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RESEARCH ARTICLE

On the lattice of sub-pseudovarieties of DA

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Abstract The wealth of information that is available on the lattice of varieties of bands, is used to illuminate the structure of the lattice of sub-pseudovarieties of **DA**, a natural generalization of bands which plays an important role in language theory and in logic. The main result describes a hierarchy of decidable sub-pseudovarieties of **DA** in terms of iterated Mal'cev products with the pseudovarieties of definite and reverse definite semigroups.

Keywords Lattice of pseudovarieties · Lattice of band varieties · Aperiodic monoids · Monoids in DA · Malcev product

The complete elucidation of the structure of the lattice $\mathcal{LB}$ of band varieties is one of the jewels of semigroup theory: this lattice turns out to be countable, with a simple structure (Birjukov [2], Fennemore [3, 4], Gerhard [6], see Sect. 2.2 below for the main features of this structure). Moreover, each of its elements can be defined by a small number of identities (at most 3), and we can efficiently solve the membership problem in each variety of bands, as well as the word problem in its free object [7].

As bands are locally finite, the lattice $\mathcal{L}(\mathbf{B})$ of pseudovarieties of finite bands is isomorphic to $\mathcal{LB}$: a class of finite bands is a pseudovariety if and only if it is the class of finite elements of a variety.

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In this paper, we discuss the structure of the lattice of sub-pseudovarieties of **DA**, which is a natural generalization of the pseudovariety **B** of bands. Indeed, **DA** is the maximum pseudovariety in which all regular elements are bands. This pseudovariety actually has several other interesting algebraic characterizations, and also many other characterizations in terms of formal languages and logic, see the survey by Tesson and Thérien [17]. This only adds to the motivation to better understand the lattice $\mathcal{L}(\mathbf{DA})$ of its sub-pseudovarieties.

In fact the authors' initial motivation regards one of the logical characterizations of **DA** by means of the 2-variable fragment of first-order theory of the linear order [19], and the main result of this paper finds an application in a paper on the language-theoretic characterizations of the quantifier alternation hierarchy within that logic [11]. The characterization in [11] can be viewed as an algebraic counterpart of recent results of Weis and Immerman's description on the 2-variable fragment of first-order logic [23], and the main result of the present paper gives a purely algebraic foundation to the results in [23] and [11].

Trotter and Weil [20] initiated the study of the structure of $\mathcal{L}(\mathbf{DA})$ by considering the map $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$, from $\mathcal{L}(\mathbf{DA})$ to $\mathcal{L}(\mathbf{B})$. They showed that, for each pseudovariety of bands $\mathbf{Y}$, the inverse image of $\mathbf{Y}$ is an interval in $\mathcal{L}(\mathbf{DA})$, with minimum element $\mathbf{Y}$ itself. They also showed how to effectively turn the identities defining $\mathbf{Y}$ as a band pseudovariety, into pseudo-identities defining $\mathbf{Y}^{\uparrow}$, the maximal element of that interval. This result uncovers the interesting role played by the lattice of decidable pseudovarieties given by the $\mathbf{Y}^{\uparrow}$, $\mathbf{Y} \in \mathcal{L}(\mathbf{B})$.

The missing element was an understanding of the fashion in which one can *climb* in that lattice. The beautiful results on $\mathcal{L}(\mathbf{B})$ include a description of the different levels of the hierarchy it forms, in terms of Mal'cev products with the pseudovarieties **RZ** and **LZ** of right zero and left zero bands. In this paper, we elucidate the structure of the sublattice of $\mathcal{L}(\mathbf{DA})$ formed by the $\mathbf{Y}^{\uparrow}$ ($\mathbf{Y} \in \mathcal{L}(\mathbf{B})$), in terms of Mal'cev products as well, with definite and reverse definite semigroups. This helps establish that the $\mathbf{Y}^{\uparrow}$ form an infinite hierarchy, whose union is all of **DA**. It follows in particular that **DA** is the least pseudovariety containing semilattices, which is closed under Mal'cev product with definite and reverse definite semigroups,—a fact with an interesting interpretation in formal language theory.

Interestingly, this last result was recently proved, independently and by completely different means (logical and language theoretical) by Lodaya, Pandya and Shah [12].

The paper is organized as follows: Sect. 1 summarizes what the reader needs to know (for the purpose of this paper!) about pseudovarieties and Mal'cev products. Section 2 discusses the known results on bands, **DA** and their respective lattice of sub-pseudovarieties, and Sect. 3 gives our main result. Its consequences are discussed in Sect. 4, in semigroup- and in language-theoretic terms.

1 Preliminaries on pseudovarieties

1.1 Pseudovarieties

Recall that a pseudovariety of semigroups (resp. monoids) is a class of finite semigroups (resp. monoids) closed under taking quotients, finite direct products and subsemigroups (resp. submonoids). If $\mathbf{V}$ is a pseudovariety of semigroups, we denote by

$\mathbf{V_M}$ the pseudovariety of monoids which consists of the monoids in $\mathbf{V}$. The pseudovariety $\mathbf{V}$ is called *monoidal* if it is generated by the monoids it contains.

If $\mathbf{W}$ is a pseudovariety of monoids, we denote by $L\mathbf{W}$ the class of semigroups S such that, for each idempotent e, the monoid $eSe \in \mathbf{W}$: $L\mathbf{W}$ forms a pseudovariety, the largest one such that $(L\mathbf{W})_\mathbf{M} \subseteq \mathbf{W}$.

There is a vast literature on pseudovarieties, and on their definition by *pseudo-identities*, see [1, 22]. For our purpose, it is enough to consider so-called *ω-pseudo-identities* of the form $u = v$, where u and v are obtained from a countable alphabet of symbols X using the operation of concatenation and formal $(\omega - 1)$-power. For instance, we will consider in the sequel identities like $(xy)^\omega x(xy)^\omega = (xy)^\omega$—where z^ω stands for $z^{\omega-1}z$. A finite semigroup S satisfies the ω-pseudo-identity $u = v$ if, for every map $\varphi\colon X \to S$, we have $\hat\varphi(u) = \hat\varphi(v)$, where $\hat\varphi$ extends φ to a monoid morphism such that $\hat\varphi(t^\omega)$ is the (unique) idempotent power of $\hat\varphi(t)$ and $\hat\varphi(t^{\omega-1})$ is the inverse of the element $\hat\varphi(t^\omega)\hat\varphi(t)$ in the minimal ideal of the subsemigroup generated by $\hat\varphi(t)$, which is a group [1, 13]. By a common abuse of notation, we also denote by s^ω ($s \in S$) the idempotent power of s, and by $s^{\omega-1}$ the inverse of $s^\omega s$ in the minimal ideal of the subsemigroup generated by s.

If $(u_i = v_i)_{i\in I}$ is a family of ω-pseudo-identities, we denote by $[\![(u_i = v_i)_{i\in I}]\!]$ the class of finite semigroups which satisfy each ω-pseudo-identity $u_i = v_i$. Such a class is always a pseudovariety.[1]

1.2 Mal'cev products

Let $\mathbf{V}$ be a pseudovariety of semigroups, $\mathbf{W}$ a pseudovariety of semigroups (resp. monoids) and M a finite semigroup (resp. monoid). We say that $M \in \mathbf{V} \circledast \mathbf{W}$ (the *Mal'cev product* of $\mathbf{V}$ and $\mathbf{W}$) if there exists a finite monoid T and onto morphisms $\alpha\colon T \to M$ and $\beta\colon T \to N$ such that $N \in \mathbf{W}$ and, for each idempotent e of N, $\beta^{-1}(e) \in \mathbf{V}$ (we say that β is a $\mathbf{V}$*-morphism)*. Then $\mathbf{V} \circledast \mathbf{W}$ is a pseudovariety of semigroups (resp. monoids), see [1, 13, 14].

In the sequel, we will consider Mal'cev products where the first component is one of the pseudovarieties **Nil**, **LZ**, **RZ**, **K**, **D** and $L\mathbf{I}$, which are defined as follows:

$$\mathbf{K} = [\![x^\omega y = x^\omega]\!], \qquad \mathbf{D} = [\![yx^\omega = x^\omega]\!],$$
$$\mathbf{Nil} = \mathbf{K} \cap \mathbf{D} = [\![x^\omega y = yx^\omega = z^\omega]\!],$$
$$L\mathbf{I} = \mathbf{K} \vee \mathbf{D} = [\![x^\omega y x^\omega = x^\omega]\!],$$
$$\mathbf{LZ} = \mathbf{K} \cap [\![x^2 = x]\!] = [\![xy = x, x^2 = x]\!],$$
$$\mathbf{RZ} = \mathbf{D} \cap [\![x^2 = x]\!] = [\![yx = x, x^2 = x]\!]$$

We will use the following fact, due to Krohn, Rhodes and Tilson [10], see [8, Corollary 4.3]. The $\mathcal{J}$-quasi-order on M is defined as follows: $x \le_\mathcal{J} y$ if and only if $x = uyv$ for some $u, v \in M \cup \{1\}$. We write $x <_\mathcal{J} y$ if $x \le_\mathcal{J} y$ but not $y \le_\mathcal{J} x$;

[1] Not all pseudovarieties are obtained this way; for a more rigorous discussion of pseudidentities, and in particular for a converse statement (involving a much larger set of pseudo-identities), see [1, 22].

that is, if the 2-sided ideal of M generated by x is properly contained in the 2-sided ideal generated by y.

Proposition 1.1 *Let M be a finite semigroup and let $\sim_{\mathbf{K}}$ and $\sim_{\mathbf{D}}$ be the equivalence relations $\sim_{\mathbf{K}}$ and $\sim_{\mathbf{D}}$ on M given, for $s, t \in M$, by*

$$s \sim_{\mathbf{K}} t \quad \text{if and only if,} \quad \text{for all } e \in E(M),\ es, et <_{\mathcal{J}} e \text{ or } es = et$$
$$s \sim_{\mathbf{D}} t \quad \text{if and only if,} \quad \text{for all } e \in E(M),\ se, te <_{\mathcal{J}} e \text{ or } es = et.$$

These two relations are congruences and $M/{\sim_{\mathbf{K}}}$ (resp. $M/{\sim_{\mathbf{D}}}$) is the least quotient of M such that the projection is a **K**- (*resp. a* **D**-) *morphism.*

If **V** *is a pseudovariety of semigroups (resp. monoids) and M is a finite semigroup (resp. monoid), then $M \in \mathbf{K} \circledm \mathbf{V}$ (resp. $M \in \mathbf{D} \circledm \mathbf{V}$) if and only if $M/{\sim_{\mathbf{K}}} \in \mathbf{V}$ (resp. $M/{\sim_{\mathbf{D}}} \in \mathbf{V}$).*

In particular, if **V** *is decidable, then so are $\mathbf{K} \circledm \mathbf{V}$ and $\mathbf{D} \circledm \mathbf{V}$.*

2 Preliminaries on bands and DA

2.1 Bands and DA

A *band* is a semigroup in which every element is idempotent. We denote by **B** the pseudovariety of bands, that is, $\mathbf{B} = [\![x^2 = x]\!] = [\![x^\omega = x]\!]$.

Let $\mathbf{DA} = [\![(xy)^\omega x (xy)^\omega = (xy)^\omega]\!]$. The following result combines several known results: we refer the reader to [17] for a synthesis on **DA** (see also [5, 16, 21]).

Proposition 2.1 *If M is a finite semigroup, the following are equivalent.*

(1) $M \in \mathbf{DA}$,
(2) *every regular element of M is idempotent,*
(3) *for every idempotent $e \in E(M)$, we have $eM_e e = e$, where M_e is the subsemigroup of M generated the elements $x \geq_{\mathcal{J}} e$,*
(4) *$M \in LI \circledm \mathbf{SL}$, where $\mathbf{SL} = [\![x^2 = x, xy = yx]\!]$ is the pseudovariety of idempotent and commutative semigroups.*

Corollary 2.2 **DA** (*resp.* $\mathbf{DA_M}$) *is the maximum pseudovariety of semigroups (resp. monoids), in which every regular semigroup (resp. monoid) is a band.*

Proof If $M \in \mathbf{DA}$ and a regular semigroup, then M is a band by Proposition 2.1(2).

Let now **V** be a pseudovariety of semigroups in which every regular element is a band. Recall that an element $s \in M$ is *regular* if there exists $t \in M$ such that $sts = s$. In particular, the regular elements of M are exactly the elements of the form $(st)^\omega s$ ($s, t \in M$). Therefore **V** satisfies the ω-pseudo-identity $(xy)^\omega x (xy)^\omega x = (xy)^\omega x$, and by right multiplication by $y(xy)^{\omega-1}$, we find that **V** satisfies $(xy)^\omega x (xy)^\omega = (xy)^\omega$, that is, **V** is contained in **DA**. □

Many more characterizations of **DA** can be found in the literature, see [17, 18]. In this paper, we will encounter one more, in Sect. 4.2 below, in relation with formal language theory.

2.2 The lattice $\mathcal{L}(\mathbf{B})$

Since the free band over a finite alphabet is finite (see [9]), the lattice $\mathcal{L}(\mathbf{B})$ of sub-pseudovarieties of $\mathbf{B}$ is isomorphic with the lattice $\mathcal{LB}$ of all band varieties. The structure of that lattice was elucidated around 1970 (Birjukov [2], Fennemore [3, 4], Gerhard [6]). The lattice $\mathcal{L}(\mathbf{B})$ turns out to be countable, with a simple structure. We summarize below the main results concerning this lattice that will be useful to us.

We define the pseudovarieties $\mathbf{BR}_m$, $\mathbf{BR}'_{m+1}$, $\mathbf{BL}_m$ and $\mathbf{BL}'_{m+1}$ $(m \geq 1)$ by letting[2]

$$\begin{aligned}
&\mathbf{BR}_1 = \mathbf{BL}_1 = \mathbf{SL}, && \\
&\mathbf{BR}_{m+1} = \mathbf{LZ} ⓜ \mathbf{BL}_m, && \mathbf{BL}_{m+1} = \mathbf{RZ} ⓜ \mathbf{BR}_m,\\
&\mathbf{BR}'_2 = [\![xyz = xzy, x^2 = x]\!], && \mathbf{BL}'_2 = [\![xyz = yxz, x^2 = x]\!],\\
&\mathbf{BR}'_{m+2} = \mathbf{LZ} ⓜ \mathbf{BL}'_{m+1}, && \mathbf{BL}'_{m+2} = \mathbf{RZ} ⓜ \mathbf{BR}'_{m+1}.
\end{aligned}$$

The following statement describes the structure of the lattice $\mathcal{L}(\mathbf{B})$, as discussed by Gerhard and Petrich [7] (the last item is due to Wismath [24]).

Theorem 2.3

(1) *The lattice* $\mathcal{L}(\mathbf{B})$ *consists of the trivial pseudovariety* $\mathbf{I}$, *the pseudovariety* $\mathbf{B}$, *the* $\mathbf{BR}_m$, $\mathbf{BR}'_{m+1}$, $\mathbf{BL}_m$, $\mathbf{BL}'_{m+1}$ $(m \geq 1)$ *and their intersections. It is depicted in Fig.* 1 (*omitting its top element* $\mathbf{B}$).
(2) *The monoidal band pseudovarieties are the trivial pseudovariety* $\mathbf{I}$, $\mathbf{B}$, *and the* $\mathbf{BR}_m$, $\mathbf{BL}_m$ *and* $\mathbf{BR}_m \cap \mathbf{BL}_m$ $(m \geq 1)$.
(3) *For each* $m \geq 1$, *we have* $\mathbf{BR}_m \vee \mathbf{BL}_m = \mathbf{BR}_{m+1} \cap \mathbf{BL}_{m+1}$.
(4) *For each* $m \geq 2$, $\mathbf{BR}'_{m+1} = L\mathbf{BR}_m \cap \mathbf{B}$ *and* $\mathbf{BL}'_{m+1} = L\mathbf{BL}_m \cap \mathbf{B}$.
(5) *For each* $m \geq 2$, *the* m*-generated free band lies in* $\mathbf{BR}'_m \vee \mathbf{BL}'_m = \mathbf{BR}'_{m+1} \cap \mathbf{BL}'_{m+1}$.
(6) $\mathcal{L}(\mathbf{B_M})$ *is isomorphic to the lattice of monoidal band pseudovarieties, with the isomorphism given by* $\mathbf{V} \mapsto \mathbf{V_M}$ ($\mathbf{V} \in \mathcal{L}(\mathbf{B})$, *monoidal*).

Gerhard and Petrich [7] also give identities defining the band pseudovarieties. Let $x_1, x_2, \ldots$ be a sequence of variables. If u is a word on that alphabet, we let $\bar{u}$ be the mirror image of u, that is, the word obtained from reading u from right to left. We let

$$G_2 = x_2x_1, \qquad I_2 = x_2x_1x_2$$
$$\text{and for } m \geq 2 \quad G_{m+1} = x_{m+1}\overline{G_m}, \qquad I_{m+1} = G_{m+1}x_{m+1}\overline{I_m}.$$

Theorem 2.4 *For each* $m \geq 2$, *we have* $\mathbf{BR}_m = [\![x^2 = x, G_m = I_m]\!]$ *and* $\mathbf{BL}_m = [\![x^2 = x, \overline{G_m} = \overline{I_m}]\!]$.

Note that Theorems 2.3 and 2.4 allow the computation of defining identities for each band pseudovariety, and indeed to show that each can be defined by a set of at most three identities.

[2] In the traditional terminology of bands, the elements of $\mathbf{BR}'_2$ and $\mathbf{BL}'_2$ are called *right normal* and *left normal* bands respectively.

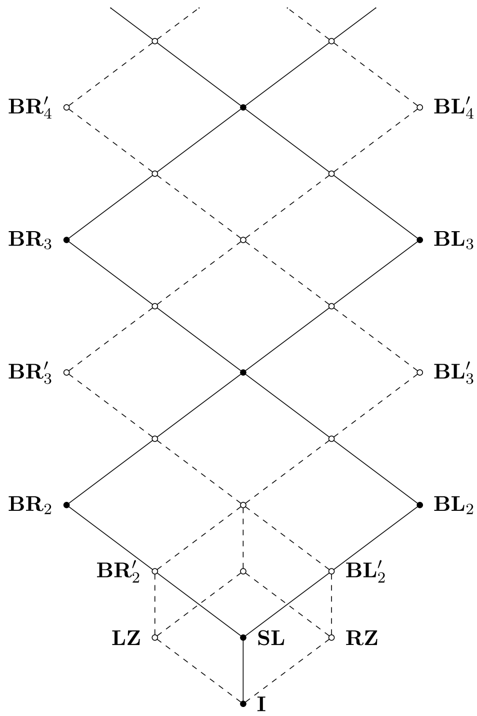

Fig. 1 The lattice $\mathcal{L}(\mathbf{B})$; *solid lines* and *bullets* denote the monoidal pseudovarieties

2.3 The map $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$

The map $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ from $\mathcal{L}(\mathbf{DA})$ to $\mathcal{L}(\mathbf{B})$ can be used to derive information on $\mathcal{L}(\mathbf{DA})$ from the information available on $\mathcal{L}(\mathbf{B})$. The following statement was proved by Trotter and Weil [20] (and, independently, by Reilly and Zhang [15] for the first item).

Theorem 2.5

(1) *The map* $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ *from* $\mathcal{L}(\mathbf{DA})$ *to* $\mathcal{L}(\mathbf{B})$ *is a complete lattice morphism, and the inverse image of a band pseudovariety* $\mathbf{Y}$ *is an interval of the form* $[\mathbf{Y}, \mathbf{Y}^{\uparrow}]$.
(2) *For each* $\mathbf{Y} \in \mathcal{L}(\mathbf{B})$, *we have* $\mathbf{Y}^{\uparrow} = (\mathbf{LZ} \,ⓜ\, \mathbf{Y})^{\uparrow} \cap (\mathbf{RZ} \,ⓜ\, \mathbf{Y})^{\uparrow}$.
(3) *The mapping* $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B_M}$ *from* $\mathcal{L}(\mathbf{DA_M})$ *to* $\mathcal{L}(\mathbf{B_M})$ *shares the properties from statement* (1). *Moreover, if* $\mathbf{Y}$ *is a monoidal band pseudovariety, then* $(\mathbf{Y_M})^{\uparrow} = (\mathbf{Y}^{\uparrow})_{\mathbf{M}}$.
(4) *If* $\mathbf{Y}$ *is a monoidal pseudovariety of bands, then* $(L\mathbf{Y_M} \cap \mathbf{B})^{\uparrow} = L(\mathbf{Y}^{\uparrow}_{\mathbf{M}}) \cap \mathbf{DA}$.
(5) *For each* $m \geq 2$, $\mathbf{BR}^{\uparrow}_m = \mathbf{DA} \cap [\![\varphi(G_m) = \varphi(I_m)]\!]$ *and* $\mathbf{BL}^{\uparrow}_m = \mathbf{DA} \cap [\![\varphi(\overline{G_m}) = \varphi(\overline{I_m})]\!]$, *where* φ *is given by*

$$\varphi(x_1) = (x_1^{\omega} x_2^{\omega} x_1^{\omega})^{\omega}, \qquad \varphi(x_2) = x_2^{\omega}$$

$$\textit{and, for } m \geq 2, \quad \varphi(x_{m+1}) = (x_{m+1}^{\omega} \varphi(\overline{G_m} G_m)^{\omega} x_{m+1}^{\omega})^{\omega}.$$

(6) *Let* $m \geq 1$. *Every* m*-generated semigroup in* $\mathbf{DA}$ *is in* $\mathbf{SL}^{\uparrow}$ *if* $m = 1$, *in* $\mathbf{BR}_3^{\uparrow} \cap \mathbf{BL}_3^{\uparrow}$ *if* $m = 2$, *and in* $L(\mathbf{BR}_m^{\uparrow} \cap \mathbf{BL}_m^{\uparrow})$ *if* $m \geq 3$. *In every case, such a semigroup is in* $\mathbf{BR}_{m+1}^{\uparrow} \cap \mathbf{BL}_{m+1}^{\uparrow}$

It is elementary to verify [20] that $\mathbf{I}^{\uparrow} = \mathbf{Nil}$, $\mathbf{LZ}^{\uparrow} = \mathbf{K}$, $\mathbf{RZ}^{\uparrow} = \mathbf{D}$ and $\mathbf{SL}^{\uparrow} = \mathbf{J}$, and that $\mathbf{BR}_2^{\uparrow}$ and $\mathbf{BL}_2^{\uparrow}$ are equal, respectively, to the pseudovarieties $\mathbf{R}$ and $\mathbf{L}$, of $\mathcal{R}$-trivial and $\mathcal{L}$-trivial semigroups.

Then Theorem 2.5 suffices to compute defining pseudo-identities for all the $\mathbf{Y}^{\uparrow}$ ($\mathbf{Y} \in \mathcal{L}(\mathbf{B})$)—and hence to prove the decidability of each of these pseudovarieties.

Example 2.6 Theorem 2.5 shows that $(\mathbf{BR}_2' \vee \mathbf{BL}_2')^{\uparrow} = L\mathbf{J} \cap \mathbf{DA}$ since $\mathbf{BR}_2' \vee \mathbf{BL}_2' = \mathbf{BR}_3' \cap \mathbf{BL}_3' = L(\mathbf{BR}_2 \cap \mathbf{BL}_2) \cap \mathbf{B} = L\mathbf{SL} \cap \mathbf{B}$ and $\mathbf{SL}^{\uparrow} = \mathbf{J}$.

It also shows that $\mathbf{BR}_2'^{\uparrow} = \mathbf{R} \cap L\mathbf{J}$, since $\mathbf{BR}_2' = \mathbf{BR}_2 \cap (\mathbf{BR}_2' \vee \mathbf{BL}_2')$.

For $\mathbf{R} = \mathbf{BR}_2^{\uparrow}$, Theorem 2.5 yields the pseudo-identity $x_2^{\omega}(x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega} = x_2^{\omega}(x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega}x_2^{\omega}$. One can verify that, together with the pseudo-identity defining $\mathbf{DA}$, this is equivalent to the usual pseudo-identity describing $\mathbf{R}$, namely $(xy)^{\omega} = (xy)^{\omega}x$.

The pseudo-identities for the $\mathbf{BR}_m^{\uparrow}$, $m \geq 3$, are naturally more complicated. For instance, for $m = 3$, we get

$$\begin{aligned}
\varphi(G_3) &= \big(x_3^{\omega}((x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega}x_2^{\omega}(x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega})^{\omega}x_3^{\omega}\big)^{\omega}\\
\varphi(I_3) &= \big(x_3^{\omega}((x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega}x_2^{\omega}(x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega})^{\omega}x_3^{\omega}\big)^{\omega}\\
&\quad \big(x_3^{\omega}((x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega}x_2^{\omega}(x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega})^{\omega}x_3^{\omega}\big)^{\omega}x_2^{\omega}(x_1^{\omega}x_2^{\omega}x_1^{\omega})^{\omega}x_2^{\omega}.
\end{aligned}$$

3 Main result

Let $m \geq 1$. It is not difficult to deduce from Theorems 2.3 and 2.5 that $\mathbf{K} \,ⓜ\, \mathbf{BL}_m^{\uparrow} \subseteq \mathbf{BR}_{m+1}^{\uparrow}$ (and it is done explicitly in the proof of Theorem 3.1 below). We prove that the equality actually holds, showing that one can climb in the lattice $\mathcal{L}(\mathbf{DA})$ in a way that directly mimics the steps in the countable lattice $\mathcal{L}(\mathbf{B})$.

Theorem 3.1 *For each* $m \geq 1$, $\mathbf{BR}_{m+1}^{\uparrow} = \mathbf{K} \,ⓜ\, \mathbf{BL}_m^{\uparrow}$ *and* $\mathbf{BL}_{m+1}^{\uparrow} = \mathbf{D} \,ⓜ\, \mathbf{BR}_m^{\uparrow}$

Proof If $m = 1$, the announced equalities are classical results, namely the facts that $\mathbf{R} = \mathbf{K} \,ⓜ\, \mathbf{J}$ and $\mathbf{L} = \mathbf{D} \,ⓜ\, \mathbf{J}$ [13]. Let us now assume that $m \geq 2$.

If M is a band in $\mathbf{K} \,ⓜ\, \mathbf{BL}_m^{\uparrow}$, then Proposition 1.1 shows that $M/{\sim_{\mathbf{K}}} \in \mathbf{BL}_m^{\uparrow}$. Since $M/{\sim_{\mathbf{K}}}$ is a band as well, we have $M/{\sim_{\mathbf{K}}} \in \mathbf{BL}_m$. Moreover, each $\sim_{\mathbf{K}}$-class is a band, and a semigroup in $\mathbf{K}$. Therefore the projection $M \to M/{\sim_{\mathbf{K}}}$ is an $\mathbf{LZ}$-morphism, and $M \in \mathbf{LZ} \,ⓜ\, \mathbf{BL}_m = \mathbf{BR}_{m+1}$. Thus $(\mathbf{K} \,ⓜ\, \mathbf{BL}_m^{\uparrow}) \cap \mathbf{B} \subseteq \mathbf{BR}_{m+1}$, and hence $\mathbf{K} \,ⓜ\, \mathbf{BL}_m^{\uparrow} \subseteq \mathbf{BR}_{m+1}^{\uparrow}$.

Conversely, let us assume that $M \in \mathbf{BR}_{m+1}^{\uparrow}$. By Theorem 2.5 (and with the notation in that statement), M satisfies the pseudo-identity $\varphi(G_{m+1}) = \varphi(I_{m+1})$. We want to show that $M/{\sim_{\mathbf{K}}} \in \mathbf{BL}_m^{\uparrow}$, that is, $M/{\sim_{\mathbf{K}}}$ satisfies $\varphi(\overline{G_m}) = \varphi(\overline{I_m})$.

It is easily verified by induction that the variables which occur in G_m are the same that occur in I_m, namely $x_1, \ldots, x_m$. We need to verify that, for each morphism $\psi\colon \{x_1, \ldots, x_m\}^* \to M$, we have $\psi(\varphi(\overline{G_m})) \sim_{\mathbf{K}} \psi(\varphi(\overline{I_m}))$.

Let $e \in M$ be an idempotent such that $e\,\psi(\varphi(\overline{G_m}))\ \mathcal{J}\ e$. Then each $\psi(x_i)$ $(1 \le i \le m)$ is in M_e, the subsemigroup of M generated by the elements that are $\mathcal{J}$-greater than or equal to e.

Let us extend ψ to $\{x_1, \ldots, x_{m+1}\}^*$ by letting $\psi(x_{m+1}) = e$. Since $\varphi(x_{m+1}) = (x_{m+1}^{\omega}\varphi(\overline{G_m}G_m)^{\omega}x_{m+1}^{\omega})^{\omega}$ and $eM_ee = e$ (Proposition 2.1), we find that $\psi(\varphi(x_{m+1})) = e$. It follows:

$$
\begin{aligned}
e\,\psi(\varphi(\overline{G_m})) &= \psi(\varphi(x_{m+1}\overline{G_m}))\\
&= \psi(\varphi(G_{m+1}))\\
&= \psi(\varphi(I_{m+1})) \quad \text{since } M \text{ satisfies } \varphi(G_{m+1}) = \varphi(I_{m+1}),\\
&= \psi(\varphi(x_{m+1}\overline{G_m}x_{m+1}\overline{I_m}))\\
&= e\,\psi(\varphi(\overline{G_m}))\,e\,\psi(\varphi(\overline{I_m})) \quad \text{since } \psi(\varphi(x_{m+1})) = e,\\
&= e\,\psi(\varphi(\overline{I_m})) \quad \text{since } eM_ee = e.
\end{aligned}
$$

By symmetry, this shows that $\psi(\varphi(\overline{G_m})) \sim_{\mathbf{K}} \psi(\varphi(\overline{I_m}))$, which concludes the proof. □

Consequences of this result are explored in the next section.

4 Applications

4.1 Semigroup-theoretic consequences

An immediate consequence of Theorem 3.1 we want to point out is that we now have explicit pseudo-identities for a number of natural pseudovarieties. For instance, no pseudo-identity was known in the literature for $\mathbf{K} ⓜ \mathbf{L} = \mathbf{BR}_3^{\uparrow}$ (even though [14] gives general tools to compute this type of pseudo-identities). We get

$$\mathbf{K} ⓜ \mathbf{L} = [\![\varphi(G_3) = \varphi(I_3),\ (xy)^{\omega}x(xy)^{\omega} = (xy)^{\omega}]\!],$$

where $\varphi(G_3)$ and $\varphi(I_3)$ were computed in Example 2.6.

For convenience, in the rest of this paper, we write $\mathbf{R}_m$ and $\mathbf{L}_m$ for $\mathbf{BR}_m^{\uparrow}$ and $\mathbf{BL}_m^{\uparrow}$ respectively. As indicated in Sect. 2.3, $\mathbf{R}_1 = \mathbf{L}_1$ is the pseudovariety of $\mathcal{J}$-trivial semigroups, $\mathbf{R}_2$ is the pseudovariety of $\mathcal{R}$-trivial semigroups and $\mathbf{L}_2$ is the pseudovariety of $\mathcal{L}$-trivial semigroups.

We note the following elementary remark (where the case $m = 2$ is well-known, see [13]).

Proposition 4.1 *For each $m \ge 2$, $\mathbf{R}_m \cap \mathbf{L}_m = \mathbf{Nil} ⓜ (\mathbf{R}_m \cap \mathbf{L}_m)$.*

Proof Observe that the operation $\mathbf{V} \mapsto \mathbf{K} ⓜ \mathbf{V}$ is idempotent. In particular, $\mathbf{K} ⓜ (\mathbf{R}_m \cap \mathbf{L}_m) \subseteq \mathbf{K} ⓜ \mathbf{R}_m = \mathbf{R}_m$. Similarly, $\mathbf{D} ⓜ (\mathbf{R}_m \cap \mathbf{L}_m) \subseteq \mathbf{L}_m$, and hence

$$\big(\mathbf{K} ⓜ (\mathbf{R}_m \cap \mathbf{L}_m)\big) \cap \big(\mathbf{D} ⓜ (\mathbf{R}_m \cap \mathbf{L}_m)\big) \subseteq \mathbf{R}_m \cap \mathbf{L}_m.$$

The result follows because $(\mathbf{K} ⓜ \mathbf{V}) \cap (\mathbf{D} ⓜ \mathbf{V}) = (\mathbf{K} \cap \mathbf{D}) ⓜ \mathbf{V}$ [14, Corollary 3.2], and $\mathbf{K} \cap \mathbf{D} = \mathbf{Nil}$. □

We now consider the sequences of pseudovarieties $(\mathbf{R}_m)_m$ and $(\mathbf{L}_m)_m$. It is clear from Theorem 3.1 that $\mathbf{R}_m \subseteq \mathbf{R}_{m+2}$, but we have a stronger result.

Proposition 4.2 *For each $m \geq 1$, we have $\mathbf{R}_m \subseteq \mathbf{R}_{m+1}$ and $\mathbf{L}_m \subseteq \mathbf{L}_{m+1}$. Moreover, both these hierarchies are infinite.*

Proof By Theorem 2.3, we have

$$\mathbf{R}_m \cap \mathbf{B} = \mathbf{BR}_m \subseteq \mathbf{BR}_m \vee \mathbf{BL}_m = \mathbf{BR}_{m+1} \cap \mathbf{BL}_{m+1} \subseteq \mathbf{BR}_{m+1}$$

and hence $\mathbf{R}_m \subseteq \mathbf{BR}_{m+1}^{\uparrow} = \mathbf{R}_{m+1}$. The dual inclusion, namely $\mathbf{L}_m \subseteq \mathbf{L}_{m+1}$ is proved in the same way.

The infinity of either hierarchy is verified by considering the sequences $(\mathbf{R}_m \cap \mathbf{B})_m = (\mathbf{BR}_m)_m$ and $(\mathbf{L}_m \cap \mathbf{B})_m = (\mathbf{BL}_m)_m$: both these hierarchies are known to be infinite. □

Remark 4.3 With the same reasoning, one can show that, if **V** is a pseudovariety containing **SL** and not containing all of **B**, then the sequence of pseudovarieties starting at **V** and obtained by applying alternately the operations $\mathbf{X} \mapsto \mathbf{K} ⓜ \mathbf{X}$ and $\mathbf{X} \mapsto \mathbf{D} ⓜ \mathbf{X}$ are infinite.

Proposition 4.4 *We have $\bigcup_m \mathbf{R}_m = \bigcup \mathbf{L}_m = \mathbf{DA}$. In particular, **DA** is the least pseudovariety of semigroups containing **SL** and closed under the operations $\mathbf{X} \mapsto \mathbf{K} ⓜ \mathbf{X}$ and $\mathbf{X} \mapsto \mathbf{D} ⓜ \mathbf{X}$. In addition, if $M \in \mathbf{DA}$ is m-generated ($m \geq 2$), then M is in the pseudovariety obtained from **SL** by m alternated applications of these operations, starting with a Mal'cev product with **K** (resp. **D**).*

Proof It is immediate that each $\mathbf{R}_m$ and each $\mathbf{L}_m$ is contained in **DA**. Conversely, let $M \in \mathbf{DA}$ and let $m \geq 1$ be such that M is m-generated. By Theorem 2.5(6), $M \in \mathbf{R}_{m+1} \cap \mathbf{L}_{m+1}$. This concludes the proof. □

Remark 4.5 Thérien's and Wilke's work [19] implicitly contains a version of the part of the statement concerning m-generated elements of **DA**, as their proof of the equivalence between **DA**-recognizability and the 2-variable fragment of first-order logic relies on an induction on the cardinality of the alphabet.

Remark 4.6 It is well known that $\mathbf{DA} = \mathbf{\mathit{LI}} ⓜ \mathbf{SL}$ [1, 13, 16], and that $\mathbf{\mathit{LI}} = \mathbf{K} \vee \mathbf{D}$. So Proposition 4.4 states the natural-sounding fact that the closure of **SL** under the repeated application of the (idempotent) operations $\mathbf{X} \mapsto \mathbf{K} ⓜ \mathbf{X}$ and $\mathbf{X} \mapsto \mathbf{D} ⓜ \mathbf{X}$ is the same as its closure under the (idempotent as well) operation $\mathbf{X} \mapsto \mathit{L}\mathbf{I} ⓜ \mathbf{X}$. Yet our proof is very specific for **DA**. It is an interesting question whether this result is in fact more general. We suspect that if $\mathbf{V} \subseteq \mathbf{DS}$ and $\mathbf{V} = \mathbf{K} ⓜ \mathbf{V} = \mathbf{D} ⓜ \mathbf{V}$, then $\mathbf{V} = \mathit{L}\mathbf{I} ⓜ \mathbf{V}$, but that **DS** is the maximal pseudovariety in which this holds.

4.2 Language-theoretic consequences

The language-theoretic corollary we want to record is a simple translation of Proposition 4.4, but one worth noting.

Recall that if A is an *alphabet* (a finite, non-empty set), we denote by A^* the free monoid over A. A *language* $L \subseteq A^*$ is *recognized* by a monoid M if there exists a morphism $\varphi: A^* \to M$ such that $L = \varphi^{-1}(\varphi(L))$. A *class of languages* $\mathcal{V}$ is a collection $\mathcal{V} = (\mathcal{V}(A))_A$, indexed by all finite alphabets A, such that $\mathcal{V}(A)$ is a set of languages in A^*. If $\mathbf{V}$ is a pseudovariety of monoids, we let $\mathcal{V}(A)$ be the set of all languages of A^* which are recognized by a monoid in $\mathbf{V}$. The class $\mathcal{V}$ has important closure properties: each $\mathcal{V}(A)$ is closed under Boolean operations and under taking residuals (if $L \in \mathcal{V}(A)$ and $u \in A^*$, then Lu^{-1} and $u^{-1}L$ are in $\mathcal{V}(A)$); and if $\varphi: A^* \to B^*$ is a morphism and $L \in \mathcal{V}(B)$, then $\varphi^{-1}(L) \in \mathcal{V}(A)$. Classes of recognizable languages with these properties are called *varieties* of languages, and Eilenberg's theorem (see [13]) states that the correspondence $\mathbf{V} \mapsto \mathcal{V}$, from pseudovarieties of monoids to varieties of recognizable languages, is a lattice isomorphism. Moreover, the decidability of membership in the pseudovariety $\mathbf{V}$, implies the decidability of the variety $\mathcal{V}$: indeed, a language is in $\mathcal{V}$ if and only if its (effectively computable) syntactic monoid is in $\mathbf{V}$.

Let K, L be languages in A^* and let $a \in A$. The product KaL is said to be *deterministic* if each word $u \in KaL$ has a unique prefix in Ka. If $k \geq 1$, $L_0, \ldots, L_k$ are languages in A^* and $a_1, \ldots, a_k \in A$, the product $L_0a_1L_1 \cdots a_kL_k$ is said to be *deterministic* if the products $L_{i-1}a_i(L_i \cdots a_nL_n)$ are deterministic, for $1 \leq i \leq n$.

Dually, the product KaL is said to be *co-deterministic* if each word $u \in KaL$ has a unique suffix in aL. The product $L_0a_1L_1 \cdots a_kL_k$ is said to be *co-deterministic* if the products $(L_0a_1 \cdots L_{i-1})a_iL_i$ are co-deterministic, for $1 \leq i \leq n$.

Finally, the product $L_0a_1L_1 \cdots a_kL_k$ is said to be *unambiguous* if every word u in this language admits a unique decomposition in the form $u = u_0a_1u_1 \cdots a_nu_n$ with each $u_i \in L_i$. It is easily verified that a deterministic or co-deterministic product is a particular case of an unambiguous product.

These operations are extended to classes of languages: If $\mathcal{V}$ is a class of languages, let $\mathcal{V}^{det}$ (resp. $\mathcal{V}^{codet}$) denote the class of languages such that, for each alphabet A, $\mathcal{V}^{det}(A)$ (resp. $\mathcal{V}^{codet}(A)$) is the set of all Boolean combinations of languages of $\mathcal{V}(A)$ and of deterministic (resp. co-deterministic) products of languages of $\mathcal{V}(A)$. Let also $\mathcal{V}^{unamb}$ be the class of languages such that, for each alphabet A, $\mathcal{V}^{unamb}(A)$ is the set of all finite unions of unambiguous products of languages of $\mathcal{V}(A)$.

Schützenberger [13, 16] gave algebraic characterizations of the closure operations $\mathcal{V} \longmapsto \mathcal{V}^{det}$, $\mathcal{V} \longmapsto \mathcal{V}^{codet}$ and $\mathcal{V} \longmapsto \mathcal{V}^{unamb}$ for varieties of languages: he showed that $\mathcal{V}^{det}$, $\mathcal{V}^{codet}$ and $\mathcal{V}^{unamb}$ are varieties of languages, and the corresponding pseudovarieties are $\mathbf{K} ⓜ \mathbf{V}$, $\mathbf{D} ⓜ \mathbf{V}$ and $\mathit{L}\mathbf{I} ⓜ \mathbf{V}$, respectively.

Proposition 4.4 now easily translates to the following statement.

Proposition 4.7 *The least variety of languages containing the languages of the form* B^* $(B \subseteq A)$ *and closed under deterministic and co-deterministic product, is the variety corresponding to* $\mathbf{DA_M}$.

More precisely, every unambiguous product of languages $B_1^* a_1 B_2^* \cdots a_k B_{k+1}^*$ *where the* B_i *are subsets of alphabet* A, *can be expressed in terms of the* B_i^* *and the* a_i *using only Boolean operations and at most* $|A|$ *alternated applications of the deterministic and co-deterministic products—starting with a deterministic* (*resp. co-deterministic*) *product.*

Remark 4.8 As in Remark 4.6, it is interesting to note that, while this result (that unambiguous products can be expressed by iterated deterministic and co-deterministic products) sounds natural, its proof is very specific for **SL** and **DA**: does it hold in general?

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References

1. Almeida, J.: Finite Semigroups and Universal Algebra. Singapore, World Scientific (1994)
2. Birjukov, P.A.: Varieties of idempotent semigroups. Algebra Log. **9**, 255–273 (1970)
3. Fennemore, C.F.: All varieties of bands I. Math. Nach. **48**, 237–252 (1971)
4. Fennemore, C.F.: All varieties of bands II. Math. Nach. **48**, 253–262 (1971)
5. Fich, F., Brzozowski, J.: A characterization of a dot-depth two analogue of generalized definite languages. In: Proc. 6th ICALP. LNCS, vol. 71, pp. 230–244. Springer, Berlin (1979)
6. Gerhard, J.A.: The lattice of equational classes of idempotent semigroups. J. Algebra **15**, 195–224 (1970)
7. Gerhard, J.A., Petrich, M.: Varieties of bands revisited. Proc. Lond. Math. Soc. **58**(3), 323–350 (1989)
8. Hall, T.E., Weil, P.: On radical congruence systems. Semigroup Forum **59**, 56–73 (1999)
9. Howie, J.M.: Fundamentals of Semigroup Theory. Oxford University Press, Oxford (1995)
10. Krohn, K., Rhodes, J., Tilson, B.: Homomorphisms and semilocal theory. In: Arbib, M. (ed.) The Algebraic Theory of Machines, Languages and Semigroups. Academic Press, San Diego (1965)
11. Kufleitner, M., Weil, P.: On $\mathbf{FO}^2$ quantifier alternation over words. In: Proceedings of the 34th Symposium on Mathematical Foundations of Computer Science (MFCS 2009). LNCS, vol. 5734, pp. 513–524. Springer, Berlin (2009)
12. Lodaya, K., Pandya, P.K., Shah, S.S.: Marking the chops: an unambiguous temporal logic. In: IFIP TCS 2008, pp. 461–476 (2008)
13. Pin, J.-E.: Varieties of Formal Languages. North Oxford Academic, London (1986)
14. Pin, J.-E., Weil, P.: Profinite semigroups, Mal'cev products and identities. J. Algebra **182**, 604–626 (1996)
15. Reilly, N.R., Zhang, S.: Complete endomorphisms of the lattice of pseudovarieties of finite semigroups. Bull. Aust. Math. Soc. **55**, 207–218 (1997)
16. Schützenberger, M.-P.: Sur le produit de concaténation non ambigu. Semigroup Forum **13**, 47–75 (1976)
17. Tesson, P., Thérien, D.: Diamonds are forever: the variety DA. In: Dos Gomes Moreira Da Cunha, G.M., Ventura Alves Da Silva, P., Pin, J.-E. (eds.) Semigroups, Algorithms, Automata and Languages, Coimbra (Portugal) 2001, pp. 475–500. World Scientific, Singapore (2002)
18. Tesson, P., Thérien, D.: Logic meets algebra: the case of regular languages. Log. Methods Comput. Sci. **3**(1), 1–37 (2007)
19. Thérien, D., Wilke, Th.: Over words, two variables are as powerful as one quantifier alternation. In: STOC, pp. 234–240 (1998)

20. Trotter, P., Weil, P.: The lattice of pseudovarieties of idempotent semigroups and a non-regular analogue. Algebra Univers. **37**, 491–526 (1997)
21. Weil, P.: Some results on the dot-depth hierarchy. Semigroup Forum **46**, 352–370 (1993)
22. Weil, P.: Profinite methods in semigroup theory. Int. J. Algebra Comput. **12**, 137–178 (2002)
23. Weis, Ph., Immerman, N.: Structure theorem and strict alternation hierarchy for FO^2 on words. Log. Methods Comput. Sci. **5**(3), 1–23 (2009)
24. Wismath, S.L.: The lattices of varieties and pseudovarieties of band monoids. Semigroup Forum **33**(2), 187–198 (1986)