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SHORT NOTE

A problem on generalized Cayley graphs of semigroups

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Abstract Zhu (Semigroup Forum 84(3), 144–156, 2012) investigated some combinatorial properties of generalized Cayley graphs of semigroups. In Remark 3.8 of (Zhu, Semigroup Forum 84(3), 144–156, 2012), Zhu proposed the following question: It may be interesting to characterize semigroups S such that $Cay(S,\omega_l) = Cay(S,\omega_r)$. In this short note, we prove that for any regular semigroup S, $Cay(S,\omega_l) = Cay(S,\omega_r)$ if and only if S is a Clifford semigroup.

Keywords Generalized Cayley graphs of semigroups · Clifford semigroups

1 Introduction and main theorem

Cayley graphs of semigroups have been studied by many authors and some important results have been obtained, Kelarev-Ryan-Yearwood [2] is a good survey in this aspect. Most recently, Zhu [3] generalized the usual *Cayley graphs* of semigroups to *generalized Cayley graphs* of them and in texts [3] and [4], Zhu investigated some algebraic and combinatorial properties for such graphs. In particular some results of the usual Cayley graphs of semigroups are generalized to generalized Cayley graphs of semigroups.

Let S be an ideal of a semigroup T and $\rho \subseteq T^1 \times T^1$. Following Zhu [3], the *generalized Cayley graph* $Cay(S,\rho)$ of S relative to ρ is defined as the graph with vertex set S and edge set

$$E\big(Cay(S,\rho)\big) = \big\{(a,b) \in S \times S \mid xay = b \text{ for some } (x,y) \in \rho\big\}.$$

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In particular, generalized Cayley graphs $Cay(S,\omega_l)$, $Cay(S,\omega_r)$ and $Cay(S,\omega)$ are called *left universal, right universal* and *universal Cayley graphs of S*, respectively, where $\omega_l = S^1 \times \{1\}$, $\omega_r = \{1\} \times S^1$ and $\omega = S^1 \times S^1$.

Zhu [3, 4] mainly investigated universal Cayley graphs of a semigroup S and obtained some useful results. On the other hand, Remark 3.8 in Zhu [4] proposes the following.

Problem It may be interesting to characterize semigroups S such that $Cay(S,\omega_l) = Cay(S,\omega_r)$.

Obviously, the above problem is trivial for commutative semigroups. As we have known, regular semigroups play a major role in the algebraic theory of semigroups. In this short note, we give an answer to this problem for regular semigroups. Recall that a semigroup is *regular* if there exists $x \in S$ such that $axa = a$ and $xax = x$ for any $a \in S$. A *Clifford semigroup* is a regular semigroup S in which $ae = ea$ for every idempotent e and every a in S. Here is our result.

Theorem *For any regular semigroup S, $Cay(S,\omega_l) = Cay(S,\omega_r)$ if and only if S is a Clifford semigroup.*

2 A proof

To give a proof of the theorem, we need to recall the following two well-known results. On one hand, from Chap. IV, Exercise 2 in Howie [1], we can obtain the following lemma.

Lemma 1 (See Howie [1, p. 125]) *Let S be a regular semigroup. Then S is a Clifford semigroup if and only if $\mathcal{L} = \mathcal{R}$.*

On the other hand, from Chap. IV, Theorem 2.1 in Howie [1], we have another characterization of Clifford semigroups as follows. On the notion of *strong semilattice of semigroups*, the reader is referred to Chap. IV in Howie [1].

Lemma 2 (See Howie [1, p. 94]) *A semigroup S is a Clifford semigroup if and only if $S = (G_\alpha, Y, \phi_{\alpha,\beta})$ is a strong semilattice of groups.*

Now we can give a proof of the Theorem.

Necessity. Assume that S is a regular semigroup and $Cay(S,\omega_l) = Cay(S,\omega_r)$. If $a, b \in S$ and $a\mathcal{L}b$, then $a = xb$ and $b = ya$ for some $x, y \in S^1$. This implies that $(a,b), (b,a) \in E(Cay(S,\omega_l))$. By hypothesis, $(a,b), (b,a) \in E(Cay(S,\omega_r))$. Therefore, there exist $x', y' \in S^1$ such that $b = ax'$ and $a = by'$. This yields that $a\mathcal{R}b$. We have shown that $\mathcal{L} \subseteq \mathcal{R}$. By a dual argument, we can obtain $\mathcal{R} \subseteq \mathcal{L}$. Thus $\mathcal{L} = \mathcal{R}$. Since S is regular, it follows that S is a Clifford semigroup from Lemma 1.

Sufficiency. Assume that $S = (G_\alpha, Y, \phi_{\alpha,\beta})$ is a Clifford semigroup by Lemma 2 and let $a, b \in S$. Suppose that $(a,b) \in E(Cay(S,\omega_l))$. Then $xa = b$ for some

$x \in S^1$. If $x = 1$, then $ax = b$ and so $(a, b) \in E(Cay(S, \omega_r))$. Now, let $x \in G_\alpha$ and $a \in G_\beta$. Then $b \in G_{\alpha\beta}$ and $b = xa = (x\phi_{\alpha,\alpha\beta})(a\phi_{\beta,\alpha\beta})$. Denote $y = (a\phi_{\beta,\alpha\beta})^{-1}(x\phi_{\alpha,\alpha\beta})(a\phi_{\beta,\alpha\beta})$, where $(a\phi_{\beta,\alpha\beta})^{-1}$ is the inverse of $a\phi_{\beta,\alpha\beta}$ in the group $G_{\alpha\beta}$. Then $y \in G_{\alpha\beta}$, and

$$\begin{aligned} ay &= (a\phi_{\beta,\beta(\alpha\beta)})(y\phi_{\alpha\beta,\beta(\alpha\beta)}) = (a\phi_{\beta,\alpha\beta})(y\phi_{\alpha\beta,\alpha\beta}) \\ &= (a\phi_{\beta,\alpha\beta})y = (a\phi_{\beta,\alpha\beta})(a\phi_{\beta,\alpha\beta})^{-1}(x\phi_{\alpha,\alpha\beta})(a\phi_{\beta,\alpha\beta}) \\ &= (x\phi_{\alpha,\alpha\beta})(a\phi_{\beta,\alpha\beta}) = xa = b. \end{aligned}$$

This implies that $(a, b) \in E(Cay(S, \omega_r))$. Therefore $E(Cay(S, \omega_l)) \subseteq E(Cay(S, \omega_r))$. By a dual argument, we can obtain $E(Cay(S, \omega_r)) \subseteq E(Cay(S, \omega_l))$. This completes our proof.

Remark 1 From the proof of "necessity" part above, we can see that $\mathcal{L} = \mathcal{R}$ for any semigroup S with $Cay(S, \omega_l) = Cay(S, \omega_r)$. The following example illustrates that there exists a semigroup S with $\mathcal{L} = \mathcal{R}$ which does not satisfy $Cay(S, \omega_l) = Cay(S, \omega_r)$. In fact, let S be the free monoid generated by the two symbols 0 and 1. Then Green's relations $\mathcal{L}$ and $\mathcal{R}$ are equal on S (both of them are identity relation on S). Obviously, $(0, 10) \in E(Cay(S, \omega_l))$. However, $(0, 10) \notin E(Cay(S, \omega_r))$.

Remark 2 Necessary and sufficient conditions for $Cay(S, \omega_l)$ and $Cay(S, \omega_r)$ to be isomorphic are not known. The following example shows that this graph isomorphism may exist for a regular semigroup which is not a Clifford semigroup.

Example Consider the 4-element rectangular band $\{e, f, g, h\}$ with $e\mathcal{R}f$, $e\mathcal{L}g$, $g\mathcal{R}h$ and $f\mathcal{L}h$. For this semigroup, $Cay(S, \omega_l)$ is the disjoint union of the complete directed graphs with vertex sets $\{e, g\}$ and $\{f, h\}$, with a loop at each vertex, while $Cay(S, \omega_r)$ is the disjoint union of the complete directed graphs with vertex sets $\{e, f\}$ and $\{g, h\}$ and with a loop at each vertex. So the two graphs are isomorphic.

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