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RESEARCH ARTICLE

Finiteness Properties of Varieties and the Restriction to Finite Algebras

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Abstract

It is known that the variety generated by a finite semigroup is finitely axiomatisable if and only if it is finitely axiomatisable in the class of finite semigroups (M. Sapir [29]). We examine similar restrictions for most other common finiteness properties of semigroup varieties.

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A variety is said to be *finitely based* (abbreviated to FB) if there is a finite subset of its identities from which all of its identities may be deduced. Likewise an algebra $\mathbf{S}$ is said to be finitely based if it generates a finitely based variety. There are many FB and many non-FB finite semigroups, and consequently the finite basis property for finite semigroups, and for finite algebras in general has been one of the most extensively studied in facets of varieties. We direct the reader to [36] for an excellent survey and introduction to this topic in the case of semigroups and their relatives.

A variety is *hereditarily finitely based* (abbreviated to HFB) if all of its subvarieties are finitely based. Every finite group generates a HFB variety [23], as does any commutative semigroup [24]. An elementary application of Zorn's Lemma shows that any variety that is not HFB contains a subvariety that is minimal with respect to being not FB. Varieties with this property are called *limit varieties*. There are very few explicitly constructed examples of limit varieties. The six element semigroup $\mathbf{B_2^1}$ generates a limit variety in the inverse semigroup signature [16] (with only 3 proper subvarieties), while examples of semigroup limit varieties are constructed in [30] and [34]. Amongst (locally finite) groups, there are known to be infinitely many limit varieties [22], however the explicit construction of such an example remains one of the foremost unsolved problems in group variety theory [17].

As much of the interest surrounding the finite basis property has centred around finite algebras, it is natural to also consider this and related properties relativised to the class of finite algebras. Formally, we will say that a locally finite variety $\underline{\mathbf{V}}$ (or an algebra generating $\underline{\mathbf{V}}$) is *finitely based within the class of finite algebras* (abbreviated to $\mathrm{FB}_{\mathrm{fin}}$) if there is an $n \in \mathbb{N}$ such that a *finite* algebra $\mathbf{A}$ is contained in $\underline{\mathbf{V}}$ if and only if its n-generated subalgebras are in $\underline{\mathbf{V}}$. Equivalently, $\underline{\mathbf{V}}$ is $\mathrm{FB}_{\mathrm{fin}}$ if there is a finite set Σ of identities satisfied by $\underline{\mathbf{V}}$ such that for a finite algebra $\mathbf{A}$ we have $\mathbf{A} \in \underline{\mathbf{V}}$ if and only if $\mathbf{A}$ satisfies Σ. The possible inequivalence of the $\mathrm{FB}_{\mathrm{fin}}$ property and the general FB property appears to be a tantalising open problem in universal algebra [3][1], however for finite semigroups, the two concepts coincide [29] (see also Lemma 1.1 below).

There now appear to be three reasonable possibilities for a relativised HFB property (we omit possibilities where the concepts of FB and $\mathrm{FB}_{\mathrm{fin}}$ are mixed). Namely, a finitely generated variety $\underline{\mathbf{V}}$ could satisfy the relativised HFB property if:

(1) all subvarieties of $\underline{\mathbf{V}}$ are $\mathrm{FB}_{\mathrm{fin}}$; or

(2) all finitely generated subvarieties of $\underline{\mathbf{V}}$ are FB; or

(3) all finitely generated subvarieties of $\underline{\mathbf{V}}$ are $\mathrm{FB}_{\mathrm{fin}}$.

There also appears to be at least three reasonable relativised versions of the notion of a limit variety. Namely, a finitely generated variety $\underline{\mathbf{V}}$ could be denoted a relativised limit variety if:

(1) $\underline{\mathbf{V}}$ is not $\mathrm{FB}_{\mathrm{fin}}$, but all proper subvarieties of $\underline{\mathbf{V}}$ are $\mathrm{FB}_{\mathrm{fin}}$; or

(2) $\underline{\mathbf{V}}$ is not FB, but all proper finitely generated subvarieties of $\underline{\mathbf{V}}$ are FB; or

(3) $\underline{\mathbf{V}}$ is not $\mathrm{FB}_{\mathrm{fin}}$, but all proper finitely generated subvarieties of $\underline{\mathbf{V}}$ are $\mathrm{FB}_{\mathrm{fin}}$.

In this paper we describe for finite semigroups, the possible discrepancies between the general and relativised notions of the above properties. Because of our restriction to finitely generated varieties of semigroups, some easy observations regarding the main result of [29] will enable some identifications between the various relativised notions introduced above. The first of the relativised notions of HFB and of limit varieties will become equivalent to their general forms, while the second and third versions will coincide (details are given in Section 4). However, we solve a problem from [36] by showing that the second and third relativised versions of the HFB property turn out to be distinct from the general version of the HFB property (Section 4). Similar behaviour is found for the relativised notions of limit varieties (Section 5). We also survey the situation

[1] If one (artificially) allows for the possibility that $\underline{\mathbf{V}}$ be not locally finite, then the two notions are distinct. To see this, let $\underline{\mathbf{V}}$ be a non-finitely based variety whose finite members are trivial—uncountably many unary varieties of this kind are constructed in Exercise II.14.8 of [2]. Then a finite algebra of appropriate signature is contained in $\underline{\mathbf{V}}$ if and only if it satisfies $x \approx y$; equivalently, if and only if its two generated subalgebras are trivial.

for some relativised versions of a number of other related finiteness properties (Section 6). In the final section we make some elementary observations and find that satisfaction of any of these finiteness properties is undecidable for finitely generated varieties.

Our main technique involves exploiting the dichotomy between the monoid variety of a monoid and the semigroup variety of a monoid. Several other results come out of this approach, including what appears to be the first explicit construction of a monoid limit variety in the usual sense (Section 5).

We begin the paper with three preliminary sections. The second of these (Section 2) is of some independent interest; namely a further characterisation of the class of aperiodic semigroups with central idempotents.

1. Preliminaries: basic definitions

For standard notions of semigroup theory we direct the reader to a book, such as [13]. Most of the essential definitions will be given here however.

A *word* in an alphabet X is an element of the free semigroup X^+, while a *possibly empty word* in X is an element of the free monoid X^*. Equality on free semigroups and free monoids will be denoted $\equiv$. We will use letters x, y, z, s, t (with or without subscripts) for single letters and p, q, u, v, w (with or without subscripts) for words or possibly empty words.

A semigroup *identity* is an expression $u \approx v$ where u and v are words. A semigroup $\mathbf{S}$ satisfies the identity $u \approx v$ (in variables X, say) if for every homomorphism $\theta : X^+ \to \mathbf{S}$, the equality $\theta(u) = \theta(v)$ holds (written $\mathbf{S} \models u \approx v$). The set of identities of $\mathbf{S}$ over some fixed countably infinite alphabet is denoted $\mathrm{Id}(\mathbf{S})$. The class of all semigroups satisfying the identities of $\mathbf{S}$ is called the *variety* generated by $\mathbf{S}$. Equivalently, the variety of $\mathbf{S}$ is the class of all homomorphic images of subalgebras of direct powers of $\mathbf{S}$. A variety is *finitely generated* if it is the variety generated by a finite algebra.

We will denote the semigroup variety of a semigroup or monoid $\mathbf{S}$ by $\mathbb{V}(\mathbf{S})$, and the monoid variety generated by a monoid by $\mathbb{V}_{\mathrm{M}}(\mathbf{S})$. In this paper we will often use the fact that a monoid $\mathbf{S}$ is contained in the semigroup variety of a monoid $\mathbf{T}$ if and only if it is contained in the monoid variety of $\mathbf{T}$. This is easily proved using the $\mathbb{HSP}$ definition of a variety; details can be found in [10].

The notation $\mathbf{S} \approxeq \mathbf{T}$ will be used to denote when $\mathrm{Id}(\mathbf{S}) = \mathrm{Id}(\mathbf{T})$ (equivalently, $\mathbb{V}(\mathbf{S}) = \mathbb{V}(\mathbf{T})$); that is, to denote *equational equivalence*. This and other definitions given above can be extended to classes of semigroups or monoids in the obvious way; for example if $\mathcal{K}$ is a class of semigroups, then $\mathrm{Id}(\mathcal{K})$ will denote the identities satisfied by all members of $\mathcal{K}$, and for classes of semigroups $\mathcal{J}$ and $\mathcal{K}$, we write $\mathcal{K} \approxeq \mathcal{J}$ when $\mathrm{Id}(\mathcal{J}) = \mathrm{Id}(\mathcal{K})$.

Let Σ be a set of semigroup identities over some fixed countably infinite alphabet of letters X. An *equational deduction* of an identity $p \approx q$ (where p and q are also words in X^+) from Σ is a sequence $p \equiv p_0 \approx p_1 \approx \ldots \approx p_n \equiv q$, where if $n \geq 1$, then for each $0 \leq i \leq n-1$, there is an identity $u_i \approx v_i$ or $v_i \approx u_i$ in Σ and a substitution $\theta_i : X^+ \to X^+$, such that p_{i+1} is obtained from

p_i by replacing the subword $\theta_i(u_i)$ of p_i by $\theta_i(v_i)$. If such a deduction exists we write $\Sigma \vdash p \approx q$ (or $\Sigma \vdash K$, if K is a set of identities equationally deducible from Σ). Thus a semigroup $\mathbf{S}$ has a finite basis of identities (or simply, $\mathbf{S}$ is FB) if there is a finite subset $\Sigma \subseteq \mathrm{Id}(\mathbf{S})$ such that $\Sigma \vdash \mathrm{Id}(\mathbf{S})$. We can give a similar definition of equational deduction for monoids, however as noted in [36], a monoid is finitely based as a monoid if and only if it is finitely based as a semigroup. The abbreviation NFB will be used to denote algebras or varieties that are not FB.

Let $\underline{\mathbf{V}}$ be a locally finite variety with only finitely many operations. A well known result of G. Birkhoff ([1]; see also [2, Theorem 4.2]) states that for each $n \in \mathbb{N}$, the class $\mathrm{Id}_n(\underline{\mathbf{V}})$ of all identities of $\underline{\mathbf{V}}$ in at most n variables can be derived from some finite subset of $\mathrm{Id}_n(\underline{\mathbf{V}})$. Hence $\underline{\mathbf{V}}$ is FB if and only if there is an $n \in \mathbb{N}$ such that the identities of $\underline{\mathbf{V}}$ in at most n-variables form a basis for $\mathrm{Id}(\underline{\mathbf{V}})$. Equivalently, $\underline{\mathbf{V}}$ is NFB if and only if for every $n \in \mathbb{N}$ there is an algebra $\mathbf{A}_n$ failing $\mathrm{Id}(\underline{\mathbf{V}})$, but satisfying $\mathrm{Id}_n(\underline{\mathbf{V}})$. If the algebras $\mathbf{A}_1$, $\mathbf{A}_2$, ... can be chosen to be finite, then we can say that $\underline{\mathbf{V}}$ is NFB within the class of finite algebras ($\mathrm{NFB}_{\mathrm{fin}}$). The reader should consult the definition of the $\mathrm{FB}_{\mathrm{fin}}$ property at the start of this paper to see that for locally finite varieties of finite type, the $\mathrm{NFB}_{\mathrm{fin}}$ property is precisely the negation of the $\mathrm{FB}_{\mathrm{fin}}$ property. The following fundamental result due to M. Sapir shows that the FB and $\mathrm{FB}_{\mathrm{fin}}$ properties coincide for any variety that lies within the variety generated by a finite semigroup or finite monoid.

Lemma 1.1. (M. Sapir [29]) *Let $\underline{\mathbf{V}}$ be a subvariety of a finitely generated variety of semigroups (monoids). Then $\underline{\mathbf{V}}$ is NFB if and only if for each $n \in \mathbb{N}$ there are finite semigroups (monoids, respectively) $\mathbf{S}_n \notin \underline{\mathbf{V}}$ such that $\mathbf{S}_n$ satisfies all n-variable identities of $\underline{\mathbf{V}}$ but not all identities of $\underline{\mathbf{V}}$.*

We note that in [29] this result is only stated and proved for the semigroup variety of a finite semigroup, however the more general result we have stated follows from only a minor modification of the proof; full details are given in [10].

We will also be needing some facts relating to formal languages and automata theory. For further details, we direct the reader to a book such as [5]. Let $\mathbf{S}$ be a monoid and $T \subseteq S$. We define the *syntactic congruence*, $\sim_T$ of T in $\mathbf{S}$ by $a \sim_T b$ if and only if

$$(\forall x, y \in S) \quad xay \in T \Leftrightarrow xby \in T.$$

If $\sim_T$ is the diagonal relation Δ on $\mathbf{S}$, then we say that T is a *syntactic subset* of $\mathbf{S}$. The quotient of a free monoid X^* by the syntactic congruence of a set (or *language*) $W \subseteq X^*$ is the *syntactic monoid*, $\mathrm{Syn}_{\mathrm{M}}(W)$, of W (in X^*). It is well known that $\mathrm{Syn}_{\mathrm{M}}(W)$ is finite if and only if W is recognised by some finite automata (Myhill's Theorem).

Let A be a generating set for a monoid $\mathbf{S}$ and $\nu : A^* \to \mathbf{S}$ be the unique surjection extending the inclusion map from A into S. If T is a syntactic subset of $\mathbf{S}$, then the quotient of A^* by the syntactic congruence of $\nu^{-1}(T)$ in A^* (that is, the syntactic monoid of $\nu^{-1}(T)$ in A^*) is isomorphic to $\mathbf{S}$.

If $\mathbf{S}$ is a semigroup that is not a monoid, and $T \subseteq S$, then one can define the syntactic congruence of T in $\mathbf{S}$ by first adjoining an identity element and then using the definition given above for monoids (of course in this case, the identity element is only used for the definition of $\sim_T$, and must be removed before taking quotient of $\mathbf{S}$ by $\sim_T$). All other notions defined above for syntactic monoids carry over to syntactic semigroups (with X^+ replacing X^*). The notation $\mathrm{Syn}(W)$ denotes the syntactic semigroup of a language W (not containing the empty word).

2. Preliminaries: aperiodic semigroups with central idempotents

In this section we develop some useful theory regarding the class of semigroups from which our main examples are taken. In the context of this paper the material is preliminary, however it is also of some independent interest and accordingly we have developed it slightly beyond what is essential for our needs.

Recall that a semigroup is *unipotent* if it contains precisely one idempotent. A finite unipotent semigroup $\mathbf{S}$ that is aperiodic (that is, has trivial subgroups) is *nilpotent*; for semigroups this simply means that there is a zero element 0, and a positive integer k (the cardinality of S will suffice) such that all products of length k of elements of S are equal to 0; in this case we may more precisely say that $\mathbf{S}$ is *k-nilpotent*. A finite monoid is said to be (k-) nilpotent if it can be obtained by adjoining an identity element to a (k-) nilpotent semigroup.

Varieties generated by finite nilpotent monoids have turned out to be a surprisingly rich source of counterexamples in the study of semigroup varieties [8, 9, 11, 24, 32] and we again use them in this paper. These structures were also investigated by Straubing [33] who proved the following result (recall that for semigroups $\mathbf{S}$, $\mathbf{T}$, we say that $\mathbf{S}$ *divides* $\mathbf{T}$ if $\mathbf{S}$ is a subsemigroup of a homomorphic image of $\mathbf{T}$).

Theorem 2.1. [33] *Let $\mathbf{S}$ be a finite semigroup with n elements. The following are equivalent:*

(i) $\mathbf{S}$ *is an aperiodic semigroup with central idempotents;*

(ii) $\mathbf{S}$ *divides a direct product of syntactic monoids of finite languages;*

(iii) $\mathbf{S}$ *divides a direct product of finite nilpotent monoids;*

(iv) $\mathbf{S}$ *satisfies* $\{x^{n+1} \approx x^n, x^n y \approx y x^n\}$;

(v) $\mathbf{S}$ *satisfies* $\{x^{n+1} \approx x^n, y_0 x y_1 x \dots x y_n \approx x^n y_0 y_1 \dots y_n \approx y_0 y_1 \dots y_n x^n\}$.

We let $\mathbb{F}$ denote the class of finite aperiodic semigroups with central idempotents. While Theorem 2.1 says much about the kind of semigroup we can expect to find in the variety of a nilpotent monoid, we will require a still finer description. In this section we are going to prove the following.

Theorem 2.2. *For every semigroup $\mathbf{S} \in \mathbb{F}$ there are finite languages U and V such that $\mathbf{S} \approxeq \{\mathrm{Syn}(U), \mathrm{Syn}_{\mathrm{M}}(V)\}$. If $\mathbf{S}$ is a monoid, then we can assume that U is empty, while if $\mathbf{S}$ is a nilpotent semigroup then we may assume that V is empty.*

This theorem is proved in Lemmas 2.3–2.9.

The following result is probably part of the folklore, but we include its proof for the sake of completeness.

Lemma 2.3. *A finite aperiodic semigroup $\mathbf{S}$ has central idempotents if and only if it is a subdirect product of a nilpotent semigroup and a finite set of monoids in $\mathbb{F}$.*

Proof. The 'if' direction is trivial since $\mathbb{F}$ is closed under finite direct products and the taking of subsemigroups. Now let E denote the set of idempotents in $\mathbf{S}$, and let $\mathbf{S}$ have central idempotents. Because the idempotents in $\mathbf{S}$ are central, the set $I := \{x \in S : (\exists e \in E)\ xe = x\}$ is an ideal of $\mathbf{S}$ containing E and therefore the Rees quotient $\mathbf{S}/I$ is a unipotent quotient of $\mathbf{S}$. We will denote $\mathbf{S}/I$ by $\mathbf{S}_I$ and let $\nu_I : \mathbf{S} \to \mathbf{S}_I$ be the natural map. For each $e \in E$ let $\mathbf{S}_e$ denote the subsemigroup of $\mathbf{S}$ on the set $\{xe : x \in S\}$. Again since idempotents are central this is a homomorphic image of $\mathbf{S}$ under the map ν_e defined by $x \mapsto xe$. Note that if x and y are such that $xe \neq ye$ for some e then $\nu_e(x) \neq \nu_e(y)$. On the other hand if $xe = ye$ for all $e \in E$ and $x \neq y$ then $\nu_I(x) \neq \nu_I(y)$. Indeed, if $xe = ye$ for all $e \in E$ but x, y both lie in I then there are $e, f \in E$ such that $xe = x$ and $yf = y$. Then $x = xe = ye = yfe = xfe = xef = xf = yf = y$.

Now there is a natural homomorphism $\phi : \mathbf{S} \to \prod_{i \in E \cup \{I\}} \mathbf{S}_i$ given by defining $\phi(s)(i) = \nu_i(s)$. Since each distinct pair $x, y \in S$ are separated by a map ν_e or the map ν_I and each of these maps is onto, ϕ is a subdirect embedding. ∎

The assumption of aperiodicity played little role in this proof. If we omit this assumption the proof shows that a semigroup with central idempotents is a subdirect product of some monoids with a unipotent semigroup.

Lemma 2.4. *Let $\mathbf{S}$ be a finite monoid with precisely two idempotents: an identity element 1 and a zero element 0. If $\mathbf{S}$ is aperiodic, then $\mathbf{S}$ is a nilpotent monoid.*

Proof. The divisors of 1 in a finite monoid lie in a subgroup containing 1. Because $\mathbf{S}$ is aperiodic, it follows that no element of $S\backslash\{1\}$ divides 1; that is $S\backslash\{1\}$ is a subuniverse of $\mathbf{S}$. Let $\mathbf{S}^-$ be the corresponding subsemigroup. Evidently $\mathbf{S}^-$ is unipotent, and hence nilpotent because $\mathbf{S}$ is finite. ∎

Lemma 2.5. *Every monoid in $\mathbb{F}$ is equationally equivalent to a finite set of finite nilpotent monoids.*

Proof. Let $\mathbf{S}$ be a monoid in $\mathbb{F}$ and let E denote the idempotents of $\mathbf{S}$. For each idempotent $e \in E$, consider the quotient $\mathbf{S}e$ of $\mathbf{S}$ obtained by collapsing each x onto the element xe (of course, $\mathbf{S}e$ is isomorphic to the subsemigroup $\mathbf{S}_e$ used in the proof of Lemma 2.3). The set $I_e := \{x \in Se : (\exists f \in Ee\backslash\{e\})\ xf = x\}$, forms an ideal in $\mathbf{S}e$ and we will denote the resulting Rees quotient by $\mathbf{S}(e)$. Note that $\mathbf{S}(e)$ has only two idempotents: the identity element e and a zero element. Lemma 2.4 then shows that $\mathbf{S}(e)$ is a nilpotent monoid.

We now claim that the map $\iota\colon \mathbf{S} \to \prod_{e\in E} \mathbf{S}(e)$ given by $\iota(s)(e) = se/I_e$ is a subdirect embedding. All but the injectivity of ι in this claim are obvious. To see that ι is injective, let a and b be distinct elements in S and let e and f be the smallest idempotents for which $ae = a$ and $bf = b$. If $e = f$ then $\iota(a)(e) = a \neq b = \iota(b)(e)$. If $e \neq f$ then one of $af = b$ or $be = a$ must fail since otherwise $a = be = afe = aef = af = b$. Say that $af \neq b$. Now $\iota(b)(f) = bf/I_f = b$ however $\iota(a)(f) = af/I_f$ which is either equal to the zero of $\mathbf{S}(f)$ (when $af \in I_f$) or it equals af. Because, $af \neq b \notin I_f$, in either case $\iota(b)(f) \neq \iota(a)(f)$. This completes the proof that ι is a subdirect embedding of $\mathbf{S}$ into a product of nilpotent monoid quotients of $\mathbf{S}$. Hence $\mathbf{S}$ satisfies the same identities as $\prod_{e\in E} \mathbf{S}(e)$, and so $\mathbb{V}(\mathbf{S})$ is generated by the finite set of finite nilpotent monoids $\{\mathbf{S}(e) : e \in E\}$. ■

Lemma 2.6. *Let $\mathbf{S}$ be a finite nilpotent monoid (semigroup). Then $\mathbf{S}$ is equationally equivalent to a finite set of finite syntactic nilpotent monoids (semigroups).*

Proof. It is well known that every monoid (semigroup) is a subdirect product of its syntactic monoid (semigroup) quotients [5]. However, every quotient of a finite nilpotent monoid (semigroup) is either trivial (and can be ignored) or is again a finite nilpotent monoid (semigroup). ■

Lemma 2.7. *Let $\mathbf{S}$ be a nilpotent monoid containing a syntactic subset T with $T \neq \{1\}$ and $T \neq S\backslash\{1\}$. Then $\sim_{T\backslash\{1\}} = \sim_{T\cup\{1\}} = \Delta$ on $\mathbf{S}$.*

Proof. Let n be such that products of length n (not involving 1) in $\mathbf{S}$ equal 0.

Claim 1. Neither $\sim_{T\backslash\{1\}}$ nor $\sim_{T\cup\{1\}}$ equals ∇.
This is because neither $T\backslash\{1\}$ nor $T \cup \{1\}$ is empty or equal to S.

Let ρ denote one of $\sim_{T\backslash\{1\}}$ or $\sim_{T\cup\{1\}}$.

Claim 2. The ρ-class containing 1 is a singleton.
Let θ be any congruence on $\mathbf{S}$ such that $a \in S\backslash\{1\}$ has $1\,\theta\, a$. Then for every $x \in S$ we have $x = x1 = x1^n\,\theta\, xa^n = x0 = 0$. Hence $\theta = \nabla$. Claim 2 follows because $\rho \neq \nabla$ by Claim 1.

Now let $a \in S\backslash\{1\}$. Because $\mathbf{S}$ is nilpotent, 1 has no proper divisors and so we have:

Claim 3. $(\forall x, y \in S)\ \ xay \in T\backslash\{1\} \Leftrightarrow xay \in T \cup \{1\} \Leftrightarrow xay \in T$.

Finally, let a and b be ρ-related elements. To complete the proof we need to show that $a=b$. By Claim 2 we have that either $a=b=1$ or $a,b\in S\backslash\{1\}$. By Claim 3 we have $a\sim_T b$, which since T is syntactic in $\mathbf{S}$, gives $a=b$. ■

Lemma 2.8. *Let $\mathbf{S}$ be a finite nilpotent monoid (semigroup) that is also a syntactic monoid (semigroup). Then $\mathbf{S}$ is the syntactic monoid (semigroup) of a finite language.*

Proof. The semigroup case is almost identical to the monoid case, but easier because we do not need to consider the identity element. We leave this case to the reader. So let us assume that $\mathbf{S}$ is a k-nilpotent monoid.

If $\mathbf{S}$ is trivial (one element), then it is the syntactic monoid of the empty language over any finite alphabet. (Note that unless we allow empty semigroups, a nilpotent monoid cannot be trivial.) Now let $\mathbf{2}$ denote the two element semilattice. If $\mathbf{S}\cong\mathbf{2}$ then $\mathbf{S}$ is the syntactic monoid of the language $\{1\}$ (over any non-empty alphabet). For the remainder of the proof we assume that $\mathbf{S}$ is neither trivial, nor isomorphic to $\mathbf{2}$ and that $\mathbf{S}$ is a finite nilpotent monoid that is a syntactic monoid with syntactic subset T. Let A be a minimal set of generators for $\mathbf{S}$ as a monoid and consider the natural map $\nu : A^*\to\mathbf{S}$. By taking complements if necessary, we may assume that $0\notin T$. Now $\mathbf{S}$ is (isomorphic to) the syntactic monoid of $\nu^{-1}(T)$ in A^*. By k-nilpotency, $\nu^{-1}(T)$ consists of a set of words of lengths at most $k-1$ and since the alphabet A is finite, it follows that $\nu^{-1}(T)$ is a finite language. ■

We note that in the semigroup case of the above proof, the finite language obtained will not include the empty word 1.

Lemma 2.9. *Let $W_1,\dots,W_n$ be a finite family of finite languages. There is a finite language W so that $\{\mathrm{Syn}_{\mathrm{M}}(W_i) : i=1,\dots,n\}\approxeq\mathrm{Syn}_{\mathrm{M}}(W)$, and if $1\notin\cup_{1\leq i\leq n}W_i$, then there is a finite language V such that $\{\mathrm{Syn}(W_i) : i=1,\dots,n\}\approxeq\mathrm{Syn}(V)$.*

Proof. This result can be gleaned from the methods given in [10], but we give the easy argument for completeness. We again leave the semigroup case to the reader. Let us assume without loss of generality that the finite languages $W_1,\dots,W_n$ are each non-empty and over pairwise disjoint alphabets $A_1,\dots,A_n$ (each finite). For each i there is a surjective morphism $\nu_i : A_i^*\to\mathrm{Syn}_{\mathrm{M}}(W_i)$ such that the kernel of ν_i is the syntactic congruence of W_i in A_i^*. We let V_i denote $\nu_i(W_i)$. Because W_i is finite, we have $0\notin V_i$. Now let W denote the union $\cup_{i\leq n}W_i$ and amalgamate the monoids $\mathrm{Syn}_{\mathrm{M}}(W_1),\dots,\mathrm{Syn}_{\mathrm{M}}(W_n)$ at $\{0,1\}$. Make the resulting object a monoid by letting any undefined products equal 0. This monoid, which we denote here by $\mathbf{S}$, is a nilpotent monoid. Clearly each $\mathrm{Syn}_{\mathrm{M}}(W_i)$ embeds into $\mathbf{S}$; in fact it does no harm to assume the convention that each $\mathrm{Syn}_{\mathrm{M}}(W_i)$ is actually a subalgebra of $\mathbf{S}$.

Now let us consider the subset $V:=\cup_{1\leq i\leq n}V_i$ of $\mathbf{S}$, and let $\sim$ denote the syntactic congruence of V in $\mathbf{S}$. Because $\mathbf{S}/{\sim}$ is a nilpotent monoid and a syntactic monoid, Lemma 2.8 shows that $\mathbf{S}/{\sim}$ is the syntactic monoid of a finite language. (In fact, it is not hard to show that it is the syntactic monoid of the language W in $(\cup_{1\leq i\leq n}A_i)^*$.) We now show that $\mathbf{S}/{\sim}$ generates the same variety as $\{\mathrm{Syn_M}(W_i) : i = 1,\ldots,n\}$.

For each i we have $V \cap \mathrm{Syn_M}(W_i) = V_i$ or $V \cap \mathrm{Syn_M}(W_i) = V_i \cup \{1\}$ and so by Lemma 2.7, the restriction of $\sim$ to $\mathrm{Syn_M}(W_i)$ is at least as fine as the syntactic congruence of V_i in $\mathrm{Syn_M}(W_i)$. However, this last congruence is the diagonal relation because V_i is a syntactic subset of $\mathrm{Syn_M}(W_i)$. Hence, each $\mathrm{Syn_M}(W_i)$ also embeds into $\mathbf{S}/{\sim}$, showing that the (semigroup or monoid) variety generated by $\mathbf{S}/{\sim}$ contains that generated by $\{\mathrm{Syn_M}(W_i) : i = 1,\ldots,n\}$. For the other direction, we observe that $\mathbf{S}$ can be identified with the subalgebra of the direct product $\prod_{1\leq i\leq n}\mathrm{Syn_M}(W_i)$, consisting of the element constantly equal to 1 along with all elements that simultaneously satisfy the properties of being equal to 0 on all but at most one coordinate and not equal to 1 on any coordinate. Therefore $\mathbf{S}/{\sim}$ is a homomorphic image of a subalgebra of a direct product of the monoids $\mathrm{Syn_M}(W_i)$, for $i = 1,\ldots,n$ and hence lies in the variety these monoids generate. This now shows that $\mathbf{S}$ and $\{\mathrm{Syn_M}(W_i) : i = 1,\ldots,n\}$ generate the same varieties (equivalently, are equationally equivalent). ∎

Now we complete the proof of Theorem 2.2. Let $\mathbf{S}$ be a member of $\mathbb{F}$. By Lemma 2.3, $\mathbf{S}$ is equationally equivalent to a set consisting of a nilpotent semigroup and a finite set of monoids from $\mathbb{F}$. By Lemma 2.5, this finite set of monoids can be replaced by a finite set of finite nilpotent monoids. Lemmas 2.6, 2.8, 2.9 show that any finite collection of finite nilpotent monoids (semigroups) is equationally equivalent to a single syntactic monoid (semigroup) of a finite language. Thus $\mathbf{S}$ is equationally equivalent to a set consisting of a syntactic semigroup of a finite language and a syntactic monoid of a finite language. This completes the proof.

3. Preliminaries: a useful construction

In this short section, we gather together some further results, definitions and constructions that are used throughout the remainder of the paper.

We will be examining the semigroup identities of monoids. The following basic lemma is very useful in this regard and we use it freely and without reference below.

Lemma 3.1. *Let $\mathbf{S}$ be a monoid in the type $\langle 2\rangle$ (that is, in the single binary operation of multiplication) and $p \approx q$ be an identity satisfied by $\mathbf{S}$. Then the semigroup identity $p_x \approx q_x$ obtained by deleting all occurrences of a variable x in $p \approx q$ is also satisfied by $\mathbf{S}$.*

Proof. The identity $p \approx q$ is also satisfied by $\mathbf{S}$ considered as a monoid (in the type $\langle 2, 0\rangle$). In the variety of monoids we may deduce by equational deduction the identity $p_x \approx q_x$ (assign 1 to the variable x and use $1y \approx y1 \approx y$) and hence this is also satisfied by $\mathbf{S}$ as a monoid or as a semigroup. ∎

We will also make use of some definitions (see [11] for example). If w is a word and x is a letter, then $\mathrm{occ}(x,w)$ is the number of occurrences of x in the word w. The *content* of the word w, denoted $\mathrm{c}(w)$ is $\{x : x \text{ is a letter and } \mathrm{occ}(x,w) > 0\}$. In the following lemma we interpret x^0 to mean the identity element 1.

Lemma 3.2. *Let $\mathbf{S} \in \mathbb{F}$ and u, v be words such that $\mathbf{S} \models u \approx v$. Say that $n := \min\{\mathrm{occ}(x,u), \mathrm{occ}(x,v)\}$ and $\mathrm{occ}(x,u) \neq \mathrm{occ}(x,v)$. Then*

(i) $\mathbf{S} \models x^n \approx x^{n+1}$ *and*

(ii) $\mathbf{S} \models x^n y \approx y x^n$.

Proof. By Lemma 3.1, $\mathbf{S} \models x^{\mathrm{occ}(x,u)} \approx x^{\mathrm{occ}(x,v)}$, which is equivalent to $x^n \approx x^{n+i}$, for some $i \geq 1$. Aperiodicity then gives $\mathbf{S} \models x^n \approx x^{n+1}$. As $\{x^n \approx x^{n+1}\} \vdash (x^n)^2 \approx x^n$, it follows that any value of x^n in $\mathbf{S}$ is idempotent and therefore commutes with any other element of S. That is, $\mathbf{S} \models x^n y \approx y x^n$. ■

The main examples in this paper are going to be equationally equivalent to syntactic monoids of finite languages, however it is more convenient to give a different description of them. The following concept was introduced by Perkins [24] (although in essence it appears in [21], where it is attributed to Dilworth). Given a set W of possibly empty words in an alphabet X, we let $\bar{W}$ denote the set of all words that are subwords of W, and $I(W)$ denote $X^*\backslash\bar{W}$. It is clear that $I(W)$ is an ideal of X^*, so that $X^*/I(W)$ (which we denote by $\mathrm{S}(W)$) is monoid with zero element (which we denote by 0). The multiplication in $\mathrm{S}(W)$ has 0 acting in the usual way, and for words $u, v \in \bar{W}$ we have $u \cdot v = uv$ if $uv \in \bar{W}$, and $u \cdot v = 0$ otherwise. Clearly $\mathrm{S}(W)$ is finite if and only if W is finite. It is shown in [11] that $\mathrm{S}(W) \approxeq \{\mathrm{S}(\{w\}) : w \in W\}$; also, when W is a singleton $\{w\}$, then $\mathrm{S}(\{w\})$ is isomorphic to $\mathrm{Syn}_{\mathrm{M}}(\{w\})$ (see [8]).

The monoids $\mathrm{S}(W)$ are often more useful for counterexamples than their syntactic monoid equivalents, because it is much easier to trace which identities hold. These are the construction used by Sapir to prove Lemma 1.1 (actually, Sapir's construction does not have an identity element, but as observed in [10], the proof still holds if identity elements are adjoined) and are the central source of examples in [8, 9, 11, 32].

A final important notion, also introduced by Perkins, is that of an isoterm. An *isoterm* for a semigroup $\mathbf{S}$ is a word w for which $\mathbf{S} \models w \approx w'$ implies $w \equiv w'$. This is one of the most useful tools in analysing the semigroup identities of monoids. There is a fundamental connection between isoterms and the constructions introduced in this section.

Lemma 3.3. *Let $\mathbf{S}$ be a monoid and let W be a set of possibly empty words. Then $\mathrm{S}(W) \in \mathbb{V}_{\mathrm{M}}(\mathbf{S})$ if and only if each word in W is an isoterm for $\mathbb{V}_{\mathrm{M}}(\mathbf{S})$*

This result is implicit in all of the above mentioned applications of the construction $\mathrm{S}(W)$. We omit the proof, but note that it is given in [8] in the case when W

is a singleton, while the extended version we state here follows from the singleton case and the fact that if W is a language, then $\mathrm{S}(W) \approxeq \{\mathrm{S}(\{w\}) : w \in W\}$.

4. Strong finite bases

At the start of this paper we have listed three possible relativised versions of the HFB property for a locally finite variety $\underline{\mathbf{V}}$:

(1) all subvarieties of $\underline{\mathbf{V}}$ are $\mathrm{FB}_{\mathrm{fin}}$; or

(2) all finitely generated subvarieties of $\underline{\mathbf{V}}$ are FB; or

(3) all finitely generated subvarieties of $\underline{\mathbf{V}}$ are $\mathrm{FB}_{\mathrm{fin}}$.

If $\underline{\mathbf{V}}$ is a finitely generated semigroup variety, then Lemma 1.1 shows the first of these properties is just the usual notion of a HFB variety, while the second and third coincide. In this section we are going to present a finite semigroup that demonstrates that these later two properties are not equivalent to the general notion of HFB (Theorem 4.3). We note that the second relativised notion of the HFB property is given the name *strongly finitely based* (abbreviated to SFB) in [36] and the possible equivalence of the SFB property with the general HFB property is questioned (Problem 5.2 of [36]); so the example we give here provides a solution to this problem. The example is also a kind of 'relativised limit variety' because we will show that it satisfies the extra property of being minimally not HFB amongst finitely generated varieties.

We note that there are existing examples of semigroup varieties that demonstrate the distinction between the properties of being SFB and HFB, however these varieties are not finitely generated (or even locally finite). The variety of all semigroups satisfying $x^2 \approx 0$ is one simply stated example; this is shown in [14] to have uncountably many non-finitely based subvarieties, while any finite member is nilpotent and therefore finitely based.

Lemma 4.1. *If* $\mathbf{T}$ *is a monoid in* $\mathbb{F}$ *whose semigroup variety does not contain* $\mathrm{S}(\{xyx\})$, *then* $\mathbf{T}$ *satisfies one of the identities* $xyx \approx xxy$ *or* $xyx \approx yxx$.

Proof. As $\mathrm{S}(\{xyx\}) \notin \mathbb{V}(\mathbf{T})$, it follows by Lemma 3.3 that xyx is not an isoterm for $\mathbf{T}$. Let $w \not\equiv xyx$ be such that $\mathbf{T} \models xyx \approx w$. We consider a number of cases.

If $\mathrm{c}(w) \neq \{x, y\}$, then Lemma 3.2 (i) shows that (as a monoid) $\mathbf{T} \models x \approx 1$, and so is trivial.

Now say that $\mathrm{c}(w) = \{x, y\}$. If $\mathrm{occ}(x, w) = 1$, or $\mathrm{occ}(y, w) > 1$, then Lemma 3.2 shows that $\mathbf{T} \models \{x \approx x^2, xy \approx yx\}$ and so also satisfies both $xyx \approx xxy$ and $xyx \approx yxx$. Thus we may assume that $\mathrm{occ}(y, w) = 1$ and $\mathrm{occ}(x, w) \geq 2$.

If $\mathrm{occ}(x, w) = 2$, then $w \in \{xxy, yxx\}$, and we are done. Now say that $\mathrm{occ}(x, w) > 2$. So there are integers $n, m \geq 0$ with $w \equiv x^n y x^m$ and such that $n+m>2$ (here x^0 denotes the empty word). By Lemma 3.2 we have $\mathbf{T}\models\{x^2\approx x^3, x^2y\approx yx^2\}$. As one of n and m is at least equal to 2, we have $\mathbf{T}\models xyx\approx w\approx x^2y\approx yx^2$, as required. ∎

It is shown in [7] that for W a finite set of words, $\mathrm{S}(W)$ is HFB if and only if $\mathrm{S}(\{xyx\})\notin\mathbb{V}(\mathrm{S}(W))$. The following result extends this.

Proposition 4.2. *Let $\mathbf{S}\in\mathbb{F}$. Then $\mathbf{S}$ is HFB if and only if $\mathrm{S}(\{xyx\})\notin\mathbb{V}(\mathbf{S})$.*

Proof. The only if direction follows because $\mathrm{S}(\{xyx\})$ is not HFB [8]. Now say that $\mathrm{S}(\{xyx\})\notin\mathbb{V}(\mathbf{S})$, where $\mathbf{S}\in\mathbb{F}$. By Theorem 2.2, we may find a nilpotent monoid $\mathbf{S}_1$ and a nilpotent semigroup $\mathbf{S}_2$ such that $\mathbf{S}\approxeq\{\mathbf{S}_1,\mathbf{S}_2\}$. By Lemma 4.1 (with $\mathbf{T}=\mathbf{S}_1$), we may assume without loss of generality that $\mathbf{S}_1\models xyx\approx yxx$.

Now $\mathbf{S}_2$ is a nilpotent semigroup, so there is a number $\ell\in\mathbb{N}$ such that $\mathbf{S}_2$ satisfies any semigroup identity in which both sides have length at least ℓ. Hence both $\mathbf{S}_1$ and $\mathbf{S}_2$ satisfy the law

$$xyxx_1x_2\dots x_\ell\approx yxxx_1x_2\dots x_\ell.$$

This identity defines a HFB variety by a result of [25] and so $\mathbf{S}\approxeq\{\mathbf{S}_1,\mathbf{S}_2\}$ is HFB. ∎

Proposition 4.2 shows that $\mathbb{V}(\mathrm{S}(\{xyx\}))$ enjoys limit type characteristics with respect to the HFB property—namely, every proper finitely generated subvariety of $\mathbb{V}(\mathrm{S}(\{xyx\}))$ is HFB, but it itself is not HFB. The proposition also gives us the main result of this section:

Theorem 4.3. *$\mathbb{V}(\mathrm{S}(\{xyx\}))$ is SFB but not HFB.*

Proof. We have already commented that $\mathbb{V}(\mathrm{S}(\{xyx\}))$ is not HFB. Let $\mathbf{S}$ be a finite semigroup in $\mathbb{V}(\mathrm{S}(\{xyx\}))$. This implies that $\mathbf{S}\in\mathbb{F}$.

The semigroup $\mathrm{S}(\{xyx\})$ is finitely based by [11], so if $\mathbf{S}\approxeq\mathrm{S}(\{xyx\})$, then $\mathbf{S}$ is also finitely based. Otherwise, $\mathbb{V}(\mathbf{S})$ is a proper finitely generated subvariety of $\mathbb{V}(\mathrm{S}(\{xyx\}))$ and hence is HFB by Proposition 4.2. ∎

With only a little extra work we can easily give a complete description of the monoid subvarieties of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$, which is useful later in this paper.

Lemma 4.4. *The proper and non-trivial subvarieties of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$ are precisely the variety of monoid semilattices, the variety $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{x\}))$ and the variety $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xy\}))$ and these form a chain under containment.*

Proof. The last claim in the lemma is trivial so we concentrate on finding the monoid subvarieties of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$.

To begin with, we recall some identity bases for the varieties mentioned (we omit the variety of semilattices).

Lemma 4.5. [11]

(i) A basis for the monoid identities of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$ *is*

$$\Sigma_{xyx} := \{xt_1xt_2x \approx xxt_1t_2, xxt \approx txx, xyt_1xt_2y \approx yxt_1xt_2y, \\ xt_1yxt_2y \approx xt_1xyt_2y, xt_1yt_2xy \approx xt_1yt_2yx\}.$$

(ii) A basis for the monoid identities of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xy\}))$ *is*

$$\Sigma_{xy} {:=} \{xtx \approx xxt \approx txx, x^2 \approx x^3\}.$$

(iii) A basis for the monoid identities of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{x\}))$ *is*

$$\Sigma_x {:=} \{xy \approx yx, x^2 \approx x^3\}.$$

Let $\mathbf{S}$ be a monoid generating a monoid variety that is properly contained in $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$. Now $\mathrm{S}(\{xyx\}) \models xxy \approx yxx$, and so $\mathbf{S}$ also satisfies $xxy \approx yxx$ as well as at least one of the identities $xyx \approx xxy$ or $xyx \approx yxx$ by Lemma 4.1. In either case we find that $\mathbf{S}$ in fact satisfies $xxy \approx xyx \approx yxx$. Now note that $\Sigma_{xyx} \cup \{xyx \approx x^2y \approx yx^2\}$ is logically equivalent to Σ_{xy}. Hence every proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$ is a subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xy\}))$.

Now let $\mathbf{S}$ be a monoid generating a monoid variety properly contained in $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xy\}))$. By Lemma 3.3, xy is not an isoterm. Applications of Lemma 3.2 (i), as in the proof of Lemma 4.1, show that $\underline{\mathbf{V}}$ satisfies one of the identities $xy \approx yx$ or $x^2 \approx x$. Adjoining the first of these to Σ_{xy} gives a system logically equivalent to Σ_x, while adjoining the second gives a system logically equivalent to $\{xy \approx yx, x^2 \approx x\}$, which defines the variety of monoid semilattices. Hence every proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xy\}))$ is a subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{x\}))$.

By Lemma 3.3, a proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{x\}))$ satisfies a non-trivial identity $x \approx w$. Lemma 3.2 (i) then shows that such a variety satisfies $x \approx x^2$, and hence is a variety of semilattice monoids. The proof is complete because of the well known fact that the variety of monoid semilattices covers the trivial variety (the reader could again use Lemmas 3.3 and 3.2 (i) to prove this). ■

5. Limit varieties

We now examine the relativised notions of limit varieties introduced at the start of this paper. Recall that a finitely generated variety $\underline{\mathbf{V}}$ could be called a relativised limit variety if:

(1) $\underline{\mathbf{V}}$ is not $\mathrm{FB}_{\mathrm{fin}}$, but all proper subvarieties of $\underline{\mathbf{V}}$ are $\mathrm{FB}_{\mathrm{fin}}$; or

(2) $\underline{\mathbf{V}}$ is not FB, but all proper finitely generated subvarieties of $\underline{\mathbf{V}}$ are FB; or

(3) $\underline{\mathbf{V}}$ is not $\mathrm{FB}_{\mathrm{fin}}$, but all proper finitely generated subvarieties of $\underline{\mathbf{V}}$ are $\mathrm{FB}_{\mathrm{fin}}$.

(We could reasonably extend the scope of this definition to include locally finite varieties.) If $\underline{\mathbf{V}}$ is a finitely generated semigroup variety, then Lemma 1.1 shows that the first of these restrictions is equivalent to the usual notion of a limit variety. Lemma 1.1 also shows that the second and third possible definitions coincide in the case when $\underline{\mathbf{V}}$ is finitely generated. We will say that a finite algebra satisfying the second of the properties is a *limit variety amongst finitely generated varieties* (abbreviated to *limit*$_{\mathit{fin}}$ *variety*).

Unlike in the case of true limit varieties, there seems to be no guarantee that $\mathrm{limit}_{\mathrm{fin}}$ varieties even exist. We note however that if $\underline{\mathbf{V}}$ is a finitely generated limit variety of semigroups, then $\underline{\mathbf{V}}$ is a $\mathrm{limit}_{\mathrm{fin}}$ variety. An example of such a variety is given in [30]. In this section we give two semigroup $\mathrm{limit}_{\mathrm{fin}}$ varieties that are not limit varieties in the true sense. Specifically, we give two examples of finite monoids whose semigroup varieties are non-finitely based and are not semigroup limit varieties but which are nevertheless $\mathrm{limit}_{\mathrm{fin}}$ varieties. As in the previous section, our technique involves an exploitation of the restricted properties of monoid varieties. If we consider our examples in the type $\langle 2, 0\rangle$ they become true limit varieties, and indeed appear to be the first known limit varieties generated by finite monoids. We now state the main result of this section.

Proposition 5.1. *The monoids* $\mathrm{S}(\{absatb, asbtab\})$ *and* $\mathrm{S}(\{asabtb\})$ *generate semigroup varieties that are limit*$_{\mathrm{fin}}$ *varieties, but are not limit varieties. As monoids, they generate monoid varieties that are limit varieties in the true sense.*

Proof. The proof of Proposition 5.1 covers most of the rest of this section.

To begin with, note that neither the semigroup variety of $\mathrm{S}(\{absatb, asbtab\})$ nor that of $\mathrm{S}(\{asabtb\})$ is a limit variety in the true sense. This is because both contain the subvariety $\mathbb{V}(\mathrm{S}(\{xyx\}))$ which has uncountably many non-finitely based subvarieties by [7].

Lemma 5.2. *A NFB monoid* $\mathbf{T} \in \mathbb{F}$ *generates a limit*$_{\mathrm{fin}}$ *variety of semigroups if all proper finitely generated subvarieties of* $\mathbb{V}_{\mathrm{M}}(\mathbf{T})$ *are FB.*

Proof. Assume that $\mathbf{T}$ is NFB (as a semigroup or monoid—as observed in [36], this makes no difference) and that any finitely generated subvariety of $\mathbb{V}_{\mathrm{M}}(\mathbf{T})$ is FB. Since $\mathbf{T}$ is NFB, it suffices to show that all proper finitely generated subvarieties of $\mathbb{V}(\mathbf{T})$ are finitely based. Let $\mathbf{S}$ be a finite semigroup generating a proper subvariety $\mathbb{V}(\mathbf{T})$. By Theorem 2.2, there are finite languages U and V such that $\mathbf{S} \approx \{\mathrm{Syn}(U), \mathrm{Syn}_{\mathrm{M}}(V)\}$.

The central result of [35] states that the join of a semigroup variety $\underline{V}$ with a variety generated by a finite nilpotent semigroup is finitely based if and only if $\underline{V}$ is finitely based. Now $\mathrm{Syn}(U)$ is a finite nilpotent semigroup and

$$\mathbb{V}(\{\mathrm{Syn}(U), \mathrm{Syn}_{\mathrm{M}}(V)\}) = \mathbb{V}(\mathrm{Syn}(U)) \vee \mathbb{V}(\mathrm{Syn}_{\mathrm{M}}(V))$$

and so $\mathbf{S}$ is finitely based if and only if $\mathrm{Syn}_{\mathrm{M}}(V)$ is finitely based as a semigroup. However we may also view $\mathrm{Syn}_{\mathrm{M}}(V)$ as a finite monoid in the monoid variety generated by $\mathbf{T}$. Now $\mathbb{V}_{\mathrm{M}}(\mathrm{Syn}_{\mathrm{M}}(V))$ is a proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathbf{T})$ because $\mathbb{V}(\mathrm{Syn}_{\mathrm{M}}(V)) \subseteq \mathbb{V}(\mathbf{S}) \subsetneq \mathbb{V}(\mathbf{T})$. Hence $\mathrm{Syn}_{\mathrm{M}}(V)$ is finitely based as a monoid, and then also as a semigroup. Therefore $\mathbf{S}$ is finitely based, as required. ∎

Lemma 5.2 shows that all remaining claims of Proposition 5.1 will follow if we can prove that $\mathrm{S}(\{absatb, asbtab\})$ and $\mathrm{S}(\{asabtb\})$ generate limit varieties of monoids.

We recall some further definitions from [11, 32].

Definition 5.3. (i) A word is *n-limited* if $\mathrm{occ}(x,w) \le n$ for all letters x.

(ii) A letter x is *n-occurring* in a word w if $\mathrm{occ}(x,w) = n$; 1-occurring letters are called *linear* letters.

(iii) If w is a word and $x_1, x_2, \ldots, x_n$ are letters then $w(x_1, x_2, \ldots, x_n)$ denotes the word obtained from w by deleting all occurrences of the letters other than $x_1, x_2, \ldots, x_n$.

(iv) If x is a letter and w is a word with $\mathrm{occ}(x,w) \ge n$ then ${}_nx$ denotes the n^{th} occurrence of x in w.

A pair of letters (x,y) is said to be *stable* in an identity $u \approx v$ if $u(x,y) \equiv v(x,y)$; otherwise (x,y) is unstable in $u \approx v$. In the case when (x,y) is unstable in $u \approx v$, it is often useful to refer to the particular occurrences of the two letters where instability is 'appearing'. To this end, we say that a pair $({}_ix, {}_jy)$ is an *unstable occurrence pair* in an identity $u \approx v$ if the order of appearance of the i^{th} occurrence of x and the j^{th} occurrence of y in u is different to that of v. A pair $({}_ix, {}_jy)$ is a *critical occurrence pair* in an identity $u \approx v$ if it is an unstable occurrence pair and ${}_ix\,{}_jy$ is a subword of u or v. An unstable pair of letters (x,y) in $u \approx v$ is *critical*, if there are numbers i and j such that $({}_ix, {}_jy)$ is a critical occurrence pair.

We may also extend these definitions to stability in words. A pair of letters (x,y) or occurrences of letters $({}_ix, {}_jy)$ is said to be stable in a word w (relative to some particular semigroup $\mathbf{S}$) if it is stable in any identity $w \approx v$ satisfied by $\mathbf{S}$. It is easily verified that every nontrivial *balanced* identity (that is, one side is a permutation of the other side) contains at least one unstable pair of letters, at least one unstable occurrence pair and at least one critical pair.

We now return to the aims of this section by proving that the two monoids under consideration are non-finitely based.

Lemma 5.4. $\mathrm{S}(\{absatb, asbtab\})$ *is NFB.*

Proof. For all integers $n>3$, let $u_{1,n}$ and $u_{2,n}$ denote the words $x_1x_2\dots x_{n^2}$ and

$$(x_1x_{n+1}\dots x_{n^2-n+1})(x_2x_{n+2}\dots x_{n^2-n+2})\dots(x_nx_{2n}\dots x_{n^2})$$

respectively (the brackets indicate 'blocks' to be used below) and let u_n denote the word $u_{1,n}tu_{2,n}$. It is routinely observed that for any permutation $\bar{u}_{1,n}$ of $u_{1,n}$ and $\bar{u}_{2,n}$ of $u_{2,n}$, $\mathrm{S}(\{absatb, asbtab\}) \models u_n \approx \bar{u}_{1,n}t\bar{u}_{2,n}$; indeed the only assignments θ for which both sides of this identity do not take the value 0, must assign all but perhaps one of the letters x_i the value 1, and then $\theta(u_n) = \theta(\bar{u}_{1,n}t\bar{u}_{2,n})$.

Now note the following two facts:

(1) If $i < j \leq n^2$ are such that $j - i < n$, then the subword of $u_{2,n}$ between x_i and x_j has at least $n-2$ letters.

(2) Every two letter subword of u_n appears just once in u_n.

Only the first of these claims requires detailed proof. Divide the word $u_{2,n}$ into blocks of length n, as indicated by the brackets in the definition of $u_{2,n}$ and let $i < j \leq n^2$ be such that $j-i<n$. Now x_i and x_j cannot lie in the same block, because this is so if and only if $i \equiv j \bmod(n)$ and then $j > i$ implies $j - i \geq n$.

Now assume x_i and x_j lie in consecutive blocks. If x_i and x_j lie in the ℓ^{th} and $(\ell+1)^{\text{th}}$ blocks respectively then there are integers $0 \leq k_1, k_2 < n$ such that $i = k_1n + \ell$ and $j = k_2n + \ell + 1$. But then, since $0 < j - i < n$, we have $0 < (k_2 - k_1)n + 1 < n$, showing that $k_1 = k_2$ and that the subword between x_i and x_j has exactly $n-1$ letters.

If x_i and x_j lie in the $(\ell+1)^{\text{th}}$ and $(\ell)^{\text{th}}$ blocks respectively, then there are $0 \leq k_1, k_2 < n$ such that $i = k_1n + \ell + 1$ and $j = k_2n + \ell$. Thus $0 < (k_2 - k_1)n - 1 < n$, showing that $k_2 = k_1 + 1$, and that the length of the subword between x_i and x_j is exactly $n-2$ letters.

Finally, if x_i and x_j lie in neither the same block nor in consecutive blocks then there is at least one block totally contained within the subword between the occurrences of x_i and x_j, which gives more than $n-2$ letters in this subword. This proves the claim.

Now let $p \approx q$ be an identity satisfied by $\mathrm{S}(\{absatb, asbtab\})$ in fewer than $n > 3$ variables. Assume (to derive a contradiction) that there is a semigroup substitution θ such that $\theta(p)$ is a subword of u_n and $\theta(p) \not\equiv \theta(q)$. Let (x,y) be an unstable pair in $p \approx q$. Evidently, both x and y are at most 2-occurring in $p \approx q$. Now since xyx is an isoterm, at least one of x and y is not a linear letter in p. If one, say y, is linear in p, then $p(x,y)$ is equal to xxy or yxx. However every 2-occurring letter in u_n has a linear letter between its occurrences. Hence there must be a linear letter, t say, between the two occurrences of x in p. Then $p(x,y,t) \equiv xtxy$ or $p(x,y,t) \equiv yxtx$, both of which are isoterms, contradicting the fact that (x, y) is unstable in $p \approx q$. It follows that neither x nor y are linear in p and thus we may assume that $\mathrm{occ}(x, p) = \mathrm{occ}(y, p) = 2$.

The form of the word u_n indicates that without loss of generality we can write p as $p_1 x p_2 y p_3 t p_4 x p_5 y p_6$ or $p_1 x p_2 y p_3 t p_4 y p_5 x p_6$ for some possibly empty words p_1, p_2, p_3, p_4, p_5, p_6, and some letter t, linear in p. Since xt_1yt_2xy and xt_1yt_2yx are isoterms, there cannot be a linear letter in p_2. By observation (5) above, for each 2-occurring letter z in $c(p)$, $\theta(z)$ is a 2-occurring letter in u_n (as opposed to some longer word). Therefore, up to a change of letter names, xp_2y is of the form $y_i y_{i+1} \dots y_{i+k}$ for some $k \geq 1$, where $k < n$ and $\theta(x_\ell) = y_\ell$ (for $\ell = i, \dots, i+k$). Hence by observation (5) above (choosing $j = i+k$), $\theta(p_5)$ contains at least $n-2$ letters and so p_5 contains a linear letter, say t' (because $|c(p) \backslash \{x, y\}| \leq n-3$). But then $p(x, y, t, t')$ is an isoterm, a contradiction.

It now follows that no finite set of identities for $\mathrm{S}(\{absatb, asbtab\})$ is an equational basis. ■

Lemma 5.5. $\mathrm{S}(\{asabtb\})$ *is NFB.*

Proof. For every $n \in \mathbb{N}$, let v_n denote the word

$$z_1 t_1 z_2 t_2 \dots z_n t_n x z_1 z_{n+1} z_2 z_{n+2} \dots z_n z_{2n} x t_{n+1} z_{n+1} \dots t_{2n} z_{2n}$$

and v_n' denote the word

$$z_1 t_1 z_2 t_2 \dots z_n t_n x x z_1 z_{n+1} z_2 z_{n+2} \dots z_n z_{2n} t_{n+1} z_{n+1} \dots t_{2n} z_{2n}.$$

It is not hard to verify that if θ is assignment such that both $\theta(v_n)$ and $\theta(v_n')$ do not take the value 0 in $\mathrm{S}(\{asabtb\})$, then $\theta(x) = 1$ and hence $\theta(v_n) = \theta(v_n')$; this is because there is no linear letter between the two occurrences of x in either word, while the only repeated subwords of $asabtb$ have linear letters between their two occurrences. It follows that $\mathrm{S}(\{asabtb\}) \models v_n \approx v_n'$ for every $n \in \mathbb{N}$.

Now consider the following observations (whose elementary proofs we leave to the reader).

(1) For all $i, j \leq 2n$, (z_i, t_j) is stable in v_n.

(2) If $i \leq n$ and $j > n$ then (z_i, z_j) is stable in v_n.

(3) Every two letter subword of v_n occurs just once in v_n.

Now if $1 \leq i < j \leq n$ we can find $k > n$ such that

$$v_n(t_i, t_j, t_k, z_i, z_j, z_k) \equiv z_i t_i z_j t_j z_i z_k z_j t_k z_k$$

which is an isoterm. Hence (z_i, z_j) is stable in v_n and by symmetry the same holds true for the case when $2n \geq i > j > n$. It follows from this and observations (1), (2) above, that the word obtained from v_n by deleting x is an isoterm for $\mathrm{S}(\{asabtb\})$. Denote this word by w_n.

Now let $p \approx q$ be an identity satisfied by $\mathrm{S}(\{asabtb\})$ and assume that there exists a substitution θ such that $\theta(p)$ is a subword of v_n and $\theta(p) \not\equiv \theta(q)$. Let (y_1, y_2) be a critical pair in $p \approx q$. Since $\mathrm{S}(\{asabtb\})$ is not commutative, at least one of y_1 and y_2, say y_1, must be 2-occurring in p and then because w_n is an isoterm, and by observation (3), we may assume without loss of generality that $\theta(y_1) = x$. We may write p in the form $p_1 y_1 p_2 y_1 p_3$ for some possibly empty words p_1, p_2, p_3. Evidently, $\theta(p_2) \equiv z_1 z_{n+1} z_2 z_{n+2} \dots z_n z_{2n}$. Now assume (to obtain a contradiction) that p contains fewer than $2n$ letters. By observation (3), p_2 contains a letter, t, that is linear in p. Note that t is distinct from y_2 since $p(y_1, t) \equiv y_1 t y_1$ showing that (y_1, t) is stable in p. If y_2 is linear in p then $p(y_1, y_2, t) \in \{y_1 t y_1 y_2, y_1 t y_2 y_1, y_1 y_2 t y_1, y_2 y_1 t y_1\}$, all of which are isoterms, contradicting the instability of (y_1, y_2) in p. If y_2 is 2-occurring, then observation (3) implies that $\theta(y_2) = z_i$ for some $i \leq 2n$. Now because (y_1, y_2) is a critical pair we have that $i = 1$ or $2n$. However, because y_2 is 2-occurring, there is a linear letter $s \in c(p)$ such that $t_i \in c(\theta(s))$. So

$$p(y_1, y_2, t, s) \in \{y_1 t y_2 y_1 s y_2, y_2 s y_1 y_2 t y_1\}$$

which are again all isoterms for $\mathrm{S}(\{asabtb\})$. This contradiction shows that that $p \approx q$ must have at least $2n$ letters. Since n was arbitrary, $\mathrm{S}(\{asabtb\})$ is non-finitely based. ■

Next we show that every monoid generating a proper subvariety of one of the two given varieties is finitely based. It turns out that $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$ presents the most difficulties and we consider this first.

We begin by giving a set of identities satisfied by $\mathrm{S}(\{absatb, asbtab\})$. We will then show that adjoining any identity that fails on $\mathrm{S}(\{absatb, asbtab\})$ to this set gives a (finitely generated) monoid variety that is either a subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{aba\}))$, and hence finitely based by Lemma 4.4, or is a subvariety of the variety generated by one of the two monoids $\mathrm{S}(\{absatb\})$ or $\mathrm{S}(\{asbtab\})$. The proof will be complete when these two monoids are shown to be HFB as monoids.

The proof of the following lemma is left to the reader.

Lemma 5.6. *Let Σ_1 denote the set of all identities obtainable from those in $\{x^2 \approx x^3, xt_1xt_2x \approx x^2t_1t_2 \approx t_1t_2x^2, xt_1xyt_2y \approx xt_1yxt_2y\}$ by deleting all occurrences of some (possibly empty) collection of letters. Then $\mathrm{S}(\{absatb, asbtab\}) \models \Sigma_1$.*

Lemma 5.7. *Let $\underline{\mathbf{V}}$ be a proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$. Then $\underline{\mathbf{V}} \models absatb \approx basatb$ or $\underline{\mathbf{V}} \models asbtab \approx asbtba$.*

Proof. Let $\underline{\mathbf{V}}$ be a proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$. By Lemma 3.3, one of $absatb$ and $asbtab$ is not an isoterm for $\underline{\mathbf{V}}$. Without loss of generality we will assume that $absatb$ is not an isoterm (the other case will follow by symmetry). If $\underline{\mathbf{V}}$ is a subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$ then Lemmas 4.4 and 4.5 show that we are done. Now assume that $\underline{\mathbf{V}}$ is not a subvariety of

$\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$. Now if $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$ is not a subvariety of $\underline{\mathbf{V}}$, then Lemma 4.1 shows that one of the identities $xyx \approx xxy$ or $xyx \approx yxx$ holds in $\underline{\mathbf{V}}$. However, in the presence of Σ_1 these identities are equivalent and imply both $asbtab \approx asbtba$ and $absatb \approx basatb$. Thus we may assume further that $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$ is a subvariety of $\underline{\mathbf{V}}$. In particular, Lemma 3.3 shows that xyx is an isoterm for $\underline{\mathbf{V}}$.

Let $absatb \approx w$ be a non-trivial identity satisfied by $\underline{\mathbf{V}}$. Since xyx is an isoterm for $\underline{\mathbf{V}}$, the pairs (a,s), (b,s) and (b,t) must be stable in $absatb \approx w$. But then (a,t) is also stable and hence the only unstable pair is (a,b). Note also that because (a,s) and (b,s) are stable we must have $\mathrm{occ}(a,w) = \mathrm{occ}(b,w) = 2$. Therefore $w \equiv basatb$, as required. ∎

Lemma 5.7 implies that any variety properly contained in $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$ satisfies at least one of the sets $\Sigma_1 \cup \{xyt_1xt_2y \approx yxt_1xt_2y\}$ or $\Sigma_1 \cup \{xt_1yt_2xy \approx xt_1yt_2yx\}$. We now show that these sets are equational bases (within the variety of all monoids) for the monoids $\mathrm{S}(\{asbtab\})$ and $\mathrm{S}(\{absatb\})$ respectively. Hence we will have shown that any proper monoid subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$ is a subvariety of one of the varieties $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asbtab\}))$ or $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb\}))$.

Proposition 5.8. *The set $\Phi := \Sigma_1 \cup \{xyt_1xt_2y \approx yxt_1xt_2y\}$ is a monoid basis for the identities of* $\mathrm{S}(\{asbtab\})$.

Proof. The identity $xt_1xt_2x \approx xxt_1t_2 \in \Phi$ can be used to reduce every word to one that is 2-limited. Thus it suffices to consider 2-limited identities.

Let x be a 2-occurring letter in a word w for which there is no linear letter t such that $w(x,t) \equiv xtx$ and let w_x be the word obtained from w by deleting x. We first show that $\Phi \vdash w \approx xxw_x$.

If xx is a subword of w, then an application of $xxt_1 \approx t_1xx \in \Phi$ transforms w into the desired form. If xx is not a subword then we may write w in the form $w_1xw_2yxw_3$ where y is 2-occurring in w (and w_2 is a possibly empty subword containing no letters that are linear in w). Applications of one of the identities $yt_1xt_2yx \approx yt_1xt_2xy$, $xt_1yxt_2y \approx xt_1xyt_2y$ or their reverse, now gives the identity $w_1xw_2yxw_3 \approx w_1xw_2xyw_3$. (Note that if $w \equiv w_1xu_1yu_2yxw_3$ then the application of $yt_1xt_2yx \approx yt_1xt_2xy$ is via the substitution θ defined by $\theta(y) = x$, $\theta(t_1) = u$, $\theta(x) = y$ and $\theta(t_2) = u_2$.) Repeating this procedure eventually gives $w \approx w_4xxw_5$, where $w_x \equiv w_4w_5$, and then $xxt \approx txx$ may be used as before.

By repeating the above paragraph for each 2-occurring letter that does not have occurrences either side of a linear letter, we eventually arrive at a word of the form uv, where u is a product of squares of letters (that can be arbitrarily commuted using $xxt_1 \approx t_1xx \in \Phi$) and every 2-occurring letter in v occurs either side of a linear letter. Now say that $w \approx w'$ is a 2-limited identity satisfied by $\mathrm{S}(\{asbtab\})$, where both w and w' have been rearranged in this way; that is $w \equiv uv$ and $w' \equiv u'v'$, where u and u' are products of squares of letters and every two occurring letter in v and v' occurs either side of a linear letter. By deleting letters it is clear that $\mathrm{S}(\{asbtab\})$ also satisfies $u \approx u'$ and $v \approx v'$. Because we can commute squares using Φ, we find that $\Phi \vdash u \approx u'$. Therefore it will now suffice to show that $\Phi \vdash v \approx v'$.

Because xyx is an isoterm for $\mathrm{S}(\{asbtab\})$, any unstable pair in $v \approx v'$ must involve only 2-occurring letters. Let $({}_ix, {}_jy)$ be a critical pair. There are several possibilities for the pattern of occurrences of x and y in v, however the identities $xyt_1xt_2y \approx yxt_1xt_2y, xt_1xyt_2y \approx xt_1yxt_2y$ enable $({}_ix, {}_jy)$ to be removed from the set of all unstable occurrence pairs in $v \approx v'$ in all but the case where v is of the form $v_1xv_2yv_3xyv_4$ or $v_1xv_2yv_3yxv_4$. In either case, the assumption that a linear letter occurs between the two occurrences of every 2-occurring letter enables us to write v more specifically in one of the forms $v_1xv_2yv_5tv_6xyv_4$ or $v_1xv_2yv_5tv_6yxv_4$ where t is linear. It will be clear below that there is no essential difference between these two cases (indeed we will use Φ to derive $v_1xv_2yv_5tv_6xyv_4 \approx v_1xv_2yv_5tv_6yxv_4$) and so we will assume that v is of the first form.

Since (x, y) is unstable while xt_1yt_2yx and xt_1yt_2xy are isoterms it follows that v_2 cannot contain any letter that is linear in v. If v_2 is empty, then we may apply the identity $xytxy \approx xytyx$ (which is obtained up to a change in letter names from $xt_1yt_2xy \approx xt_1yt_2yx \in \Phi$ by deletion of the letter t_1). Otherwise v_2 is nonempty and so there is some 2-occurring letter z such that $v \equiv v_1xzv_7yv_5tv_6xyv_4$ (where v_7 is such that $zv_7 \equiv v_2$). Regardless of the position of the other occurrence of z, the identities $xyt_1xt_2y \approx yxt_1xt_2y, yt_1yxt_2x \approx yt_1xyt_2x$ enable us to derive $v_1xzv_7yv_5tv_6xyv_4 \approx v_1zxv_7yv_5tv_6xyv_4$. Since v_7 contains no letters that are linear in w, we may repeat this procedure until finally arriving at the word $v_1v_2xyv_5tv_6xyv_4$. The identity $xytxy \approx xytyx$ now gives us $v_1v_2xyv_5tv_6xyv_4 \approx v_1v_2xyv_5tv_6yxv_4$.

We may now reverse the procedure above, eventually returning us the word $v_1xv_2yv_5tv_6yxv_4$. That is, we have derived

$$v \equiv v_1xv_2yv_5tv_6xyv_4 \approx v_1xv_2yv_5tv_6yxv_4.$$

The two words differ only in that the second occurrences of x and y have been switched. No new unstable occurrence pairs have been created and in fact one unstable occurrence pair in $v \approx v'$ has been removed. Clearly the number of unstable occurrence pairs in $v \approx v'$ is finite and therefore if this procedure is performed sufficiently many times we eventually obtain a word v'' for which $\Phi \vdash v \approx v''$ and such that $v'' \approx v'$ has no unstable occurrence pairs. That is, we have $v'' \equiv v'$ and therefore $\Phi \vdash v \approx v'$ as required. ■

By symmetry, a similar result holds for $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb\}))$.

Lemma 5.9. *Every proper monoid subvariety of* $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asbtab\}))$ *is a monoid subvariety of* $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{aba\}))$.

Proof. Let $\underline{\mathbf{V}}$ be a proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asbtab\}))$. By Lemma 3.3, we may assume that $asbtab$ is not an isoterm for $\underline{\mathbf{V}}$ and that $\underline{\mathbf{V}}$ is not a

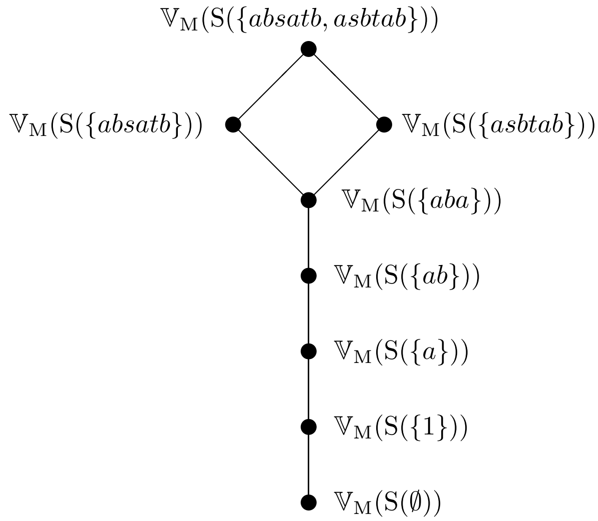

Figure 1: The lattice of subvarieties of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$.

proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{aba\}))$. Arguments using Lemma 3.2 (i) as in the proof of Lemma 5.7 then easily show that $\underline{\mathbf{V}}$ satisfies $asbtab \approx asbtba$. However $\Phi \cup \{asbtab \approx asbtba\}$ is a basis for the monoid identities of $\mathrm{S}(\{aba\})$ (it contains a copy of each of the identities given in the basis Σ_{xyx} of Lemma 4.5). Hence $\underline{\mathbf{V}} = \mathbb{V}(\mathrm{S}(\{aba\}))$, and therefore every proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asbtab\}))$ is a subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{aba\}))$ and therefore HFB by Lemmas 4.4 and 4.5. ■

Again, a similar result holds for $\mathrm{S}(\{asbtab\})$. We now complete the proof of the first case in Proposition 5.1.

Lemma 5.7, Proposition 5.8, Lemma 5.9 and their duals combine with Lemma 4.4 to show that the monoid subvarieties of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$ are precisely those shown in Figure 1. Proposition 5.8, its dual and Lemma 4.4 show that the proper subvarieties $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$ are all FB, while Lemma 5.4 shows that $\mathrm{S}(\{absatb, asbtab\})$ is NFB (as a monoid or as a semigroup). All this shows that $\mathrm{S}(\{absatb, asbtab\})$ generates a limit variety of monoids, and so Lemma 5.2 shows that $\mathrm{S}(\{absatb, asbtab\})$ generates a limit$_{\mathrm{fin}}$ variety of semigroups.

Note that $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$ has the apparently unusual property of being a NFB variety that is the join of two finitely based aperiodic semigroup varieties; the only other examples with this property are given in [11].

Now we consider the subvarieties of $\mathrm{S}(\{asabtb\})$.

Lemma 5.10. *Every proper monoid subvariety of* $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asabtb\}))$ *is a monoid subvariety of* $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$.

Proof. This proof is almost identical to that of Lemma 5.9. First note that $\mathrm{S}(\{asabtb\})$ satisfies the set Σ_2 consisting of all identities obtainable from

$$\Sigma_2 := \{x^2 \approx x^3, xt_1xt_2x \approx x^2t_1t_2 \approx t_1t_2x^2, xt_1yt_2yx \approx xt_1yt_2xy, \\ xyt_1xt_2y \approx yxt_1xt_2y\},$$

by deleting all occurrences of some (possibly empty) collection of letters. Let $\underline{\mathbf{V}}$ define a proper subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asabtb\}))$. If $\mathrm{S}(\{xyx\})$ is not in $\underline{\mathbf{V}}$, then by Lemma 4.1, $\underline{V}$ satisfies one of $xyx \approx xxy$ or $xyx \approx yxx$. However, in the presence of Σ_2, these imply all of the identities in the set Σ_{xy} of Lemma 4.5. But then $\underline{V}$ is a subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xy\}))$, and so certainly a subvariety of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$. Now say that $\mathrm{S}(\{xyx\}) \in \underline{\mathbf{V}}$. By Lemma 3.3, xyx is an isoterm for $\underline{\mathbf{V}}$, but by Lemma 3.3, $asabtb$ is not an isoterm for $\underline{\mathbf{V}}$. Thus $\underline{\mathbf{V}} \models asabtb \approx asbatb$, and more particularly $\underline{V} \models \Sigma_2 \cup \{asabtb \approx asbatb\}$. Each identity from the set Σ_{xyx} of Lemma 4.5 can be derived from $\Sigma_2 \cup \{asabtb \approx asbatb\}$ (we leave this easy check to the reader), and hence $\underline{\mathbf{V}} = \mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{xyx\}))$ as required. ∎

This lemma and Lemma 4.4 combine to completely describe the proper monoid subvarieties of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asabtb\}))$, which are all FB—their bases are given in Lemma 4.5. (Thus the lattice of monoid subvarieties of $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asabtb\}))$ is the six element chain.) Hence $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asabtb\}))$ is a limit variety of monoids. Lemma 5.2 now shows that $\mathbb{V}(\mathrm{S}(\{asabtb\}))$ is a limit$_{\mathrm{fin}}$ variety.

This completes the proof of Proposition 5.1. ∎

We have been unable to find any other examples of finitely generated limit varieties of aperiodic monoids with central idempotents. Certainly, all non-finitely based monoids found in [8, 11, 32] generate varieties containing at least one of the two cases given above. Even outside of the class $\mathbb{F}$, the two examples given here appear to be very 'small'; for example, it follows from [27] that all three of $asabtb$, $absatb$, $asbtab$ are isoterms for inherently non-finitely based finite semigroups, which in the monoid case (every inherently non-finitely based semigroup contains an inherently non-finitely based submonoid [28]) implies that $\mathrm{S}(\{asabtb\})$ and $\mathrm{S}(\{absatb, asbtab\})$ are contained in the corresponding varieties (by Lemma 3.3). On the other hand, there are known to be infinitely many limit varieties of groups [22], though none of these lie in any finitely generated semigroup variety and none have been explicitly described. We also note that a finite completely regular monoid without a finite basis of identities would establish the existence of a limit variety other than those presented above because such an example would satisfy the law $x \approx x^{n+1}$ for some $n \in \mathbb{N}$, a property not shared by $\mathrm{S}(\{asabtb\})$ and $\mathrm{S}(\{absatb, asbtab\})$. Certain monoids are conjectured to have this property in [36], Problem 6.1. We pose the following questions.

Question 1. Are $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{asabtb\}))$ and $\mathbb{V}_{\mathrm{M}}(\mathrm{S}(\{absatb, asbtab\}))$ the only limit varieties generated by finite aperiodic monoids with central idempotents?

Are there any finitely generated but non-finitely based aperiodic monoid varieties that do not contain these varieties?

Note that any counterexample to the first of these questions would have to satisfy at least one of the following two sets of identities: $\{xsxyty \approx xsyxty, xsytxy \approx xsytyx\}$ or $\{xsxyty \approx xsyxty, xysxty \approx yxsxty\}$.

6. Other finiteness properties

In this section we briefly discuss the behaviour of some other finiteness properties with known connections to the finite basis problem. In addition to the FB and HFB properties, the most popular finiteness properties are probably:

(1) all subvarieties of $\underline{\mathbf{V}}$ are finitely generated;

(2) $\underline{\mathbf{V}}$ contains (up to isomorphism) only finitely many subdirectly irreducible members;

(3) $\underline{\mathbf{V}}$ has only finitely many subvarieties;

(4) $\underline{\mathbf{V}}$ is contained in a finitely based locally finite variety.

The first of these properties does not appear to be of any interest upon relativising to the class of finitely generated varieties.

The obvious relativised version of Property (2) is covered by a well known result of Quackenbush; namely a locally finite variety with finitely many subdirectly irreducibles has no infinite subdirectly irreducibles [26, 4]. (Note that because each variety is determined by its subdirectly irreducible members, we have the following implications between the first three properties on locally finite varieties: (2)$\Rightarrow$(1); and (2)$\Rightarrow$(3).) While Property (2) is of deep importance in universal algebra, it has not played such a significant role in semigroup theory because so few varieties actually satisfy it; a complete description can be found in [6] or [18].

Relativising Property (3) to finite algebras presents an interesting question; namely is there a locally finite variety with infinitely many subvarieties, but only finitely many finitely generated subvarieties? The answer (which is negative) is probably folklore, but to complete the picture we give its proof.

In the following lemma, and elsewhere to follow, we say that a variety is *non-finitely generated* if it cannot be generated by any finite algebra.

Lemma 6.1. *Let $\underline{\mathbf{V}}$ be a non-finitely generated locally finite variety of arbitrary type. The poset of finitely generated subvarieties of $\underline{\mathbf{V}}$ (partially ordered by containment) does not satisfy the ascending chain condition. Hence $\underline{\mathbf{V}}$ contains infinitely many finitely generated subvarieties.*

Proof. Let $\underline{\mathbf{V}}$ be a non-finitely generated locally finite variety. For $n \in \mathbb{N}$, let $\underline{\mathbf{V}}_n$ denote the variety generated by the n-generated $\underline{\mathbf{V}}$-free algebra. Note that each $\underline{\mathbf{V}}_n$ is finitely generated since finitely generated algebras in locally finite varieties are finite. Also, $\underline{\mathbf{V}}_1 \subseteq \underline{\mathbf{V}}_2 \subseteq \dots$ is a chain.

Now $\underline{\mathbf{V}}$ is non-finitely generated and hence not equal to $\underline{\mathbf{V}}_n$ for any $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$ there is an identity $p_n \approx q_n$ that fails in $\underline{\mathbf{V}}$ but is satisfied by $\underline{\mathbf{V}}_n$. If m is the number of variables in $p_n \approx q_n$, then, by freeness, $p_n \approx q_n$ fails on the m-generated free algebra in $\underline{\mathbf{V}}$ and therefore is not satisfied by $\underline{\mathbf{V}}_m$. Hence for all $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that the containment $\underline{\mathbf{V}}_n \subset \underline{\mathbf{V}}_m$ is proper, whence the chain $\underline{\mathbf{V}}_1 \subseteq \underline{\mathbf{V}}_2 \subseteq \dots$ is infinite and has no maximum member. ∎

Note that this shows that Property (3) above implies Property (1) for locally finite varieties.

Proposition 6.2. *Let $\underline{\mathbf{V}}$ be a locally finite variety of arbitrary type. Then $\underline{\mathbf{V}}$ has infinitely many subvarieties if and only if it has infinitely many finitely generated subvarieties.*

Proof. The 'if' direction is trivial. So assume that $\underline{\mathbf{V}}$ has infinitely many subvarieties. If all subvarieties of $\underline{\mathbf{V}}$ are finitely generated we are done. Otherwise, $\underline{\mathbf{V}}$ contains a (locally finite) subvariety that is non-finitely generated, which by Lemma 6.1 contains infinitely many finitely generated subvarieties. ∎

Only Property (4) remains. A locally finite variety (or an algebra generating such a variety) failing Property (4) is said to be *inherently non-finitely based* (INFB), while otherwise such a variety is said to be *weakly finitely based*. The most natural relativised version of this property would be as follows: a finite algebra is inherently non-finitely based in the class of finite algebras if it is not contained in any finitely generated, finitely based variety. Following [36], we will say that such an algebra is *strongly not finitely based* (SNFB). While every INFB finite algebra is SNFB, there are no known examples contradicting the reverse implication. The possible existence of a SNFB but not INFB finite semigroup is Problem 4.4 in [36]. A similar open problem exists for inverse semigroups: in [16] Kleiman asked whether or not every finite inverse semigroup (in the type $\langle 2, 1 \rangle$) whose variety contains $\mathbf{B}_2^1$ is NFB. M. Sapir [31] has proved that there are no INFB finite inverse semigroups, so if the answer to Kleiman's problem is positive, then $\mathbf{B}_2^1$ is SNFB amongst finite inverse semigroups but not INFB. Kleiman's problem is restricted to inverse semigroups so is perhaps weaker than the general SNFB implies INFB problem. A partial solution to this problem has recently been obtained by Kad'ourek [15] who has shown that $\mathbf{B}_2^1$ is SNFB within any (sufficiently large) inverse semigroup variety whose groups are solvable. The author and T. Stokes have also recently constructed a 9-element semigroup with additional binary operation that is not INFB but is SNFB within a certain finitely based variety [12]. Finally, we note that the corresponding notions of SNFB and INFB for quasivarieties are known to be distinct (see [36] for discussion).

7. Decidability

To conclude this paper we explain how to obtain the undecidability of all of the finiteness properties we have considered in this paper, at least in the case of universal algebras. Within semigroups, corresponding solutions are known for only two of the properties: it is decidable whether or not a finite semigroup generates a weakly finitely based variety (Property (4) of Section 6; see [27, 28]) or a variety with only finitely many subdirectly irreducibles (Property (2) of Section 6; see [6, 18]). A solution to the question of decidability for any of the remaining properties would be of substantial interest.

A celebrated result of R. McKenzie is the undecidability of the FB property for finite algebras [20]. That is, there is no algorithm to decide whether or not an arbitrary finite algebra has a finite basis of identities. McKenzie's proof of this result involves a modification of a related undecidability result (also by McKenzie, and equally celebrated) in [19]. Soon after the appearance of these results, R. Willard [37] showed that the undecidability of the finite basis problem can be established directly from [19], without the need of modification. While the undecidability of the problems we are interested in will follow from only brief (even trivial) observations of the combined results of [37] and [19], they do not seem to have been stated elsewhere.

In [19], McKenzie shows how to effectively associate with each Turing machine program $\mathcal{T}$, a finite algebra $\mathbf{A}(\mathcal{T})$ with the property that $\mathbf{A}(\mathcal{T})$ generates a residually very finite variety (that is, satisfying Property (2) of Section 6) if and only if $\mathcal{T}$ eventually halts when started on the blank tape. (We do not require a precise definition of a Turing machine here, but the reader unfamiliar with the concept may think of it as some kind of computer program, where 'started on the blank tape' corresponds to the program being run without any input). The main difficulties of [37] are showing that in the halting case, the algebra $\mathbf{A}(\mathcal{T})$ is finitely based. McKenzie already proves in [19] that $\mathbf{A}(\mathcal{T})$ is (inherently) non-finitely based in the case when $\mathcal{T}$ does not halt and so the undecidability of the finite basis problem (as well as the problems of deciding satisfaction of Properties (2) and (4) of Section 6) then follows because of the undecidability of the halting problem for Turing machines started on blank tapes.

We will say that a variety $\underline{\mathbf{V}}$ is *hereditarily finitely generated* (HFG) if every subvariety is finitely generated (Property (1) of Section 6). Recall that a variety is *small* if it has only finitely many subvarieties (Property (3) of Section 6) and HFB if all its subvarieties are finitely based. In this section our algebras are not semigroups but we nevertheless use the notation $\mathbb{V}(\mathbf{A})$ to denote the variety generated by an algebra $\mathbf{A}$.

Theorem 7.1. *The following decision problems are undecidable: given an arbitrary finite algebra $\mathbf{A}$, decide whether or not $\mathbb{V}(\mathbf{A})$ is HFB, HFG or small.*

Proof. Let $\mathcal{T}$ be a Turing machine that eventually halts when started on the blank tape, and consider $\mathbb{V}(\mathbf{A}(\mathcal{T}))$. This variety is finitely based by [37] and contains only finitely many isomorphism types of subdirectly irreducibles (all finite) by [19]. Every subvariety of $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ is determined by the subdirectly irreducibles it contains and therefore $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ is small and HFG. It is well known and easy to prove that a finitely based variety that contains a non-finitely based subvariety has infinitely many subvarieties. As $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ is small and finitely based it must also be HFB.

Now consider the case when $\mathcal{T}$ does not halt on the blank tape. In this case $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ is not HFB since it is non-finitely based by [19]. We now need to show that $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ is neither HFG nor small. By Lemma 6.1 it suffices to show that $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ contains a non-finitely generated subvariety.

McKenzie [19] shows that $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ contains a particular algebra $\mathbf{Q}_{\mathbb{Z}}$ which can be defined as follows. The universe of $\mathbf{Q}_{\mathbb{Z}}$ is the set $\{0, a_i, b_i : i \in \mathbb{Z}\}$. The operations are: a product $\cdot$ defined by $a_i \cdot b_{i+1} = b_i$ and $x \cdot y = 0$ otherwise; a meet operation $\wedge$ given by $x \wedge x = x$ and $x \wedge y = 0$ otherwise; the nullary operation 0 (actually the algebra has more operations than this, but all others are term operations in these three). We show that $\mathbb{V}(\mathbf{Q}_{\mathbb{Z}})$ is non-finitely generated.

Consider a finite algebra $\mathbf{A}$ in the variety $\mathbb{V}(\mathbf{Q}_{\mathbb{Z}})$ with, say, n elements. We need to find an identity satisfied by $\mathbf{A}$ that fails on $\mathbf{Q}_{\mathbb{Z}}$. For any i, j with $1 \le i < j \le n+1$, the identity

$$x_1(\dots(x_i(\dots(x_{j-1}(x_i(x_{j+1}(\dots(x_n x_{n+1})\dots))))\dots))\dots) \approx 0$$

can be seen to hold on $\mathbf{Q}_{\mathbb{Z}}$, while $x_1(x_2(\dots(x_n x_{n+1})\dots)) \approx 0$ fails on $\mathbf{Q}_{\mathbb{Z}}$ because $a_1(a_2(\dots(a_n b_{n+1})\dots)) = b_1 \neq 0$. Let $c_1, \dots, c_{n+1} \in A$ and consider the product $c_1(c_2(\dots(c_n c_{n+1})\dots))$. Since $|A| = n$, there is $i \neq j$ such that $c_i = c_j$ and therefore

$$c_1(c_2(\dots(c_n c_{n+1})\dots)) = c_1(\dots(c_{j-1}(c_i(c_{j+1}(\dots(c_n c_{n+1})\dots))))\dots) = 0.$$

Hence $\mathbf{A} \models x_1(x_2(\dots(x_n x_{n+1})\dots)) \approx 0$ and so $\mathbb{V}(\mathbf{A}) \neq \mathbb{V}(\mathbf{Q}_{\mathbb{Z}})$ as required.

The undecidability of the three problems now follows immediately from the undecidability of the halting problem for Turing machines started on blank tapes. ∎

This proof also shows that the non-halting case is equivalent to $\mathbb{V}(\mathbf{A}(\mathcal{T}))$ failing to be SFB and failing to have its lattice of subvarieties satisfy the ascending chain condition; hence these properties are also undecidable for finite algebras.

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