Semigroup Forum (2010) 80: 405–415
DOI 10.1007/s00233-009-9197-2

RESEARCH ARTICLE

Stabilization through viscoelastic boundary damping: a semigroup approach

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Received: 15 June 2009 / Accepted: 25 November 2009 / Published online: 12 December 2009

Abstract The undamped linear wave equation on a bounded domain in $\mathbb{R}^n$ with C^2 boundary is considered. The interaction of the interior waves and the viscoelastic boundary material is modeled by convolution boundary conditions. It is assumed that the convolution kernel is integrable and completely monotonic. The main result is that the derivatives of all solutions tend to zero. The proof is given by an application of the Arendt-Batty-Lyubic-Vu Theorem. To this end, the model is reformulated as an abstract first order Cauchy problem in an appropriate Hilbert space, including the memory of the boundary as a state component. It is shown that the differential operator of the Cauchy problem is the generator of a contraction semigroup on the state space by establishing the range condition for the Lumer-Phillips Theorem using

Communicated by Rainer Nagel.

This work was supported by the grant Österreich–Tschechische Republik Kontakt 2004/8; W.D. was supported by Spezialforschungsbereich Optimierung und Kontrolle F-003; E.F. and J.M. were supported by the grant of Ministry of Education, Youth and Sports of the Czech Republic No. MSM 0021620839.

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a generalized Lax-Milgram argument and Fredholm's alternative. Furthermore, it is shown that neither the generator nor its adjoint have purely imaginary eigenvalues.

Keywords Wave equation · Boundary conditions of memory type · Viscous damping · $\mathcal{C}_0$-semigroup · Stability

1 Introduction

Consider a model for the evolution of the sound in a compressible fluid with viscoelastic surface (cf. [7]):

$$\begin{aligned} p_{tt}(t,x) - \Delta p(t,x) &= 0, \quad x \in \Omega,\ t>0, \\ \frac{\partial p}{\partial n}(t,x) + a * p_t(t,x) &= 0, \quad x \in \partial\Omega,\ t>0 \end{aligned} \tag{1}$$

supplemented with initial conditions $p(0,.) = p_0$, $p_t(0,.) = p_1$, where $p(t,x) \in \mathbb{C}$ denotes the acoustic pressure at time t, $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary and $n(x)$ the outer normal to $\partial\Omega$ at x. The convolution is $a * v(t,.) := \int_0^t a(t-s)v(s,.)\,ds$.

Our goal is to prove stabilization through the viscoelastic boundary damping $a * p_t$, i.e. to prove that the derivatives of solutions to (1) converge to zero in the sense $\int_\Omega (|(p_t(t,x)|^2 + |\nabla p(t,x)|^2)\,dx \to 0$ when time tends to infinity.

We will rewrite the model as an evolution problem $x_t = \mathfrak{A}x$, $x(0) = x_0$, with $\mathfrak{A}$ being a generator of a $\mathcal{C}_0$-semigroup on a space $\mathfrak{X}^\perp$, and use the Arendt-Batty-Lyubic-Vu theorem (see e.g. [2, Theorem 5.5.5] or [9]) to obtain the following result.

Theorem 1 *Let $S(t)$ be the semigroup generated by $\mathfrak{A}$. Then $\lim_{t\to\infty} S(t)x = 0$ for all $x \in \mathfrak{X}^\perp$.*

Our approach thus rests on $\mathcal{C}_0$-semigroup theory, and we are not aware of analogous pathways in the theory of integral equations (cf. [7]). It is shown in [3] by methods of complex analysis, that the rate of exponential decay might be arbitrarily low; this is so, because there are eigenvalues arbitrarily close to the imaginary axis.

2 Construction of the semigroup

We will consider the n-dimensional version of (1), i.e. the case when $p(t,x) \in \mathbb{C}^n$. The basic model (1) can be rewritten as the system

$$\begin{aligned} \frac{\partial v}{\partial t} &= \operatorname{div} \sigma \quad \text{in}\, \Omega,\ t>0, \\ \frac{\partial \sigma}{\partial t} &= \nabla v \quad \text{in}\, \Omega,\ t>0, \\ \sigma \cdot n + a * v &= 0 \quad \text{on}\, \partial\Omega,\ t>0, \\ \text{where } v := \frac{\partial p}{\partial t}, &\quad \sigma := \nabla p. \end{aligned} \tag{2}$$

One obtains solutions of (1) by considering just the first component of solutions of (2).

To rewrite the system (2) in the form of an evolution equation we need certain assumptions on the kernel a. We will suppose that

(H1) the kernel a is a completely monotonic function, i.e. there exists a locally finite positive measure $\nu \neq 0$ on $[0, \infty)$ such that

$$a(t) = \int_0^\infty e^{-st} d\nu(s), \quad t > 0;$$

and that

(H2) $a \in L^1(0, \infty)$.

Under these assumptions the Laplace transform $\hat{a}$ of a is given by

$$\hat{a}(\lambda) = \int_0^\infty \frac{1}{\lambda + s} d\nu(s), \quad \Re\lambda > 0,$$

and has a holomorphic extension to $\mathbb{C}\backslash(-\infty, 0]$. We note that $a \in L^1(0, \infty)$ if and only if

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_0^\infty \frac{d\nu(s)}{s} < \infty$$

(see e.g. [10] or [5]). The convolution $a * v$ can be expressed in the following form

$$(a * v)(t, x) = \int_0^t a(t - \tau)(\Gamma v)(\tau, x) d\tau = \int_0^\infty \psi(t, s, x) d\nu(s)$$

on $\partial\Omega$, for any $v \in C(\mathbb{R}^+, H^1(\Omega, \mathbb{C}^n))$, where

$$\psi(t, s, x) := \int_0^t e^{-s(t-\tau)}(\Gamma v)(\tau, x) d\tau,$$

and Γ is the trace operator (see below). Note that ψ satisfies the equation

$$\frac{\partial\psi}{\partial t}(t, s, x) = (\Gamma v)(t, x) - s\psi(t, s, x), \quad t, s, \in \mathbb{R}^+, \ x \in \partial\Omega.$$

We consider the Hilbert space

$$\mathfrak{X} := L^2(\Omega; \mathbb{C}^n) \times L^2(\Omega; \mathbb{C}^{n\times n}) \times L^2_\nu,$$

where we use the abbreviation $L^2_\nu := L^2_\nu(\mathbb{R}^+; L^2(\partial\Omega, \mathbb{C}^n))$. Let

$$Y := \{\sigma \in L^2(\Omega; \mathbb{C}^{n\times n}); \ \mathrm{DIV}\sigma \in L^2(\Omega; \mathbb{C}^{n\times n})\},$$

where the divergence DIV is to be understood in the sense of distributions. Note that for $\sigma \in Y$ there is a generalized trace $\sigma \cdot n$ in the sense that

$$\langle \sigma \cdot n, \Gamma u\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} := \int_\Omega (\mathrm{DIV}\ \sigma)\overline{u} + \int_\Omega \sigma.\overline{\nabla u}, \tag{3}$$

for all $u \in H^1(\Omega;\mathbb{C}^n)$. Here $\Gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, $u \mapsto u|_{\partial\Omega}$ is the trace operator. For details see e.g. [8]. The dot in the last term of (3) denotes the inner product in $\mathbb{C}^{n\times n}$.

We define the operator

$$\widetilde{\mathfrak{A}} : \mathcal{D}(\widetilde{\mathfrak{A}}) \subset \mathfrak{X} \to \mathfrak{X}$$
$$\widetilde{\mathfrak{A}}\begin{pmatrix} v \\ \sigma \\ \psi \end{pmatrix} := \begin{pmatrix} \mathrm{DIV}\,\sigma \\ \nabla v \\ \Gamma v - s\psi(s) \end{pmatrix}$$

with domain

$$\mathcal{D}(\widetilde{\mathfrak{A}}) := \Big\{ (v,\sigma,\psi) \in \mathfrak{X};\ v \in H^1(\Omega;\mathbb{C}^n),\ \sigma \in Y,\ \Gamma v - s\psi(s) \in L^2_\nu$$
$$\text{and } \sigma\cdot n = -\int_0^\infty \psi(s)d\nu(s) \text{ with } \sigma\cdot n \text{ defined by (3)} \Big\}.$$

Note that the assumption $\Gamma v - s\psi(s) =: \widetilde{\psi}(s) \in L^2_\nu$ implies that $\psi \in L^1_\nu$, since we can write $\psi(s) = \frac{\lambda}{\lambda+s}\psi(s) + \frac{\Gamma v}{\lambda+s} - \frac{\widetilde{\psi}(s)}{\lambda+s}$, $\lambda > 0$, and $\frac{1}{\lambda+s} \in L^1_\nu \cap L^2_\nu(\mathbb{R}^+)$.

We will use the Helmholtz orthogonal decomposition

$$L^2(\Omega;\mathbb{C}^{n\times n}) = X \oplus X^\perp,$$

where X is the L^2-closure of such $\sigma \in C^\infty(\overline{\Omega};\mathbb{C}^{n\times n})$ for which $\operatorname{div}\sigma = 0$ and $\sigma \cdot n = 0$ (n is the outer normal to the C^2 boundary $\partial\Omega$). It follows that $\mathrm{DIV}\,\sigma = 0$ for the distributional divergence of $\sigma \in X$. Gradient fields are dense in $X^\perp$. For details, see e.g. [8]. According to this decomposition we have

$$\mathfrak{X} = \mathfrak{X}_0 \oplus \mathfrak{X}^\perp,$$

where

$$\mathfrak{X}_0 = \{0\} \times X \times \{0\}, \qquad \mathfrak{X}^\perp = L^2(\Omega;\mathbb{C}^n) \times X^\perp \times L^2_\nu.$$

Proposition 2 $\mathfrak{X}_0 = \operatorname{Ker}\widetilde{\mathfrak{A}}$, $\widetilde{\mathfrak{A}}(\mathfrak{X}^\perp \cap \mathcal{D}(\widetilde{\mathfrak{A}})) \subset \mathfrak{X}^\perp$.

Proof Only the inclusion $\operatorname{Ker}\widetilde{\mathfrak{A}} \subset \mathfrak{X}_0$ is not obvious. Let $\mathrm{DIV}\,\sigma = 0, \nabla v = 0, \psi(s) = \frac{\Gamma v}{s}$. By the generalized trace and the boundary condition for σ we have

$$0 = \int_\Omega (\mathrm{DIV}\sigma)\overline{v} = \langle \sigma\cdot n, \Gamma v\rangle - \int_\Omega \sigma.\overline{\nabla v} = -\Big\langle \int_0^\infty \psi(s)d\nu(s), \Gamma v\Big\rangle$$
$$= -|\Gamma v|^2 \int_0^\infty \frac{d\nu(s)}{s}.$$

Thus $\Gamma v = 0$ and $\psi = 0$. For $v \in H^1_0$ with $\nabla v = 0$, we obtain that $v = 0$. Moreover, $\mathrm{DIV}\,\sigma = 0$ and $\sigma\cdot n = 0$, i.e. $\sigma \in X$. $\square$

With respect to this proposition we will consider the restriction $\mathfrak{A} := \widetilde{\mathfrak{A}}|_{\mathfrak{X}^\perp}$. Note that

$$\mathcal{D}(\mathfrak{A}) = \Big\{(v, \sigma, \psi);\ v \in H^1(\Omega; \mathbb{C}^n),\ \sigma \in Y \cap X^\perp,\ \psi \in L^2_\nu,$$
$$\Gamma v - s\psi(s) \in L^2_\nu \text{ and } \sigma \cdot n = -\int_0^\infty \psi(s) d\nu(s)\Big\}.$$

Instead of the system (2) we consider the following evolution problem

$$\frac{d}{dt}\begin{pmatrix} v \\ \sigma \\ \psi \end{pmatrix} = \mathfrak{A}\begin{pmatrix} v \\ \sigma \\ \psi \end{pmatrix}, \quad t > 0 \tag{4}$$

on the space $\mathfrak{X}^\perp$. Solutions of (2) with initial condition (v_0, σ_0) correspond to solutions of (4) with initial condition $(v_0, \sigma_0, 0)$.

Theorem 3 *$\mathfrak{A}$ is a generator of a contractive C_0-semigroup on $\mathfrak{X}^\perp$.*

Proof We will show that the assumptions of the Lumer-Phillips theorem (see e.g. [4]) are satisfied.

(i) $\mathcal{D}(\mathfrak{A})$ is dense in $\mathfrak{X}^\perp$.

Let $(v_0, \sigma_0, \psi_0) \in \mathfrak{X}^\perp$. Choose $v \in \mathcal{D}(\Omega, \mathbb{C}^n)$ close to v_0, and $s_0 > 0$ such that

$$\int_{s_0}^\infty |\psi_0(s)|^2 d\nu(s) < \varepsilon.$$

Take $\psi \in L^2_\nu(\mathbb{R}^+, H^{\frac{1}{2}}(\partial\Omega, \mathbb{C}^n))$ such that

$$\psi(s) := \frac{1}{s}\Gamma v \text{ for } s \in [s_0, \infty) \quad \text{and} \quad \int_0^{s_0} |\psi_0(s) - \psi(s)|^2 d\nu(s) < \varepsilon.$$

Then $\Gamma v - s\psi(s) \in L^2_\nu$ and $\int_0^\infty \psi(s) d\nu(s) =: w \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{C}^n)$. Since $\sigma_0 \in X^\perp$, there is a $p_1 \in H^2(\Omega, \mathbb{C}^n)$ such that

$$|\sigma_0 - \nabla p_1|_{L^2(\Omega)} < \varepsilon.$$

Since $\partial\Omega$ is C^2, the Neumann trace operator $\frac{\partial}{\partial n} : H^2(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is surjective (see [6, 8]), so there is a $p_2 \in H^2(\Omega, \mathbb{C}^n)$ and a $q \in \mathcal{D}(\Omega, \mathbb{C}^n)$ such that

$$\frac{\partial p_2}{\partial n} = -\frac{\partial p_1}{\partial n} - w \quad \text{and} \quad |\nabla p_2 - \nabla q|_{L^2(\Omega)} < \varepsilon.$$

For $p = p_1 + p_2 - q$ we have $\frac{\partial p}{\partial n} = -w$. Therefore, $(v, \nabla p, \psi) \in \mathcal{D}(\mathfrak{A})$ and

$$|\sigma_0 - \nabla p| \leq |\sigma_0 - \nabla p_1| + |\nabla p_2 - \nabla q| < 2\varepsilon.$$

(ii) $\mathfrak{A}$ is dissipative.

Indeed,

$$\Re\langle\mathfrak{A}(v,\sigma,\psi),(v,\sigma,\psi)\rangle = \int_\Omega (\mathrm{DIV}\sigma)\overline{v} + \int_\Omega \overline{\sigma}.\nabla v + \int_0^\infty \langle\Gamma v - s\psi(s), \psi(s)\rangle d\nu(s)$$
$$= -\int_0^\infty s|\psi(s)|^2_{L^2(\partial\Omega)} d\nu(s) \le 0,$$

since $s|\psi(s)|^2 = \langle s\psi(s) - \Gamma v, \psi(s)\rangle + \langle \Gamma v, \psi(s)\rangle \in L^1_\nu$.

(iii) $\mathcal{R}(\lambda - \mathfrak{A}) = \mathfrak{X}^\perp$ for $\lambda > 0$.

This range condition can be proved via the method of sesquilinear forms. We do that in several steps. Fix some $\lambda > 0$ and define

$$a(v,u) := \lambda^2 \int_\Omega v\overline{u} + \int_\Omega \nabla v.\overline{\nabla u} + \lambda\hat{a}(\lambda)\int_{\partial\Omega} \Gamma v \overline{\Gamma u}, \tag{5}$$

a continuous bilinear form on $H^1(\Omega;\mathbb{C}^n) \times H^1(\Omega;\mathbb{C}^n)$, and fix $(w,\kappa,\varphi) \in \mathfrak{X}^\perp$ and define

$$F(\overline{u}) := \lambda\int_\Omega w\overline{u} - \int_\Omega \kappa\overline{\nabla u} - \lambda\int_{\partial\Omega}\left[\int_0^\infty \frac{\varphi(s)}{\lambda + s} d\nu(s)\right]\overline{\Gamma u}, \tag{6}$$

a continuous linear form on $H^1(\Omega;\mathbb{C}^n)$. We will decompose

$$a(v,u) = a_1(v,u) + a_2(v,u)$$

according to

$$a_1(v,u) := \langle v,u\rangle_{H^1} + \lambda\hat{a}(\lambda)\int_{\partial\Omega}\Gamma v\overline{\Gamma u},$$
$$a_2(v,u) := (\lambda^2 - 1)\int_\Omega v\overline{u}.$$

The idea is to reformulate the resolvent equation in terms of a and F, to show that a_1 defines a bijective operator and a_2 a compact one and to use the Fredholm alternative.

1st step
Let $(w,\kappa,\varphi) \in \mathfrak{X}^\perp$ and let $\lambda > 0$. We shall prove that the equation

$$(\lambda - \mathfrak{A})\begin{pmatrix} v \\ \sigma \\ \psi \end{pmatrix} = \begin{pmatrix} w \\ \kappa \\ \varphi \end{pmatrix} \tag{7}$$

has a solution (v,σ,ψ) if and only if v satisfies

$$a(v,u) = F(\overline{u}), \text{ for all } u \in H^1(\Omega;\mathbb{C}^n), \tag{8}$$

where F is defined in (6).

Let $(v, \sigma, \psi) \in \mathcal{D}(\mathfrak{A})$ solve (7). Then

$$\lambda v - \mathrm{DIV}\sigma = w, \tag{9}$$

$$\lambda\sigma - \nabla v = \kappa, \tag{10}$$

$$(\lambda + s)\psi(s) - \Gamma v = \varphi(s). \tag{11}$$

By multiplying equation (9) by $\overline{u} \in H^1(\Omega; \mathbb{C}^n)$ and integrating over Ω we get

$$\lambda \int_\Omega v\overline{u} - \int_\Omega (\mathrm{DIV}\sigma)\overline{u} = \int_\Omega w\overline{u}.$$

The second integral can be rewritten with help of the boundary condition for σ and with the substitution

$$\psi(s) = \frac{\Gamma v + \varphi(s)}{\lambda + s} \tag{12}$$

from (11). This yields

$$\int_\Omega (\mathrm{DIV}\sigma)\overline{u} = -\int_{\partial\Omega}\left[\int_0^\infty \frac{\varphi(s)}{\lambda + s} d\nu(s)\right]\overline{\Gamma u} - \hat{a}(\lambda)\int_{\partial\Omega} \Gamma v \overline{\Gamma u} - \int_\Omega \sigma.\overline{\nabla u}.$$

Using (10) for σ we finally obtain (8).

Conversely, let $v \in H^1$ satisfy (8). Define ψ by (12) and $\sigma \in \mathfrak{X}^\perp$ by $\sigma := \frac{\kappa + \nabla v}{\lambda}$. Then $\psi \in L^2_\nu \cap L^1_\nu$ and $\Gamma v - s\psi(s) \in L^2_\nu$. By substitution for ∇v and φ in (8) we have

$$\lambda \int_\Omega v\overline{u} + \int_\Omega \sigma \cdot \overline{\nabla u} + \int_{\partial\Omega}\left[\int_0^\infty \psi(s) d\nu(s)\right]\overline{\Gamma u} = \int_\Omega w\overline{u} \tag{13}$$

for all $u \in H^1(\Omega, \mathbb{C}^n)$. In particular, if $u \in \mathcal{D}(\Omega, \mathbb{C}^n)$, this condition reads as follows

$$\int_\Omega (\lambda v - w)\overline{u} = -\int_\Omega \sigma \overline{\nabla u},$$

i.e. $\mathrm{DIV}\sigma = \lambda v - w \in L^2$ and $\sigma \in Y$. Therefore, from (13) we obtain that

$$-\int_{\partial\Omega}\left[\int_0^\infty \psi(s) d\nu(s)\right]\overline{\Gamma u} = \int_\Omega (\mathrm{DIV}\sigma)\overline{u} + \int_\Omega \sigma \cdot \overline{\nabla u} = \langle \sigma \cdot n, \Gamma u\rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}.$$

Since $\Gamma(H^1(\Omega)) = H^{\frac{1}{2}}(\partial\Omega)$, this implies that $\sigma \cdot n = -\int_0^\infty \psi(s) d\nu(s)$. Therefore $(v, \sigma, \psi) \in \mathcal{D}(\mathfrak{A})$ and (7) holds.

2nd step

Note that since $|a_1(v, u)| \le c|v|_{H^1}|u|_{H^1}$, there exists $T_1 \in \mathcal{L}(H^1, (H^1)^*)$ such that $a_1(v, u) = (T_1 v, \overline{u})_{(H^1)^* \times H^1}$. We will prove that T_1 is a bijection of H^1 onto $(H^1)^*$.

In fact, since

$$|T_1 u|_{(H^1)^*} = \sup_{|v|_{H^1} \le 1} |(T_1 u, v)| \ge \frac{1}{|u|_{H^1}} |(T_1 u, \overline{u})| = \frac{|u|^2_{H^1} + \lambda\hat{a}(\lambda)|\Gamma u|^2}{|u|_{H^1}} \ge |u|_{H^1},$$

T_1 is injective and $\mathcal{R}(T_1)$ is closed. If $\mathcal{R}(T_1) \neq (H^1)^*$, then there is $w \in H^1$, $w \neq 0$, such that $(T_1 u, \overline{w}) = 0$ for all $u \in H^1$. In particular, $0 = (T_1 w, \overline{w}) \geq |w|_1^2$, which is a contradiction.

3rd step

In this step we will prove that either (8) has a unique solution $v \in H^1$ for any $F \in (H^1)^*$ or (8) with $F = 0$ has a nontrivial solution $v \in H^1$.

To show this, define T_2 by

$$a_2(v, u) = (T_2 v, \overline{u})_{(H^1)^* \times H^1}.$$

Since $T_2 \in \mathcal{L}(L^2(\Omega), H^1(\Omega)^*)$ and $H^1(\Omega, \mathbb{C}^n)$ is compactly embedded into $L^2(\Omega, \mathbb{C}^n)$ (Ω is bounded), T_2 is a compact operator from H^1 into $(H^1)^*$. Equation (8) has the equivalent form

$$T_1 v + T_2 v = F,$$

or, using the result of the second step,

$$(I + (T_1)^{-1} T_2) v = (T_1)^{-1} F.$$

This proves the assertion.

4th step

Here we prove that if $a(v, u) = 0$ for all $u \in H^1$, then $v = 0$.

It follows from the first step that $a(v, u) = 0$ for all $u \in H^1$ if and only if

$$\begin{aligned} \lambda v - \mathrm{DIV}\sigma &= 0, \\ \lambda \sigma - \nabla v &= 0, \\ (\lambda + s)\psi(s) - \Gamma v &= 0, \end{aligned}$$

and $(v, \sigma, \psi) \in \mathcal{D}(\mathfrak{A})$. The first and second equations imply that $\lambda^2 v = \triangle v$. In particular, $v \in H^2$. By the second equation and the boundary condition,

$$\begin{aligned} &\frac{1}{\lambda} \frac{\partial v}{\partial n} = \sigma \cdot n = -\hat{a}(\lambda) \Gamma v, \quad \text{i.e.} \\ &\frac{\partial v}{\partial n} + \lambda \hat{a}(\lambda) \Gamma v = 0. \end{aligned}$$

So we have

$$\lambda^2 |v|_{L^2}^2 = \int_\Omega \triangle v \overline{v} = \int_{\partial\Omega} \frac{\partial v}{\partial n} \overline{\Gamma v} - |\nabla v|_{L^2}^2 = -\lambda \hat{a}(\lambda) |\Gamma v|^2 - |\nabla v|^2 \leq 0.$$

Since $\lambda > 0$, this implies that $v = 0$.

Thus we have finished the proof of the range condition, and in turn, the proof of the theorem. $\square$

3 The spectrum of $\mathfrak{A}$

Since the semigroup generated by $\mathfrak{A}$ is contractive, we obtain immediately the corollary.

Corollary 4 $\sigma(\mathfrak{A}) \subset \{\Re\lambda \le 0\}$.

Proposition 5 $\sigma(\mathfrak{A}) \cap (\mathbb{C}\backslash(-\infty, 0]) = P_\sigma(\mathfrak{A})$, *where* P_σ *denotes the point spectrum of the operator.*

Proof This can be proved by examining the part (iii) of the proof of Theorem 3. Indeed, the assertion of the first step remains true for $\lambda \in \mathbb{C}$ provided $\int_0^\infty \frac{d\nu(s)}{\lambda+s} = \hat{a}(\lambda)$ exists and $\int_0^\infty \frac{\varphi(s)}{\lambda+s} d\nu(s)$ exists for any $\varphi \in L^2_\nu$. The second expression exists provided $\frac{1}{\lambda+.} \in L^2_\nu(\mathbb{R}^+, \mathbb{C})$, i.e. $\int_0^\infty \frac{d\nu(s)}{(\lambda+s)^2} = -\widehat{a}'(\lambda)$ exists. This is certainly true for $\lambda \in \mathbb{C}\backslash(-\infty, 0]$.

The estimate in the second step can be improved as follows. Since

$$\lambda\widehat{a}(\lambda) = \int_0^\infty \frac{\lambda}{\lambda+s} d\nu(s) = \int_0^\infty \frac{\Re\lambda(\Re\lambda+s) + \Im^2\lambda + is\Im\lambda}{(\Re\lambda+s)^2 + \Im^2\lambda} d\nu(s),$$

for $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ we obtain that $\lambda\widehat{a}(\lambda) \in \mathbb{C}\backslash(-\infty, 0]$. On the other hand, for this $\lambda\widehat{a}(\lambda)$ one has $\inf_{q\ge 0} |1 + q\lambda\widehat{a}(\lambda)| > 0$, hence there exists a constant $C > 0$ such that

$$|T_1 u|_{(H^1)^*} \ge \frac{1}{|u|_{H^1}} |(T_1 u, \overline{u})| = \frac{1}{|u|_{H^1}} \big| \, |u|^2_{H^1} + \lambda\hat{a}(\lambda)|\Gamma u|^2 \big| \ge C|u|_{H^1}.$$

So the third step can be interpreted as follows: $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ belongs either to the point spectrum of $\mathfrak{A}$ or to the resolvent set of $\mathfrak{A}$. □

Proposition 6 $P_\sigma(\mathfrak{A}) \cap i\mathbb{R} = \emptyset$.

Proof Suppose $i\tau \in P_\sigma(\mathfrak{A})$. As in the above proposition and the fourth step in the proof of Theorem 3, there is a $v \in H^2$ such that

$$-\tau^2 v = \triangle v \quad \text{and} \quad \frac{\partial v}{\partial n} + i\tau\hat{a}(i\tau)\Gamma v = 0.$$

This implies that

$$-\tau^2 |v|^2_{L^2} = \int_\Omega \triangle v \overline{v} = \int_{\partial\Omega} \frac{\partial v}{\partial n} \overline{\Gamma v} - |\nabla v|^2_{L^2} = -i\tau\hat{a}(i\tau)|\Gamma v|^2 - |\nabla v|^2.$$

If $\tau \ne 0$ then, by taking the imaginary part we obtain that $\Gamma v = 0$ and, therefore, $\frac{\partial v}{\partial n} = 0$. This means that v can by extended by zero outside Ω and such an extension solves the equation $-\tau^2 v = \triangle v$ in $\mathbb{R}^n$. But the Laplacian in $\mathbb{R}^n$ has empty point spectrum, i.e. $v = 0$, and also $\psi = 0$ ((v, σ, ψ) as in the first step in the proof of Theorem 3). Moreover, DIV$\sigma = 0$ and $\sigma \cdot n = 0$. Thus $\sigma \in X^\perp \cap X$ and $\sigma = 0$. This proves that $i\tau \notin P_\sigma(\mathfrak{A})$ for $\tau \ne 0$.

Suppose now that $0 \in P_\sigma(\mathfrak{A})$, i.e. that there is a $v \in H^2$ such that $\triangle v = 0$, $\frac{\partial v}{\partial n} = 0$. As above this implies $\nabla v = 0$. We have also

$$\mathrm{DIV}\sigma = 0, \qquad \sigma \cdot n = -\int_0^\infty \psi(s) d\nu(s) = -\hat{a}(0)\Gamma v$$

and

$$0 = \int_\Omega (\mathrm{DIV}\sigma)\overline{v} = -\hat{a}(0)|\Gamma v|^2 - \int_\Omega \sigma.\overline{\nabla v} = -\hat{a}(0)|\Gamma v|^2.$$

Since $\hat{a}(0) = \int_0^\infty \frac{d\nu(s)}{s} \neq 0$, $\Gamma v = 0$. It follows that $v = 0$, $\psi = 0$ and $\sigma \in X^\perp \cap X = \{0\}$, i.e. $0 \notin P_\sigma(\mathfrak{A})$. □

Note that arbitrarily small $\lambda \in (-\infty, 0)$ might be in $\sigma(\mathfrak{A})$ (not necessarily all, if the support of ν has holes). Therefore $\mathfrak{A}$ does not generate a $\mathcal{C}_0$-group (see e.g. [4]).

4 The operator $\mathfrak{A}^*$ and its spectrum

In the Hilbert space $\mathfrak{X}^\perp$ the assertion of Theorem 9 below can be obtained by Corollary V.2.22 of [4] without reference to the adjoint $\mathfrak{A}^*$ of $\mathfrak{A}$. Nevertheless, $\mathfrak{A}^*$ is of interest in itself.

Proposition 7 *The adjoint of the operator $\mathfrak{A}$ is given by*

$$\mathcal{D}(\mathfrak{A}^*) = \Big\{(w, \kappa, \varphi);\ w \in H^1(\Omega; \mathbb{C}^n),\ \kappa \in Y \cap X^\perp,\ \varphi \in L^2_\nu,$$
$$\Gamma w + s\varphi(s) \in L^2_\nu,\ \kappa \cdot n = -\int_0^\infty \varphi(s) d\nu(s)\Big\} \tag{14}$$

and

$$\mathfrak{A}^* \begin{pmatrix} w \\ \kappa \\ \varphi \end{pmatrix} = \begin{pmatrix} -\mathrm{DIV}\,\kappa \\ -\nabla w \\ -\Gamma w - s\varphi(s) \end{pmatrix}. \tag{15}$$

Proof Denote the set on the right-hand side of (14) by $\mathfrak{M}$ and define an operator $\mathfrak{B}$ by the right hand side of (15) with domain $\mathcal{D}(\mathfrak{B}) = \mathfrak{M}$. The isomorphism $(v, \sigma, \psi) \mapsto (-v, \sigma, \psi)$ takes $\mathfrak{A}$ into $\mathfrak{B}$, therefore $\mathfrak{B}$ is m-dissipative and $P_\sigma(\mathfrak{B}) \cap i\mathbb{R} = \emptyset$.

Now we are left to prove that $\mathfrak{B} = \mathfrak{A}^*$. To show the inclusion $\mathfrak{B} \subset \mathfrak{A}^*$, choose $(w, \kappa, \varphi) \in \mathfrak{M}$ and take any $(v, \sigma, \psi) \in \mathcal{D}(\mathfrak{A})$. Then the calculation

$$\left(\mathfrak{A}\begin{pmatrix} v \\ \sigma \\ \psi \end{pmatrix}, \begin{pmatrix} w \\ \kappa \\ \varphi \end{pmatrix}\right)$$
$$= \int_\Omega (\mathrm{DIV}\sigma)\overline{w} + \int_\Omega \nabla v.\overline{\kappa} + \int_{\partial\Omega}\int_0^\infty (\Gamma v - s\psi(s))\overline{\varphi(s)} d\nu(s)$$

$$
\begin{aligned}
&= -\int_{\partial\Omega}\left(\int_0^\infty \psi(s)d\nu(s)\right)\overline{\Gamma w} - \int_\Omega \sigma.\overline{\nabla w} - \int_{\partial\Omega}\Gamma v\left(\overline{\int_0^\infty \varphi(s)d\nu(s)}\right)\\
&\quad - \int_\Omega v\overline{\mathrm{DIV}\kappa} + \int_{\partial\Omega}\int_0^\infty (\Gamma v - s\psi(s))\overline{\varphi(s)}d\nu(s)\\
&= -\int_\Omega v\overline{\mathrm{DIV}\kappa} - \int_\Omega \sigma.\overline{\nabla w} - \int_{\partial\Omega}\int_0^\infty \psi(s)\left(\overline{\Gamma w + s\varphi(s)}\right)d\nu(s)
\end{aligned}
$$

implies that $(w,\kappa,\varphi)\in\mathcal{D}(\mathfrak{A}^\star)$ and proves the inclusion. On the other hand, $\mathfrak{A}^*$ has to be m-dissipative, since it is the adjoint of a generator of a $\mathcal{C}_0$-semigroup of contractions. So $\mathfrak{A}^*$ cannot strictly contain an m-dissipative $\mathfrak{B}$, which proves $\mathcal{D}(\mathfrak{A}^*)=\mathfrak{M}$ and $\mathfrak{B}=\mathfrak{A}^*$. □

Corollary 8 $P_\sigma(\mathfrak{A}^*)\cap i\mathbb{R}=\emptyset$.

5 Conclusion

Now we can obtain Theorem 1 by applying the following theorem.

Theorem 9 [2, *Theorem* 5.5.5] *Let T be a bounded $\mathcal{C}_0$-semigroup on a Banach space X with generator A, and suppose that $\sigma(A)\cap i\mathbb{R}$ is countable. If $P_\sigma(A^*)\cap i\mathbb{R}$ is empty, then $\|T(t)x\|\to 0$ as $t\to\infty$, for each $x\in X$.*

Acknowledgements We thank the referee for his careful reading and his helpful and stimulating comments.

Note added in proof While the present article was under review, we learned about [1] where a similar problem is treated by different methods.

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