

Generalized Cayley graphs of semigroups II

Yongwen Zhu

Received: 2 October 2010 / Accepted: 30 November 2011 / Published online: 29 December 2011
© Springer Science+Business Media, LLC 2011

Abstract Following Zhu (Semigroup Forum, 2011, doi:10.1007/s00233-011-9368-9), we study generalized Cayley graphs of semigroups. The Cayley D -saturated property, a particular combinatorial property, of generalized Cayley graphs of semigroups is considered and most of the results in Kelarev and Quinn (Semigroup Forum 66:89–96, 2003), Yang and Gao (Semigroup Forum 80:174–180, 2010) are extended. In addition, for some basic graphs and their complete fission graphs, we describe all semigroups whose universal Cayley graphs are isomorphic to these graphs.

Keywords Cayley graph · Complete fission graph · D -saturated property · \mathcal{J} -partial order of complete graphs with loops · Semigroup

1 Introduction

The Cayley graphs of semigroups have been investigated by many authors, see, for example, [1, 4, 6, 7, 9, 12–14, 17]. Some of the earliest references on this subject are [2] and [18]. The Cayley graphs of semigroups are closely related to finite state automata and have many valuable applications (see [5, 11]). Combinatorial properties of semigroups defined in terms of their Cayley graphs were considered in [8, 10].

The concept of generalized Cayley graphs of semigroups was first introduced by the author in [19], where some fundamental properties of generalized Cayley graphs of semigroups were studied.

Following [19], in the present paper, we continue discussion of generalized Cayley graphs of semigroups. In Sect. 3, we describe the semigroups whose universal Cayley graphs are some basic graphs or their complete fission graphs, and in Sect. 4, we study

Communicated by Tom Hall.

Y. Zhu (✉)

School of Mathematics and Information Science, Yantai University, Yantai City 264005, P.R. China
e-mail: zyw@ytu.edu.cn

the D -saturated property, a particular combinatorial property, of generalized Cayley graphs of semigroups. Some results of [19] are used in this paper.

2 Preliminaries

Recall that if S is an ideal of a semigroup T , then we call T an *ideal extension* of S . Let T^1 be the semigroup T with an identity adjoined if necessary.

Definition 2.1 [19] Let T be an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$. The *Cayley graph* $\text{Cay}(S, \rho)$ of S relative to ρ is defined as the graph with vertex set S and edge set $E(\text{Cay}(S, \rho))$ consisting of those ordered pairs (a, b) , where $xay = b$ for some $(x, y) \in \rho$. We also call the Cayley graphs defined in this way the *generalized Cayley graphs*, in order to distinguish them from the usual ones.

Assume that S is a semigroup and $a \in S$. Then $J(a) = S^1aS^1$ ($L(a) = S^1a$, $R(a) = aS^1$) is the principal (left, right) ideal generated by a . For other notations and notions of semigroup theory, we refer the reader to [3, 15, 19, 20]. Let $\omega_l = S^1 \times \{1\}$ (the left universal relation on S^1), $\omega_r = \{1\} \times S^1$ (the right universal relation on S^1) and $\omega = S^1 \times S^1$ (the universal relation on S^1). Then the generalized Cayley graphs $\text{Cay}(S, \omega_l)$, $\text{Cay}(S, \omega_r)$ and $\text{Cay}(S, \omega)$ are called *left universal*, *right universal* and *universal Cayley graphs* of S , respectively.

We use standard terminology of graph theory following, e.g., [16]. Throughout the paper, graphs are directed graphs without multiple edges, but possibly with loops, or digraphs in terms of [1, 13]. A directed graph without loops and multiple edges is called *simple*. We may equate two graphs each of which is isomorphic to the other one. For a graph Γ , denote by $V(\Gamma)$ and $E(\Gamma)$ its vertex set and edge set, respectively. For any $a \in V(\Gamma)$, Let

$$\vec{a} = \{b \in V(\Gamma) | (a, b) \in E(\Gamma)\}, \quad \overleftarrow{a} = \{b \in V(\Gamma) | (b, a) \in E(\Gamma)\}. \tag{2.1}$$

Assume that T is an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$. Let

$$\rho(a) = \{xay | (x, y) \in \rho\}, \tag{2.2}$$

$$\rho(a)^1 = \rho(a) \cup \{a\}. \tag{2.3}$$

We call $\rho(a)^1$ the ρ -class of a . It is evident that $(a, b) \in E(\text{Cay}(S, \rho))$ if and only if $b \in \rho(a)$. So by (2.1), we obtain

$$\rho(a) = \vec{a}. \tag{2.4}$$

Lemma 2.2 [19] Let T be an ideal extension of a semigroup S , $\rho \subseteq T^1 \times T^1$ and $a, b \in S$. If $\rho^1(a) \supseteq \rho^1(b)$ and $a \neq b$, then $(a, b) \in E(\text{Cay}(S, \rho))$.

Definition 2.3 [19] Let T be a semigroup and $\rho \subseteq T \times T$. If $(a, b), (c, d) \in \rho$ for any $a, b, c, d \in T$ always implies $(ca, bd) \in \rho$, then we call ρ *inversely compatible*, or briefly, *I-compatible*.

Definition 2.4 [19] Let Γ be a graph. If $(a, b), (b, c) \in E(\Gamma)$ for $a, b, c \in V(\Gamma)$ always implies that $(a, c) \in E(\Gamma)$, then Γ is said to be *edge-transitive*.

Proposition 2.5 [19] Assume that T is an ideal extension of a semigroup S . If $\rho \subseteq T^1 \times T^1$ is I -compatible, then $\text{Cay}(S, \rho)$ is *edge-transitive*.

Corollary 2.6 [19] Assume that T is an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$ is I -compatible. If $(a, b) \in E(\text{Cay}(S, \rho))$, then $\rho(a) \supseteq \rho(b)$ and $\rho^1(a) \supseteq \rho^1(b)$.

Definition 2.7 [19] Assume that T is an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$. An element a of S is called *stable* under ρ if $a \in \rho(a)$. If all elements of S are stable under ρ , then S is called *stable* under ρ .

Let (Y, \geq) be a partially ordered set. For any $a \in Y$, set

$$a \downarrow = \{b \in Y \mid a \geq b\}, \quad a \uparrow = \{b \in Y \mid b \geq a\}. \tag{2.5}$$

The relation graph of the inverse relation \geq of the natural order \leq on a semilattice is called a *semilattice graph*. A *complete graph with loops* is defined as a graph Γ with the edge set $E(\Gamma) = \{(x, y) \mid x, y \in V(\Gamma)\}$. Assume that T is an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$. If $\rho^1(a) = \rho^1(b)$ for all $a, b \in S$, then S is called ρ -*simple*.

Recall that a totally ordered set is also called a *chain*. The relation graph of a chain is called a *linear graph* or *chain graph*. A finite chain graph of length n could be indicated by the set $L_n = \{0, 1, \dots, n\}$ with edge set $\{(i, j) \mid i \geq j\}$, where \geq is the usual order on natural numbers.

Theorem 2.8 [19] Assume that T is an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$ is an I -compatible relation such that S is stable under ρ . Then the following conditions are equivalent:

- (1) $\text{Cay}(S, \rho)$ is a chain of complete graphs with loops;
- (2) S is a chain of ρ -simple semigroups.

Lemma 2.9 [19] For any semigroup S , the following conditions are equivalent:

- (1) $(a, b) \in E(\text{Cay}(S, \omega))$;
- (2) $J(a) \supseteq J(b)$.

A partially ordered set (Y, \geq) is called a *strong partially ordered set* if for any $\alpha, \beta \in X$, there is $\gamma \in Y$ such that $\alpha \geq \gamma$ and $\beta \geq \gamma$. If (Y, \geq) is a partially ordered set and Y is a semigroup such that for any $\alpha, \beta \in Y$, $\alpha, \beta \geq \alpha\beta$, then we call the semigroup Y a *partially ordered semigroup*, and we call the partially ordered set Y a *semigroup-partially-ordered set*. If B is a partially ordered semigroup that is also a band (i.e., an idempotent semigroup), then we call B a *band-partially-ordered set* or a *partially ordered band*. If (Y, \geq) is a partially ordered set, Y is a semigroup, T is an ideal extension of Y and $\rho \subseteq T^1 \times T^1$ such that for any $\alpha \in Y$, $\rho^1(\alpha) = \alpha \downarrow$ where

by $\alpha \downarrow$ we denote the set $\{\beta | \alpha \geq \beta\}$, then we call the semigroup Y a ρ -semigroup, and we call the partially ordered set Y a ρ -partially-ordered set. For example, if $\rho = S^1 \times S^1$, a ρ -partially-ordered set or ρ -semigroup is called a \mathcal{J} -partially-ordered set or \mathcal{J} -semigroup. The relation graph of a strong partially ordered set (resp. semigroup-partially-ordered set, \mathcal{J} -partially-ordered set, etc.) is called a strong partially ordered graph (resp. semigroup-partially-ordered graph, \mathcal{J} -partially-ordered graph, etc.)

For a graph Γ define a relation \sim on $V(\Gamma)$ by setting $a \sim b$ if and only if $(a, b) \in V(\Gamma)$ and $(b, a) \in V(\Gamma)$. If \sim is an equivalence such that the quotient graph Γ / \sim is a strong partially ordered graph (resp. semigroup-partially-ordered graph, \mathcal{J} -partially-ordered graph, semilattice graph, etc.), then we call Γ a strong partial order (resp. semigroup-partial order, \mathcal{J} -partial-order, semilattice, etc.) of complete graphs with loops. A strong partial order Γ of complete graphs Γ_α ($\alpha \in Y$) with loops is called nonidempotent-trivial if Γ_α is trivial (i.e., $|V(\Gamma_\alpha)| = 1$) for every nonidempotent α of Y . It is easy to show that $\text{Cay}(S, \omega) / \sim$ is a strong partially ordered graph.

Lemma 2.10 [19] *For any semigroup S , $\text{Cay}(S, \omega)$ is a strong partial order of complete graphs with loops.*

Theorem 2.11 [19] *The following statements are valid:*

- (1) *if S is a completely regular semigroup, then $\text{Cay}(S, \omega)$ is a semilattice of complete graphs with loops;*
- (2) *if Γ is a semilattice of complete graphs with loops, then there is a (commutative) completely regular semigroup such that $\text{Cay}(S, \omega) = \Gamma$.*

Theorem 2.12 [19] *The following statements hold:*

- (1) *For any semigroup S , $\text{Cay}(S, \omega)$ is a strong partial order of complete graphs with loops, and for each $a \in S$ and all $b \in J(a)$, $(a, b) \in E(\text{Cay}(S, \omega))$;*
- (2) *Let Γ be a nonidempotent-trivial \mathcal{J} -partial order Y of complete graphs with loops. Then there is a semigroup S such that $\text{Cay}(S, \omega) = \Gamma$. Moreover, if Y is a commutative semigroup, then S can be chosen to be commutative.*

3 Some basic graphs as universal Cayley graphs of semigroups

In this section, for some simple and important special kinds of graphs such as k -partite complete chain graph with loops and their complete fission graphs, we construct the corresponding semigroups whose generalized Cayley graphs are the given ones. At last, we give a necessary and sufficient condition for the universal Cayley graph of a semigroup to be a finite linear graph.

First, we introduce two new notions.

Definition 3.1 Assume that A_1, A_2, \dots, A_k are some pairwise disjoint nonempty sets. Let Γ be a graph with vertex set $V(\Gamma) = \bigcup_{i=1}^k A_i$ and edge set

$$E(\Gamma) = \{(a, b) | a \in A_i, b \in A_j, \text{ and } i < j\} \cup \{(a, a) | a \in V(\Gamma)\}.$$

Table 1 Cayley table used to prove Theorem 3.3

	0	1	2	...	n	a_1	...	a_m
0	0	0	0	...	0	0	...	0
1	0	1	0	...	0	1	...	1
2	0	0	2	...	0	2	...	2
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\ddots	\vdots
n	0	0	0	...	n	n	...	n
a_1	0	1	2	...	n	0	...	0
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\ddots	\vdots
a_m	0	1	2	...	n	0	...	0

Then we call Γ a *k-partite complete chain graph with loops*, denoted by $K(A_1, \dots, A_k)$. If $|A_i| = \alpha_i$ for $i = 1, \dots, k$, then $K(A_1, A_2, \dots, A_k)$ is also denoted by $K(\alpha_1, \dots, \alpha_k)$. For an arbitrary cardinality α , $K(1, \alpha)$ is also called a *down-claw with α toes*, and $K(\alpha, 1)$ is called an *up-claw with α toes*. For an infinite cardinality α , we can define similarly α -partite complete chain graphs with loops.

Definition 3.2 Let Γ be a graph and $A = \{a_i | i \in I\} \subseteq V(\Gamma)$, and $\{A_i\}_{i \in I}$ a class of pairwise disjoint sets, each of which is disjoint with $V(\Gamma)$. A *complete fission graph* by splitting Γ at points of A is defined as a graph Γ^* with the vertex set $V(\Gamma^*) = (V(\Gamma) \setminus A) \cup (\bigcup_{i \in I} A_i)$, and edge set

$$\begin{aligned}
 E(\Gamma^*) = & \{(x, y) | x, y \notin A, (x, y) \in E(\Gamma)\} \\
 & \cup \{(x, y) | x, y \in A_i \text{ for some } i \in I\} \\
 & \cup \{(x, y) | x \in A_i, y \in A_j, (a_i, a_j) \in E(\Gamma) \text{ for some } i, j \in I\} \\
 & \cup \{(x, y) | y \in A_j, (x, a_j) \in E(\Gamma) \text{ for some } j \in I\} \\
 & \cup \{(x, y) | x \in A_i, (a_i, y) \in E(\Gamma) \text{ for some } i \in I\}.
 \end{aligned}$$

Note that if $|A|, |B| > 1$, then $K(A, B, C)$ is a partially ordered graph but not a semilattice graph, so it is worth being considered. If $K(A, B, C)$ is a generalized Cayley graph of a semigroup, then by Lemma 2.10, it is a strong partially ordered graph which implies that $|C| = 1$. Thus we may suppose that $|C| = 1$ when studying $K(A, B, C)$, as we do in the sequel.

Theorem 3.3 *For any positive integers m and n , there is a commutative semigroup S such that $\text{Cay}(S, \omega) = K(m, n, 1)$. Moreover, if Γ is a fission graph of $K(A, B, C)$ with sets A, B, C and that $|C| = 1$ by splitting up at any points of $B \cup C$, then there exists a commutative semigroup T such that $\text{Cay}(T, \omega) \simeq \Gamma$.*

Proof Let $S = \{0, 1, 2, \dots, n, a_1, a_2, \dots, a_m\}$. Define the multiplication on S by the Cayley table, Table 1.

Table 2 Cayley table used to prove Theorem 3.4

	0	1	...	n	a_1	a_2	...	a_n	...	a_m
0	0	0	...	0	0	0	...	0	...	0
1	0	0	...	0	0	0	...	0	...	0
2	0	0	...	0	0	0	...	0	...	0
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots
n	0	0	...	0	0	0	...	0	...	0
a_1	0	0	...	0	1	2	...	n	...	n
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots
a_m	0	0	...	0	1	2	...	n	...	n

It is easy to verify that S is a semigroup. Since Table 1 is symmetric, S is commutative. Simple computation shows that $J(0) = \{0\}$, $J(k) = \{0, k\}$ for $k = 1, 2, \dots, n$, and $J(a_k) = \{0, 1, \dots, n, a_k\}$ for $k = 1, 2, \dots, m$. Thus by Lemma 2.9, $\text{Cay}(S, \omega) = K(m, n, 1)$.

Suppose without loss of generality that $A = \{a_1, a_2, \dots, a_m\}$, $B = \{1, 2, \dots, n\}$ and $C = \{0\}$. Then $K(A, B, C) = K(m, n, 1) = S$ is a semigroup in view of the above. According to Lemma 2.10, $K(A, B, C)$ is a strong-partially-ordered graph. Furthermore, a direct verification shows that $K(A, B, C)$ is a \mathcal{J} -partially-ordered graph. Since Γ is a complete fission graph of $K(A, B, C)$ by splitting up at the idempotents $\{0, 1, 2, \dots, n\} = B \cup C$, Γ is a nonidempotent-trivial \mathcal{J} -partial order of complete graphs with loops by Definition 3.2. In light of Theorem 2.12, there exists a commutative semigroup T such that $\text{Cay}(T, \omega) \simeq \Gamma$, which completes the proof. □

Theorem 3.4 *If m and n are positive integers such that $m \geq n \geq 2$, then there exists a noncommutative semigroup S with a zero 0 such that $\text{Cay}(S, \omega) = K(m, n, 1)$. Moreover, if Γ is a fission graph of $K(A, B, C)$ with sets A, B, C and that $C = \{0\}$ by splitting up at 0 , then there exists a noncommutative semigroup T such that $\text{Cay}(T, \omega) \simeq \Gamma$.*

Proof The arguments are similar to those in the proof of Theorem 3.3, so we just give the Cayley table, Table 2. □

Similarly, we have

Theorem 3.5 *For any positive integer n , there is a noncommutative semigroup S such that $\text{Cay}(S, \omega) = K(2, n, 1)$. Moreover, if Γ is a fission graph of $K(A, B, C)$ with sets A, B, C and that $|C| = 1$ by splitting up at any points of $A \cup C$, then there exists a noncommutative semigroup T such that $\text{Cay}(T, \omega) \simeq \Gamma$.*

Recall that if $\{S_\alpha\}_{\alpha \in I}$ is a class of 0-subsemigroups of S satisfying that $S = \bigcup_{\alpha \in I} S_\alpha$ and for $\alpha \neq \beta$, $S_\alpha \cap S_\beta = S_\alpha S_\beta = \{0\}$, then we say that S is the 0-direct union of S_α 's. If E is a semilattice, i.e., a commutative idempotent semigroup, then

the natural order on E is defined by setting $e \leq f$ if and only if $ef = fe = e$. A semilattice E with a zero 0 is called *0-primitive*, if every nonzero element of E is 0-primitive, that is, minimal in $E \setminus \{0\}$. Now we can characterize the up-claws $K(\alpha, 1)$ as follows.

Theorem 3.6 *Let S be a semigroup whose cardinality is $\alpha + 1$ with $\alpha > 1$. Then $\text{Cay}(S, \omega) = K(\alpha, 1)$ if and only if S is either a 0-primitive semilattice, or a null semigroup, or a 0-direct union of both of them.*

Proof Sufficiency: Without loss of generality, let S be a 0-direct union of a 0-primitive semilattice and a null semigroup with zero 0 . Then $J(0) = \{0\}$, and $J(a) = \{0, a\}$ for each $a \in S \setminus \{0\}$. Hence, by Lemma 2.9, $\text{Cay}(S, \omega) = K(\alpha, 1)$.

Necessity: Assume that $\text{Cay}(S, \omega) = K(\alpha, 1)$. Then $\text{Cay}(S, \omega)$ has a lower bound which we denote by 0 , i.e., $(a, 0) \in E(\text{Cay}(S, \omega))$ for all $a \in S$. By (2.4), $J(a) = \vec{a}$ for every $a \in S$. It follows that $J(0) = \{0\}$ which means that 0 is a zero of S . Also $J(a) = \{0, a\}$ for all $a \in S \setminus \{0\}$. Hence for all $a \neq b \in S$, $ab \in J(a) \cap J(b) = \{0\}$. For any $a \in S \setminus \{0\}$, $a^2 \in J(a) = \{0, a\}$. Thus $a^2 = 0$ or $a^2 = a$. If the first case occurs for all $a \in S$, then S is null. Otherwise, there exists $a \in S$ such that $a^2 = a \neq 0$. Let E be the set of all idempotents of S and let $F = (S \setminus E) \cup \{0\}$. Then F is a null semigroup. Let $e, f \in E$ such that $e \neq 0$ and $e \neq f$. Then $e^2 = e$ and $ef = 0 = fe$, and so e is a 0-primitive idempotent. Furthermore, E is a 0-primitive semilattice. It is clear that for all $e \in E$ and $n \in F$, $en = ne = 0$. Thus S is a 0-direct union of E and F . In particular, if $F = \{0\}$, then $S = E$ is a 0-primitive semilattice. So the theorem is proved. □

Theorem 3.7 *For a semigroup $S = L_n$, $\text{Cay}(S, \omega) = L_n$ if and only if there exists a total order \geq on S such that the following conditions are satisfied:*

- (1) for any $i, j \in L_n$, $\min(i, j) \geq ij$;
- (2) for all $i \neq j$, $|J(i)| \neq |J(j)|$.

Proof Necessity: Assume that $S = L_n$ is a semigroup such that $\text{Cay}(S, \omega) = L_n$. The natural order \geq for natural numbers may be taken as the total order on S . Then for any $i \in L_n$, $J(i) = \vec{i} = i \downarrow = \{0, 1, \dots, i\}$. Thus that $|J(i)| = |J(j)|$ implies that $i = j$, and (2) follows. For any $i, j \in L_n$, $ij \in J(i) \cap J(j) = J(\min(i, j)) = \min(i, j) \downarrow$, hence $\min(i, j) \geq ij$ and (1) follows.

Sufficiency: Assume $S = \{0 < 1 < 2 < \dots < n\}$ is a total set with the total order \leq such that the two conditions presented in the theorem are satisfied. Then $J(0) = \{0\}$, $J(1) = \{0, 1\}$, $J(2) = \{0, 1, 2\}$, and $J(n) = \{0, 1, 2, \dots, n\}$. Thus by Lemma 2.9, $\text{Cay}(S, \omega) = L_n$ as desired. □

Remark 3.8

- (1) The next question remains open: Is it true that the countably-infinite-partite complete chain graph with loops

$$K(\{a_1, a_2\}, \dots, \{3\}, \{2\}, \{1\}, \{0\}, \{-1\}, \{-2\}, \{-3\}, \dots)$$

is a \mathcal{J} -partially-ordered graph, where $\{a_1, a_2\} \cap \mathbb{Z} = \emptyset$?

- (2) It may be interesting to characterize semigroups S such that $\text{Cay}(S, \omega_l) = \text{Cay}(S, \omega_r)$.

4 D -saturated property of the Cayley graphs of semigroups

The following combinatorial property was introduced in [9] and was investigated further in [17] for the Cayley graphs of semigroups. A semigroup S is said to be *Cayley D -saturated* with respect to a subset T of S if, for all infinite subsets V of S , there exists a subgraph of $\text{Cay}(S, T)$ isomorphic to D with all vertices in V . In this section, we study the Cayley D -saturated property of a generalized Cayley graph and generalize the corresponding results of [9, 17].

Let \mathbb{N} be the set of all natural numbers. An *infinite ascending* (resp. *descending*) *chain* A_∞ (resp. D_∞) is the graph with the set \mathbb{N} as its vertex set and with edges (i, j) such that $i < j$ (resp. $i > j$). An *infinite complete symmetric graph* K_∞ is the graph with the set \mathbb{N} as vertex set and with edges (i, j) for all distinct $i, j \in \mathbb{N}$. The *null graph* is a graph with no edges. Recall that a finite graph D has no cycles if and only if D can be embedded in A_∞ or D_∞ , and that any finite graph can be embedded in K_∞ .

Lemma 4.1 [9] *Every infinite graph contains an infinite set of vertices which induces a null subgraph, A_∞ , D_∞ or K_∞ .*

If S is a finite semigroup, then it is vacuously true that S is Cayley D -saturated with respect to all relations ρ on T^1 , where T is an ideal extension of S . If D is null, then all semigroups S are Cayley D -saturated with respect to all relations ρ on T^1 . If S is an infinite semigroup, ρ is the empty relation but D is not null, then it is obvious that S is not Cayley D -saturated with respect to ρ . So in the sequel, we only need to consider the case where S is an infinite semigroup, D is not a null graph and ρ is a nonempty relation. We first consider the case where D has no cycles.

Theorem 4.2 *Assume that D is a finite simple graph with no cycles, T is an ideal extension of an infinite semigroup S and $\rho \subseteq T^1 \times T^1$ is a nonempty relation. Then the following conditions are equivalent:*

- (1) S is Cayley D -saturated with respect to ρ ;
- (2) every infinite subset of $V(\text{Cay}(S, \rho))$ induces a subgraph of $\text{Cay}(S, \rho)$ which is not null.

Proof (1) \Rightarrow (2): Suppose towards a contradiction that there is an infinite subset $V \subseteq S$ such that V induces a null subgraph in $\text{Cay}(S, \rho)$. Then D is not embeddable in the subgraph of $\text{Cay}(S, \rho)$ induced by V , which contradicts the assumption that S is Cayley D -saturated with respect to ρ .

(2) \Rightarrow (1): Let V be an infinite subset of S . Then V induces an infinite subgraph Γ of $\text{Cay}(S, \rho)$. By Lemma 4.1, there is an infinite subset $U \subseteq V$ which induces a subgraph of Γ that is either null, or isomorphic to D_∞ , A_∞ or K_∞ . Since the first case is excluded by condition (2) and D can be embedded in D_∞ , A_∞ or K_∞ ,

the graph D can be embedded in the subgraph Γ induced by U . Thus D can be embedded in the graph induced by V in $\text{Cay}(S, \rho)$ and so S is D -saturated with respect to ρ . \square

Two sets are *incomparable* if neither one is contained in the other. As a consequence of Theorem 4.2, we have the following

Corollary 4.3 *Assume that D is a finite simple graph with no cycles, T is an ideal extension of an infinite semigroup S and $\rho \subseteq T^1 \times T^1$ is a nonempty I -compatible relation. Then the following conditions are equivalent:*

- (1) S is Cayley D -saturated with respect to ρ ;
- (2) S does not have an infinite set $\{\rho^1(s_i) \mid i \in \mathbb{N}\}$ such that $\rho^1(s_i)$ and $\rho^1(s_j)$ are incomparable for all distinct $i, j \in \mathbb{N}$.

Proof (1) \Rightarrow (2): Suppose by way of contradiction that there is an infinite set $A = \{\rho^1(s_i) \mid i \in \mathbb{N}\}$ such that $\rho^1(s_i)$ and $\rho^1(s_j)$ are incomparable for all distinct $i, j \in \mathbb{N}$. Then there is an infinite set V such that V induces a subgraph of $\text{Cay}(S, \rho)$ which is not null by (2) of Theorem 4.2, since S is Cayley D -saturated with respect to ρ . Thus there are distinct $i, j \in \mathbb{N}$ such that $(s_i, s_j) \in E(\text{Cay}(S, \rho))$. Since ρ is I -compatible, it follows from Corollary 2.6 that $\rho^1(s_i) \supseteq \rho^1(s_j)$, which contradicts that $\rho^1(s_i)$ and $\rho^1(s_j)$ are incomparable.

(2) \Rightarrow (1): Let $V = \{v_i \mid i \in \mathbb{N}\}$ be an infinite subset of S . From (2), it follows that there are $i \neq j \in \mathbb{N}$ such that $\rho^1(v_i) \supseteq \rho^1(v_j)$. By Lemma 2.2, $(v_i, v_j) \in E(\text{Cay}(S, \rho))$ since $v_i \neq v_j$. Thus V does not induce a null subgraph of $\text{Cay}(S, \rho)$ and so S is Cayley D -saturated with respect to ρ by (2) of Theorem 4.2. \square

Corollary 4.4 *Let D be a finite simple graph with no cycles, S an infinite semigroup. Then the following conditions are equivalent:*

- (1) S is Cayley D -saturated with respect to $S \times S$ (resp. $S \times \{1\}, \{1\} \times S$);
- (2) S is Cayley D -saturated with respect to $S^1 \times S^1$ (resp. $S^1 \times \{1\}, \{1\} \times S^1$);
- (3) S does not have an infinite set of pairwise incomparable principal ideals (resp. principal left ideals, principal right ideals).

Remark 4.5 The corresponding results remain correct if $S^1 \times S^1$ is replaced by $S^1 \times S$ or by $S \times S^1$ in Corollary 4.4.

Next we consider the case where the graph D has a cycle.

Theorem 4.6 *Assume that D is a finite simple graph which has a cycle, T is an ideal extension of an infinite semigroup S and $\rho \subseteq T^1 \times T^1$ is a nonempty relation. Then the following conditions are equivalent:*

- (1) S is Cayley D -saturated with respect to ρ ;
- (2) every subgraph of $\text{Cay}(S, \rho)$ induced by an infinite subset of S contains a subgraph isomorphic to K_∞ ;

(3) for any infinite subset $U \subseteq S$, there is an infinite subset $W \subseteq U$ such that $W \subseteq \rho(a)$ for any $a \in W$.

Proof (1) \Rightarrow (2): Let V be an infinite subset of S . Then V induces an infinite subgraph Γ of $\text{Cay}(S, \rho)$. By Lemma 4.1, there is an infinite subset $U \subseteq V$ such that U induces a subgraph of Γ that is a null graph, D_∞ , A_∞ , or K_∞ . Since D has a cycle and S is Cayley D -saturated with respect to ρ , the first, second and third cases are excluded. Thus U induces K_∞ in $\text{Cay}(S, \rho)$ and so the subgraph of $\text{Cay}(S, \rho)$ induced by V contains K_∞ .

(2) \Rightarrow (3): Let U be an infinite subset of S . Then by (2) there is an infinite subset $W \subseteq U$ such that W induces K_∞ in $\text{Cay}(S, \rho)$. Let $a, b \in W$. Then $(a, b) \in E(\text{Cay}(S, \rho))$ and so $b \in \rho(a)$. Thus $W \subseteq \rho(a)$, as required.

(3) \Rightarrow (1): Let V be an infinite subset of S . Then by (3), there is an infinite subset $W \subseteq V$ such that $W \subseteq \rho(a)$ for any $a \in W$. Thus $(a, b) \in E(\text{Cay}(S, \rho))$ for any $b \in W$ and so W induces K_∞ in $\text{Cay}(S, \rho)$. Since D can be embedded in K_∞ , D can be embedded in the subgraph of K_∞ induced by V . Therefore, S is Cayley D -saturated with respect to ρ and the theorem is proved. \square

Corollary 4.7 *Let D be a finite simple graph which has a cycle, T an ideal extension of an infinite semigroup S and $\rho \subseteq T^1 \times T^1$ a nonempty relation. If S is Cayley D -saturated with respect to ρ , then there exist elements $s_0, s_1, \dots, s_n \in S$ such that $S = \rho^1(s_0) \cup \rho^1(s_1) \cup \dots \cup \rho^1(s_n)$.*

Proof Let $s_0 \in S$ and consider the set $S \setminus \rho^1(s_0)$. If the set $S \setminus \rho^1(s_0)$ is finite, then the result is true. If the set $S \setminus \rho^1(s_0)$ is infinite, let $s_1 \in S \setminus \rho^1(s_0)$ and consider the set $S \setminus (\rho^1(s_0) \cup \rho^1(s_1))$. Repeating the same process, if the process terminates after some finite steps, then the result follows. Otherwise, we can get an infinite set $\{\rho^1(s_i) \mid i \in \mathbb{N}\}$ such that $s_i \in S \setminus \bigcup_{0 \leq j < i} \rho^1(s_j)$ for all $i > 0$. Let $V = \{s_i \mid i \in \mathbb{N}\}$. Since S is Cayley D -saturated with respect to ρ , the subgraph induced by V in $\text{Cay}(S, \rho)$ contains a subgraph K isomorphic to K_∞ by (2) of Theorem 4.6. Without loss of generality, let $s_0 \in V(K)$. Then there is a positive i such that $s_i \in V(K)$. It follows that $(s_0, s_i) \in E(\text{Cay}(S, \rho))$, and so $s_i \in \rho(s_0)$. Therefore, $s_i \in \rho^1(s_0)$ which contradicts the choice of s_i . \square

Corollary 4.8 *Let D be a finite simple graph which has a cycle, T an ideal extension of an infinite semigroup S and $\rho \subseteq T^1 \times T^1$ a nonempty relation. If S is Cayley D -saturated with respect to ρ , then the following statements hold:*

- (1) the set ρ is infinite;
- (2) if ρ is 1-compatible, then the set of ρ -classes of S is finite;
- (3) if ρ is 1-compatible, then the set $\{\rho(a) \mid a \in S\}$ is finite.

Proof Let U be an infinite subset of S . Since S is Cayley D -saturated with respect to ρ , there is an infinite subset $W \subseteq U$ such that $W \subseteq \rho(a)$ for any $a \in W$ by (3) of Theorem 4.6. Thus ρ must be infinite and (1) holds.

To show (2), suppose toward a contradiction that the set A of ρ -classes of S is infinite. Let $A = \{\rho^1(s_i) \mid i \in \mathbb{N}\}$ be an infinite set, and set $V = \{s_i \mid i \in \mathbb{N}\}$. Then V

induces an infinite subgraph of $\text{Cay}(S, \rho)$, and so there is an infinite subset $U \subseteq V$ such that U induces a subgraph in $\text{Cay}(S, \rho)$ which is isomorphic to K_∞ by (2) of Theorem 4.6. Thus there are $s_i \neq s_j \in U$ such that $(s_i, s_j), (s_j, s_i) \in E(\text{Cay}(S, \rho))$. Hence by Corollary 2.6, $\rho^1(s_i) = \rho^1(s_j)$, which contradicts that s_i and s_j are in different ρ -classes. Therefore (2) holds. Similarly, we can prove (3). \square

Recall that the union of two graphs Γ_1 and Γ_2 , denoted by $\Gamma_1 \cup \Gamma_2$, is the graph with the vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and the edge set $E(\Gamma_1) \cup E(\Gamma_2)$. Furthermore, if $V(\Gamma_1) \cap V(\Gamma_2) = \emptyset$, we call $\Gamma_1 \cup \Gamma_2$ the disjoint union of Γ_1 and Γ_2 .

Theorem 4.9 *Let D be a finite simple graph which has a cycle, T an ideal extension of an infinite semigroup S and $\rho \subseteq T^1 \times T^1$ a nonempty I-compatible relation. Then the following conditions are equivalent:*

- (1) S is Cayley D -saturated with respect to ρ ;
- (2) $\text{Cay}(S, \rho)$ is the finite disjoint union of complete symmetric graphs;
- (3) the set $\{\rho^1(s) | s \in S\}$ is finite.

Proof (1) \Rightarrow (2): Since S is Cayley D -saturated with respect to ρ , every subgraph of $\text{Cay}(S, \rho)$ induced by an infinite subset of S contains a subgraph isomorphic to K_∞ by (2) of Theorem 4.6. Let A_0 be the set of complete symmetric subgraphs of $\text{Cay}(S, \rho)$. For $\Gamma_1, \Gamma_2 \in A_0$, define a partial order \leq on A_0 by setting $\Gamma_1 \leq \Gamma_2$ if and only if $V(\Gamma_1) \subseteq V(\Gamma_2)$. Suppose that $\Gamma_1 \leq \Gamma_2 \leq \dots$ is a chain of A_0 . Then $\bigcup_{i \in \mathbb{N}} \Gamma_i$ is an upper bound of the chain in A_0 . So by Zorn's Lemma, there is a maximal element $H_0 \in A_0$. Now we consider the set $S \setminus V(H_0)$. If the set $S \setminus V(H_0)$ is finite, then the result is true. If the set $S \setminus V(H_0)$ is infinite, let A_1 be the set of complete symmetric subgraphs of the subgraph induced by $S \setminus V(H_0)$ in $\text{Cay}(S, \rho)$. With a similar argument, there is a maximal element H_1 in A_1 . Repeating the same process, if the process terminates after some finite steps, then the result follows. Otherwise, we can obtain an infinite set $\{H_i | i \in \mathbb{N}\}$, where each H_i is maximal in the set of complete symmetric subgraphs of the subgraph induced by $S \setminus \bigcup_{1 \leq j < i} V(H_j)$ in $\text{Cay}(S, \rho)$. Let $v_i \in H_i$ and $V = \{v_i | i \in \mathbb{N}\}$. Since S is Cayley D -saturated with respect to ρ , V contains an infinite complete symmetric subgraph K by (2) of Theorem 4.6. Let $I = \{i | v_i \in V(K)\}$. Without loss of generality, let $0 \in I$. Then there is a positive integer $j \in I$, which means that $v_j \in V(K)$ and $(v_0, v_j), (v_j, v_0) \in E(\text{Cay}(S, \rho))$. Since $\rho \subseteq T^1 \times T^1$ is I-compatible, then $\text{Cay}(S, \rho)$ is edge-transitive according to Proposition 2.5. Now for any $v \in V(H_0) \setminus \{v_0\}$, we have $(v, v_0), (v_0, v) \in E(\text{Cay}(S, \rho))$ and thus $(v, v_j), (v_j, v) \in E(\text{Cay}(S, \rho))$. It follows that $V(H_0) \cup \{v_j\}$ induces a complete symmetric subgraph in $\text{Cay}(S, \rho)$, which contradicts that H_0 is maximal in A_0 since $v_j \notin H_0$.

(2) \Rightarrow (3): Let $\text{Cay}(S, \rho) = K_0 \cup K_1 \cup \dots \cup K_n$ where n is finite and all K_i 's are complete symmetric subgraphs. Let $u, v \in K_i$ and $u \neq v$. Then $(u, v), (v, u) \in E(\text{Cay}(S, \rho))$. In light of Corollary 2.6, we have $\rho^1(u) \supseteq \rho^1(v)$ and $\rho^1(v) \supseteq \rho^1(u)$. Immediately, $\rho^1(u) = \rho^1(v)$, which implies that the cardinality of the set $\{\rho^1(s) | s \in S\}$ is at most n . Thus the set $\{\rho^1(s) | s \in S\}$ is finite.

(3) \Rightarrow (1): Let V be an infinite subset of S . Since the set $\{\rho^1(s) | s \in S\}$ is finite, there is an infinite subset $U \subseteq V$ such that $\rho^1(u) = \rho^1(v)$ for any $u \neq v \in U$.

By Lemma 2.2, $(u, v), (v, u) \in E(\text{Cay}(S, \rho))$. So U induces a complete symmetric subgraph K in $\text{Cay}(S, \rho)$. Consequently, V contains K and D can be embedded in the subgraph of $\text{Cay}(S, \rho)$ induced by V . Therefore, S is Cayley D -saturated with respect to ρ . \square

Remark 4.10 If we drop the assumption that ρ is I-compatible from the above theorem, then the implication (3) \Rightarrow (1) remains true.

Corollary 4.11 *Let D be a finite simple graph which has a cycle, T an ideal extension of an infinite semigroup S and $\rho \subseteq T^1 \times T^1$ a nonempty I-compatible relation. Then the following conditions are equivalent:*

- (1) S is Cayley D -saturated with respect to $S \times S$ (rep. $S \times \{1\}, \{1\} \times S$);
- (2) S is Cayley D -saturated with respect to $S^1 \times S^1$ (rep. $S^1 \times \{1\}, \{1\} \times S^1$);
- (3) the number of principal ideals (rep. principal left ideals, principal right ideals) of S is finite.

Remark 4.12 The corresponding results remain correct if $S^1 \times S^1$ is replaced by $S^1 \times S$ or by $S \times S^1$ in the Corollary 4.11.

Corollary 4.13 *Let T be an ideal extension of an infinite semigroup S and let $\rho \subseteq T^1 \times T^1$ be a nonempty I-compatible relation. If for a finite simple graph D_1 with a cycle, S is Cayley D_1 -saturated with respect to ρ , then for any finite simple graph D , S is Cayley D -saturated with respect to ρ .*

Proof Since D_1 has a cycle and S is Cayley D_1 -saturated with respect to ρ , $\text{Cay}(S, \rho)$ is a finite disjoint union of complete symmetric graphs by (2) of Theorem 4.9. Let D be a finite graph and V an infinite subset of S . Then there is an infinite subset $U \subseteq V$ such that the elements of U lie in a same complete symmetric subgraph, and so U induces a subgraph in $\text{Cay}(S, \rho)$ which is isomorphic to K_∞ . This implies that D can be embedded in the subgraph of $\text{Cay}(S, \rho)$ induced by U . Therefore D can be embedded in the subgraph of $\text{Cay}(S, \rho)$ induced by V and so S is Cayley D -saturated with respect to ρ . \square

Acknowledgements The author thanks Tom Hall for his help and also Andrei Kelarev for guidance and valuable suggestions. The work was supported by National Natural Foundation of China (No. 10571005) and Shandong Province Natural Science Foundation ZR2011AM005.

References

1. Arworn, Sr., Knauer, U., Chiangmai, N.N.: Characterization of digraphs of right (left) zero unions of groups. *Thai J. Math.* **1**(1), 131–140 (2003)
2. Dénes, J.: Connections between transformation semigroups and graphs. In: *Theory of Graphs*. Gordon and Breach, New York (1967)
3. Howie, J.M.: *Fundamentals of Semigroup Theory*. Clarendon Press, Oxford (1995)
4. Kelarev, A.V.: On undirected Cayley graphs. *Aust. J. Comb.* **25**, 73–78 (2002)
5. Kelarev, A.V.: *Graph Algebras and Automata*. Marcel Dekker, New York (2003)
6. Kelarev, A.V.: On Cayley graphs of inverse semigroups. *Semigroup Forum* **72**, 411–418 (2006)

7. Kelarev, A.V., Praeger, C.E.: On transitive Cayley graphs of groups and semigroups. *Eur. J. Comb.* **24**(1), 59–72 (2003)
8. Kelarev, A.V., Quinn, S.J.: Directed graphs and combinatorial properties of semigroups. *J. Algebra* **251**(1), 16–26 (2002)
9. Kelarev, A.V., Quinn, S.J.: A combinatorial property and Cayley graphs of semigroups. *Semigroup Forum* **66**, 89–96 (2003)
10. Kelarev, A.V., Quinn, S.J.: A combinatorial property and power graphs of semigroups. *Comment. Math. Univ. Carol.* **45**(1), 1–7 (2004)
11. Kelarev, A., Ryan, J., Yearwood, J.: Cayley graphs as classifiers for data mining: the influence of asymmetries. *Discrete Math.* **309**, 5360–5369 (2009)
12. Panma, S., Knauer, U., Arworn, S.: On transitive Cayley graphs of right (left) groups and of Clifford semigroups. *Thai J. Math.* **2**(1), 183–195 (2004)
13. Panma, S., Chiangmai, N.N., Knauer, U., Arworn, S.: Characterizations of Clifford semigroup digraphs. *Discrete Math.* **306**(12), 1247–1252 (2006)
14. Panma, S., Knauer, U., Arworn, S.: On transitive Cayley graphs of strong semilattices of right (left) groups. *Discrete Math.* **309**(17), 5393–5403 (2009)
15. Petrich, M., Reilly, N.: *Completely Regular Semigroups*. Wiley, New York (1999)
16. Wilson, R.J.: *Introduction to Graph Theory*, 3rd edn. Longman, New York (1982)
17. Yang, D., Gao, X.: D-saturated property of the Cayley graphs of semigroups. *Semigroup Forum* **80**, 174–180 (2010)
18. Zelinka, B.: Graphs of semigroups. *Cas. Pest. Mat.* **106**, 407–408 (1981)
19. Zhu, Y.: Generalized Cayley graphs of semigroups I. *Semigroup Forum* (2011). doi:[10.1007/s00233-011-9368-9](https://doi.org/10.1007/s00233-011-9368-9)
20. Zhu, Y.: On (n, m) -semigroups. *Semigroup Forum* (2011). doi:[10.1007/s00233-011-9360-4](https://doi.org/10.1007/s00233-011-9360-4)