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RESEARCH ARTICLE

Continuity and equicontinuity of semigroups on norming dual pairs

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Abstract We study continuity and equicontinuity of semigroups on norming dual pairs with respect to topologies defined in terms of the duality. In particular, we address the question whether continuity of a semigroup already implies (local/quasi) equicontinuity. We apply our results to transition semigroups and show that, under suitable hypothesis on E, every transition semigroup on $C_b(E)$ which is continuous with respect to the strict topology β_0 is automatically quasi-equicontinuous with respect to that topology. We also give several characterizations of β_0-continuous semigroups on $C_b(E)$ and provide a convenient condition for the transition semigroup of a Banach space valued Markov process to be β_0-continuous.

Keywords Norming dual pairs · Transition semigroups · Equicontinuity · Strict topology

Introduction

An object of central interest in the study of Markov processes is the transition semigroup of the process. If the Markov process $(X_t)_{t\geq 0}$ takes values in the measurable space (E,Σ), the state space of the process, then the transition semigroup $\mathbf{T} = (T(t))_{t\geq 0}$ is a positive contraction semigroup on the space $B_b(E)$ of all bounded, measurable functions on E. This semigroup contains all information about the transition probabilities of X_t. More precisely, for $t, s \geq 0$ and $A \in \Sigma$ we have $P(X_{t+s} \in A | X_s = x) = (T(t)\mathbb{1}_A)(x)$.

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Whereas the orbits of the semigroup $\mathbf{T}$ usually bear no continuity properties, often the restriction of $\mathbf{T}$ to certain invariant subspaces is continuous in one way or other. The best known example for this is that of a *Feller semigroup*. Here, E is a locally compact Hausdorff space endowed with the Borel σ-algebra.

If $C_0(E)$, the space of all continuous functions on E which vanish at infinity, is invariant under E, then often $\mathbf{T}|_{C_0(E)}$ is strongly continuous. This can be used to great effect in the study of Markov processes, see [9, 15]. If E is not locally compact or if $C_0(E)$ is not invariant, then one can consider other, invariant subspaces. Of particular interest is the space $C_b(E)$ of all bounded, continuous functions on E. However, even if $C_b(E)$ is invariant under $\mathbf{T}$, the restriction of $\mathbf{T}$ to $C_b(E)$ is in general not strongly continuous. But in many cases, see e.g. [7, 13, 14], the restriction is continuous with respect to the so-called *strict* (or *mixed*) topology β_0, cf. [3, 4, 24] and Sect. 1.3.

For locally compact spaces E, Sentilles [23] has studied β_0-continuous semigroups on $C_b(E)$ in the framework of equicontinuous semigroups on locally convex spaces [25]. Since β_0 agrees with the compact-open topology τ_{co} on $\|\cdot\|_\infty$-bounded subsets of $C_b(E)$, it is also possible to treat β_0-continuous semigroups as τ_{co}-continuous semigroups. This point of view was taken by Cerrai in [5] and led to the concept of bi-continuous semigroups introduced by Kühnemund in [19]. Farkas [10] has used the theory of bi-continuous semigroups to study transition semigroups on $C_b(E)$, where E is a Polish space, i.e. the topology of E is induced by a complete, separable metric. It should be noted that transition semigroups on $C_b(E)$ in general do not satisfy the equicontinuity assumption of [25] with respect to τ_{co}, see [19, Example 6]. However, in the examples in [13, 14] local equicontinuity with respect to β_0 holds.

In this paper, we study continuity and equicontinuity of semigroups in the framework of semigroups on norming dual pairs introduced in [20]. Thus, in addition to a Banach space X, we are given a closed subspace Y of X^*, the norm dual of X, which is norming for X. We then study semigroups $\mathbf{T}$ on X such that the adjoint semigroup $\mathbf{T}^*$ leaves the space Y invariant. In applications to transition semigroups we will choose $X = C_b(E)$, here the state space E is assumed to be a completely regular Hausdorff space, and $Y = \mathcal{M}_0(E)$, the space of all bounded Radon measures on E. In this context the assumption that $\mathbf{T}^*\mathcal{M}_0(E) \subset \mathcal{M}_0(E)$ is quite natural and has a stochastic interpretation. Namely, if $\mathbf{T}$ is the transition semigroup of the Markov process $(X_t)_{t\geq 0}$ and we put $\mathbf{T}' = \mathbf{T}^*|_{\mathcal{M}_0(E)}$, then $\mathbf{T}'$ gives the distribution of the random elements X_t, i.e. if $X_s \sim \mu \in \mathcal{M}_0(E)$ then $X_{t+s} \sim T(t)'\mu$.

In Sects. 2 and 3 we will work on general norming dual pairs and study continuity and equicontinuity with respect to general locally convex topologies defined in terms of the duality, see Sect. 1.1. This generality allows us to consider continuity with respect to various topologies. In particular, if we choose $\tau = \|\cdot\|$, then we obtain strongly continuous semigroups. On the norming dual pair $(C_b(E), \mathcal{M}_0(E))$ not only the strict topology β_0 but also the weak topology $\sigma(C_b(E), \mathcal{M}_0(E))$ is of interest. This topology is connected with the concept of *bounded and pointwise convergence*, see [9, Sect. 3.4]. Priola [21] has used this continuity concept to study transition semigroups.

If we additionally impose certain equicontinuity assumptions, then it is not surprising that we can prove a generation theorem for such semigroups. The more interesting question is whether equicontinuity assumptions are restrictive or whether, at least for certain topologies, these assumptions are satisfied automatically. We will address this question in Sect. 3 and give some abstract examples where this is the case. In Sect. 4 we apply our results to transition semigroups. We will prove that if E is a Polish space, then every β_0-continuous semigroup on $(C_b(E), \mathcal{M}_0(E))$ is locally β_0-equicontinuous. A variant of this result has been obtained independently by Farkas [10]. However, we will also prove that this result remains valid for *positive* semigroups, whenever E is a so-called T-space (see the definition in Sect. 4). In the main result of Sect. 4, Theorem 4.4, we give various equivalent conditions for a semigroup on $(C_b(E), \mathcal{M}_0(E))$ to be β_0-continuous. In the concluding Sect. 5 we discuss several examples and give a convenient condition for the transition semigroup of a Banach space valued Markov process to be β_0-continuous.

1 Preliminaries and notation

1.1 Dual pairs

Throughout this paper we will be working on dual pairs and use locally convex topologies defined in terms of the duality. We briefly recall some results from the theory and fix some notation. Our main reference are Chaps. 20 and 21 of [17]. A *dual pair* is a triple $(X, Y, \langle \cdot\, , \cdot \rangle)$ where X and Y are vector spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\langle \cdot\, , \cdot \rangle$ is a bilinear form from $X \times Y$ to $\mathbb{K}$ which separates points, i.e. $\langle x\, , y \rangle = 0$ for all $x \in X$ implies $y = 0$ and $\langle x\, , y \rangle = 0$ for all $y \in Y$ implies $x = 0$. We may define locally convex topologies on X as follows. If $M \subset Y$ is bounded, i.e. $\sup_{y \in M} |\langle x\, , y \rangle| < \infty$ for all $x \in X$, then $p_M(x) := \sup_{y \in M} |\langle x\, , y \rangle|$ defines a seminorm on X. If $\mathfrak{M}$ is a collection of bounded subsets of Y, then the collection of seminorms $(p_M)_{M \in \mathfrak{M}}$ defines a locally convex topology on X if and only if for every $x \in X$ there exists some $M \in \mathfrak{M}$ such that $p_M(x) \neq 0$ (we say that $\mathfrak{M}$ is *separating*). If $\mathfrak{M}$ is a separating collection of bounded subsets of Y, then $\tau_{\mathfrak{M}}$ denotes the locally convex topology induced by the seminorms $(p_M)_{M \in \mathfrak{M}}$.

A locally convex topology τ on X is called *consistent* if $(X, \tau)' = Y$, i.e. every τ-continuous linear functional φ on X is of the form $\varphi(x) = \langle x\, , y \rangle$ for some $y \in Y$. By the Mackey-Arens theorem, [17, 21.4 (2)], every consistent topology is of the form $\tau_{\mathfrak{M}}$ for a suitable collection $\mathfrak{M}$. Furthermore, there exists a coarsest consistent topology, namely the *weak topology* $\sigma(X, Y) = \tau_{\mathfrak{F}}$, where $\mathfrak{F}$ denotes the collection of all finite subsets of Y, and a finest consistent topology, namely the *Mackey topology* $\mu(X, Y) = \tau_{\mathfrak{K}}$, where $\mathfrak{K}$ denotes the collection of all absolutely convex, $\sigma(Y, X)$-compact subsets of Y. We note that every topology $\tau_{\mathfrak{M}}$ is finer than the weak topology $\sigma(X, Y)$. To simplify notation, we will write σ (resp. μ) for $\sigma(X, Y)$ (resp. $\mu(X, Y)$) and denote σ-convergence on X by $\rightharpoonup$. We will write σ' (resp. μ') for $\sigma(Y, X)$ (resp. $\mu(Y, X)$) and denote σ'-convergence on Y by $\rightharpoonup'$.

1.2 Norming dual pairs

Definition 1.1 A *norming dual pair* is a dual pair $(X, Y, \langle \cdot\, , \cdot \rangle)$ where X and Y are Banach spaces and we have $\|x\| = \sup\{|\langle x\, , y \rangle| : y \in Y, \|y\| \leq 1\}$ and $\|y\| = \sup\{|\langle x\, , y \rangle| : x \in X, \|x\| \leq 1\}$.

In what follows, we will often write (X, Y) instead of $(X, Y, \langle\cdot\,,\,\cdot\rangle)$ if the duality pairing is understood. It is easy to see that if (X, Y) is a norming dual pair, then Y is isometrically isomorphic to a closed subspace of X^*, the norm dual of X. We will often identify Y with this closed subspace of X^*.

It is an easy but crucial consequence of the definition that if (X, Y) is a norming dual pair, then on X and Y the notions of weak (i.e. σ- or σ'-) boundedness and of norm boundedness coincide, cf. [20]. It follows that the norm topology on X is equal to $\tau_{\mathfrak{B}}$, where $\mathfrak{B}$ denotes the collection of all bounded subsets of Y. The norm topology is in general not consistent, but it is finer than any topology $\tau_{\mathfrak{M}}$. It is proved in [20] that a σ-continuous linear operator is automatically $\|\cdot\|$-continuous. Furthermore, a $\|\cdot\|$-continuous linear operator T is σ-continuous if and only if its norm-adjoint T^* leaves the space Y invariant. By [17, 21.4 (6)], a linear operator is σ-continuous if and only if it is μ-continuous. If τ is a consistent locally convex topology, then every τ-continuous linear operator is σ-continuous. The converse is not true in general. If τ is any (not necessarily consistent) locally convex topology on X, we write $L(X, \tau)$ for the algebra of τ-continuous linear operators on X. For $\tau = \|\cdot\|$ we merely write $L(X)$ instead of $L(X, \|\cdot\|)$. If $T \in L(X, \sigma)$, we write T^* for its norm adjoint and T' for its σ-adjoint. Note that $T' = T^*|_Y$.

1.3 The dual pair $(C_b(E), \mathcal{M}_0(E))$

Our main example for applications is the norming dual pair $(C_b(E), \mathcal{M}_0(E))$. Here, E is a completely regular Hausdorff space and $C_b(E)$ denotes the Banach space of all bounded, continuous functions form E to $\mathbb{C}$ endowed with the supremum norm. A positive measure μ, defined on the Borel σ-algebra $\mathcal{B}(E)$, is called a *Radon measure* if for all $A \in \mathcal{B}(E)$, we have $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{compact}\}$. If μ is a complex measure on $\mathcal{B}(E)$, then its total variation $|\mu|$ is defined by $|\mu|(A) = \sup_{\mathcal{Z}} \sum_{B\in\mathcal{Z}} |\mu(B)|$, where the supremum is taken over all finite partitions $\mathcal{Z}$ of A into pairwise disjoint measurable sets. A complex measure μ is called a Radon measure, if $|\mu|$ is a Radon measure. Note that if E is a Polish space, then every measure on $\mathcal{B}(E)$ is a Radon measure. $\mathcal{M}_0(E)$ denotes the Banach space of all bounded Radon measures on E, endowed with the total variation norm $\|\mu\| := |\mu|(E)$. It is proved in [20] that $(C_b(E), \mathcal{M}_0(E))$ is a norming dual pair with respect to the duality $\langle f\,,\,\mu\rangle = \int_E f\,d\mu$. If $T \in L(C_b(E), \sigma)$, then T has the representation $Tf(x) = \int_E f(y)k(x, dy)$. Here, $k(x, \cdot) = T'\delta_x$, where δ_x denotes the Dirac measure in x. We will call k the *kernel associated with* T. The question whether $k(\cdot, A)$ is measurable for all $A \in \mathcal{B}(E)$ is discussed in [20].

The strict topology β_0 on $C_b(E)$ is defined as follows:

Denote by $\mathcal{F}_0(E)$ the space of all bounded functions on E which vanish at infinity, i.e. given $\varepsilon > 0$, we find a compact set $K \subset E$ such that $|f(x)| \le \varepsilon$ for all $x \notin K$. The *strict topology* β_0 on $C_b(E)$ is the locally convex topology generated by the set of seminorms $(p_\varphi)_{\varphi\in\mathcal{F}_0(E)}$, where $p_\varphi(f) := \|\varphi f\|_\infty$.

This definition is taken from [16]. It generalizes the definition given by Buck [3, 4] for locally compact spaces E. By [16, Theorem 7.6.3], $(C_b(E), \beta_0)' = \mathcal{M}_0(E)$, i.e. β_0 is a consistent topology. Furthermore, $(C_b(E), \beta_0)$ is complete if and only if $C(E)$, the space of all continuous functions on E, is complete with respect to τ_{co}, see

Theorems 4 and 9 in Sect. 3.6 of [16]. In particular, if E is a metric space or a locally compact space, then $(C_0(E), \beta_0)$ is complete. Sentilles [24] has considered several strict topologies yielding different spaces of measures as dual spaces. We will recall some results from [24] in Sect. 4.

2 Semigroups and their generators

We now study semigroups on norming dual pairs. As a matter of fact, several interesting properties of such semigroups can be proved without continuity assumptions, merely imposing integrability assumptions. This leads to the concept of integrable semigroups on norming dual pairs. Such semigroups are studied in [20] and we content ourselves with recalling the definition and collecting some of the results from [20] in Propositions 2.3 and 2.4 below.

Definition 2.1 Let (X, Y) be a norming dual pair. A *semigroup* on (X, Y) is a family $\mathbf{T} = (T(t))_{t\geq 0} \subset L(X, \sigma)$ such that

(1) $\mathbf{T}$ is a *semigroup*, i.e. $T(0) = id$ and $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$.
(2) $\mathbf{T}$ is *exponentially bounded*, i.e. there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. We then say that $\mathbf{T}$ is of *type* (M, ω).

A semigroup $\mathbf{T}$ of type (M, ω) is called *integrable* if

(3) for all λ with $\operatorname{Re}\lambda > \omega$, there exists an operator $R(\lambda) \in L(X, \sigma)$ such that

$$\langle R(\lambda)x\ ,\ y\rangle = \int_0^\infty e^{-\lambda t}\langle T(t)x\ ,\ y\rangle dt \quad \forall x \in X, y \in Y. \tag{2.1}$$

In particular, we assume that all the integrals on the right hand side exist. $\mathbf{R} = (R(\lambda))_{\operatorname{Re}\lambda>\omega}$ is called the *Laplace transform of* $\mathbf{T}$.

Remark 2.2 Even though in our definition of a semigroup on (X, Y) only a semigroup on X appears, we are actually dealing with *two* semigroups simultaneously—a semigroup on X and a semigroup on Y. This resembles the situation for transition semigroups which serves as a leitmotif here.

Indeed since $\mathbf{T} \subset L(X, \sigma)$, it follows from the remarks in section 1.2 that $\mathbf{T}' = (T(t)')_{t\geq 0} \subset L(Y, \sigma')$. Since also $\mathbf{R} \subset L(X, \sigma)$, it follows that $\mathbf{T}$ is an integrable semigroup on (X, Y) if and only if $\mathbf{T}'$ is an integrable semigroup on (Y, X). In this case, $\mathbf{R}'$ is the Laplace transform of $\mathbf{T}'$.

Proposition 2.3 *Let* $\mathbf{T}$ *be an integrable semigroup of type* (M, ω) *with Laplace transform* $\mathbf{R}$.

(1) $\mathbf{R}$ *is a pseudoresolvent and every* $R(\lambda)$ *commutes with every* $T(t)$.
(2) *We have* $\|(\operatorname{Re}\lambda - \omega)^k R(\lambda)^k\| \leq M$ *for all* $\operatorname{Re}\lambda > \omega$ *and* $k \in \mathbb{N}$.
(3) *If* $\operatorname{rg}\mathbf{R}$ *is* σ*-dense in* X, *then* $\mathbf{R}$ *determines* $\mathbf{T}$ *uniquely, i.e. if* $\tilde{\mathbf{T}}$ *is a second integrable semigroup on* (X, Y) *having the same Laplace transform* $\mathbf{R}$, *then* $\mathbf{T} = \tilde{\mathbf{T}}$.

Recall that a pseudoresolvent is a map R from some nonempty set $\Omega \subset \mathbb{C}$ to $L(X, \|\cdot\|)$, such that $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$ for all $\lambda, \mu \in \Omega$. It is well known that for a given pseudoresolvent $(R(\lambda))_{\lambda\in\Omega}$ there exists a unique multivalued operator $\mathcal{A}$ such that $R(\lambda) = (\lambda - \mathcal{A})^{-1}$ for all $\lambda \in \Omega$. In particular, the range $\operatorname{rg} R(\lambda)$ and the kernel $\ker R(\lambda)$ are independent of $\lambda \in \Omega$.

The following proposition gives a characterization of the operator $\mathcal{A}$. The integrals appearing are to be understood as Y-integrals, see [20]. More precisely, if $f : I \to X$ is a function defined on some interval I such that $\langle f(\cdot), y\rangle$ is integrable for every $y \in Y$, then the Y-integral $\int_I f(t)dt$ denotes the unique element $\varphi \in Y^*$ such that $\varphi(y) = \int_I \langle f(t), y\rangle dt$ for all $y \in Y$. In the proposition below, $\int_I f(t)dt$ will actually be an element of X which is considered as a closed subspace of Y^*. Hence, $\int_0^t f(t)dt = x \in X$ if and only if $\langle x, y\rangle = \int_I \langle f(t), y\rangle dt$ for all $y \in Y$. However, even if $\int_I f(t)dt \in X$, the integral in general does not exist as a Bochner or as a Pettis integral.

Proposition 2.4 *Let* $\mathbf{T}$ *be an integrable semigroup on the norming dual pair* (X, Y) *with Laplace transform* $R(\lambda) = (\lambda - \mathcal{A})^{-1}$.

(1) *The following are equivalent.*
  (a) $x \in D(\mathcal{A})$ *and* $z \in \mathcal{A}x$;
  (b) *for every* $t > 0$ *and* $y \in Y$ *we have*

$$\int_0^t T(s)zds = T(t)x - x. \tag{2.2}$$

(2) *For* $x \in X$ *and* $t > 0$ *we have* $\int_0^t T(s)xds \in D(\mathcal{A})$ *and*

$$T(t)x - x \in \mathcal{A}\int_0^t T(s)xds.$$

Remark 2.5 It follows from (2.2) that $t \mapsto T(t)x$ is $\|\cdot\|$-continuous for every $x \in D(\mathcal{A})$. To see this, let $x \in D(\mathcal{A})$ and $z \in \mathcal{A}x$ be given. For every $t_0 > 0$ we have $C := \sup_{t\le t_0} \|T(t)\| < \infty$ and hence (2.2) implies that

$$|\langle T(t)x - T(s)x, y\rangle| \le \int_s^t |\langle T(r)z, y\rangle| dr \le |t - s| 2C\|z\| \cdot \|y\|$$

for $t, s \le t_0$ and $y \in Y$. Taking the supremum over $y \in Y$ with $\|y\| \le 1$, $\|\cdot\|$-continuity of $t \mapsto T(t)x$ follows.

Definition 2.6 Let $\mathbf{T}$ be an integrable semigroup on the norming dual pair (X, Y) such that the Laplace transform $\mathbf{R}$ of $\mathbf{T}$ is injective. Then the unique (single valued) operator A such that $R(\lambda) = (\lambda - A)^{-1}$ is called the *generator* of $\mathbf{T}$. In this case we say that $\mathbf{T}$ *has a generator* or that $\mathbf{T}$ is a *semigroup with generator* A.

If τ is a locally convex topology on X, then, as usual, an operator A on X is called *τ-closed* if the graph of A is closed in $X \times X$ with respect to $\tau \times \tau$. If τ is a consistent topology, then, by the Hahn-Banach theorem, an operator A is τ-closed if and only if it is σ-closed. Furthermore, a σ-closed operator is automatically norm closed. For an operator A we denote its resolvent set by $\rho(A)$ and for $\lambda \in \rho(A)$ we write $R(\lambda, A)$ for the resolvent of A in λ. We define

$$\rho_\sigma(A) := \{\lambda \in \rho(A) : R(\lambda, A) \in L(X, \sigma)\}.$$

It is an open question whether $\rho_\sigma(A) = \rho(A)$ for a σ-closed operator A. For a σ-densely defined, σ-closed operator, the σ-adjoint of A is denoted by A'.

Proposition 2.7 *Let* $\mathbf{T}$ *be an integrable semigroup of type* (M, ω) *with generator* A. *Then* A *is a* σ*-closed operator with* $\{\operatorname{Re}\lambda > \omega\} \subset \rho_\sigma(A)$. *Furthermore, for* $\operatorname{Re}\lambda > \omega$ *and* $k \in \mathbb{N}_0$ *we have*

$$\|R(\lambda, A)^k\| \le \frac{M}{(\operatorname{Re}\lambda - \omega)^k}. \tag{2.3}$$

The operator A *is* σ*-densely defined if and only if* $\mathbf{T}'$ *has a generator.*

Proof Since the resolvent of A is the Laplace transform of $\mathbf{T}$ and since the Laplace transform consists of σ-continuous operators, $\{\operatorname{Re}\lambda > \omega\} \subset \rho_\sigma(A)$. In particular, A is σ-closed. Estimate (2.3) follows from Proposition 2.3. Now assume that A is σ-densely defined. In this case, the σ-adjoint A' of A is well defined and $R(\lambda, A)' = R(\lambda, A')$, as is easy to see. Since clearly $\langle x, R(\lambda, A)'y\rangle = \int_0^\infty e^{-\lambda t}\langle x, T(t)'y\rangle dt$ for all $x \in X$ and $y \in Y$, it follows that A' is the generator of $\mathbf{T}'$. Conversely assume that $\mathbf{T}'$ has a generator B. As the Laplace transform of $\mathbf{T}'$ is $R(\lambda, A)'$, we find $R(\lambda, A)' = R(\lambda, B)$. If $y \in Y$ vanishes on $D(A)$, then $0 = \langle R(\lambda, A)x, y\rangle = \langle x, R(\lambda, B)y\rangle$ for all $x \in X$. It follows that $R(\lambda, B)y = 0$. Since $R(\lambda, B)$ is injective by hypothesis, $y = 0$ follows. By the Hahn-Banach theorem, $D(A)$ is σ-dense in X. □

We now turn to continuous semigroups.

Definition 2.8 Let $\mathbf{T}$ be a semigroup on (X, Y) and τ be a locally convex topology on X. Then $\mathbf{T}$ is called τ*-continuous* (*at 0*) if for all $x \in X$ the map $t \mapsto T(t)x$ is τ-continuous (at 0).

Remark 2.9

(1) Using the uniform boundedness principle, it can be shown that if $\mathbf{T}$ is a semigroup on (X, Y) which is σ-continuous at 0, then $\mathbf{T}$ is automatically exponentially bounded, i.e. condition (2) in Definition 2.1 is automatically satisfied, see [20].
(2) In the remainder of this section, we will assume integrability of semigroups, i.e. condition (3) in Definition 2.1, also under continuity assumptions. This is due to the fact that σ-continuity at 0 in general does not imply integrability of a semigroup, see the example in Sect. 5.2. However, it is proved in [20] that if E is a complete metric space, then every semigroup on $(C_b(E), \mathcal{M}_0(E))$ which is σ-continuous at 0 is integrable.

Our definition of the generator via the Laplace transform is in the spirit of [1]. The following theorem shows that, under continuity assumptions, this "integral definition" coincides with the "differential definition" of the generator, see e.g. [8].

Theorem 2.10 *Let* $\mathbf{T}$ *be an integrable semigroup on* (X, Y) *of type* (M, ω) *and* $\mathfrak{M}$ *be a separating collection of bounded subsets of* Y*. If* $\mathbf{T}$ *is* $\tau_{\mathfrak{M}}$*-continuous at* 0*, then* $\tau_{\mathfrak{M}}\text{ - }\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$*. In particular, the Laplace transform of* $\mathbf{T}$ *is injective and* $\mathbf{T}$ *has a generator* A *such that* $D(A)$ *is sequentially* $\tau_{\mathfrak{M}}$*-dense in* X*. Furthermore, the generator is given by*

$$D(A) = \left\{ x \in X : \tau_{\mathfrak{M}}\text{ - }\lim_{h\downarrow 0} \Delta_h x \text{ exists} \right\}, \qquad Ax = \tau_{\mathfrak{M}}\text{ - }\lim_{h\downarrow 0} \Delta_h x,$$

where $\Delta_h x := h^{-1}(T(h)x - x)$*.*

Proof Let $x \in X$, $S \in \mathfrak{M}$ and $\varepsilon > 0$ be given. Since S is bounded, there exists $C > 0$ such that $\|y\| \leq C$ for all $y \in S$. By $\tau_{\mathfrak{M}}$-continuity at 0, there exists $t_0 > 0$ such that $|\langle T(t)x - x\,,\, y\rangle| \leq \varepsilon$ for all $t \leq t_0$ and $y \in S$. Now for $\lambda > \max\{\omega, 0\}$ and $y \in S$ we have

$$\begin{aligned}
\sup_{y\in S} |\langle \lambda R(\lambda)x - x\,,\, y\rangle| &= \sup_{y\in S} \left| \int_0^\infty \left\langle \lambda e^{-\lambda t} T(t)x - \lambda e^{-\lambda t} x\,,\, y \right\rangle dy \right| \\
&\leq \sup_{y\in S} \int_0^{t_0} \lambda e^{-\lambda t} |\langle T(t)x - x\,,\, y\rangle| dt \\
&\quad + \int_{t_0}^\infty \lambda e^{-\lambda t}(1 + Me^{\omega t})C \cdot \|x\| dt \\
&\leq \varepsilon \left(1 - e^{-\lambda t_0}\right) + C \cdot \|x\| \left(e^{-\lambda t_0} + \frac{\lambda \cdot M}{\lambda - \omega} e^{(\omega-\lambda)t_0} \right) \\
&\to \varepsilon
\end{aligned}$$

as $\lambda \to \infty$. Since $S \in \mathfrak{M}$ was arbitrary, the first part is proved.

Now denote the generator of $\mathbf{T}$ (in the sense of Definition 2.6) by B and let A be the operator in the statement. If $x \in D(B)$, then, by Proposition 2.4, we have

$$|\langle \Delta_h x - Bx\,,\, y\rangle| \leq \frac{1}{h} \int_0^h |\langle T(s)Bx - Bx\,,\, y\rangle| ds,$$

for every $y \in Y$. Now let $S \in \mathfrak{M}$ and $\varepsilon > 0$ be given. Choose $t_0 > 0$ such that $p_S(T(s)Bx - Bx) \leq \varepsilon$, for all $0 \leq s \leq t_0$. Then,

$$p_S\left(\Delta_h x - Bx\right) \leq \frac{1}{h} \int_0^h \varepsilon ds = \varepsilon,$$

for all $0 \leq s \leq t_0$. This proves that $x \in D(A)$ and that $Ax = Bx$. Conversely suppose that $x \in D(A)$. Since $\tau_{\mathfrak{M}}$ is finer than σ it follows that $\Delta_h x \rightharpoonup Ax$ as $h \downarrow 0$. Since every operator $T(s)$ is σ-continuous, $T(s)\Delta_h x \rightharpoonup T(s)Ax$ for every $s \geq 0$. Furthermore, $(\Delta_h x)_{h\leq 1}$ is norm bounded. Indeed, for every $y \in Y$, the set $\{\langle \Delta_h x\,,\, y\rangle : h \leq 1\}$ is bounded. Hence, by the uniform boundedness principle, $\sup_{h\leq 1} \|\Delta_h x\|_{Y^*} < \infty$. However, since X embeds isometrically into Y^*, we have $\|\Delta_h x\|_{Y^*} = \|\Delta_h x\|_X$ for every $h > 0$.

Now fix $t > 0$ and $y \in Y$. Put $I_t := \int_0^t T(s)x\,ds$. Then

$$\int_0^t \langle T(s)Ax\,,\, y\rangle ds = \lim_{h\downarrow 0} \int_0^t \langle T(s)\Delta_h x\,,\, y\rangle ds = \lim_{h\downarrow 0} \langle \Delta_h I_t\,,\, y\rangle,$$

by the dominated convergence theorem. Note that $I_t \in D(B)$ and $\langle BI_t\,,\, y\rangle = \langle T(t)x - x\,,\, y\rangle$ by Proposition 2.4. Since $B \subset A$, it follows that

$$\int_0^t \langle T(s)Ax\,,\, y\rangle = \lim_{h\downarrow 0} \langle \Delta I_t\,,\, y\rangle = \langle BI_t\,,\, y\rangle = \langle T(t)x - x\,,\, y\rangle.$$

Thus Proposition 2.4 implies that $x \in D(B)$ and $Bx = Ax$. □

Remark 2.11 Assume in addition to the hypothesis of Theorem 2.10 that the semigroup $\mathbf{T}$ is $\tau_{\mathfrak{M}}$-continuous. Then, arguing similar as in the proof of Theorem 2.10, it is easy to see that $x \in D(A)$ if and only if $t \mapsto T(t)x$ is $\tau_{\mathfrak{M}}$-differentiable. In this case we have $\frac{d}{dt}T(t)x = T(t)Ax$. Note however that $\tau_{\mathfrak{M}}$-continuity at 0 does not imply $\tau_{\mathfrak{M}}$-continuity.

We now give a characterization of continuous semigroups.

Proposition 2.12 *Let (X, Y) be a norming dual pair and $\mathfrak{M}$ be a separating collection of bounded subsets of Y. Furthermore, let $\mathbf{T}$ be an integrable semigroup on (X, Y) with generator. Then the following are equivalent*:

(1) $\mathbf{T}$ *is $\tau_{\mathfrak{M}}$-continuous.*
(2) *For all $t_0 > 0$ and every $x \in X$ the set $\{T(t)x : t \in [0, t_0]\}$ is $\tau_{\mathfrak{M}}$-compact.*
(3) *For some $t_0 > 0$ and every $x \in X$ the set $\{T(t)x : t \in [0, t_0]\}$ is relatively countably $\tau_{\mathfrak{M}}$-compact.*

Proof (1) ⇒ (2) and (2) ⇒ (3) are trivial. For (3) ⇒ (1) suppose that $t \mapsto T(t)x$ is not $\tau_{\mathfrak{M}}$-continuous at $t \in [0, t_0]$. Then there exists a $\tau_{\mathfrak{M}}$-continuous seminorm p, an $\varepsilon > 0$ and a sequence $(t_n) \subset [0, t_0]$ converging to t such that $p(T(t_n)x - T(t)x) \geq \varepsilon$ for all $n \in \mathbb{N}$. By hypothesis, the sequence $T(t_n)x$ has an accumulation point $z \in X$. Thus there exists a subnet t_α of t_n such that $T(t_\alpha)x \stackrel{\tau_{\mathfrak{M}}}{\to} z$. Since $\tau_{\mathfrak{M}}$ is finer than σ we have $T(t_\alpha)x \rightharpoonup z$. As $R(\lambda)$ is σ-continuous and commutes with the semigroup, we have

$$R(\lambda)z = \sigma\text{ - }\lim R(\lambda)T(t_\alpha)x = \sigma\text{ - }\lim T(t_\alpha)R(\lambda)x = T(t)R(\lambda)x = R(\lambda)T(t)x,$$

since $s \mapsto T(s)R(\lambda)x$ is $\|\cdot\|$-continuous. As $R(\lambda)$ is injective, it follows that $z = T(t)x$. But then $p(T(t_\alpha)x - T(t)x) \to 0$, a contradiction. This proves that $t \mapsto T(t)x$ is $\tau_{\mathfrak{M}}$-continuous on $[0, t_0]$. Using the semigroup law and that $\{T(t)T(t_0)x : t \in$

$[0, t_0]\}$ is relatively countably compact, it follows that $t \mapsto T(t)x$ is $\tau_{\mathfrak{M}}$-continuous on $[0, 2t_0]$. Inductively we obtain continuity for all times. $\square$

3 Equicontinuity

In the context of semigroups, several types of equicontinuity assumptions have been discussed in the literature. We briefly recall the definitions.

Definition 3.1 Let (X, τ) be a locally convex space. A set $\mathcal{S} \subset L(X, \tau)$ is called *equicontinuous*, if for every τ-continuous seminorm p, there exists a τ-continuous seminorm q such that $p(Tx) \le q(x)$ for all $x \in X$ and $T \in \mathcal{S}$. A semigroup $\mathbf{T}$ of τ-continuous operators is called *locally τ-equicontinuous*, if $\{T(t) : t \in [0, t_0]\}$ is τ-equicontinuous for all $t_0 > 0$. It is called *(globally) τ-equicontinuous*, if $\{T(t) : t \ge 0\}$ is τ-equicontinuous. If for some $\alpha \in \mathbb{R}$ the rescaled semigroup $\mathbf{T}_\alpha := (e^{-\alpha t}T(t))_{t\ge 0}$ is τ-equicontinuous, then $\mathbf{T}$ is called $_\alpha$*quasi-τ-equicontinuous*. We will say that $\mathbf{T}$ is *quasi-τ-equicontinuous*, if it is $_\alpha$quasi-τ-equicontinuous for some $\alpha \in \mathbb{R}$.

Obviously, every quasi-τ-equicontinuous semigroup (in particular, every τ-equicontinuous semigroup) is locally equicontinuous. The converse is not true in general.

Example 3.2 Consider the norming dual pair $(C_b(\mathbb{R}), \mathcal{M}_0(\mathbb{R}))$. The compact open topology τ_{co} is of the form $\tau_{\mathfrak{M}}$. More precisely, $\mathfrak{M}$ is the separating collection of sets of the form $\{\delta_x : x \in K\}$, where δ_x denotes the Dirac measure in x and K is a compact subset of $\mathbb{R}$. The shift semigroup $\mathbf{T}$, defined by $T(t)f(x) = f(x+t)$ is locally τ_{co}-equicontinuous but not quasi-τ_{co}-equicontinuous.

Proposition 3.3 *Let $\mathbf{T}$ be an integrable semigroup on (X, Y) with generator A and $\mathfrak{M}$ be a separating collection of bounded subsets of Y. If $\mathbf{T}$ is locally $\tau_{\mathfrak{M}}$-equicontinuous and $D(A)$ is $\tau_{\mathfrak{M}}$-dense in X, then $\mathbf{T}$ is $\tau_{\mathfrak{M}}$-continuous.*

Proof We first prove that $X_0 := \{x \in X : t \mapsto T(t)x \text{ is } \tau_{\mathfrak{M}}\text{-continuous}\}$ is $\tau_{\mathfrak{M}}$-closed in X. Let x be an accumulation point of X_0, $t_0 \ge 0$ and p be a $\tau_{\mathfrak{M}}$-continuous seminorm. Pick a $\tau_{\mathfrak{M}}$-continuous seminorm q such that $p(T(t)z) \le q(z)$ for all $t \in [0, t_0 + 1]$ and $z \in X$. Given $\varepsilon > 0$, we find $x_0 \in X_0$ such that $q(x - x_0) \le \varepsilon$. Since $x_0 \in X_0$, there exists $0 < \delta < 1$ such that $p(T(t_0)x_0 - T(t)x_0) \le \varepsilon$ for all $|t - t_0| \le \delta$. Now

$$\begin{aligned}p(T(t_0)x - T(t)x) &\le p(T(t_0)x - T(t_0)x_0) + p(T(t_0)x_0 - T(t)x_0)\\&\quad + p(T(t)x_0 - T(t)x)\\&\le 2q(x - x_0) + p(T(t_0)x_0 - T(t)x_0) \le 3\varepsilon,\end{aligned}$$

for all $|t - t_0| \le \delta$. This proves that $x \in X_0$ whence X_0 is $\tau_{\mathfrak{M}}$-closed. For $x \in D(A)$, $t \mapsto T(t)x$ is $\|\cdot\|$-continuous and hence $\tau_{\mathfrak{M}}$-continuous. Thus $D(A) \subset X_0$. As $D(A)$ is $\tau_{\mathfrak{M}}$-dense, $X_0 = X$ follows. $\square$

For $\tau_{\mathfrak{M}} = \|\cdot\|$, we note that local norm-equicontinuity of a semigroup is equivalent with exponential boundedness. Hence from Theorem 2.10 and Proposition 3.3 we immediately obtain the following characterization.

Corollary 3.4 *Let* $\mathbf{T}$ *be an integrable semigroup on* (X, Y). *The following are equivalent*:

(1) $\mathbf{T}$ *is strongly continuous*;
(2) $\mathbf{T}$ *has a* $\|\cdot\|$*-densely defined generator.*

For quasi-equicontinuous semigroups, we obtain the following generation result.

Theorem 3.5 *Let* (X, Y) *be a norming dual pair,* τ *be a consistent topology on* X *which is sequentially complete, and* A *be a* σ*-closed operator on* X. *The following are equivalent.*

(1) *A is the generator of a τ-continuous, ${}_{\alpha}$quasi-τ-equicontinuous, integrable semigroup* $\mathbf{T}$ *on* (X, Y) *of type* (M, ω);
(2) *A is a sequentially τ-densely defined operator such that*
   (a) $\{\lambda \in \mathbb{R} : \lambda > \omega\} \subset \rho_\sigma(A)$ *and*

$$\|(\lambda - \omega)^k R(\lambda, A)^k\| \leq M \quad \forall \lambda > \omega, k \in \mathbb{N}.$$

   (b) *The set*

$$\left\{(\lambda - \alpha)^k R(\lambda, A)^k : \lambda > \alpha, k \in \mathbb{N}\right\}$$

   is τ-equicontinuous.

Proof (1) $\Rightarrow$ (2): A is sequentially τ-densely defined by Theorem 2.10. Condition (2)(a) follows directly from Proposition 2.7. As a resolvent, $R(\lambda, A)$ satisfies $\frac{d^k}{d\lambda^k} R(\lambda, A) = (-1)^k k! R(\lambda, A)^{k+1}$. Interchanging differentiation and integration in the formula for the Laplace transform yields

$$\big\langle R(\lambda, A)^k x\,,\, y\big\rangle = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \langle T(t)x\,,\, y\rangle dt, \tag{3.1}$$

for all $x \in X$ and $y \in Y$. Now let p be a τ-continuous seminorm. Since $\{e^{-\alpha t} T(t) : t \geq 0\}$ is τ-equicontinuous, we find a τ-continuous seminorm q such that $p(e^{-\alpha t} T(t)x) \leq q(x)$ for all $t \geq 0$ and $x \in X$. Since τ is consistent, it follows that $\tau = \tau_{\mathfrak{M}}$ for a suitable separating collection of bounded subsets of Y. We may thus assume that $p = p_S$ and $q = p_R$ for certain $S, R \in \mathfrak{M}$. For $y \in S$, $k \in \mathbb{N}$ and $\lambda > \alpha$ we obtain from (3.1)

$$\begin{aligned}
\big|\big\langle(\lambda - \alpha)^k R(\lambda, A)^k x\,,\, y\big\rangle\big| &\leq \frac{(\lambda - \alpha)^k}{(k-1)!} \int_0^\infty t^{k-1} e^{(\alpha - \lambda)t} \big|\big\langle e^{\alpha t} T(t)x\,,\, y\big\rangle\big| dt \\
&\leq \frac{(\lambda - \alpha)^k}{(k-1)!} \int_0^\infty t^{k-1} e^{(\alpha - \lambda)t} q(x) dt \\
&= q(x).
\end{aligned}$$

Taking the supremum over $y \in S$, (2)(b) follows.

(2) $\Rightarrow$ (1) Let $B = A - \alpha$. Since τ is sequentially complete, it follows from (2)(b) and the theorem in Sect. IX.7 of [25] that B generates a τ-equicontinuous, τ-continuous semigroup $\mathbf{S}$ on X. Since τ is a consistent topology, $L(X, \tau) \subset L(X, \sigma)$ and hence $\mathbf{S}$ is a semigroup on (X, Y) (note that $\mathbf{S}$ is exponentially bounded by Remark 2.9(1)). Furthermore,

$$R(\lambda, B) = R\text{-}\int_0^\infty e^{-\lambda t} S(t)x\,dt,$$

where $R\text{-}\int_0^\infty$ denotes the improper Riemannian integral with respect to τ. However, since the map $x \mapsto \langle x\, ,\, y\rangle$ is τ-continuous for every $y \in Y$, it follows that $\langle R(\lambda, B)x\, ,\, y\rangle = \int_0^\infty e^{-\lambda t}\langle S(t)x\, ,\, y\rangle dt$ for all $x \in X$ and $y \in Y$. Thus, $\mathbf{S}$ is an integrable semigroup on (X, Y) with generator B. Now put $T(t) = e^{\alpha t} S(t)$. It is routine to check that $\mathbf{T}$ is an integrable semigroup with generator A. It remains to prove that $\mathbf{T}$ is of type (M, ω). To that end, consider the rescaled semigroup $\mathbf{T}_\omega$. Note that the generator of $\mathbf{T}_\omega$ is $A - \omega =: C$. Now fix $x \in X$ and $y \in Y$. The function $\varphi_{x,y} : t \mapsto \langle e^{-\omega t} T(t)x, y\rangle$ has Laplace transform $\langle R(\lambda, C)x\, ,\, y\rangle$. Since $\varphi_{x,y}$ is continuous, every point $t \geq 0$ is a Lebesgue point of $\varphi_{x,y}$ and we infer from the Post-Widder inversion formula [1, Theorem 1.7.7] and the equation $\frac{d^k}{d\lambda^k} R(\lambda, C) = (-1)^k k! R(\lambda, C)^{k+1}$ that

$$\varphi_{x,y}(t) = \lim_{n\to\infty} \left\langle \left[\frac{n}{t} R\left(\frac{n}{t}, C\right)\right]^n x, y \right\rangle \quad \forall t \geq 0.$$

However, since $\|\lambda^n R(\lambda, C)^n\| \leq M$, it follows that $|\langle e^{-\omega t} T(t)x, y\rangle| \leq M\|x\| \cdot \|y\|$. Since Y is norming for X, it follows that $\mathbf{T}$ has type (M, ω). $\square$

Remark 3.6

(1) The assumption that τ is sequentially complete and consistent is not needed in the implication (1) $\Rightarrow$ (2) in Theorem 3.5. In the implication (2) $\Rightarrow$ (1), the sequential completeness is needed to apply the Theorem from [25], whereas the consistency was used to ensure that $L(X, \tau) \subset L(X, \sigma)$.
(2) The proof of Theorem 3.4 shows that if τ is sequentially complete and consistent, then a τ-continuous, quasi-τ-equicontinuous semigroup is integrable.

The question remains whether quasi-equicontinuity is a mere technical assumption in order to prove a Hille-Yosida type theorem or whether there are interesting cases in which continuity in a certain topology already implies quasi-equicontinuity. In [18] it is proved that on a barreled locally convex space (X, τ) (recall that (X, τ) is called *barreled* if every absorbing, absolutely convex and closed subset of X is a τ-neighborhood of zero) every τ-continuous semigroup is locally τ-equicontinuous. The following proposition shows that on a norming dual pair *consistent* topologies are never barreled, except when the norm topology is consistent. The special case $(X, Y) = (C_b(E), \mathcal{M}_0(E))$ was considered in [24, Theorem 4.8].

Proposition 3.7 *Let (X, Y) be a norming dual pair and τ be a consistent topology on X. The following are equivalent.*

(1) (X, τ) *is barreled*;
(2) $\tau = \|\cdot\|$ *and thus* $Y = X^*$.

Proof (1) $\Rightarrow$ (2) If (X, τ) is barreled, then every weakly bounded subset of $Y = (X, \tau)'$ is relatively σ'-compact and $\tau = \mu$, see [17, 21.4 (4)]. However, if every weakly bounded subset of Y is relatively σ'-compact, then $\|\cdot\| = \sup_{\{y:\|y\|\leq 1\}} |\langle \cdot\,,\, y\rangle|$ is a μ-continuous seminorm and hence $\mu = \|\cdot\|$.

(2) $\Rightarrow$ (1) Is clear, since every normed space is barreled. □

However also in our general setting there are interesting examples in which continuity with respect to $\tau_{\mathfrak{M}}$ of a semigroup on (X, Y) implies quasi-$\tau_{\mathfrak{M}}$-equicontinuity. We begin with the following

Lemma 3.8 *Let* (X, Y) *be a norming dual pair and* $\mathfrak{M}$ *be a separating collection of* σ'*-compact subsets of* Y*. Further, let* Ω *be a compact Hausdorff space and* $F : \Omega \to L(X, \sigma)$ *be strongly* $\tau_{\mathfrak{M}}$*-continuous. Then for every* $\mathcal{K} \in \mathfrak{M}$ *the set*

$$\mathcal{L}_{\mathcal{K}} := \{F(t)'y : t \in \Omega,\, y \in \mathcal{K}\}$$

is σ'*-compact. If for every such* $\mathcal{L}_{\mathcal{K}}$ *there exists a set* $\mathcal{K}_0 \in \mathfrak{M}$ *such that* $\mathcal{L}_{\mathcal{K}} \subset \mathcal{K}_0$*, then* $\{F(t) : t \in \Omega\}$ *is* $\tau_{\mathfrak{M}}$*-equicontinuous.*

Proof We fix $\mathcal{K} \in \mathfrak{M}$ and write for simplicity $\mathcal{L}$ instead of $\mathcal{L}_{\mathcal{K}}$. Let a net $z_\alpha = F(t_\alpha)'y_\alpha \in \mathcal{L}$ be given. Since Ω is compact, there exists a subnet t_β and some t_0 such that $t_\beta \to t_0$ in Ω. Since $\mathcal{K}$ is compact, there is a subnet y_γ of y_β and an element $y_0 \in \mathcal{K}$ such that $y_\gamma \rightharpoonup' y_0$. We claim that $z_\gamma = F(t_\gamma)y_\gamma \rightharpoonup' z_0 := F(t_0)y_0$. To see this, let $x \in X$ be given. Then

$$\begin{aligned}
\big|\big\langle x\,,\, z_\gamma - z_0\big\rangle\big| &\leq \big|\big\langle F(t_\gamma)x - F(t_0)x\,,\, y_\gamma\big\rangle\big| + \big|\big\langle F(t_0)x\,,\, y_\gamma - y_0\big\rangle\big| \\
&\leq p_{\mathcal{K}}(F(t_\gamma)x - F(t_0)x) + \big|\big\langle F(t_0)x\,,\, y_\gamma - y_0\big\rangle\big| \to 0
\end{aligned}$$

by the $\tau_{\mathfrak{M}}$-continuity of $F(\cdot)x$ and the weak convergence of y_γ. This shows that $\mathcal{L}$ is σ'-compact. We now prove the addendum. If $\mathcal{L} \subset \mathcal{K}_0 \in \mathfrak{M}$, then

$$p_{\mathcal{K}}(F(t)x) = \sup_{y\in\mathcal{K}} \big|\big\langle x\,,\, F(t)'y\big\rangle\big| \leq \sup_{y\in\mathcal{L}} |\langle x\,,\, y\rangle| \leq p_{\mathcal{K}_0}(x)$$

for all $x \in X$ and $t \in \Omega$. Hence, if for every $\mathcal{K} \in \mathfrak{M}$ we find a set $\mathcal{K}_0 \in \mathfrak{M}$ such that the above holds, it follows that $F(\Omega)$ is $\tau_{\mathfrak{M}}$-equicontinuous. □

We immediately obtain:

Theorem 3.9 *Let* (X, Y) *be a norming dual pair and let* $\tau_c := \tau_{\mathfrak{C}}$*, where* $\mathfrak{C}$ *is the collection of all* σ'*-compact subsets of* Y*. If* $\mathbf{T}$ *is a semigroup of type* (M, ω) *which is* τ_c*-continuous, then* $\mathbf{T}$ *is* ${}_\alpha$*quasi-*τ_c*-equicontinuous for every* $\alpha > \omega$*.*

Proof For $\alpha > \omega$ and $x \in X$ we have $e^{-\alpha t}T(t)x \to 0$ in norm and hence with respect to the coarser topology τ_c. It follows that the map

$$F : [0,\infty] \to L(X,\sigma), \qquad F(t) = \begin{cases} e^{-\alpha t}T(t), & t \in [0,\infty), \\ 0, & t = \infty \end{cases}$$

is strongly τ_c-continuous. Now the statement follows from Lemma 3.8. □

We note that the topology τ_c is in general not consistent. However, it can happen that the Mackey topology μ coincides with this topology ([6, 24] then say μ is the *strong* Mackey topology of the pair (X,Y)). This is the case if and only if for every σ'-compact subset $\mathcal{K}$ of Y also its σ'-closed, absolutely convex hull $\overline{\mathrm{aco}}\,\mathcal{K}$ is σ'-compact. By Krein's theorem [17, 24.5 (4)], if $\mathcal{K}$ is σ'-compact then $\overline{\mathrm{aco}}\,\mathcal{K}$ is σ'-compact if and only if $\overline{\mathrm{aco}}\,\mathcal{K}$ is μ'-complete. In particular, if μ' is quasicomplete, i.e. μ' is complete on every bounded, μ'-closed subset of X, then every μ-continuous semigroup on X is quasi-μ-equicontinuous.

Corollary 3.10 *If (X,Y) is a norming dual pair such that μ' is quasicomplete, then every μ-continuous semigroup $\mathbf{T}$ on (X,Y) is quasi-μ-equicontinuous. In particular*

(1) *If $\mathbf{T}$ is a norm continuous semigroup on a Banach space X, then its adjoint semigroup $\mathbf{T}^*$ on X^* is $\mu(X^*,X)$-continuous if and only if it is quasi-$\mu(X^*,X)$-equicontinuous.*
(2) *If E is a completely regular Hausdorff space such that $(C_b(E),\beta_0)$ is complete, then every $\mu(\mathcal{M}_0(E),C_b(E))$-continuous integrable semigroup on $(\mathcal{M}_0(E), C_b(E))$ is quasi-$\mu(\mathcal{M}_0(E),C_b(E))$-equicontinuous.*

Proof The proof of the general statement was explained above. For (1) we note that $\mu' = \mu(X,X^*) = \|\cdot\|$ is complete whence every $\mu(X^*,X)$-continuous adjoint semigroup is quasi-$\mu(X^*,X)$-equicontinuous. The converse follows from Proposition 3.3 since adjoint semigroups have a $\sigma(X^*,X)$-densely defined generator and hence, by the Hahn-Banach theorem, a $\mu(X^*,X)$-densely defined generator. For (2) observe that, as a consequence of Grothendieck's completeness theorem [17, 21.9 (4)], the Mackey topology $\mu(C_b(E),\mathcal{M}_0(E))$ is complete, since there exists a complete, consistent topology, namely β_0, on $C_b(E)$. □

We will now apply Lemma 3.8 in the context of positive semigroups. We introduce the following notation. An *ordered norming dual pair* is a norming dual pair (X,Y) where X is an ordered Banach space with σ-closed positive cone X^+. Note that in this case the dual cone $Y^+ := \{y \in Y : \langle x\,,\,y\rangle \geq 0\ \forall x \in X^+\}$ is σ'-closed. As usual, we call $T \in L(X,\sigma)$ *positive* if $TX^+ \subset X^+$. Note that in this case also $T'Y^+ \subset Y^+$.

Theorem 3.11 *Let (X,Y) be an ordered norming dual pair and τ_+ be the topology of uniform convergence on the σ'-compact subsets of Y^+. If $\mathbf{T}$ is a positive, τ_+-continuous semigroup of type (M,ω) on (X,Y), then $\mathbf{T}$ is ${}_{\alpha}$quasi- τ_+-equicontinuous for every $\alpha > \omega$.*

Proof This follows from Lemma 3.8 as in the proof of Theorem 3.9, noting that for $\alpha > \omega$ and $\mathcal{K} \subset Y^+$ the set $\{e^{-\alpha t}T(t)'y : t \geq 0, y \in \mathcal{K}\}$ is not only compact but also a subset of Y^+ by the positivity of the operators $T(t)$. □

4 Applications to transition semigroups

In this section, we apply the results of the previous sections to semigroups on the norming dual pair $(C_b(E), \mathcal{M}_0(E))$. Here, and throughout this section, E will be a completely regular Hausdorff space. The consistent topology we are interested in is the strict topology β_0. In order to apply our results, we need additional information about β_0 and the dual pair $(C_b(E), \mathcal{M}_0(E))$.

It is well known, see [17, 21.3 (2)], that if (X, τ) is any locally convex space then τ is the topology of uniform convergence on the τ-equicontinuous subsets of $(X, \tau)'$. For the strict topology, we have the following description of the β_0-equicontinuous subsets of $\mathcal{M}_0(E)$.

Theorem 4.1 (Sentilles [24, Theorem 5.1]) *A set $\mathcal{H} \subset \mathcal{M}_0(E)$ is β_0-equicontinuous if and only if* (1) $\sup_{\mu\in\mathcal{H}} |\mu|(E) < \infty$ *and* (2) *for every* $\varepsilon > 0$ *there exists a compact set $K \subset E$ such that $|\mu|(E \setminus K) \leq \varepsilon$ for all $\mu \in \mathcal{H}$.*

Condition (2) says that $\mathcal{H}$ is a *tight* set of measures. From Theorem 4.1 we infer the following description of β_0-equicontinuous sets of linear operators.

Proposition 4.2 *Let $\mathcal{S} = \{T_\alpha : \alpha \in I\} \subset L(C_b(E), \sigma)$ be a bounded family of operators on $C_b(E)$ with associated kernels p_α. The following are equivalent.*

(1) *$\mathcal{S}$ is β_0-equicontinuous*;
(2) *given a compact subset $K \subset E$ and* $\varepsilon > 0$, *there exists a compact subset L of E such that*

$$|p_\alpha|(x, E \setminus L) \leq \varepsilon \quad \forall x \in K, \alpha \in I.$$

Proof (1) ⇒ (2): Let $K \subset E$ be compact. Then $\mathcal{K} := \{\delta_x : x \in K\}$ is β_0-equicontinuous by Theorem 4.1. In particular, $p_{\mathcal{K}}$ is a β_0-continuous seminorm. Since $\mathcal{S}$ is β_0-equicontinuous, we find a β_0-continuous seminorm q such that $p_{\mathcal{K}}(T_\alpha f) \leq q(f)$ for all $f \in C_b(E)$ and $\alpha \in I$. Note that $q = p_{\mathcal{L}}$ for some β_0-equicontinuous set $\mathcal{L}$. We may assume that $\mathcal{L}$ is σ'-closed and absolutely convex. But then it follows that $p_\alpha(x, \cdot) = T'_\alpha \delta_x \in \mathcal{L}$ for all $x \in K$. Indeed, if $T'_{\alpha_0}\delta_{x_0} \notin \mathcal{L}$ for some $\alpha_0 \in I$ and $x_0 \in K$ then, as a consequence of the Hahn-Banach theorem, we can strictly separate $T'_{\alpha_0}\delta_{x_0}$ from $\mathcal{L}$, i.e. we find $f \in C_b(E) = (\mathcal{M}_0(E), \sigma')'$ and $\varepsilon > 0$ such that $|\langle f, \mu\rangle| + \varepsilon \leq |\langle f, T'_{\alpha_0}\delta_{x_0}\rangle|$ for all $\mu \in \mathcal{L}$. But then $p_{\mathcal{K}}(T_{\alpha_0} f) \geq p_{\mathcal{L}}(f) + \varepsilon$, a contradiction to the choice of $\mathcal{L}$. Hence, the set $\{p_\alpha(x, \cdot) : \alpha \in I, x \in K\}$ is a subset of $\mathcal{L}$ and thus β_0-equicontinuous. Theorem 4.1 yields (2).

(2) ⇒ (1): Let $\mathcal{H}$ be a β_0-equicontinuous subset of $\mathcal{M}_0(E)$. Then there exists $C > 0$ such that $\|\mu\| = |\mu|(E) \leq C$ for all $\mu \in \mathcal{H}$. If we choose $M > 0$ such that $\|T_\alpha\| \leq M$ for all $\alpha \in I$, then

$$\left|\langle f, T'_\alpha\mu\rangle\right| \leq M \cdot C \cdot \|f\| \quad \forall f \in C_b(E), \alpha \in I, \mu \in \mathcal{H}.$$

Taking the supremum over f with $\|f\|_\infty \leq 1$, it follows that $|T'_\alpha\mu|(E) \leq C \cdot M < \infty$ for all $\alpha \in I$ and $\mu \in \mathcal{H}$. Furthermore, given $\varepsilon > 0$, we find a compact set K such that $|\mu|(E \setminus K) \leq \frac{\varepsilon}{2M}$ for all $\mu \in \mathcal{H}$. By (2), we find $L \subset E$ compact such that $|p_\alpha|(x, E \setminus L) \leq \frac{\varepsilon}{2C}$ for all $\alpha \in I$ and $x \in K$. It follows that for $\mu \in \mathcal{H}$ and $\alpha \in I$ we have

$$\begin{aligned} |T'_\alpha\mu|(E \setminus L) &\leq \int_K |p_\alpha|(x, E \setminus L)d|\mu|(x) + \int_{K^c} |p_\alpha|(x, E \setminus L)|d|\mu|(x) \\ &\leq \frac{\varepsilon}{2C}\|\mu\| + \frac{\varepsilon}{2M}M = \varepsilon. \end{aligned}$$

Hence $\mathcal{L} := \{T'_\alpha\mu : \alpha \in I, \mu \in \mathcal{H}\}$ is β_0-equicontinuous and thus $p_\mathcal{L}$ is a β_0-continuous seminorm. However, $p_\mathcal{H}(T_\alpha f) \leq p_\mathcal{L}(f)$ for all $f \in C_b(E)$. Since $\mathcal{H}$ was arbitrary, it follows that $\mathcal{S}$ is β_0-equicontinuous. □

Let us recall the following definition from [24]. A completely regular space E is called a *T-space* if every σ'-compact set of positive Radon measures is tight. The celebrated Prokhorov theorem, see [22], asserts that every Polish space is a T-space. More generally, every complete metric space and every locally compact space is a T-space, see [24, Theorem 5.4]. If E is an infinite dimensional separable Hilbert space endowed with the weak topology, then E is *not* a T-space, cf. [12, Example I.6.4].

Theorem 4.3 *Let τ_+ denote the topology of uniform convergence on the σ'-compact subsets of $\mathcal{M}_0^+(E)$.*

(1) *$\beta_0 = \tau_+$ iff E is a T-space.*
(2) *If E is a T-space and every measure on E is a Radon measure, then $\beta_0 = \mu(C_b(E), \mathcal{M}_0(E))$. In this case, every σ'-compact subset of $\mathcal{M}_0(E)$ is tight, i.e. β_0 is the topology of uniform convergence on the σ'-compact subsets of $\mathcal{M}_0(E)$.*

Proof (1) is [24, Theorem 5.3], (2) follows from Theorems 5.8 and 4.5 of that paper. □

We note that Conway [6] has proved that if $E = [0, \omega_1)$, where ω_1 is the first uncountable ordinal and E is endowed with the order topology, then β_0 is not the Mackey topology of the pair $(C_b(E), \mathcal{M}_0(E))$. However, also in this case $\beta_0 = \tau_+$ since E, being locally compact, is a T-space.

We now come to the main result of this section.

Theorem 4.4 *Let E be a T-space and let $\mathbf{T}$ be an integrable semigroup on $(C_b(E), \mathcal{M}_0(E))$. If there exists a measure μ on $\mathcal{B}(E)$ which is not a Radon measure, then additionally assume that $\mathbf{T}$ is positive. We denote the kernel associated to $T(t)$ by p_t. Consider the following statements*

(1) *For every $f \in C_b(E)$, the map $(t, x) \mapsto T(t)f(x)$ is continuous;*
(2) *For every $f \in C_b(E)$ we have $T(t)f \to T(s)f$ for $t \to s$ uniformly on the compact subsets of E;*
(3) *$\mathbf{T}$ is a β_0-continuous semigroup;*

(4) $\mathbf{T}$ *is a quasi-*β_0*-equicontinuous*, β_0*-continuous semigroup*;
(5) $\mathbf{T}$ *has a* σ*-densely defined generator and, given a compact subset* $K \subset E$ *and constants* $t_0, \varepsilon > 0$ *there exists a compact subset* $L \subset E$ *such that*

$$|p_t|(x, E \setminus L) \leq \varepsilon \quad \forall x \in K, t \in [0, t_0].$$

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). *If* E *is either locally compact or a metric space, all five statements are equivalent.*

Proof (1) $\Rightarrow$ (2): Fix $f \in C_b(E)$ and $s \geq 0$. By assumption, for every $\varepsilon > 0$ and $x \in E$ we find $\delta = \delta(s, x)$ and a neighborhood $U = U(s, x)$ of x such that

$$|T(s)f(x) - T(t)f(y)| < \varepsilon \quad \forall (t, y) \in B(s, \delta) \times U.$$

Now let $K \subset E$ compact. Then $\{s\} \times K \subset \bigcup_{x \in K} B(s, \delta(s, x)) \times U(s, x)$. As $\{s\} \times K$ is compact in $[0, \infty) \times E$ we find finitely many $x_1, \ldots, x_n$ and $\delta_i := \delta(s, x_i)$ such that $\{s\} \times K \subset \bigcup_{i=1}^n B(s, \delta_i) \times U(x_i)$. Put $\delta = \min\{\delta_1, \ldots, \delta_n\}$. For $x \in K$, there exists i_0 such that $x \in U(x_{i_0})$. For $|t - s| < \delta$ we have

$$|T(t)f(x) - T(s)f(x)| \leq |T(t)f(x) - T(s)f(x_{i_0})| + |T(s)f(x_{i_0}) - T(s)f(x)| < 2\varepsilon,$$

since $(t, x), (s, x) \in B(s, \delta_{i_0}) \times U(x_{i_0})$. As $x \in K$ was arbitrary, we have $\sup_{x \in K} |T(t)f(x) - T(s)f(x)| \leq 2\varepsilon$ for $|t - s| < \delta$. This proves (2).

(2) $\Rightarrow$ (1) If E is a metric space, then $(t, x) \mapsto T(t)f(x)$ is continuous iff it is sequentially continuous. So let $(t_n, x_n) \to (s, x_0)$. By (2), $T(t_n)f \to T(s)f$ uniformly on the compact set $K = \{x_n : n \in \mathbb{N}_0\}$. But then $T(t_n)f(x_n) \to T(s)f(x_0)$ follows using the continuity of $T(s)f$.

Now assume that E is locally compact. Fix $(s, x_0) \in [0, \infty) \times E$ and $f \in C_b(E)$. Since $T(s)f$ is continuous, given $\varepsilon > 0$, there is a neighborhood $U(x_0)$ such that $|T(s)f(x) - T(s)f(x_0)| < \varepsilon$ for all $x \in U(x_0)$. It is no restriction to assume that $U(x_0)$ is relatively compact. By (2) we find $\delta > 0$ such that $|T(t)f(x) - T(s)f(x)| < \varepsilon$ for all $x \in \overline{U(x_0)}$ and all $|t - s| < \delta$. Thus $|T(t)f(x) - T(s)f(x_0)| < 2\varepsilon$ for all $(t, x) \in B(s, \delta) \times U(x_0)$. This proves (1).

(2) $\Leftrightarrow$ (3): Is clear since $\mathbf{T}$ is locally bounded and since the strict topology coincides with the compact-open topology on norm-bounded sets.

(3) $\Leftrightarrow$ (4): If every measure on E is a Radon measure, then, by Theorem 4.3 (2), β_0 is the topology of uniform convergence on the σ'-compact subsets of $\mathcal{M}_0(E)$ and (3) $\Rightarrow$ (4) follows from Theorem 3.9. In the other case, β_0 is the topology of uniform convergence on the σ'-compact subsets of $\mathcal{M}_0(E)^+$ by Theorem 4.3 (1). Thus $\mathbf{T}$ is quasi-β_0-equicontinuous as a consequence of the positivity of $\mathbf{T}$ and Theorem 3.11. This shows (3) $\Rightarrow$ (4), the converse implication is trivial.

(4) $\Rightarrow$ (5): Follows from Theorem 2.10 and Proposition 4.2.

(5) $\Rightarrow$ (4): Is a consequence of Propositions 4.2 and 3.3. □

Remark 4.5 The assumption in Theorem 4.4 that $\mathbf{T}$ is an *integrable* semigroup is only needed for the equivalence (4) $\Leftrightarrow$ (5).

Theorem 4.4 can be used to establish that a given transition semigroup on $C_b(E)$ is β_0-continuous. In [2], transition semigroups are constructed from the solutions of parabolic partial differential equations. Here E is a subset of $\mathbb{R}^d$. For such transition semigroups, condition (1) can easily be verified, as the PDE techniques yield solutions of the PDE which are continuous in both time and space variables. If one follows [5, 11, 19] and prefers to think about semigroups on $C_b(E)$ which have τ_{co}-continuous orbits, then of course Condition (2) is satisfied. In the next section, we will show that if $\mathbf{T}$ is the transition semigroup of a Markov process obtained from solving a stochastic differential equation, then Condition (5) can often be verified.

We note that the crucial assumption in Theorem 4.4 is that $\mathbf{T}$ consists of $\sigma(C_b(E), \mathcal{M}_0(E))$-continuous operators. Under suitable assumption on E, e.g. if E is a Polish space, an operator T on $C_b(E)$ is $\sigma(C_b(E), \mathcal{M}_0(E))$-continuous if and only if it is a kernel operator, see [20]. If one follows the approach of [2], then it is a consequence of the PDE techniques that the operators of the semigroup are represented by a *Green function* and thus are kernel operators.

If E is a Polish space and $\mathbf{T}$ is a τ_{co}-bi-continuous semigroup on $C_b(E)$, then it follows from the definition of a bi-continuous semigroup that every operator $T(t)$ is sequentially τ_{co}-continuous on normbounded sets and hence sequentially β_0-continuous. By [24, Corollary 8.4] it follows that $T(t) \in L(C_b(E), \beta_0) \subset L(C_b(E), \sigma)$. Farkas [10, Example 3.9] has given an example of a τ_{co}-bi-continuous semigroup which does not consist of β_0-continuous operators and is thus, in particular, not locally β_0-equicontinuous. In that example, $E = [0, \omega_1)$ with the order topology, where ω_1 is the first uncountable ordinal. Note that since $[0, \omega_1)$ is locally compact and hence a T-space, it follows from Theorem 4.4 that every positive β_0-continuous semigroup of operators in $L(C_b(E), \sigma)$ is quasi-β_0-equicontinuous.

5 Examples

5.1 The case $E = \mathbb{N}$

If $E = \mathbb{N}$ endowed with the discrete topology, then $C_b(E) = \ell^\infty$ and $\mathcal{M}_0(E) = \ell^1$. Thus in this case, $\mathcal{M}_0(E)$ is the predual of $C_b(E)$. The weak topology $\sigma = \sigma(C_b(E), \mathcal{M}_0(E))$ is merely the weak*-topology of ℓ^∞ whereas the weak topology $\sigma' = \sigma(\mathcal{M}_0(E), C_b(E))$ is the weak topology (in the Banach space sense) of ℓ^1. A bounded operator T on ℓ^∞ is σ-continuous if and only if it is the adjoint of a bounded operator on ℓ^1. We now have the following result.

Proposition 5.1 *If $E = \mathbb{N}$ endowed with the discrete topology, then every semigroup $\mathbf{T}$ on $(C_b(E), \mathcal{M}_0(E))$ which is σ-continuous at 0 is β_0-continuous and quasi-β_0-equicontinuous.*

Proof If $\mathbf{T}$ is a semigroup on (ℓ^∞, ℓ^1) then $\mathbf{T} = \mathbf{S}^*$ for some semigroup $\mathbf{S}$ on ℓ^1. Now $\mathbf{T}$ is $\sigma(\ell^\infty, \ell^1)$-continuous (at 0) if and only if $\mathbf{S}$ is $\sigma(\ell^1, \ell^\infty)$-continuous (at 0). It is well known (but also follows from Theorem 2.10 and Corollary 3.4) that a semigroup of bounded operators on a Banach space X which is weakly continuous at 0 is already $\|\cdot\|$-continuous. It follows that $\mathbf{T}$ is $\sigma(\ell^\infty, \ell^1)$-continuous. In particular, $t \mapsto T(t)f(n)$ is continuous for every $n \in \mathbb{N}$ and every $f \in C_b(\mathbb{N})$. However, since every compact subset of $\mathbb{N}$ is finite, it follows that $t \mapsto T(t)f$ is τ_{co}-continuous and hence β_0-continuous for every $f \in C_b(\mathbb{N})$. Since $\mathbb{N}$ is locally compact and since every measure on $\mathbb{N}$ is a Radon measure, the assertion follows from Theorem 4.4. $\square$

5.2 The Sorgenfrey line

Let us consider the real line $\mathbb{R}$ endowed with the Sorgenfrey topology τ_s, i.e. τ_s is generated by the intervals of the form $[a, b)$. It follows that the Borel σ-algebra of $(\mathbb{R}, \tau_s)$ is the usual Borel σ-algebra of $\mathbb{R}$ (with the metric $|\cdot|$). It is well known that every compact subset of $(\mathbb{R}, \tau_s)$ is countable. However, as the example $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$ shows, not every countable subset of $\mathbb{R}$ is compact in $(\mathbb{R}, \tau_s)$. It follows that every Radon measure on $(\mathbb{R}, \tau_s)$ is concentrated on a countable set. Hence $\mathcal{M}_0(\mathbb{R}, \tau_s) = \ell^1(\mathbb{R})$, the space of all discrete measures on $\mathbb{R}$. A function f on $\mathbb{R}$ is continuous with respect to τ_s if and only if it is right-continuous. Thus in this situation $(C_b(\mathbb{R}, \tau_s), \mathcal{M}_0(\mathbb{R}, \tau_s)) = (C_r(\mathbb{R}), \ell^1(\mathbb{R}))$, where $C_r(\mathbb{R})$ denotes the space of all bounded, right continuous functions on $\mathbb{R}$.

Consider the shift semigroup $\mathbf{T}$ given by $T(t)f(x) = f(x + t)$. Then $\mathbf{T}$ is a positive contraction semigroup on $(C_r(\mathbb{R}), \ell^1(\mathbb{R}))$. However, $\mathbf{T}$ is not integrable. Indeed, it is easy to see that the Laplace transform of $\mathbf{T}$ is given by

$$R(\lambda)f(x) = e^{\lambda x} \int_x^\infty e^{-\lambda y} f(y)dy.$$

But this operator is not σ-continuous since $R(\lambda)^*\delta_0 = e^{-\lambda t}\mathbb{1}_{(0,\infty)}dt \notin \ell^1(\mathbb{R})$. Furthermore, $\mathbf{T}$ is σ-continuous at 0 but not σ-continuous. Since a continuous function is uniformly continuous on compact sets, it follows that $T(t)f \overset{\tau_{\text{co}}}{\to} f$ as $t \downarrow 0$ for every $f \in C_r(\mathbb{R})$. Hence $\mathbf{T}$ is β_0-continuous at 0.

5.3 Applications to Markov processes

If E is a Banach space, then some Markov processes are obtained as solutions of stochastic differential equations, see e.g. [13, 14]. The transition semigroup of such a Markov process is defined as follows. If $X(t, x)$ denotes the solution of the stochastic differential equation with initial datum $x \in E$, then one defines $T(t)f(x) = \mathbb{E}(f(X(t, x)))$. It is natural to ask for a condition for $\mathbf{T}$ to be β_0-continuous in terms of properties of the map $(t, x) \mapsto X(t, x)$. In applications, it frequently happens that $X(t, x) \in L^p(\Omega, E)$, where Ω is the underlying probability space and $1 \le p < \infty$, and the map $(t, x) \mapsto X(t, x)$ is continuous. The following theorem shows that in this case the semigroup $\mathbf{T}$ is indeed β_0-continuous. However, this result remains true in a even more general setting.

If $(\Omega, \mathcal{F}, P)$ is a probability space and (E, ρ) is a complete metric space, then $L^0(\Omega, E)$ denotes the space of all strongly measurable maps $X : \Omega \to E$ (modulo equality P-almost everywhere) endowed with the topology of convergence in measure.

Theorem 5.2 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, (E, ρ) be a complete metric space and $X : [0, \infty) \times E \to L^0(\Omega, E)$ be a continuous map. Define $T(t)f(x) = \mathbb{E}(f(X(t,x))$ for $f \in C_b(E)$. Then, for every $t_0 > 0$, the set $\{T(t) : 0 \le t \le t_0\}$ is a β_0-equicontinuous family of operators on $(C_b(E), \mathcal{M}_0(E))$. If $(T(t))_{t\ge0}$ is a semigroup, then it is β_0-continuous and quasi-β_0-equicontinuous.*

Proof Consider the map $\Phi : L^0(\Omega, E) \to \mathcal{M}_0(E)$ given by $\Phi(X) = \mu_X$, where μ_X denotes the distribution of X. Note that for $X \in L^0(\Omega, E)$ the distribution μ_X is indeed a Radon measure since X has separable range. The map Φ is continuous. Indeed, if $X_n \to X$ in measure then, passing to a subsequence, we have $X_n \to X$ almost everywhere. But then

$$\langle f\,,\,\mu_{X_n}\rangle = \int_\Omega f(X_n)dP \to \int_\Omega f(X)dP = \langle f\,,\,\mu_X\rangle.$$

Thus, every subsequence of $\Phi(X_n)$ has a subsequence which converges to $\Phi(X)$ with respect to $\sigma(\mathcal{M}_0(E), C_b(E))$.

Since $T(t)f(x) = \langle f\,,\,\mu_{X(t,x)}\rangle$, the continuity of $x \mapsto X(t,x)$ for fixed t implies that every operator $T(t)$ maps $C_b(E)$ into $C_b(E)$. It follows from the joint continuity of $X(\cdot,\cdot)$ that for every $t_0 > 0$ and every compact set $K \subset E$ the set $\{X(t,x) : 0 \le t \le t_0, x \in K\}$ is compact in $L^0(\Omega, E)$. Hence the set $\{\mu_{X(t,x)} : 0 \le t \le t_0, x \in K\}$ is σ'-compact in $\mathcal{M}_0(E)^+$. Since E is a complete metric space and hence a T-space, it follows that the latter set is tight. Hence Proposition 4.2 implies that the set $\{T(t) : 0 \le t \le t_0\}$ is β_0-equicontinuous. In particular, every single operator $T(t)$ is β_0-continuous and hence an element of $L(C_b(E), \sigma)$. Now assume that $(T(t))_{t\ge0}$ is a semigroup. Since $t \mapsto X(t,x)$ is continuous, it follows that $t \mapsto T(t)f(x)$ is continuous for every $x \in E$ and hence $(T(t))_{t\ge0}$ is σ-continuous. Taking Remark 2.9 into account, it follows from Proposition 2.10 that $\mathbf{T}$ has a σ-densely defined generator. The claim follows from Theorem 4.4. □

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