = \emptyset$ for distinct $(\eta,\zeta), (\eta',\zeta') \in H$, so $V = \bigsqcup_{(\eta,\zeta)\in H} Z(\eta,\zeta)$. Hence $1_U = \sum_{(\eta,\zeta)\in H} 1_{Z(\eta,\zeta)}$, and it follows that ϕ is surjective. □

Remark 4.4 When $k = 1$ in the preceding proposition, Λ is the path category of the directed graph $E = (\Lambda^0, \Lambda^1, r, s)$ and, in this case, the proposition specialises to the statement that $A(G)$ is isomorphic to the Leavitt path algebra of [1].

5 The uniqueness theorems

Interestingly, in the situation of groupoids, the graded uniqueness theorem is a corollary of the natural generalisation of the Cuntz–Krieger uniqueness theorem. This in turn is essentially Renault's structure theorem for the reduced C^*-algebra of a groupoid in which the units with trivial isotropy are dense in the unit space. This condition has been referred to, variously, as "topologically free", "topologically principal", "essentially free."

Given a unit u, it is standard to denote the isotropy subgroup $\{\alpha \in G : r(\alpha) = s(\alpha) = u\}$ by either $G(u)$ or G^u_u. Here we have chosen the more suggestive notation uGu, which is in keeping with the notation established in (2.1). Likewise, we write Gu for $s^{-1}(u)$.

Theorem 5.1 *Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. Suppose that $\{u \in G^{(0)} : uGu = \{u\}\}$ is dense in $G^{(0)}$. Let $\pi : A(G) \to B$ be a *-homomorphism into a complex *-algebra B. Suppose that* $\ker(\pi) \neq \{0\}$. *Then there is a compact open subset $K \subseteq G^{(0)}$ such that* $\pi(1_K) = 0$.

Remark 5.2 To see why the hypothesis that the units with trivial isotropy are dense is needed in Theorem 5.1, consider the situation where $G = \mathbb{Z}/2\mathbb{Z}$ regarded as a groupoid with one unit 0. Then $A(G)$ is the group algebra $\mathbb{C}\delta_0 + \mathbb{C}\delta_1$, and the map $\pi : A(G) \to \mathbb{C}$ such that $\pi(\delta_0) = \pi(\delta_1) = 1$ is a *-homomorphism of $A(G)$ which is not injective, but which restricts to an injective representation of $C_c(G^{(0)}) = \mathbb{C}\delta_0$. A related construction applies for arbitrary G—see [4, Proposition 4.4].

To prove Theorem 5.1, we need a technical lemma.

Lemma 5.3 *Let G be a locally compact, Hausdorff, étale groupoid. Fix $\alpha \in G$ and a precompact neighbourhood V of α. Suppose that $r(\alpha)Gs(\alpha) = \{\alpha\}$. Then there exist neighbourhoods X of $r(\alpha)$ and Y of $s(\alpha)$ such that XVY is a precompact open bisection.*

Proof Suppose, to the contrary, that for every neighbourhood X of $r(\alpha)$ and every neighbourhood Y of $s(\alpha)$, XVY fails to be a bisection. Let U be an open bisection containing α. Fix a fundamental sequence of neighbourhoods $(Y_i)_{i=1}^\infty$ of $s(\alpha)$, and for each i, let $X_i := r(UY_i)$, so that $(X_i)_{i=1}^\infty$ forms a fundamental sequence of neighbourhoods of $r(\alpha)$. Since each $X_i V Y_i$ fails to be a bisection, for each i there exist $\beta_i, \gamma_i \in X_i V Y_i$ with $\beta_i \neq \gamma_i$ such that either $s(\beta_i) = s(\gamma_i)$ or $r(\beta_i) = r(\gamma_i)$ for all i. The sequence $\big((\beta_i, \gamma_i)\big)_{i=1}^{\infty}$ belongs to the precompact set $V \times V$, so by passing to a subsequence and relabelling we may assume that $\beta_i \to \beta$ and $\gamma_i \to \gamma$. Since the X_i and Y_i are fundamental sequences of neighbourhoods, it follows that $r(\beta_i), r(\gamma_i) \to r(\alpha)$ and $s(\beta_i), s(\gamma_i) \to s(\alpha)$. Since $r, s : G \to G^{(0)}$ are continuous and $G^{(0)}$ is Hausdorff, $r(\beta) = r(\alpha) = r(\gamma)$ and $s(\beta) = s(\alpha) = s(\gamma)$. By hypothesis, $s(\alpha)Gr(\alpha) = \{\alpha\}$, so we have $\beta = \gamma = \alpha$. Since U is a neighbourhood of α, we then have $\beta_i, \gamma_i \in U$ for large i. Fix i such that $\beta_i, \gamma_i \in U$. Then $\beta_i \neq \gamma_i$ but either $r(\beta_i) = r(\gamma_i)$ or $s(\beta_i) = s(\gamma_i)$, contradicting that U is a bisection. □

Proof of Theorem 5.1 Fix $f \in \ker(\pi) \setminus \{0\}$. Since s is a local homeomorphism, it is an open map, and since f is locally constant, we deduce that $s(\operatorname{supp}(f)) \subseteq G^{(0)}$ open. Because $\{u \in G^{(0)} : uGu = \{u\}\}$ is dense in $G^{(0)}$, there exists $u \in s(\operatorname{supp}(f))$ such that $uGu = \{u\}$. Fix $\alpha \in \operatorname{supp}(f)$ with $s(\alpha) = u$. Then $r(\alpha)Gs(\alpha) = \alpha(\alpha^{-1}Gu) \subseteq \alpha(uGu) = \{\alpha\}$.

By Lemma 5.3, there exist compact open neighbourhoods X of $r(\alpha)$ and Y of $s(\alpha)$ such that $X \operatorname{supp}(f) Y$ is a bisection containing α. Because r and s are continuous, $X \operatorname{supp}(f) Y = r^{-1}(X) \cap \operatorname{supp}(f) \cap s^{-1}(Y)$ is compact. Since f is locally constant, $X \operatorname{supp}(f) Y$ is also open and there exist subneighbourhoods $X_0 \subseteq r(X \operatorname{supp}(f) Y)$ of $r(\alpha)$ and $Y_0 \subseteq s(X \operatorname{supp}(f) Y)$ of $s(\alpha)$ such that $X_0 \operatorname{supp}(f) Y_0$ is a compact open bisection and $f(\beta) = f(\alpha)$ for all $\beta \in X_0 \operatorname{supp}(f) Y_0$.

We have $1_{X_0}, 1_{Y_0} \in A(G)$. By Lemma 3.3, f may be written as a linear combination of characteristic functions of compact open bisections. [22, Proposition 4.5 (3)] together with bilinearity of multiplication implies that for $\beta \in G$,

$$
\begin{aligned}
(1_{X_0} * f * 1_{Y_0})(\beta) &= 1_{X_0}(r(\beta)) f(\beta) 1_{X_0}(s(\beta)) \\
&= 1_{X_0 \operatorname{supp}(f) Y_0}(\beta) f(\beta) = 1_{X_0 \operatorname{supp}(f) Y_0}(\beta) f(\alpha).
\end{aligned}
$$

Thus $f_0 := 1_{X_0} * f * 1_{Y_0} = f(\alpha) 1_{X_0 \operatorname{supp}(f) Y_0}$. Since $\pi(f) = 0$, we have $\pi(f_0) = 0$. We have $(X_0 \operatorname{supp}(f) Y_0)^{-1}(X_0 \operatorname{supp}(f) Y_0) = Y_0$ because $X_0 \operatorname{supp}(f) Y_0$ is a bisection. Proposition 4.5 (3) [22] implies that

$$
f_0^* * f_0 = |f(\alpha)|^2 1_{(X_0 \operatorname{supp}(f) Y_0)^{-1}(X_0 \operatorname{supp}(f) Y_0)} = |f(\alpha)|^2 1_{Y_0}.
$$

Hence $K := Y_0$ satisfies $\pi(1_K) = \frac{1}{|f(\alpha)|^2} \pi(f_0^* * f_0) = 0$ as required. □

Our graded uniqueness theorem now follows from a bootstrapping argument.

Theorem 5.4 *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Suppose that $\{u \in G^{(0)} : uG_e u = \{u\}\}$ is dense in $G^{(0)}$. Let B be a complex *-algebra and let $\pi : A(G) \to B$ be a graded *-homomorphism. Suppose that* $\ker(\pi) \neq \{0\}$. *Then there is a compact open subset $K \subseteq G^{(0)}$ such that $\pi(1_K) = 0$.*

Proof We first claim that there exists nonzero $f \in A_e$ such that $\pi(f) = 0$. To see this, choose $g \in \ker(\pi) \setminus \{0\}$. Since g is an element of the graded algebra $A(G)$, g can be expressed as a finite sum of graded components $g = \sum_{h \in F} g_h$ where $F \subseteq \Gamma$ and each $g_h \in A_h$. Now $\pi(g) = \sum_{h \in F} \pi(g_h) = 0$, and each $\pi(g_h) \in B_h$ because π is a graded homomorphism. Because the graded subspaces of B are linearly independent, it follows that each $\pi(g_h) = 0$. Since $g \neq 0$, there exists $k \in F$ such that $g_k \neq 0$. By Lemma 3.5, we can write g_k as $\sum_{V \in K} a_V 1_V$ where K is a finite set of mutually disjoint elements of $B_k^{\mathrm{co}}(G)$. Note that $g_k^* = \sum_{V \in K} \overline{a_V} 1_{V^{-1}}$; define $f := g_k^* * g_k$. We claim that $f \in A_e \setminus \{0\}$ and $\pi(f) = 0$. To see this, first notice that

$$f = \left(\sum_{V \in K} \overline{a_V} 1_{V^{-1}} \right) * \left(\sum_{W \in K} a_W 1_W \right) = \sum_{V, W \in K} \overline{a_V} a_W 1_{V^{-1}} * 1_W$$
$$= \sum_{V, W \in K} \overline{a_V} a_W 1_{V^{-1}W}$$

by [22, Proposition 4.5 (3)]. Now, because each $V \in K$ is a subset of G_k, each $V^{-1}W \subseteq G_{k^{-1}k} = G_e$, and thus $f \in A_e$ as claimed. We have $\pi(f) = 0$ because $\pi(g_k) = 0$.

To show that f is nonzero, fix $\alpha \in G_k$ such that $g(\alpha) \neq 0$. Since the elements of K are mutually disjoint, there is a unique $V_\alpha \in K$ such that $\alpha \in V_\alpha$, and then $a_{V_\alpha} = g(\alpha) \neq 0$. Since s is a local homeomorphism, $Gs(\alpha)$ is a discrete space. Write $C_c(Gs(\alpha))$ for the space of finitely supported functions from $Gs(\alpha)$ to $\mathbb{C}$ and for each $\beta \in Gs(\alpha)$ let δ_β denote the point-mass at β so that $C_c(Gs(\alpha)) = \operatorname{span}\{\delta_\beta : \beta \in Gs(\alpha)\}$. For $f \in C_c(G)$, let $\rho(f)$ be the linear map on $C_c(Gs(\alpha))$ determined by

$$\rho(f)\delta_\beta = \sum_{s(\alpha) = r(\beta)} f(\alpha)\delta_{\alpha\beta}.$$

Let $(\cdot \mid \cdot)$ be the standard inner product on $C_c(Gs(\alpha))$, that is $(f|g) = \sum_\beta \overline{f(\beta)} g(\beta)$. Since the elements of K are mutually disjoint, $\big(\rho(1_V)\delta_{s(\alpha)} \mid \rho(1_W)\delta_{s(\alpha)}\big) = 0$ for distinct $V, W \in K$. A calculation shows that for $V \in K$ and $\beta, \gamma \in Gs(\alpha)$, we have $(\delta_\beta | \rho(1_{V^{-1}})\delta_\gamma) = (\rho(1_V)\delta_\beta | \delta_\gamma)$. Hence

$$\begin{aligned}\big(\rho(f)\delta_{s(\alpha)} \mid \delta_{s(\alpha)}\big) &= \big(\rho(g_k)\delta_{s(\alpha)} \mid \rho(g_k)\delta_{s(\alpha)}\big) \\ &= \sum_{V, W \in K} \overline{a_V} a_W \big(\rho(1_W)\delta_{s(\alpha)} \mid \rho(1_V)\delta_{s(\alpha)}\big) \\ &= \sum_{\substack{V \in K, \\ s(\alpha) \in s(V)}} |a_V|^2 \geq |a_{V_\alpha}|^2.\end{aligned}$$

Hence $\rho(f) \neq 0$ which forces $f \neq 0$.

By hypothesis $\{u \in G^{(0)} : uG_e u = \{u\}\}$ is dense in $G^{(0)}$. By definition, A_e is equal to the space of locally constant, continuous, compactly supported functions on G_e, so we may apply Theorem 5.1 to see that $\pi|_{A_e} : A_e \to B$ annihilates 1_K for some compact open $K \subseteq G_e^{(0)} = G^{(0)}$. □

Corollary 5.5 *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Suppose that $\{u \in G^{(0)} : uG_e u = \{u\}\}$ is dense in $G^{(0)}$. Let B be a Γ-graded complex algebra and let $\{t_U : U \in B_*^{\text{co}}(G)\}$ be a representation of $B_*^{\text{co}}(G)$ in B. Suppose that $t_U \in B_n$ whenever $U \in B_n^{\text{co}}(G)$ and that $t_K \neq 0$ for each compact open $K \subseteq G^{(0)}$. Then the homomorphism $\pi : A(G) \to B$ obtained from Theorem 3.10 is injective.*

Proof Since each $A(G)_n$ is spanned by $\{1_U : U \in B_n^{\text{co}}(G)\}$, the homomorphism π is graded. Since $\pi(1_K) = t_K \neq 0$ for all compact open $K \subseteq G^{(0)}$, it follows from Theorem 5.4 that $\ker(\pi) = \{0\}$. □

Remark 5.6 Suppose that G is a locally compact, Hausdorff, étale groupoid with totally disconnected unit space such that $\{u \in G^{(0)} : uGu = \{u\}\}$ is dense in $G^{(0)}$. We may apply Corollary 5.5 with c the trivial cocycle to prove that $A(G)$ is the unique algebra generated by nonzero elements $\{t_U : U \text{ is a compact open bisection of } G\}$ satisfying

(1) $t_\emptyset = 0$;
(2) $t_U t_V = t_{UV}$ for all compact open bisections U, V; and
(3) $t_U + t_V = t_{U \cup V}$ whenever U and V are disjoint compact open bisections whose union is a bisection.

Remark 5.7 In the proof of Theorem 5.4, to see that the function $g_k^* * g_k$ was nonzero, we really just checked that its image under Renault's left-regular representation of G associated to the unit $s(\alpha)$ is nonzero. However, since we are not working in a C^*-completion, we can do everything at the level of linear algebra rather than on Hilbert space. We could instead have appealed to the C^*-identity by regarding $A(G)$ as a subalgebra of $C_r(G)$, but chose a more elementary argument: our argument is essentially that used by Renault to show that the reduced norm is positive definite on $C_c(G)$.

Remark 5.8 Recall from [8] that if Λ is a finitely aligned k-graph, then the k-graph groupoid G_Λ is totally disconnected and locally compact, and carries a $\mathbb{Z}^k$-grading such that $\{u \in G^{(0)} : uG_e u = \{u\}\}$ is dense in $G^{(0)}$. So our graded uniqueness theorem applies to $A(G_\Lambda)$ for any finitely aligned k-graph. Likewise, Remark 5.6 suggests a Cuntz–Krieger uniqueness theorem for $A(G_\Lambda)$. But in practise the relations described in Definition 3.9 and Remark 5.6 are much harder to verify than those of [3, Definition 3.1].

We do not, at this stage, have any invariants at our disposal to decide whether, given groupoids G and G' satisfying our hypotheses, the algebras $A(G)$ and $A(G')$ are or are not isomorphic. It would be very interesting to develop computable algebraic invariants of $A(G)$ for this purpose, but it is beyond the scope of this paper.

However, as an indication that our construction is more flexible the construction of Kumjian–Pask algebras in [3], we describe a class of groupoids that satisfy our hypotheses but do not obviously arise from k-graphs.

Example 5.9 Let $T : X \to X$ be a surjective local homeomorphism of a totally disconnected, compact, Hausdorff space X. Define $T^0 := \text{id}$ and for $k \geq 2$ let $T^k :=$

$T \circ \cdots \circ T$ be the k-fold self-composite of T. Let G be the Deaconu-Renault groupoid defined in [9, Sect. 3]. So

$$G = \{(x, n, y) \in X \times \mathbb{Z} \times X \ : \ T^k(x) = T^l(y), n = k - l\}.$$

Let $G^{(0)}$ be the subset $\{(x, 0, x) : x \in X\}$, which we identify with X in the obvious way. The range and source maps are given by $r(x, n, y) = x$ and $s(x, n, y) = y$. Hence triples (x_1, n_1, y_1) and (x_2, n_2, y_2) are composable if and only if $x_2 = y_1$, in which case $(x_1, n_1, y_1)(x_2, n_2, y_2) := (x_1, n_1 + n_2, y_2)$. The inverse of (x, n, y) is $(y, -n, x)$. For open subsets $U, V \subseteq X$ and $k, l \geq 0$ such that $T^k|_U$ and $T^l|_V$ are homeomorphisms and $T^k(U) = T^l(V)$, define

$$Z(U, V, k, l) := \{(x, k - l, y) \in G : x \in U, y \in V\}.$$

Then

$$\{Z(U, V, k, l) : U, V \subseteq X \text{ are compact open}, k, l \geq 0, \\ T^k|_U \text{ and } T^l|_V \text{ are homeomorphisms and } T^k(U) = T^l(V)\}$$

is a basis of compact open sets for a topology on G under which it becomes a locally compact, Hausdorff groupoid with totally disconnected unit space X. Fix $(x, n, y) \in G$ and k, l such that $k - l = n$ and $T^k(x) = T^l(y)$. The source map on G restricts to a homeomorphism on each basic open set $Z(U, V, k, l)$ so is a local homeomorphism. Moreover, the map $c : G \to \mathbb{Z}$ defined by $c((x, n, y)) = n$ is a cocycle and is continuous because each basic open set belongs to some $c^{-1}(n)$. Hence (G, c) satisfies our hypotheses, and $A(G)$ is a sensible candidate for the Leavitt algebra of (X, T).

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