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RESEARCH ARTICLE

Ascending chain conditions on principal left and right ideals for semidirect products of semigroups

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Abstract In this paper we investigate the ascending chain conditions on principal left and right ideals for semidirect products of semigroups and show how this is connected to the corresponding problem for rings of skew generalized power series. Let S be a left cancellative semigroup with a unique idempotent e, T a right cancellative semigroup with an idempotent f and $\omega : T \to \mathrm{End}(S)$ a semigroup homomorphism such that $\omega(f) = id_S$. We show that in this case the semidirect product $S \rtimes_\omega T$ satisfies the ascending chain condition for principal left ideals (resp. right ideals) if and only if S and T satisfy the ascending chain condition for principal left ideals (resp. right ideals and $\mathrm{Im}\,\omega(t)$ is closed for complete inverses for all $t \in T$). We also give several examples to show that for more general semigroups these implications may not hold.

Keywords Ascending chain condition · Principal ideal · Semidirect product · Semigroup · Cancellation property

1 Introduction

The aim of this article is to investigate how ascending chain conditions for principal left and right ideals behave in regard to semidirect products of semigroups. The motivation comes from the following theorem for rings.

Theorem 1.1 [5, Theorem 3.3] *Let R be a ring, $(S, \cdot, \leq)$ a strictly totally ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism.*

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(i) *$R[[S,\omega]]$ is an ACCPL-domain if and only if R is an ACCPL-domain, S is an ACCPL-monoid and ω_s is injective for any $s \in S$.*
(ii) *$R[[S,\omega]]$ is an ACCPR-domain if and only if R is an ACCPR-domain, S is an ACCPR-monoid and ω_s is injective and preserves nonunits of R for any $s \in S$.*

We are able to prove a similar result for semidirect products of semigroups (see Theorem 3.11 and Corollary 3.12) from where the main parts of the above theorem follow as a corollary.

Let S be a semigroup. If S is not a monoid we may adjoint an identity 1 to form a monoid which will be denoted by S^1. If S is already a monoid we define $S^1 = S$. Recall that a *left ideal* of S is any nonempty subset of S that is closed under multiplication from the left with elements of S. Right ideals are defined analogously. A principal left (resp. right) ideal of S generated by an element $a \in S$ is given by S^1a (resp. aS^1).

An element $b \in S$ is an *inverse* of $a \in S$ if $aba = a$ and $bab = b$. If such an element b exists then a is said to be a *regular* element of S. If there exists an inverse b of a such that $ab = ba$ then a is *completely regular* and we shall say that b is a *complete inverse* of a. If such b exists then it is unique. The set of all regular elements of S will be denoted by $\operatorname{Reg}(S)$.

A semigroup S is said to satisfy the ascending chain condition on principal left ideals (ACCPL) if there does not exist an infinite strictly ascending chain of principal left ideals of S. The ascending chain condition on principal right ideals (ACCPR) if defined analogously. Semigroups that satisfy the ACCPL (resp. ACCPR) are sometimes called ACCPL-semigroups (resp. ACCPR-semigroups). It has been shown in [5, Example 2.6] that these two conditions are independent. For any undefined concepts on semigroups we refer the reader to [1].

In Sect. 2 we present some results involving cancellativity that we need later on. In particular in the class of all left cancellative semigroups we give a characterization of those semigroups that contain at most one idempotent. The main results are contained is Sect. 3 where we study in detail under what conditions the semidirect product of semigroups will satisfy ACCPL and ACCPR. We divide the proof of the main theorem into several lemmas and give numerous examples to demonstrate that the conditions involved are essential. At the end of Sect. 3 we show how our results are connected to the skew generalized power series rings. We postpone the more technical proofs of our examples to Sect. 4.

2 Cancellativity and chain conditions

Ascending chain conditions for principal one-sided ideals are essentially conditions on sequences of elements. We put this simple fact into a lemma since we will use it throughout this paper. The point of this lemma is also that even though one-sided ideals are given via S^1 we need only work with sequences in S.

Lemma 2.1 *For an arbitrary semigroup S the following are equivalent*:

(1) *S is an ACCPR-semigroup.*

(2) *For any sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *in* S *such that* $a_n = a_{n+1}b_n$ *for all* $n \in \mathbb{N}$, *there exists* $N \in \mathbb{N}$ *and a sequence* $(c_n)_{n\in\mathbb{N}}$ *in* S *such that* $a_{n+1} = a_n c_n$ *for all* $n \geq N$.
(3) *For any sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *in* S *such that* $a_n = a_{n+1}b_n$ *for all* $n \in \mathbb{N}$, *there exists* $N \in \mathbb{N}$ *and* $c_N \in S$ *such that* $a_{N+1} = a_N c_N$.

Proof (1) ⇒ (2): If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are as in (2) then we have a chain $a_1S^1 \subseteq a_2S^1 \subseteq a_3S^1 \subseteq \cdots$ of principal right ideals, which must stabilize say at N. Hence there exists a sequence $(c_n)_{n\in\mathbb{N}} \subseteq S^1$ such that $a_{n+1} = a_n c_n$ for all $n \geq N$. If $c_n = 1$ for some n, then $a_{n+1} = a_n$ and hence $a_{n+1} = a_n = a_{n+1}b_n = a_n b_n$. So we may take $c_n = b_n$ instead of $c_n = 1$. The new sequence $(c_n)_{n\in\mathbb{N}}$ then lies in S.

(2) ⇒ (3): This is obvious.

(3) ⇒ (1): Suppose there exists a strictly increasing chain $a_1S^1 \subsetneq a_2S^1 \subsetneq a_3S^1 \subsetneq \cdots$ of principal right ideals of S. Then there is a sequence $(b_n)_{n\in\mathbb{N}} \subseteq S^1$ such that $a_n = a_{n+1}b_n$ for all $n \in \mathbb{N}$. Infact b_n must lie in S since the inclusions in the chain are strict. By assumption $a_{N+1} = a_N c_N \in a_N S^1$ for some N and $c_N \in S$, which is a contradiction. □

Of course with obvious changes we also get a version of the above lemma for ACCPL.

Left and right cancellativity of semigroups is often very helpful when considering ACCPR and ACCPL. Recall that a semigroup S is left (resp. right) cancellative if $ax = ay$ (resp. $xa = ya$) implies $x = y$ for all $a, x, y \in S$. The following is a generalization of [5, Proposition 2.1] to semigroups and only one-sided cancellativity is assumed.

Proposition 2.2 *For a left cancellative semigroup* S *the following are equivalent*:

(1) S *is an ACCPR-semigroup*.
(2) *For any sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *in* S *such that* $a_n = a_{n+1}b_n$ *for all* $n \in \mathbb{N}$, *there exists* $N \in \mathbb{N}$ *such that* b_n *is regular and* $a_n \in a_n S$ *for all* $n \geq N$.
(3) *For any sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *in* S *such that* $a_n = a_{n+1}b_n$ *for all* $n \in \mathbb{N}$, *there exists* $N \in \mathbb{N}$ *such that* b_N *is regular and* $a_{N+1} \in a_{N+1}S$.

Proof (1) ⇒ (2): Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be as in (2). Then by Lemma 2.1 $a_{n+1} = a_n c_n$ for some $c_n \in S$ for all $n \geq N$. Choose any $n \geq N$. Then $a_n = a_{n+1}b_n = a_n c_n b_n \in a_n S$, which implies $a_n c_n = a_n c_n b_n c_n$ and by left cancellativity $c_n = c_n b_n c_n$. From this we get $c_n b_n = c_n b_n c_n b_n$ and again by left cancellativity $b_n = b_n c_n b_n$. This shows that b_n is regular.

(2) ⇒ (3): This is obvious.

(3) ⇒ (1): Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subseteq S$ be sequences such that $a_n = a_{n+1}b_n$ for all $n \in \mathbb{N}$. By assumption there exists $N \in \mathbb{N}$ such that b_N is regular and $a_{N+1} = a_{N+1}x$ for some $x \in S$. Let c be an inverse of b_N. Then $cb_N cx = cx$ and by left cancellativity $b_N cx = x$. Hence $a_{N+1} = a_{N+1}x = a_{N+1}b_N cx = a_N cx$. By Lemma 2.1 S satisfies ACCPR. □

Next we characterize left cancellative semigroups in which at most one idempotent exists.

Proposition 2.3 *For a left cancellative semigroup S the following are equivalent*:

(1) *S has at most one left identity.*
(2) *S has at most one idempotent.*
(3) *Every inverse in S is complete.*
(4) *Each element in S has at most one inverse.*
(5) *If $s = sab$ for some elements a, b, s in S then a is a complete inverse of b.*
(6) $\mathrm{Reg}(S)$ *is either empty or a subgroup of S.*

Proof Let s and x be elements of S such that $s = sx$. Then for all $a \in S$ we have $sa = sxa$ and left cancellativity implies $a = xa$. So x is a left identity.

(1) $\Leftrightarrow$ (2): Any left identity is an idempotent and any idempotent $e = e^2$ is a left identity by the above.

(2) $\Rightarrow$ (3): Let a' be an inverse of a. Then aa' and $a'a$ are idempotents, hence they are equal.

(3) $\Rightarrow$ (4): This is obvious since complete inverses are unique.

(4) $\Rightarrow$ (1): Let e and f be two left identities of S. Then $efe = e$ and $fef = f$ and also $eee = e$. So e and f are both inverses of e, hence $e = f$.

(1) $\Rightarrow$ (5): If $s = sab$ then also $sa = saba$, so by the above ab and ba are left identities. This implies $aba = a$, $bab = b$ and by assumption $ab = ba$.

(5) $\Rightarrow$ (1): Let e and f be two left identities of S. Then $e = efe$ so by assumption e and f are each others complete inverses. In particular they commute, hence $e = fe = ef = f$.

(2) $\Rightarrow$ (6): Every regular element is L-related to some idempotent and R-related to some idempotent. So either $\mathrm{Reg}(S)$ is empty or every element in $\mathrm{Reg}(S)$ is H-related to a unique idempotent of S. In the second case $\mathrm{Reg}(S)$ is an H-class, because every element that is related to an idempotent is automatically regular. Since this H-class contains an idempotent it is a subgroup of S.

(6) $\Rightarrow$ (2): This follows from the fact that all idempotents lie in $\mathrm{Reg}(S)$ and a group has only one idempotent. □

Observe that conditions (2) and (6) are equivalent in any semigroup. Note also that in a left cancellative monoid S the identity element 1 is a unique left identity and regular elements are exactly the invertible elements of S in the group sense. The following example shows that a left cancellative semigroup with a unique left identity need not be a left cancellative monoid.

Example 2.4 Let $\mathbb{N}$ denote the set of positive integers. Define multiplication on $S = \mathbb{N} \times \mathbb{N}$ by

$$(n, m) \cdot (n', m') = (nmn', m').$$

It is easy to see that $(S, \cdot)$ is a left cancellative semigroup with a unique left identity $(1, 1)$. Since $(1, 1)$ is not a right identity, $(S, \cdot)$ is not a monoid. This is an example with trivial group $\mathrm{Reg}(S)$. However if G is any group then $S \times G$ is an example with $\mathrm{Reg}(S \times G)$ isomorphic to G.

Proposition 2.2 has a simple corollary (cf. [5, Corollary 2.2] for part (2)).

Corollary 2.5 *Let T be a left cancellative ACCPR-semigroup with at most one idempotent.*

(1) *Let $\psi : S \to T$ be a semigroup homomorphism such that $\psi^{-1}(\mathrm{Reg}(T)) \subseteq \mathrm{Reg}(S)$. If ψ is injective or if S is a left cancellative monoid then S is an ACCPR-semigroup as well.*
(2) *If S is a subsemigroup of T such that $\mathrm{Reg}\, S = \mathrm{Reg}\, T \cap S$ then S is an ACCPR-semigroup as well.*

Proof (1): If ψ is injective then S is also a left cancellative semigroup. So in both cases we may use Proposition 2.2. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in S such that $a_n = a_{n+1}b_n$ for all $n \in \mathbb{N}$. Since $\psi(a_n) = \psi(a_{n+1})\psi(b_n)$ for all $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ and $t \in T$ such that $\psi(b_N)$ is regular and $\psi(a_{N+1}) = \psi(a_{N+1})t$. By assumption b_N is also regular. Let c be an inverse of b_N. Then $\psi(c)$ is an inverse of $\psi(b_N)$. As in the proof of Proposition 2.3 this implies that t and $\psi(b_N)\psi(c)$ are idempotents in T, so they are equal. Hence $\psi(a_{N+1}) = \psi(a_{N+1})\psi(b_N)\psi(c) = \psi(a_{N+1}b_N c)$. If ψ is injective then $a_{N+1} = a_{N+1}b_N c$ and so $a_{N+1} \in a_{N+1}S$. If S is a monoid then we also have $a_{N+1} \in a_{N+1}S$. So S satisfies condition (3) of Proposition 2.2 hence it is an ACCPR-semigroup.

(2): In this case the inclusion of S into T is an injective homomorphism that satisfies the condition in (1). □

3 Semidirect product of semigroups

Let S and T be two semigroups and $\omega : T \to \mathrm{End}(S)$ a semigroup homomorphism where $\mathrm{End}(S)$ is the monoid of all endomorphisms of S. The image of an element $t \in T$ under ω will be denoted by ω_t. Now define multiplication on the Cartesian product $S \times T$ by

$$(s_1, t_1)(s_2, t_2) = \big(s_1\omega_{t_1}(s_2), t_1t_2\big).$$

Then the set $S \times T$ together with this multiplication becomes a semigroup called the *semidirect product* of semigroups S and T over ω. This semigroup will be denoted by $S \rtimes_\omega T$.

The following lemma is easy to verify. Some parts of it can be found in [2, Lemmas 3.1 and 3.2].

Lemma 3.1 *Let S and T be semigroups and $\omega : T \to \mathrm{End}\, S$ a semigroup homomorphism. Then the following holds.*

(1) *If S and T are right cancellative then $S \rtimes_\omega T$ is right cancellative.*
(2) *If S and T are left cancellative and ω_t is injective for all $t \in T$ then $S \rtimes_\omega T$ is left cancellative.*

If there exists $u \in T$ such that ω_u is surjective then the implications go the other way as well.

If we take $\omega_t = id_S$ for all $t \in T$ then $S \ltimes_\omega T$ is the direct product of S and T. It is known that for cancellative monoids S and T the direct product $S \times T$ satisfies $ACCPL$ if and only if S and T satisfy ACCPL (this is true also if S and T are not cancellative, see Theorem 3.13). The same holds even for infinite direct sums of cancellative monoids (see [3] and [5]). The aim of this section is to investigate how ACCPL and ACCPR behave in regard to semidirect products of semigroups.

For the rest of this section, let S and T be semigroups and $\omega : T \to \mathrm{End}(S)$ a semigroup homomorphism. First we consider when the fact that the semidirect product satisfies ACCPL or ACCPR implies the same for each factor.

Lemma 3.2 *If $S \ltimes_\omega T$ satisfies ACCPL (ACCPR) and there exists $f \in T$ such that $f^2 = f$ and $\omega_f = id_S$ then S satisfies ACCPL (ACCPR) as well.*

Proof Denote $Q = S \ltimes_\omega T$ and let $s_1, s_2 \in S$. If $s_1 = ss_2$ for some $s \in S$ then $(s_1, f) = (s, f)(s_2, f)$. If $(s_1, f) = (s, t)(s_2, f)$ for some $s \in S$ and $t \in T$ then $f = tf$ and $s_1 = s\omega_t(s_2)$. Hence $\omega_t = \omega_t\omega_f = \omega_{tf} = \omega_f = id_S$ and so $s_1 = ss_2$. This shows that

$$S^1 s_1 \subseteq S^1 s_2 \quad \text{iff} \quad Q^1(s_1, f) \subseteq Q^1(s_2, f). \tag{1}$$

If $s_1 = s_2 s$ for some $s \in S$ then $(s_1, f) = (s_2, f)(s, f)$. If $(s_1, f) = (s_2, f)(s, t)$ for some $s \in S$ and $t \in T$ then $s_1 = s_2\omega_f(s) = s_2 s$. This shows that

$$s_1 S^1 \subseteq s_2 S^1 \quad \text{iff} \quad (s_1, f)Q^1 \subseteq (s_2, f)Q^1. \tag{2}$$

The conclusion of the lemma follows easily from (1) and (2). □

Note that the existence of an idempotent f as in Lemma 3.2 gives us an inclusion $S \hookrightarrow S \ltimes_\omega T$ defined by $s \mapsto (s, f)$. As the next example demonstrates the existence of an idempotent alone does not suffice for the conclusion of the lemma, not even if there are other elements in T that act on S as the identity map.

Example 3.3 Let $S = (0, 1] \times (0, \infty)$ be equipped with multiplication and $T = \mathbb{N} \cup \{\infty\}$ be equipped with addition (here $n + \infty = \infty + n = \infty$ and $\infty + \infty = \infty$). Define $\omega : T \to \mathrm{End}(S)$ by

$$\omega_n = id_S \quad \text{for all } n \in \mathbb{N},$$
$$\omega_\infty(x, y) = (1, xy) \quad \text{for all } (x, y) \in S.$$

So ∞ is an idempotent in T but $\omega_\infty \neq id_S$. The commutativity of multiplication in $(0, \infty)$ ensures that ω_∞ is an endomorphism of S. Obviously ω is a homomorphism. Simple calculations show that for all $(a, b) \in S$ and all $n \in \mathbb{N}$ we have

$$(S \ltimes_\omega T)^1\big((a, b), \infty\big) = \big((0, 1] \times (0, \infty)\big) \times \{\infty\},$$
$$(S \ltimes_\omega T)^1\big((a, b), n\big) = \Big(\big((0, a] \times (0, \infty)\big) \times \{n + 1, n + 2, \dots\}\Big)$$
$$\cup\Big(\big((0, 1] \times (0, \infty)\big) \times \{\infty\}\Big) \cup \Big\{\big((a, b), n\big)\Big\},$$

and

$$\big((a,b),\infty\big)(S \rtimes_\omega T)^1 = \big(\{a\} \times (0,\infty)\big) \times \{\infty\},$$
$$\big((a,b),n\big)(S \rtimes_\omega T)^1 = \Big(\big((0,a] \times (0,\infty)\big) \times \{n+1, n+2, \ldots\}\Big) \cup \Big(\big((0,a] \times (0,\infty)\big) \times \{\infty\}\Big) \cup \Big\{\big((a,b),n\big)\Big\}.$$

Suppose there is a strictly increasing chain of principal left or right ideals in $S \rtimes_\omega T$ generated by $((a_k, b_k), t_k)$. Then $t_k = \infty$ for at most one k and from this k on the sequence t_k must be decreasing and hence finite. So $S \rtimes_\omega T$ satisfies *ACCPL* and *ACCPR* (and so does T). However S does not satisfy *ACCPL* nor *ACCPR*, since $(0, 1]$ does not (see Theorem 3.13).

Lemma 3.4 *If $S \rtimes_\omega T$ satisfies ACCPL (ACCPR) and there exists $e \in S$ such that $e^2 = e$ then T satisfies ACCPL (ACCPR) as well.*

Proof Let $S \rtimes_\omega T$ be an ACCPL-semigroup. Let t_n and u_n be elements of T such that $t_n = u_n t_{n+1}$ for all $n \in \mathbb{N}$. Define $x_n = (\omega_{t_n}(e), t_n) \in S \rtimes_\omega T$ and $y_n = (\omega_{t_n}(e), u_n) \in S \rtimes_\omega T$. Then

$$\begin{aligned} y_n x_{n+1} &= \big(\omega_{t_n}(e), u_n\big)\big(\omega_{t_{n+1}}(e), t_{n+1}\big) = \big(\omega_{t_n}(e)\omega_{u_n t_{n+1}}(e), u_n t_{n+1}\big) \\ &= \big(\omega_{t_n}(e)^2, t_n\big) = \big(\omega_{t_n}(e), t_n\big) = x_n \end{aligned}$$

for all $n \in \mathbb{N}$. Since $S \rtimes_\omega T$ is an ACCPL-semigroup there exist $N \in \mathbb{N}$ and $(s, t) \in S \rtimes_\omega T$ such that $x_{N+1} = (s, t)x_N$. In particular $t_{N+1} = t t_N$. Therefore T is an ACCPL-semigroup.

Now let $S \rtimes_\omega T$ be an ACCPR-semigroup. Let t_n and v_n be elements of T such that $t_n = t_{n+1} v_n$ for all $n \in \mathbb{N}$. Define $x_n = (\omega_{t_1}(e), t_n) \in S \rtimes_\omega T$ and $y_n = (\omega_{v_n v_{n-1} \ldots v_1}(e), v_n) \in S \rtimes_\omega T$. Then

$$\begin{aligned} x_{n+1} y_n &= \big(\omega_{t_1}(e), t_{n+1}\big)\big(\omega_{v_n v_{n-1} \ldots v_1}(e), v_n\big) \\ &= \big(\omega_{t_1}(e)\omega_{t_{n+1} v_n v_{n-1} \ldots v_1}(e), t_{n+1} v_n\big) \\ &= \big(\omega_{t_1}(e)\omega_{t_1}(e), t_n\big) = \big(\omega_{t_1}(e), t_n\big) = x_n \end{aligned}$$

for all $n \in \mathbb{N}$. As above this implies $t_{N+1} = t_N t$ for some $N \in \mathbb{N}$ and $t \in T$. Hence T is an ACCPR-semigroup. □

The following example shows that the lemma does not hold if S has no idempotent.

Example 3.5 Let $S = \mathbb{N}$ be equipped with addition and $T = (0, 1]$ with multiplication. Let ω be trivial, so $S \rtimes_\omega T = S \times T$. For all $(n, t) \in S \times T$ we have

$$(S \times T)^1(n,t) = (n,t)(S \times T)^1 = \big(\{n+1, n+2, \ldots\} \times (0,t]\big) \cup \big\{(n,t)\big\}.$$

From this it is obvious that $S \times T$ satisfies ACCPL and ACCPR (and so does S), however T does not satisfy ACCPL nor ACCPR.

Now we investigate when the semidirect product inherits the property of satisfying ACCPL and ACCPR from its factors. As it turns out these two conditions for semidirect products are very different, so we treat them separately.

Lemma 3.6 *If S and T satisfy ACCPL and for all $t, u \in T$ with $ut = t$ there exists $v \in T$ with $vt = t$ such that $\omega_v\omega_u = \omega_u\omega_v = id_S$ then $S \rtimes_\omega T$ satisfies ACCPL as well.*

Proof Suppose $s_n, r_n \in S$ and $t_n, u_n \in T$ are such that we have $(s_n, t_n) = (r_n, u_n)(s_{n+1}, t_{n+1})$ in $S \rtimes_\omega T$ for all $n \in \mathbb{N}$. So $s_n = r_n\omega_{u_n}(s_{n+1})$ and $t_n = u_n t_{n+1}$ for all $n \in \mathbb{N}$. Since T is an ACCPL-semigroup there exists $m \in \mathbb{N}$ such that $t_{n+1} = v_n t_n$ for some $v_n \in T$ for all $n \geq m$. For $n > m$ define $s'_n = \omega_{u_m u_{m+1} \dots u_{n-1}}(s_n) \in S$. Then

$$s'_n = \omega_{u_m u_{m+1} \dots u_{n-1}}\big(r_n\omega_{u_n}(s_{n+1})\big) = \omega_{u_m u_{m+1} \dots u_{n-1}}(r_n)s'_{n+1}$$

and since S is an ACCPL-semigroup there exists $k > m$ and $s \in S$ such that $s'_{k+1} = ss'_k$. Since $t_{n+1} = v_n u_n t_{n+1}$ for all $n \geq m$ by assumption there exist $w_n \in T$ with $w_n t_{n+1} = t_{n+1}$ such that $\omega_{w_n}\omega_{v_n u_n} = id_S$ for all $n \geq m$. If we denote $z_n = w_n v_n$ then ω_{z_n} is a left inverse of ω_{u_n} for all $n \geq m$. Hence

$$s_{k+1} = \omega_{z_k z_{k-1} \dots z_m}\big(s'_{k+1}\big) = \omega_{z_k z_{k-1} \dots z_m}(s)\omega_{z_k z_{k-1} \dots z_m}\big(s'_k\big) = s'\omega_{z_k}(s_k),$$

where $s' = \omega_{z_k z_{k-1} \dots z_m}(s)$. In addition $z_k t_k = w_k v_k t_k = w_k t_{k+1} = t_{k+1}$, so $(s_{k+1}, t_{k+1}) = (s', z_k)(s_k, t_k)$. This shows that $S \rtimes_\omega T$ is an ACCPL-semigroup. □

Without additional assumptions about elements of T the conclusion of the lemma may not hold. In fact even if we only exclude the assumption $vt = t$ the semidirect product may not satisfy ACCPL.

Example 3.7 Let $S = \langle \dots, x_{-2}, x_{-1}, x_0, x_1, x_2 \dots \rangle$ be the free monoid over the set $\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$, $\langle v \rangle$ the free monoid over $\{v\}$ and T_1 a monoid with representation $T_1 = \langle t, u \mid tu = ut = t \rangle$. Define the monoid T by $T = T_1 \times \langle v \rangle = \langle t, u, v \mid tu = ut = t, tv = vt, uv = vu \rangle$. First note that $T_1 = \{1, u, u^2, u^3, \dots\} \cup \{t, t^2, t^3, \dots\}$ and

$$T_1^1 u^n = \{u^n, u^{n+1}, u^{n+2}, \dots\} \cup \{t, t^2, t^3, \dots\} \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$

$$T_1^1 t^n = \{t^n, t^{n+1}, t^{n+2}, \dots\} \quad \text{for all } n \in \mathbb{N}.$$

So it is obvious that T_1 satisfies ACCPL. Since any free monoid satisfies ACCPL, both S and T (by Theorem 3.13) satisfy ACCPL. Now define endomorphisms of S by

$$\omega_1 = id_S,$$
$$\omega_t(s) = 1 \quad \text{for all } s \in S,$$
$$\omega_u(1) = 1, \qquad \omega_u(x_n) = x_{n-1} \quad \text{for all } n \in \mathbb{Z},$$
$$\omega_v(1) = 1, \qquad \omega_v(x_n) = x_{n+1} \quad \text{for all } n \in \mathbb{Z}.$$

Since $\omega_t\omega_u = \omega_u\omega_t = \omega_t$, $\omega_t\omega_v = \omega_v\omega_t$ and $\omega_u\omega_v = \omega_v\omega_u = id_S$, these endomorphisms induce a homomorphism $\omega : T \to \mathrm{End}(S)$. If $\overline{ut} = \overline{t}$ in T then $\overline{u} = u^n$ for some $n \in \mathbb{N} \cup \{0\}$ and for $\overline{v} = v^n$ we have $\omega_{\overline{v}}\omega_{\overline{u}} = \omega_{\overline{u}}\omega_{\overline{v}} = id_S$. The semigroup $S \rtimes_\omega T$ does not satisfy ACCPL since the sequence of principal left ideals generated by elements (x_n, t) is an infinite strictly increasing chain. This is because $(x_n, t) = (1, u)(x_{n+1}, t)$ for all $n \in \mathbb{Z}$. However if $(x_{n+1}, t) = (s, a)(x_n, t) = (s\omega_a(x_n), at)$ then $at = t$ implies $a = u^k$ for some $k \in \mathbb{N} \cup \{0\}$ and hence $s\omega_a(x_n) = sx_{n-k} \neq x_{n+1}$ which is a contradiction.

Lemma 3.8 *If S and T satisfy ACCPR and for all $p, r, s \in S$ and $t, u \in T$ $sr\omega_t(p) = s$ and $t = tu$ imply $r \in \mathrm{Im}\,\omega_t$ then $S \rtimes_\omega T$ satisfies ACCPR as well.*

Proof Let $s_n, r_n \in S$ and $t_n, u_n \in T$ be such that $(s_n, t_n) = (s_{n+1}, t_{n+1})(r_n, u_n)$ for all $n \in \mathbb{N}$. So $s_n = s_{n+1}\omega_{t_{n+1}}(r_n)$ and $t_n = t_{n+1}u_n$ for all $n \in \mathbb{N}$. Since S and T are ACCPR-semigroups there exist $k \in \mathbb{N}$, $s \in S$ and $t \in T$ such that $s_{k+1} = s_k s$ and $t_{k+1} = t_k t$. Then $s_k = s_k s\omega_{t_{k+1}}(r_k)$ and $t_{k+1} = t_{k+1}u_k t$, hence by assumption there exists $r \in S$ such that $s = \omega_{t_{k+1}}(r)$. This implies

$$(s_{k+1}, t_{k+1}) = \big(s_k\omega_{t_{k+1}}(r), t_{k+1}\big) = \big(s_k\omega_{t_k t}(r), t_k t\big) = (s_k, t_k)\big(\omega_t(r), t\big),$$

which means that $S \rtimes_\omega T$ satisfies ACCPR. □

We will say that a subset M of a semigroup S is *closed for complete inverses* if $r \in M$ whenever r is a complete inverse of some element of M. In the main theorem the above lemma will be used in the following way. Let S and T satisfy ACCPR and suppose in addition that for all $p, r, s \in S$ and $t, u \in T$ $sr\omega_t(p) = s$ and $t = tu$ imply that r is a complete inverse of $\omega_t(p)$. If $\mathrm{Im}\,\omega_t$ is closed for complete inverses for all $t \in T$ with $t = tu$ for some $u \in T$ then by the above lemma $S \rtimes_\omega T$ satisfies ACCPR. The next lemma shows that the inverse implication holds as well.

Lemma 3.9 *If $S \rtimes_\omega T$ satisfies ACCPR and for all $p, r, s \in S$ and $t, u \in T$ $sr\omega_t(p) = s$ and $t = tu$ imply that r is a complete inverse of $\omega_t(p)$ then $\mathrm{Im}\,\omega_t$ is closed for complete inverses for all $t \in T$ with $t = tu$ for some $u \in T$.*

Proof Let $t = tu$ in T and let $r \in S$ be a complete inverse of $\omega_t(p)$ for some $p \in S$. Define $x_n = (r^n, t) \in S \rtimes_\omega T$ and $y = (p, u) \in S \rtimes_\omega T$. Then

$$x_{n+1}y = \big(r^{n+1}, t\big)(p, u) = \big(r^{n+1}\omega_t(p), tu\big) = \big(r^n\omega_t(p)r, t\big) = \big(r^n, t\big) = x_n$$

for all $n \in \mathbb{N}$ and since $S \rtimes_\omega T$ is an ACCPR-semigroup there exists $k \in \mathbb{N}$, $q \in S$ and $v \in T$ such that $x_{k+1} = x_k \cdot (q, v)$ in $S \rtimes_\omega T$. The first component of this equation gives us $r^{k+1} = r^k\omega_t(q)$. If we multiply this by $\omega_t(p)$ from the right side and use the fact that r is a complete inverse of $\omega_t(p)$ we get

$$r^k\omega_t(q)\omega_t(p) = r^{k+1}\omega_t(p) = r^k.$$

By assumption this means that $\omega_t(q)$ in a complete inverse of $\omega_t(p)$. So $r = \omega_t(q)$, since complete inverses are unique. □

Observe that in the paragraph before this lemma we could exchange the term "complete inverse" with term "unique inverse" or even "inverse" and the implications would still hold. However in this case the inverse implication would fail to hold. Namely, the following example shows that we can not replace "complete inverse" with "unique inverse" in the above lemma. Since the proof of the example is rather technical we will present it in the last section.

Example 3.10 Let $S = S_1 \times S_2$ where S_1 and S_2 are semigroups with representations $S_1 = \langle z, w \mid w^2 = w\rangle$ and $S_2 = \langle x, y \mid xyx = x,\ yxy = y\rangle$. Let $T = \{0, 1, 2\}$ be a submonoid of the multiplicative monoid $(\mathbb{Z}_4, \cdot)$. Define a homomorphism $\omega : T \to \mathrm{End}(S)$ by

$$\omega_0(s_1, s_2) = (w, xy) \quad \text{for all } (s_1, s_2) \in S,$$

$$\omega_1 = id_S,$$

$$\omega_2(z, s_2) = (w, x), \qquad \omega_2(w, s_2) = (w, xy) \quad \text{for all } s_2 \in S_2.$$

It is clear that the above induces an endomorphism ω_2. It is easy to check that ω is indeed a homomorphism. As it turns out (see last section) the semigroup $S \rtimes_\omega T$ satisfies ACCPR and for all $s, r, p \in S$ $srp = s$ implies that r is a unique inverse of p (in particular all inverses in S are unique). However even though $2 = 2 \cdot 1$ in T, $\mathrm{Im}\,\omega_2$ is not closed for unique inverses since $(w, y) \notin \mathrm{Im}\,\omega_2$ is a unique inverse of $(w, x) \in \mathrm{Im}\,\omega_2$.

Now we assemble everything into our main theorem.

Theorem 3.11 *Let S and T be semigroups with idempotents e and f respectively and $\omega : T \to \mathrm{End}\,S$ a semigroup homomorphism.*

(1) *Suppose that for all $t, u \in T$ with $ut = t$ there exists $v \in T$ with $vt = t$ such that $\omega_v\omega_u = \omega_u\omega_v = id_S$. Then $S \rtimes_\omega T$ satisfies ACCPL if and only if S and T satisfy ACCPL.*

(2) *Suppose that $\omega_f = id_S$ and that for all $p, r, s \in S$ and $t, u \in T$ $sr\omega_t(p) = s$ and $t = tu$ imply that r is a complete inverse of $\omega_t(p)$. Then $S \rtimes_\omega T$ satisfies ACCPR if and only if S and T satisfy ACCPR and $\mathrm{Im}\,\omega_t$ is closed for complete inverses for all $t \in T$ with $t = tu$ for some $u \in T$.*

Proof (1): Since $f^2 = f$ by assumption ω_f is invertible. But $\omega_f\omega_f = \omega_f$, hence $\omega_f = id_S$. The rest follows directly from Lemmas 3.2, 3.4 and 3.6.

(2): This follows from Lemmas 3.2, 3.4, 3.8 and 3.9. □

Assuming some cancellativity simplifies the formulation of the above theorem.

Corollary 3.12 *Let S be a left cancellative semigroup with a unique idempotent e and T a right cancellative semigroup with an idempotent f such that $\omega_f = id_S$. Then the following holds.*

(1) *$S \rtimes_\omega T$ satisfies ACCPL if and only if S and T satisfy ACCPL.*

(2) *$S \rtimes_\omega T$ satisfies ACCPR if and only if S and T satisfy ACCPR and* $\operatorname{Im}\omega_t$ *is closed for complete inverses for all t in T.*

Proof (1): If $ut = t$ in T then $fut = ft$ and right cancellativity implies $fu = f$. Using ω on this we get $\omega_u = id_S$. Now apply Theorem 3.11.

(2): Proposition 2.3 ensures that the assumptions of Theorem 3.11 are satisfied. Since T is right cancellative the idempotent f is a right identity. So $t = tf$ for all $t \in T$. Now apply Theorem 3.11. □

For a semigroup S we will denote by SEnd(S) the monoid of all surjective endomorphisms of S.

Theorem 3.13 *Let S and T be semigroups with idempotents and let $\omega : T \to$* SEnd(S) *be a semigroup homomorphism. Then $S \rtimes_\omega T$ satisfies ACCPL* (*ACCPR*) *if and only if S and T satisfy ACCPL* (*ACCPR*).

Proof If $ut = t$ in T then $\omega_u = id_S$ since ω_t is surjective. In particular $\omega_f = id_s$ for any idempotent $f \in T$. Also $\operatorname{Im}\omega_t = S$ for all $t \in T$. Now the theorem follows from Lemmas 3.2, 3.4, 3.6 and 3.8. □

In particular the theorem states that for semigroups S and T with idempotents $S \times T$ satisfies ACCPL (ACCPR) if and only if S and T satisfy ACCPL (ACCPR).

In [4] the ring of skew generalized power series was introduced. The construction is as follows. Let $(S, \cdot, \leq)$ be a strictly ordered monoid, R a ring with identity and $\omega : S \to \operatorname{End} R$ a monoid homomorphism where End(R) is the monoid of all ring endomorphisms of R that preserve the identity. Let $R[[S, \omega, \leq]]$ be the set of all maps $f : S \to R$ whose support $\operatorname{supp}(f) = \{s \in S;\ f(s) \neq 0\}$ is artinian and narrow (for details see [4]). For two such maps f and g the set $X_s(f, g) = \{(x, y) \in S \times S;\ xy = s,\ f(x) \neq 0,\ g(y) \neq 0\}$ turns out to be finite for all $s \in S$. Thus one can define a multiplication on $R[[S, \omega, \leq]]$ by

$$(fg)(s) = \sum_{(x,y)\in X_s(f,g)} f(x)\omega_x\big(g(y)\big)$$

if $X_s(f, g) \neq \emptyset$ and $(fg)(s) = 0$ otherwise. Then $R[[S, \omega, \leq]]$ together with pointwise addition and above multiplication becomes a ring called the *ring of skew generalized power series* with coefficients in R and exponents in S.

Conditions ACCPL and ACCPR can be defined for rings analogously to those for semigroups. In fact a ring $(R, +, \cdot)$ satisfies ACCPL (ACCPR) if and only if the multiplicative monoid $(R, \cdot)$ does. A domain R satisfies ACCPL (ACCPR) if and only if the cancellative monoid $R^* = (R\backslash\{0\}, \cdot)$ does. In [5, Theorem 3.3] the authors have characterized skew generalized power series rings with exponents in a strictly totally ordered monoid that are domains satisfying ACCPL (resp. ACCPR). Now we show how part of this result can be derived from the above. In fact this was the motivation for our considerations (compare Corollary 3.12 with [5, Theorem 3.3]).

Let R be a domain with identity, S a strictly totally ordered monoid and $\omega : S \to$ End(R) a monoid homomorphism such that ω_s is injective for all $s \in S$. Then by [5, Proposition 3.1(ii)] $R[[S,\omega,\leq]]$ is a domain, hence $R[[S,\omega,\leq]]^*$ is a cancellative monoid. For any $f \in R[[S,\omega,\leq]]^*$ the set $\mathrm{supp}(f) \subseteq S$ is well ordered and so it has a smallest element that we denote by $\pi(f)$. By [5, Proposition 3.1(i)] the map

$$\psi : R\big[[S,\omega,\leq]\big]^* \to R^* \rtimes_\omega S$$
$$f \mapsto \big(f\big(\pi(f)\big), \pi(f)\big)$$

is a monoid homomorphism. Since S is strictly totally ordered, it is cancellative. Lemma 3.1 then implies that $R^* \rtimes_\omega S$ is a cancellative monoid. If f is in $\psi^{-1}(\mathrm{Reg}(R^* \rtimes_\omega S))$ then $(f(\pi(f)), \pi(f))$ is invertible in $R^* \rtimes_\omega S$. It is easy to see that this implies that both $\pi(f)$ and $f(\pi(f))$ are invertible, so by [5, Proposition 3.2] f is invertible in $R[[S,\omega,\leq]]^*$. Thus $\psi^{-1}(\mathrm{Reg}(R^* \rtimes_\omega S)) \subseteq \mathrm{Reg}(R[[S,\omega,\leq]]^*)$. If S and R satisfy ACCPL then by Corollary 3.12 so does $R^* \rtimes_\omega S$. Similarly if S and R satisfy ACCPR and ω_s preserves nonunits of R for all $s \in S$ (that is $\mathrm{Im}\,\omega_s$ is closed for complete inverses, since R^* is cancellative monoid) then by Corollary 3.12 $R^* \rtimes_\omega S$ satisfies ACCPR as well. Corollary 2.5 now implies that $R[[S,\omega,\leq]]^*$ satisfies ACCPL, respectively ACCPR, hence so does the ring $R[[S,\omega,\leq]]$.

4 Some proofs

In this section we present the proofs needed for Example 3.10. Assume that all the notations are as in the example. By definition $S_2 = F/\sim$, where F is the free semigroup over the set $\{x, y\}$ and $\sim$ is the least congruence relation on F such that $xyx \sim x$ and $yxy \sim y$. First we need a more explicit description of the relation $\sim$.

A word in F is called *alternating* if it does not contain xx or yy as a subword (u is a *subword* of v if $v \in F^1uF^1$). For an alternating word a define

$$\bar{a} = \begin{cases} x; & \text{if } a \text{ starts and ends with } x,\\ y; & \text{if } a \text{ starts and ends with } y,\\ xy; & \text{if } a \text{ starts with } x \text{ and ends with } y,\\ yx; & \text{if } a \text{ starts with } y \text{ and ends with } x. \end{cases}$$

Note that $\bar{a}$ is again an alternating word that starts and ends with the same letters as a and $\bar{\bar{a}} = \bar{a}$. An arbitrary word in F is a unique product of its maximal alternating subwords. If $u = u_1u_2\ldots u_n$, where u_i are the maximal alternating subwords of u then define

$$\bar{u} = \bar{u}_1\bar{u}_2\ldots\bar{u}_n.$$

Note that $\bar{u}_i$ are exactly the maximal alternating subwords of $\bar{u}$, since $\bar{\ }$ does not change the beginning or the end of the word. So $\bar{\bar{u}} = \bar{u}$.

Step 1 *For $u, v \in F$ we have $u \sim v$ if and only if $\bar{u} = \bar{v}$.*

For an arbitrary word $u \in F$ assume the following notations

b(u) the first letter of u,
e(u) the last letter of u,
B(u) the first maximal alternating subword of u,
E(u) the last maximal alternating subword of u,
$|u|$ the length of u,
$\overleftarrow{u}$ the word $\bar{u}$ without the first letter,
$\overrightarrow{u}$ the word $\bar{u}$ without the last letter,

where we allow $\overleftarrow{u}$ and $\overrightarrow{u}$ to be empty. By considering all possible cases it can be seen that for two alternating words a and c we have $\overline{ac} = \overline{\bar{a}\bar{c}}$. Since we only have four possibilities for $\bar{a}$ and $\bar{c}$ it is easy to check that

$$\overline{\bar{a}\bar{c}} = \begin{cases} \bar{a}\bar{c}; & \text{if } \mathrm{e}(a) = \mathrm{b}(c) \text{ or } |\bar{a}| = |\bar{c}| = 1, \\ \overrightarrow{a}\overleftarrow{c}; & \text{if } \mathrm{e}(a) \neq \mathrm{b}(c) \text{ and } |\bar{a}||\bar{c}| \neq 1. \end{cases}$$

Let $u = u_1 u_2 \dots u_n$ and $v = v_1 v_2 \dots v_m$ be arbitrary words in F, where u_i and v_i are maximal alternating subwords. Then all u_i and v_i except perhaps u_n and v_1 are maximal alternating subwords of uv, thus

$$\overline{uv} = \left. \begin{cases} \bar{u}_1 \dots \bar{u}_n \bar{v}_1 \dots \bar{v}_m; & \text{if } \mathrm{e}(u) = \mathrm{b}(v) \\ \bar{u}_1 \dots \bar{u}_{n-1} \overline{u_n v_1} \bar{v}_2 \dots \bar{v}_m; & \text{if } \mathrm{e}(u) \neq \mathrm{b}(v) \end{cases} \right\} = \bar{u}_1 \dots \bar{u}_{n-1} \overline{\bar{u}_n \bar{v}_1} \bar{v}_2 \dots \bar{v}_m$$

and hence by the above

$$\overline{uv} = \begin{cases} \bar{u}\bar{v}; & \text{if } \mathrm{e}(u) = \mathrm{b}(v) \text{ or } |\mathrm{E}(\bar{u})| = |\mathrm{B}(\bar{v})| = 1, \\ \overrightarrow{u}\overleftarrow{v}; & \text{if } \mathrm{e}(u) \neq \mathrm{b}(v) \text{ and } |\mathrm{E}(\bar{u})||\mathrm{B}(\bar{c})| \neq 1. \end{cases} \tag{3}$$

Now define a relation $\approx$ in F by $u_1 \approx u_2$ if and only if $\bar{u}_1 = \bar{u}_2$. Clearly this is an equivalence relation. If $u_1 \approx u_2$ then $\mathrm{b}(u_1) = \mathrm{b}(u_2)$, $\mathrm{e}(u_1) = \mathrm{e}(u_2)$, $\mathrm{B}(\bar{u}_1) = \mathrm{B}(\bar{u}_2)$ and $\mathrm{E}(\bar{u}_1) = \mathrm{E}(\bar{u}_2)$. So $\overline{u_1 v}$ and $\overline{u_2 v}$ (resp. $\overline{vu_1}$ and $\overline{vu_2}$) will calculate in the same way. Hence $u_1 v \approx u_2 v$ (resp. $vu_1 \approx vu_2$). Thus $\approx$ is in fact a congruence. For arbitrary $u \in F$ we have $\bar{u} \sim u$ (for alternating words this is clear, for arbitrary words this is a consequences of $\sim$ being a congruence), so the relation $\approx$ is contained in the relation $\sim$. Since $xyx \approx x$ and $yxy \approx y$, the minimality of $\sim$ implies that relations $\sim$ and $\approx$ are the same.

Step 2 *If $srp \sim s$ in F then one of the following holds*: ($r \sim x$ *and* $p \sim y$) *or* ($r \sim y$ *and* $p \sim x$) *or* ($r \sim xy \sim p$) *or* ($r \sim yx \sim p$).

If we denote $e = rp$ then $\overline{se} = \bar{s}$. Since $\overline{se} = \bar{s}\bar{e}$ would lead to a contradiction $e = \emptyset$ we must have $\overline{se} = \overrightarrow{s}\overleftarrow{e} = \bar{s}$. This means that $\overleftarrow{e} = \mathrm{e}(s)$, hence $\bar{e} = \mathrm{b}(e)\,\mathrm{e}(s)$, where $\mathrm{b}(e) \neq \mathrm{e}(s)$ by (3). This implies $e \sim xy$ or $e \sim yx$. By symmetry we may assume $e \sim xy$, so $\overline{rp} = xy$. If $\overline{rp} = \bar{r}\bar{p}$ then $\bar{r} = x$ and $\bar{p} = y$. Now let $\overline{rp} = \overrightarrow{r}\overleftarrow{p}$, hence by (3) $\mathrm{e}(r) \neq \mathrm{b}(p)$ and $|\mathrm{E}(\bar{r})||\mathrm{B}(\bar{p})| \neq 1$. If $\overrightarrow{r} = \emptyset$ and $\overleftarrow{p} = xy$ then $|\mathrm{E}(\bar{r})| = 1$, which implies $|\mathrm{B}(\bar{p})| > 1$ and so $\bar{p} = yxy$. This leads to a contradiction $\bar{p} = \bar{\bar{p}} = y$. Similarly $\overrightarrow{r} = xy$ and $\overleftarrow{p} = \emptyset$ would lead to a contradiction. Thus $\overrightarrow{r} = x$ and $\overleftarrow{p} = y$. Now $\bar{r} = xx$ and $\bar{p} = yy$ would imply $|\mathrm{E}(\bar{r})||\mathrm{B}(\bar{p})| = 1$, which is not true. Therefore $\bar{r} = xy$ and $\bar{p} = xy$, since $\mathrm{e}(r) \neq \mathrm{b}(p)$.

Step 3 *If $srp = s$ in S then r is a unique inverse of p.*

Suppose $(s_1, s_2)(r_1, r_2)(p_1, p_2) = (s_1, s_2)$ is S. Then $s_1 r_1 p_1 = s_1$ in S_1 and $s_2 r_2 p_2 = s_2$ in S_2. By *Step* 2 clearly r_2 is an inverse of p_2 in S_2. *Step* 2 also shows that inverses in S_2 are unique, so r_2 is a unique inverse of p_2. Since in S_1 length in z (the number of letters z occurring in the element) is clearly well defined, r_1 and p_1 can not contain any z, so $r_1 = p_1 = w$. Since w is a unique inverse of w in S_1, (r_1, r_2) is a unique inverse of (p_1, p_2) in S.

Step 4 *Semigroup $S \rtimes_\omega T$ satisfies ACCPR.*

Suppose $s_n = s_{n+1} q_n$ in S_2, that is $\bar{s}_n = \overline{s_{n+1} q_n}$ in F. Closer examination of (3) shows that we have the following possible cases:

(1) $\bar{s}_n = \bar{s}_{n+1} \bar{q}_n$, hence $|\bar{s}_{n+1}| < |\bar{s}_n|$,
(2) $\bar{s}_n = \overrightarrow{s}_{n+1} \overleftarrow{q}_n$ and $|\bar{q}_n| \geq 3$, hence $|\bar{s}_{n+1}| < |\bar{s}_n|$,
(3) $\bar{s}_n = \overrightarrow{s}_{n+1} \overleftarrow{q}_n$ and $|\bar{q}_n| = 2$, hence $|\bar{s}_{n+1}| = |\bar{s}_n|$ and one of the following holds:
    (3.1) $\bar{q}_n = xx,\ \bar{s}_{n+1} = axy,\ \bar{s}_n = axx$ for some $a \in F^1$,
    (3.2) $\bar{q}_n = yy,\ \bar{s}_{n+1} = ayx,\ \bar{s}_n = ayy$ for some $a \in F^1$,
    (3.3) $\bar{q}_n = xy,\ \bar{s}_{n+1} = ay,\ \bar{s}_n = ay$ for some $a \in F^1$,
    (3.4) $\bar{q}_n = yx,\ \bar{s}_{n+1} = ax,\ \bar{s}_n = ax$ for some $a \in F^1$,
(4) $\bar{s}_n = \overrightarrow{s}_{n+1} \overleftarrow{q}_n$ and $|\bar{q}_n| = 1$ and one of the following holds:
    (4.1) $\bar{q}_n = x,\ \bar{s}_{n+1} = axy,\ \bar{s}_n = ax$ for some $a \in F^1$,
    (4.2) $\bar{q}_n = y,\ \bar{s}_{n+1} = ayx,\ \bar{s}_n = ay$ for some $a \in F^1$.

To prove that $S \rtimes_\omega T$ satisfies ACCPR let $(r_n)_n, (p_n)_n \subseteq S_1$, $(s_n)_n, (q_n)_n \subseteq S_2$ and $(t_n)_n, (u_n)_n \subseteq T$ be sequences such that

$$\big((r_n, s_n), t_n\big) = \big((r_{n+1}, s_{n+1}), t_{n+1}\big)\big((p_n, q_n), u_n\big)$$

for all $n \in \mathbb{N}$. Then $(r_n, s_n) = (r_{n+1}, s_{n+1})\omega_{t_{n+1}}(p_n, q_n)$ and $t_n = t_{n+1} u_n$. If 1 appears in $(t_n)_n$ then all consequent terms must be 1. If 1 does not appear in $(t_n)_n$ but 2 does then all consequent terms must be 2. If 1 and 2 do not appear in $(t_n)_n$ then all terms are 0. So t_n must be constant from some term on.

Let first $t_n = 0$ for all $n \in \mathbb{N}$ big enough. Then for these n we get $(r_n, s_n) = (r_{n+1}, s_{n+1})(w, xy)$. But (w, xy) is an idempotent in S, so $(r_{n+1}, s_{n+1}) = (r_{n+2}, s_{n+2})(w, xy) = (r_{n+2}, s_{n+2})(w, xy)^3 = (r_n, s_n)(w, xy)$. This implies

$$\big((r_{n+1}, s_{n+1}), t_{n+1}\big) = \big((r_n, s_n), t_n\big)\big((w, x), 1\big)$$

for all n big enough.

Now let $t_n = 1$ for all $n \in \mathbb{N}$ big enough. Then for these n we have $(r_n, s_n) = (r_{n+1}, s_{n+1})(p_n, q_n) = (r_{n+1} p_n, s_{n+1} q_n)$. For $a \in S_1$ let $|a|_z$ denote the number of letters z in a (this is well defined). Then for n big enough $|r_n|_z = |r_{n+1} p_n|_z = |r_{n+1}|_z + |p_n|_z \geq |r_{n+1}|_z$. Thus the sequence $|r_n|_z$ is nonincreasing, hence it must be constant from some n on. Therefore $|p_n|_z = 0$ and so $p_n = w$ for n big enough. Then $r_n = r_{n+1} w$ and as above (w is an idempotent in S_1) this implies $r_{n+1} = r_n w$ for n big enough. Now for these n consider what can happen with $s_n = s_{n+1} q_n$ (see the cases above). If (3.3), (3.4), (4.1) and (4.2) never happen then the sequence $|\bar{s}_n|$ is nonincreasing and hence constant from some n on. So for n big enough only (3.1) and (3.2) can happen, but this is impossible since on one hand all s_n should end with two equal letters on the other hand all s_n should end with two different letters. So for some n one of the (3.3), (3.4), (4.1) and (4.2) happens. In any case there exists $s \in S_2$ such that $s_{n+1} = s_n s$ ($s = xy$ in case (3.3), $s = yx$ in case (3.4), $s = y$ in case (4.1) and $s = x$ in case (4.2)). For this n we have $(r_{n+1}, s_{n+1}) = (r_n, s_n)(w, s)$, thus

$$\big((r_{n+1}, s_{n+1}), t_{n+1}\big) = \big((r_n, s_n), t_n\big)\big((w, s), 1\big).$$

Now let $t_n = 2$ for all $n \in \mathbb{N}$ big enough. Then for these n we have $(r_n, s_n) = (r_{n+1}, s_{n+1})(w, a_n)$, where $\bar{a}_n$ is of the form x^m or $x^m y$ for some $m \in \mathbb{N}$ (see the definition of ω_2). As above $r_n = r_{n+1}w$ implies $r_{n+1} = r_n w$ for n big enough. Now as above consider what can happen with $s_n = s_{n+1}a_n$ for these n (a_n takes the place of q_n). Since a_n starts with x, cases (3.2), (3.4) and (4.2) can never happen. Suppose that (3.3) never happens. For $u \in F$ let $|u|_x$ denote the number of letters x in u. In case (1) we have $|\bar{s}_{n+1}|_x < |\bar{s}_n|_x$, since $\bar{a}_n$ contains at least one x. In case (2) element $\overleftarrow{a}_n$ contains at least one x and since $\bar{a}$ starts with x, $\bar{s}_{n+1}$ has to end with y. Hence in this case we also have $|\bar{s}_{n+1}|_x < |\bar{s}_n|_x$. In case (3.1) we also have $|\bar{s}_{n+1}|_x < |\bar{s}_n|_x$ and in case (4.1) we have $|\bar{s}_{n+1}|_x = |\bar{s}_n|_x$. Thus the sequence $|\bar{s}_n|_x$ is nonincreasing, so it must be constant from some term on. So for n big enough only (4.1) can happen, but this is impossible since on one hand all s_n should end with x on the other hand all s_n should end with y. So for some n big enough (3.3) must happen. For this n we have $s_{n+1} = s_n xy$ and $(r_{n+1}, s_{n+1}) = (r_n, s_n)(w, xy)$, thus

$$\big((r_{n+1}, s_{n+1}), t_{n+1}\big) = \big((r_n, s_n), t_n\big)\big((w, x), 1\big).$$

We have shown that in any case there exists $n \in \mathbb{N}$ and $((r, s), t) \in S \rtimes_\omega T$ such that $((r_{n+1}, s_{n+1}), t_{n+1}) = ((r_n, s_n), t_n)((r, s), t)$. So $S \rtimes_\omega T$ satisfies ACCPR.

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