Semigroup Forum Vol. 68 (2004) 357–364

DOI: 10.1007/s00233-003-0016-x

RESEARCH ARTICLE

Some Remarks on Some Second-order Hyperbolic Differential Equations

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Communicated by Jerome A. Goldstein

Abstract

We are concerned with the almost automorphic solutions to the second-order hyperbolic differential equations of type $\ddot{u}(s) + 2B\dot{u}(s) + Au(s) = f(s) \quad (*)$, where A, B are densely defined closed linear operators acting in a Hilbert space $\mathbb{H}$, and $f : \mathbb{R} \mapsto \mathbb{H}$ is a vector-valued almost automorphic function. Using invariant subspaces, it will be shown that under appropriate assumptions, every solution to $(*)$ is almost automorphic.

Key words and phrases: Almost automorphic functions; invariant subspace; reducing subspace.

2000 AMS Subject Classification: 34G10; 47B44.

1. Introduction

This paper deals with the almost automorphic solutions to the homogeneous second-order hyperbolic differential equation of the form

$$\frac{d^2}{ds^2}u(s) + 2B\ \frac{d}{ds}u(s) + A\ u(s) = 0, \tag{1}$$

and the associated nonhomogeneous differential equation

$$\frac{d^2}{ds^2}u(s) + 2B\ \frac{d}{ds}u(s) + A\ u(s) = f(s), \tag{2}$$

where A, B are densely defined closed linear operators acting in a Hilbert space $\mathbb{H}$ and $f : \mathbb{R} \mapsto \mathbb{H}$ is an almost automorphic vector-valued function.

We use *invariant subspaces theory* to show that under appropriate assumptions, every solution to the equations (1) and/or (2) is an *almost automorphic* vector-valued function. The idea of using the method of invariant subspaces to study the existence of almost automorphic solutions is recent and due to Diagana and N'Guerekata[3].

This paper is dedicated to Abdel Nasser Ould Othman Sid'Ahmed Yessa, within the framework of our friendship and the common values that we both share.

Let us indicate that the invariant subspaces method works smoothly in the framework of abstract differential equations involving the algebraic sum of unbounded linear operators.

The existence and uniqueness of a solutions to equations (1)-(2) have been of great interest for many mathematicians in the past decades. Note the pioneer work of S. G. Krein[6] regarding the solvability to (1)-(2) over $s \in [0,1]$. Recently, many important contributions to this problem have been made in ([4], [7]). Our interest in paper is to focus on solutions of it that are almost automorphic.

Now setting $v(s) = \dfrac{d}{ds}u(s)$, the problem (1)-(2) can be rewritten in $\mathbb{H}\times\mathbb{H}$ of the form

$$\frac{d}{ds}\mathcal{U}(s) = (\mathcal{A} + \mathcal{B})\,\mathcal{U}(s), \tag{3}$$

and

$$\frac{d}{ds}\mathcal{U}(s) = (\mathcal{A} + \mathcal{B})\,\mathcal{U}(s) + F(s), \tag{4}$$

where $\mathcal{U}(s) = (u(s), v(s))$, $F(s) = (0, f(s))$ and $\mathcal{A}, \mathcal{B}$ are the operator matrices of the form

$$\mathcal{A} = \begin{pmatrix} O & I \\ -A & O \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} O & O \\ O & -2B \end{pmatrix},$$

on $\mathbb{H}\times\mathbb{H}$ with $D(\mathcal{A}) = D(A)\times\mathbb{H}$, $D(\mathcal{B}) = \mathbb{H}\times D(B)$, and O, I denote the zero and identity operators on $\mathbb{H}$, respectively. Since (1)-(2) is equivalent to (3)-(4), instead of studying (1)-(2), we will focus on the characterization of almost automorphic solutions to (3)-(4).

First we recall some tools in section 2, then we use them to prove our main results in section 3.

2. Preliminaries

2.1. Invariant subspaces

Let $\mathbb{H}$ be a Hilbert space and let $S \subset \mathbb{H}$ be a closed subspace. Let A be a densely defined closed unbounded linear operator on $\mathbb{H}$ and let P_S denote the orthogonal projection onto the closed subspace S.

Definition 2.1. S is said to be an *invariant subspace* for A if we have the inclusion $A(D(A)\cap S) \subset S$.

Example 2.2. Let us mention the following classical invariant subspaces for a given linear operator A defined in a Hilbert space $\mathbb{H}$.

1. $S = N(A) = \{x \in D(A) : \; Ax = 0\}$ is an invariant subspace for A.

2. Assume that the operator A is self-adjoint, we may take S as any eigenspace $N(\lambda I - A)$. Thus, $S = N(\lambda I - A)$ is invariant for A.

Theorem 2.3. *The equality $P_S A P_S = A P_S$ is a necessary and sufficient condition for a subspace S to be invariant for a linear operator A.*

Proof. Assume $P_S A P_S = A P_S$ and if $x \in D(A) \cap S$, then $x = P_S x \in D(A)$ and $Ax = A P_S x = P_S A P_S x \in S$.

Conversely, if S is invariant for A; let $x \in \mathbb{H}$ such that $P_S x \in D(A)$. Then $A P_S x \in S$ and then $P_S A P_S x = A P_S x$. Therefore $A P_S \subset P_S A P_S$. Since $D(A P_S) = D(P_S A P_S)$, it turns out that $A P_S = P_S A P_S$. ■

Definition 2.4. A closed proper subspace S of the Hilbert space $\mathbb{H}$ is said to reduce an operator A if $P_S D(A) \subset D(A)$ and both S and $H \ominus S$, the orthogonal complement of S, are invariant for A.

Using Theorem 2.3, the following key result can be proved.

Theorem 2.5. *A closed subspace S of $\mathbb{H}$ reduces an operator A if and only if $P_S A \subset A P_S$.*

Proof. See the proof in [8, Theorem 4.11., p. 29]. ■

Remark 2.6. In fact the meaning of the inclusion $P_S A \subset A P_S$ is that: if $x \in D(A)$, then $P_S x \in D(A)$ and $P_S A x = A P_S x$.

Throughout the paper $\mathbb{H}$, $D(C)$, $R(C)$ and $N(C)$, denote a Hilbert space, the the domain of C, the range and the kernel of the linear operator C, respectively.

Let A and B be densely defined closed unbounded linear operators on $\mathbb{H}$. Recall that their algebraic sum is defined by

$$D(A+B) = D(A) \cap D(B) \text{ and } (A+B)x = Ax + Bx, \quad \forall x \in D(A) \cap D(B).$$

Since both A and B are densely defined, then the algebraic sum of $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{S} = \mathcal{A} + \mathcal{B}$ is also a densely defined operator and

$$D(\mathcal{A}+\mathcal{B}) = D(A) \times D(B), \quad \text{and} \quad (\mathcal{A}+\mathcal{B})\mathcal{U} = \mathcal{A}\mathcal{U} + \mathcal{B}\mathcal{U}.$$

Throughout the paper, $\mathcal{A}$ and $\mathcal{B}$ will play similar roles.

2.2. Almost automorphic functions

Definition 2.7. A continuous function $f : \mathbb{R} \mapsto \mathbb{H}$ is said to be almost automorphic if for every sequence of real numbers (σ_n), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n \mapsto \infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n \mapsto \infty} g(t - s_n)$$

for each $t \in \mathbb{R}$.

The range of an almost automorphic function is relatively compact on $\mathbb{H}$, therefore it is bounded. Almost automorphic functions constitute a Banach space $\mathbb{A}A(\mathbb{H})$ under the supnorm. They generalize naturally the concept of almost periodic functions as introduced by Bochner in the early sixties. For applications to differential equations, it is necessary to study derivatives and integrals of almost automorphic functions. This is well presented in [10]. We recall some results we need in the sequel:

Theorem 2.8. *Let $f: \mathbb{R} \mapsto \mathbb{R}$ be an almost automorphic function. Then the function F defined by $F(t) = \int_0^t f(s)ds, \quad t \in \mathbb{R}$ is almost automorphic iff its range is bounded in $\mathbb{H}$.*

The integral here is understood in Bochner's sense for vector-valued functions. Detailed proofs of these results can be found in [10].

Setting our main result, instead of assuming that similar assumptions hold as in [10, Theorem 4.4.1], the following assumptions will be made:

The operators $\mathcal{A}$ and $\mathcal{B}$ are infinitesimal generators of C_0-groups of bounded operators $(\mathcal{T}(t))_{s\in\mathbb{R}}$, $(\mathcal{R}(t))_{s\in\mathbb{R}}$, respectively, such that

(i) $\mathcal{T}(s)\mathcal{U} : s \mapsto \mathcal{T}(s)\mathcal{U}$ is almost automorphic for each $\mathcal{U} \in \mathbb{H}\times\mathbb{H}$, $\mathcal{R}(s)\mathcal{V}$: $s \mapsto \mathcal{R}(s)\mathcal{V}$ is almost automorphic for each $\mathcal{V} \in \mathbb{H}\times\mathbb{H}$, respectively.

(ii) there exists $S \subset \mathbb{H}\times\mathbb{H}$, a closed subspace that reduces both $\mathcal{A}$ and $\mathcal{B}$.

We denote by P_S, $Q_S = (I \times I - P_S) = P_{[\mathbb{H}\times\mathbb{H}]\ominus S}$, the orthogonal projections onto S and $[\mathbb{H}\times\mathbb{H}]\ominus S$, respectively.

(iii) $R(\mathcal{A}) \subset R(P_S) = N(Q_S)$

(iv) $R(\mathcal{B}) \subset R(Q_S) = N(P_S)$

Remark 2.9. 1. Recall that if A, B generate C_0-groups, their sum $\mathcal{A}+\mathcal{B}$ need not be a C_0-group generator.

2. The assumption (ii) implies that both S and $[\mathbb{H}\times\mathbb{H}\ominus S]$ are invariant for the algebraic sum (it is well-defined as stated above) $\mathcal{A}+\mathcal{B}$.

3. The main results

3.1. Almost automorphic solutions to (3)-(4)

We have the following result.

Theorem 3.1. *Under assumptions (i)-(ii)-(iii)-(iv), every solution to the differential equation (3) is almost automorphic.*

Proof. Let $X(s)$ be a solution to (3). Clearly $X(s) \in D(\mathcal{A})\cap D(\mathcal{B}) = D(A)\times D(B)$ (notice that the algebraic sum $\mathcal{S} = \mathcal{A}+\mathcal{B}$ exists since $\overline{D(\mathcal{S})} = \mathbb{H}\times\mathbb{H}$).

Now decompose $X(s)$ as follows

$$X(s) = P_S X(s) + (I \times I - P_S) X(s), \tag{5}$$

where $P_S X(s) \in R(P_S) = N(Q_S)$, and $Q_S X(s) \in N(P_S) = R(Q_S)$.

We have

$$\begin{aligned}
\frac{d}{ds}(P_S X(s)) &= P_S \frac{d}{ds} X(s) \\
&= P_S \mathcal{A} X(s) + P_S \mathcal{B} X(s) \\
&= \mathcal{A} P_S X(s) + P_S \mathcal{B} X(s) \quad (\text{according to}(ii)) \\
&= \mathcal{A} P_S X(s) \quad (\text{according to}(iv))
\end{aligned}$$

From $\frac{d}{ds}(P_S X(s)) = \mathcal{A} P_S X(s)$, it follows that

$$P_S X(s) = \mathcal{T}(t) P_S X(0). \tag{6}$$

Now according to (i), the vector-valued function $s \mapsto P_S X(s) = \mathcal{T}(t) P_S X(0)$ is almost automorphic.

In the same way, since $[\mathbb{H} \times \mathbb{H}] \ominus ([\mathbb{H} \times \mathbb{H}] \ominus S) = S$. It follows that the closed subspace S reduces $\mathcal{A}$ and $\mathcal{B}$ if and only if $[\mathbb{H} \times \mathbb{H}] \ominus S$ does. In other words, $[\mathbb{H} \times \mathbb{H}] \ominus S$ reduces $\mathcal{A}$ and $\mathcal{B}$. That is, a similar remark as remark 2.6 holds when S is replaced by $[\mathbb{H} \times \mathbb{H}] \ominus S$. Thus, we have

$$\begin{aligned}
\frac{d}{ds}(Q_S X(s)) &= Q_S \frac{d}{ds} X(s) \\
&= Q_S \mathcal{A} X(s) + Q_S \mathcal{B} X(s) \\
&= Q_S \mathcal{A} X(s) + \mathcal{B} Q_S X(s) \quad (\text{according to}(ii)) \\
&= \mathcal{B} Q_S X(s) \quad (\text{according to}(iii))
\end{aligned}$$

From the equation $\frac{d}{dt}(Q_S X(s)) = \mathcal{B} Q_S X(s)$, it follows that $s \mapsto Q_S X(s) = \mathcal{R}(s) Q_S X(0)$ is almost automorphic (according to (i)).

Therefore $X(s) = P_S X(s) + Q_S X(s)$ is also almost automorphic as the sum of almost automorphic vector-valued functions. ■

We now get a slightly different version of N'Guerekata's result (see [10, Theorem 4.4. 1, p. 84]); in the case where $B : \mathbb{H} \mapsto \mathbb{H}$ is a bounded linear operator on $\mathbb{H}$ (this implies that $\mathcal{B}$ is bounded on $\mathbb{H} \times \mathbb{H}$).

Corollary 3.2. *Let $B : \mathbb{H} \mapsto \mathbb{H}$ be a bounded linear operator in the Hilbert space $\mathbb{H}$. Under assumptions (i)-(ii)-(iii)-(iv). Then every solution to the equation* (3) *is almost automorphic.*

Proof. This an immediate consequence of the Theorem 3.1 to the case where $\mathcal{B}$ is a bounded linear operator, it is straightforward. ■

Consider the nonhomogeneous equation (4). Assume that the vector valued function $f:\ \mathbb{R} \mapsto \mathbb{H}$ is almost automorphic. In fact, this implies that $F:\ s \mapsto (0, f(s))$ is in $\mathbb{A}A(\mathbb{H}\times\mathbb{H})$.

We have

Theorem 3.3. *Under assumption (i)-(ii)-(iii)-(iv), assume that* $f \in \mathbb{A}A(\mathbb{H}) \cap L^1(\mathbb{R}, \mathbb{H})$. *Then every solution to the equation* (4) *is almost automorphic.*

Proof. Let $X(s)$ be a solution to (4). As in the proof of Theorem 3.1, the solution $X(s) \in D(\mathcal{A}) \cap D(\mathcal{B})$. Now express $X(s)$ as $X(s) = P_S X(s) + Q_S X(s)$, where P_S, $Q_S = (I\times I - P_S) = P_{[\mathbb{H}\times\mathbb{H}]\ominus S}$ are the orthogonal projections defined above.

We have

$$\begin{aligned}
\frac{d}{ds}(P_S X(s)) &= P_S \frac{d}{ds} X(s) \\
&= P_S \mathcal{A} X(s) + P_S \mathcal{B} X(s) + P_S F(s) \\
&= \mathcal{A} P_S X(s) + P_S \mathcal{B} X(s) + P_S F(s) \quad \text{(according to}(ii)) \\
&= \mathcal{A} P_S X(s) + P_S F(s) \quad \text{(according to}(iv))
\end{aligned}$$

From $\frac{d}{ds}(P_S X(s)) = \mathcal{A} P_S X(s) + P_S F(s)$; it follows that

$$P_S X(s) = \mathcal{T}(s) P_S X(0) + \int_0^s \mathcal{T}(s-\sigma) P_S F(\sigma) d\sigma.$$

Set $G(s) = \int_0^s \mathcal{T}(s-\sigma) P_S F(\sigma) d\sigma$. First observe that $\sigma \mapsto \mathcal{T}(-\sigma)\ P_S\ F(\sigma)$ is almost automorphic. Moreover the function $x(s) =: \int_0^s\ \mathcal{T}(-\sigma) P_S F(\sigma)\ d\sigma$ is bounded (as it can be easily proved), thus it is almost automorphic by Theorem 2.8 above. Now $\mathcal{T}(s)\ x(s) = G(s)$ is almost automorphic.

According to assumption (i), the vector-valued function $s \mapsto P_S X(s) = \mathcal{T}(s) P_S X(0)$ is almost automorphic. Therefore $s \mapsto P_S X(s)$ is almost automorphic as the sum of almost automorphic vector-valued functions.

In the same way, it is not hard to see that

$$\frac{d}{ds}(Q_S X(s)) = \mathcal{B} Q_S X(s) + Q_S F(s),$$

and that $Q_S X(s)$ can be expressed as

$$Q_S X(s) = \mathcal{R}(s) Q_S X(0) + \int_0^s \mathcal{R}(s-\sigma) Q_S F(\sigma) d\sigma.$$

Using similar arguments as above, it can be shown that $s \mapsto Q_S X(s)$ is almost automorphic. Therefore $X(s) = P_S X(s) + Q_S X(s)$ is also an almost automorphic vector-valued function. ∎

Remark 3.4. Let us notice that the previous results (Theorem 3.1 and Theorem 3.3) still hold in the case where $\mathcal{A}, \mathcal{B} : \mathbb{H} \times \mathbb{H} \mapsto \mathbb{H} \times \mathbb{H}$ are bounded linear operator matrices on $\mathbb{H} \times \mathbb{H}$. In such a case, the similar assumptions are required, that is, (i)-(ii)-(iii) and (iv).

Acknowledgement

The author wants to express many thanks to the referee for his/her comments and suggestions on the first version of this paper. Also, the author wants to thank Professor Goldstein for his latest suggestions that significantly improved the paper.

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Received August 9, 2003
and in final form October 16, 2003
Online publication January 12, 2004