Heat and Mass Transfer 36 (2000) 159–163 © Springer-Verlag 2000

Thermal instability in an anisotropic rotating porous medium

S. M. Alex, P. R. Patil

Abstract Influenced by the article of Vadasz [1], an analysis has been carried out to investigate convective instability due to centrifugal acceleration in an anisotropic porous medium. Results reveal that anisotropy in thermal diffusivity destabilizes the system whereas that in permeability has the opposite effect.

1
Introduction

The onset of convection in a fluid saturated, rotating porous medium is of focal interest to geophysicists and engineers, since it has a wide spectrum of applications. Flows in porous geological formations, the flow of magma in the earth's crust, and packed bed mechanically agitated vessels used in the food processing and chemical industries serve as examples, [2]. The effect of rotation on convective flow during the solidification of binary alloys has been investigated in [3], viewing the dendritic mushy zone formed as a porous medium, so that the topic finds application in the materials processing industry too.

Thermal instability in a rotating system consisting of horizontally superposed fluid and porous layers has been researched in [4]. The effect of rotation on natural convection in a horizontal porous cylinder and an annular porous layer are presented in [5] and [6], respectively. The stability of a rotating double diffusive fluid saturated porous medium has been analysed in [7]. The flow and heat transfer characteristics of laminar mixed convection in a radially rotating semiporous channel has been investigated in [8]. The effect of rotation on the natural convective flow of an incompressible viscous fluid between two heated vertical walls in a rotating porous system has been analysed in [9].

The thermal instability of a rotating fluid saturated heterogenous porous channel has been studied in [2]. The effect of the Coriolis force on convection in fluid saturated porous boxes has been focused on in [10–12]. Natural convection in rotating porous layers and porous boxes, adjacent to and at a distance from the axis of rotation, due only to the centrifugal body force has been analysed in [1, 13–17]. The window of parameters within which the Coriolis force and gravitational body force can Ce neglected, so that the convection is due to the centrifugal force alone, has been obtained in [15]. The onset of convection in a rotating fluid saturated porous layer subject to the gravitational and centrifugal body forces is investigated analytically in [18]. An experimental and analytical investigation of the temperature and flowfields resulting from centrifugally driven free convection in a rotating Hele Shaw cell has been presented in [19].

As most naturally occurring porous media are anisotropic, the aim of this article is to evaluate the effect of the centrifugal body force alone, in generating convective instability in a rotating fluid saturated anisotropic porous medium. The validity of the Boussinesq approximation is assumed. Both Darcy and Brinkman models have been considered.

2
Mathematical formulation

A fluid saturated anisotropic porous layer, rotating with constant angular velocity ω about the Z axis, whose positive direction is vertically upwards, opposing the direction of gravity, is considered. The porous layer is bounded by impermeable, perfectly conducting planes at $x = 0$ and $x = L$ and extends to infinity in the Y and Z directions, where the principle directions of permeability and thermal diffusivity are assumed to be coincident and are taken to be the Cartesian directions. The planes $x = 0$ and $x = L$ are maintained at constant temperatures T_C and T_H respectively, $T_C < T_H$. This sets up a temperature gradient collinear with the centrifugal body force. The gravity force which is orthogonal to the temperature gradient is neglected. For simplicity it has been assumed that the permeabilities are equal in the Y and Z directions, and so also the thermal diffusivities. That is, $k_y = k_z \neq k_x$ and $\kappa_y = \kappa_z \neq \kappa_x$ where k_x, k_y and k_z denote the permeabilities in the X, Y and Z directions respectively and κ_x, κ_y and κ_z the diffusivities in these directions.

2.1
Darcy model

Assuming the Boussinesq approximation, and neglecting the Coriolis force and the component of the centrifugal acceleration in the Y direction, the governing equations are,

$$\nabla \cdot \vec{q} = 0 \tag{1}$$

$$\frac{\mu\vec{q}}{(k_x, k_y, k_y)} = -\nabla p - \rho_0\beta(T - T_C)\omega^2 x\hat{i} \tag{2}$$

Received on 26 February 1999

S. M. Alex, P. R. Patil
Department of Mathematics
Anna University, Chennai-600 025, India

Correspondence to: P. R. Patil

$$\frac{\partial T}{\partial t} + M_f \vec{q} \cdot \nabla T = (\kappa_x, \kappa_y, \kappa_y)\nabla^2 T \tag{3}$$

$$\rho = \rho_0[1 - \beta(T - T_C)] \tag{4}$$

where $\vec{q} = (u, v, w)$, p, ρ, ρ_0, β and T are the fluid velocity, pressure, density, density at temperature T_C, coefficient of volume expansion and temperature respectively and $\hat{i}$ the unit vector in the X direction. M_f is the ratio of the heat capacity of the fluid to the porous matrix and t denotes time.

Since the boundaries $x = 0$ and $x = L$ are impermeable and perfectly conducting and are maintained at temperatures T_C and T_H respectively, the boundary conditions are $u = 0$ on $x = 0$ and $x = L$, $T = T_C$ on $x = 0$ and $T = T_H$ on $x = L$.

It may be noted that this problem is analogous to the Bénard problem of a fluid layer heated from below, and subject only to the gravity force.

The nondimensional governing equations (with $M_f = 1$) and boundary conditions are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

$$u = -\frac{\partial p}{\partial x} - \mathrm{R}_T x T \tag{6}$$

$$\frac{v}{k_1} = -\frac{\partial p}{\partial y} \tag{7}$$

$$\frac{w}{k_1} = -\frac{\partial p}{\partial z} \tag{8}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \kappa_1\frac{\partial^2 T}{\partial y^2} + \kappa_1\frac{\partial^2 T}{\partial z^2} \tag{9}$$

$$u = 0 \text{ on } x = 0 \text{ and } x = 1, \quad T = 0 \text{ on } x = 0,\ T = 1 \text{ on } x = 1\ . \tag{10}$$

The scaling has been done using

$$(x^*, y^*, z^*) = \frac{1}{L}(x, y, z), \quad (u^*, v^*, w^*) = \frac{L}{\kappa_x}(u, v, w)$$

$$t^* = \frac{\kappa_x}{L^2}t, \quad p^* = \frac{k_x}{\mu\kappa_x}p, \quad T^* = \frac{T - T_C}{T_H - T_C} = \frac{T - T_C}{\Delta T} \tag{11}$$

and the asterisks dropped for simplicity. k_1 and κ_1 are the anisotropy parameters of permeability and diffusivity and have values k_y/k_x and κ_y/κ_x respectively.

$$\mathrm{R}_T = \mathrm{R}_\omega K' \quad \text{where } \mathrm{R}_\omega = \frac{\beta\Delta T\omega^2 L^4}{\upsilon\kappa_x}$$

is the dimensionless Rayleigh number due to the centrifugal acceleration, υ denotes the kinematic viscosity of the fluid and $K' = k_x/L^2$ is the Brinkman number which assumes values less than 10^{-3} for the Darcy model and exceeds 10^{-3} for the Brinkman model [20].

2.2 Basic state

The basic state is assumed to be quiescent in which there are temperature gradients only in the x direction so that,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = \frac{\partial}{\partial t} = 0, \quad \frac{\partial}{\partial x} \neq 0 \tag{12}$$

This leads to

$$\vec{q}_b = 0, \ \ T_b = x \ \text{ and } \ p_b = -\mathrm{R}_T\frac{x^3}{3} + \text{constant} \tag{13}$$

where the suffix 'b' denotes the basic state quantities.

2.3 Perturbed state

The modified flow field when small perturbations are imposed on the basic state are given by

$$\begin{aligned} \vec{q} &= \hat{\vec{q}} = (\hat{u}, \hat{v}, \hat{w}) \\ T &= T_b + \hat{T} \\ p &= p_b + \hat{p} \end{aligned} \tag{14}$$

where the carets denote perturbed quantities.

Assuming a normal mode expansion in the form

$$(\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{T}) = (u(x), v(x), w(x), p(x), \theta(x))\mathrm{e}^{\mathrm{i}[ky + mz - \sigma t]} \tag{15}$$

the linearised equations for velocity and temperature perturbations are

$$(D^2 - k_1\alpha^2)u(x) - k_1\alpha^2\mathrm{R}_T x\theta(x) = 0 \tag{16}$$

$$u(x) - (D^2 + \mathrm{i}\sigma - \kappa_1\alpha^2)\theta(x) = 0 \tag{17}$$

where $D = \mathrm{d}/\mathrm{d}x$, $\alpha = (k^2 + m^2)^{1/2}$ is the dimensionless wave number and σ is the frequency of disturbance, subject to the boundary conditions

$$\begin{aligned} u(x) &= 0 \quad \text{on } x = 0 \text{ and } x = 1 \\ \theta(x) &= 0 \quad \text{on } x = 0 \text{ and } x = 1\ . \end{aligned} \tag{18}$$

It can be noted that the system of Eqs. (16) and (17) subject to the boundary conditions (18), is an eigenvalue problem for R_T.

2.4 Method of solution

It can be shown that the principle of exchange of stabilities is valid. Hence the onset of the stationary mode of convection ($\sigma = 0$), has been analysed.

The Galerkin method which has been found to yield excellent results even at lower orders [21], is adopted to obtain the critical Rayleigh number. Expanding $u(x)$ and $\theta(x)$, in a series of the orthogonal trial functions $u_i = \theta_i = \sin i\pi x,\ i = 1, 2, \ldots,$ which satisfy the boundary conditions, and restricting to two terms of the expansion, we have

$$\begin{aligned} u(x) &= a_1 u_1 + a_2 u_2 \\ \theta(x) &= a_3\theta_1 + a_4\theta_2 \end{aligned} \tag{19}$$

where a_1, a_2, a_3, a_4 are constants.

Substituting for u and θ in (16) and (17), multiplying each of the resulting equations by each of the trial functions $u_1 = \theta_1 = \sin \pi x, u_2 = \theta_2 = \sin 2\pi x$ and then integrating these equations between $x = 0$ and $x = 1$, leads to,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = 0 \ . \tag{20}$$

Equation (20) is a homogeneous set of linear algebraic equations in a_1, a_2, a_3 and a_4 which yield a nontrivial solution only for particular values of R_T provided $\det(A_{ij}) = 0$, where

$$\begin{aligned} A_{11} &= \frac{\pi^2 + k_1\alpha^2}{2}, \\ A_{12} &= A_{21} = A_{32} = A_{34} = A_{41} = A_{43} = 0 \\ A_{13} &= A_{24} = \frac{\mathrm{R}_T k_1 \alpha^2}{4}, \quad A_{14} = A_{23} = \frac{-8\mathrm{R}_T k_1 \alpha^2}{9\pi^2}, \\ A_{22} &= \frac{4\pi^2 + k_1\alpha^2}{2}, \\ A_{31} &= A_{42} = \frac{1}{2}, \quad A_{33} = \frac{\pi^2 + \kappa_1\alpha^2}{2}, \text{ and} \\ A_{44} &= \frac{4\pi^2 + \kappa_1\alpha^2}{2} \ . \end{aligned} \tag{21}$$

3 Brinkman model

Using the same assumptions as used for the Darcy model, analysis has been carried out for the Brinkman model where the viscous resistance term $\mu\nabla^2\vec{q}$ is added to the right hand side of Eq. (2), the Eqs. (1), (3) and (4), remaining unaltered. The boundaries at $x = 0$ and $x = L$ are impermeable and perfectly conducting and are maintained at temperatures T_C and T_H respectively, $T_C < T_H$.

On simplifying and expressing as normal modes as before, we arrive at

$$(D^2 - k_1\alpha^2)u(x) - \mathrm{R}_T k_1 \alpha^2 x \theta(x) - K' k_1 (D^2 - \alpha^2)^2 u(x) = 0 \tag{22}$$

$$u(x) - (D^2 + \mathrm{i}\sigma - \kappa_1\alpha^2)\theta(x) = 0 \tag{23}$$

subject to the boundary conditions

$$\begin{aligned} &u(x) = 0 \quad \text{and} \quad Du = 0 \quad \text{on } x = 0 \text{ and } x = 1 \\ &\theta(x) = 0 \quad \text{on } x = 0 \text{ and } x = 1 \ . \end{aligned} \tag{24}$$

It can be proved that the principle of exchange of stabilities is valid in this case too. The Galerkin method is applied with orthogonal trial functions $u_i = \theta_i = \sin \pi x \sin i\pi x, \ i = 1, 2, \ldots,$ which satisfy the boundary conditions identically. Adopting a two term approximation, writing

$$u = a_1 u_1 + a_2 u_2$$
$$\theta = a_3 \theta_1 + a_4 \theta_2$$

in Eqs. (22) and (23), multiplying by each of the trial functions and integrating the resulting equation between $x = 0$ and $x = 1$, we obtain Eq. (20) which is a system of homogeneous equations in the constants a_1, a_2, a_3, a_4 yielding a non trivial solution for certain values of R_T if $\det(A_{ij}) = 0$, where

$$\begin{aligned} A_{11} &= \left(\frac{4\pi^2 + 3k_1\alpha^2}{8}\right) + K'k_1\left(\frac{3\alpha^4 + 8\pi^2\alpha^2 + 16\pi^4}{8}\right), \\ A_{12} &= A_{21} = A_{32} = A_{34} = A_{41} = A_{43} = 0, \\ A_{13} &= 3\mathrm{R}_T k_1 \alpha^2/16 \\ A_{14} &= A_{23} = -\frac{32\mathrm{R}_T k_1 \alpha^2}{75\pi^2}, \\ A_{22} &= \left(\frac{5\pi^2 + k_1\alpha^2}{4}\right) + K'k_1\left(\frac{\alpha^4 + 10\pi^2\alpha^2 + 41\pi^4}{4}\right) \\ A_{24} &= \frac{\mathrm{R}_T k_1 \alpha^2}{8}, \quad A_{31} = 3/8, \quad A_{33} = \frac{4\pi^2 + 3\kappa_1\alpha^2}{8} \\ A_{42} &= 1/4 \quad \text{and} \quad A_{44} = \frac{5\pi^2 + \kappa_1\alpha^2}{4} \end{aligned} \tag{25}$$

4 Results and discussion

The value of α which gives the minimum value of R_T, that is the critical wave number α_{crit} and the corresponding value of R_T that is the critical Rayleigh number $\mathrm{R}_{T_{\text{crit}}}$ are determined from $\det(A_{ij}) = 0$, with $Aij's$ given by (21) for the Darcy model and (25) for the Brinkman model. The results for different values of the anisotropy parameters which have been varied under the assumption that k_x is greater than k_y and κ_x is greater than κ_y, [22] are presented in Table 1 for the Darcy model, and in Tables 2–4 for the Brinkman model, for Brinkman numbers 0.001, 0.01 and 0.1, and are presented graphically in Figs. 1–4.

From the Tables 1–4 and Figs. 1–4 it is seen that the critical values of the Rayleigh number for the Brinkman model, for all values of the Brinkman number considered, exceeds that of the Darcy model for each of the values of the anisotropy parameters, thereby implying that the Brinkman model is more stable than the Darcy model. Further an increase in the permeability imparts stability to the system.

Table 1. Critical Rayleigh numbers and wave numbers: Darcy model

κ_1	1		0.5		0.1	
k_1	$\mathrm{R}_{T_{\text{crit}}}$	α_{crit}	$\mathrm{R}_{T_{\text{crit}}}$	α_{crit}	$\mathrm{R}_{T_{\text{crit}}}$	α_{crit}
1	77.082747	3.19491	56.142315	3.800959	33.285647	5.706091
0.2	201.49549	4.787864	128.320825	5.685483	56.142315	8.499203
0.1	332.85647	5.706091	201.49549	6.771062	77.082747	10.103194

Table 2. Critical Rayleigh numbers and wave numbers: Brinkman model ($K' = 0.001$)

κ_1	1		0.5		0.1	
k_1	$R_{T_{crit}}$	α_{crit}	$R_{T_{crit}}$	α_{crit}	$R_{T_{crit}}$	α_{crit}
1	107.710398	3.64344	78.700839	4.30927	47.494395	6.25253
0.2	276.928064	5.37617	177.704534	6.3352	80.619796	9.10108
0.1	456.092169	6.36258	278.44957	7.48135	111.194127	10.68491

Table 3. Critical Rayleigh numbers and wave numbers: Brinkman model ($K' = 0.01$)

κ_1	1		0.5		0.1	
k_1	$R_{T_{crit}}$	α_{crit}	$R_{T_{crit}}$	α_{crit}	$R_{T_{crit}}$	α_{crit}
1	144.909459	3.50665	107.574415	4.04357	69.411476	5.298
0.2	326.476534	4.78047	218.398061	5.48397	114.234608	7.17367
0.1	520.367082	5.52779	332.33029	6.32386	155.976528	8.2658

Table 4. Critical Rayleigh numbers and wave numbers: Brinkman model ($K' = 0.1$)

κ_1	1		0.5		0.1	
k_1	$R_{T_{crit}}$	α_{crit}	$R_{T_{crit}}$	α_{crit}	$R_{T_{crit}}$	α_{crit}
1	511.091811	3.32001	387.778305	3.72294	267.7227	4.51466
0.2	723.862401	3.77134	526.346473	4.2284	336.611609	5.16049
0.1	963.2016	4.14126	678.469835	4.64254	407.859687	5.69392

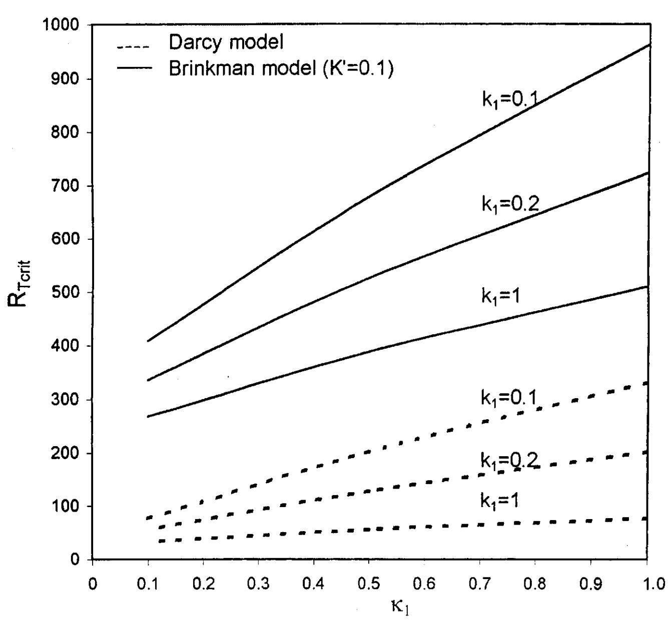

Fig. 1. The effect of anisotropy in permeability on the critical Rayleigh no. for the Darcy and Brinkman models ($K' = 0.1$)

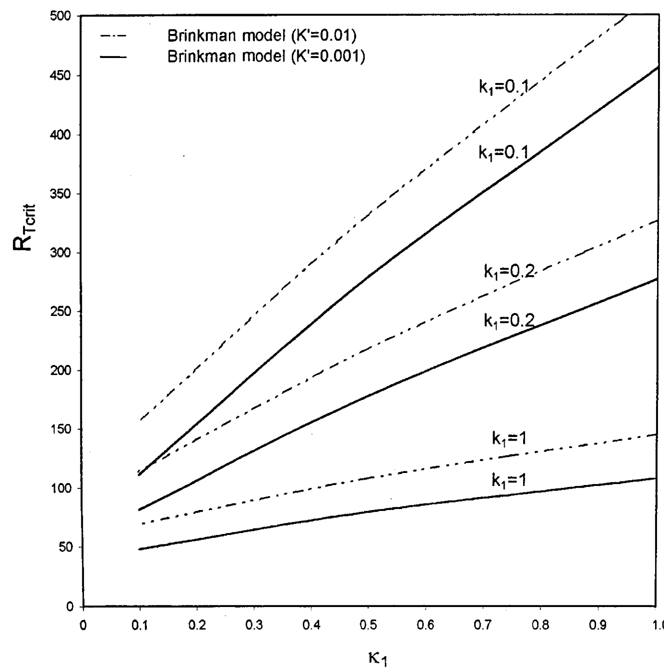

Fig. 2. The effect of anisotropy in permeability on the critical Rayleigh no. for the Brinkman models ($K' = 0.01$ and $K' = 0.001$)

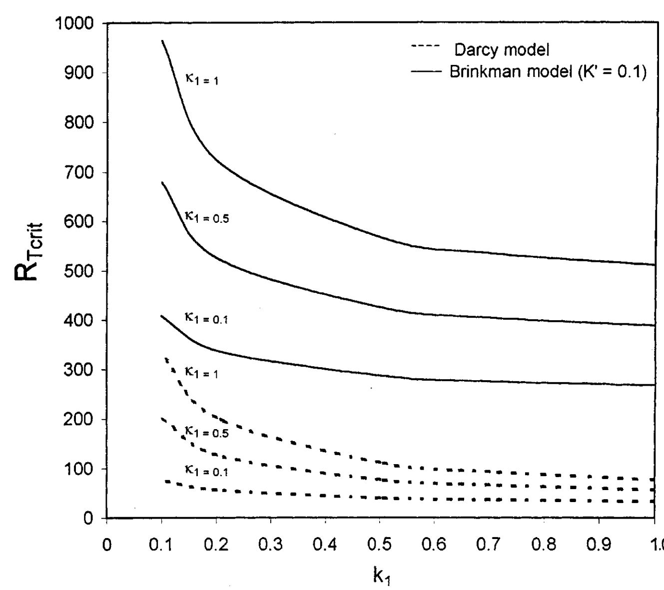

Fig. 3. The effect of anisotropy in thermal diffusivity on the critical Rayleigh no. for the Darcy and Brinkman models ($K' = 0.1$)

From Figs. 1 and 2, and Tables 1–4 it is noticed that (for both the models) when the permeability is isotropic ($k_1 = 1$) and when it is anisotropic ($k_1 = 0.1$ and $k_1 = 0.2$), the critical Rayleigh number decreases as κ_1 decreases. That is an increase in the thermal diffusivity in the X direction, namely the direction collinear with the centrifugal acceleration, advances the onset of convection.

On the other hand, from Figs. 3 and 4 and Tables 1–4 it is seen (for both the models) that when the anisotropy parameter of diffusivity κ_1 takes values 1, 0.5 and 0.1 the critical Rayleigh number increases as k_1 decreases. That is an increase in the permeability in the direction collinear with the centrifugal acceleration has the effect of stabilizing the system.

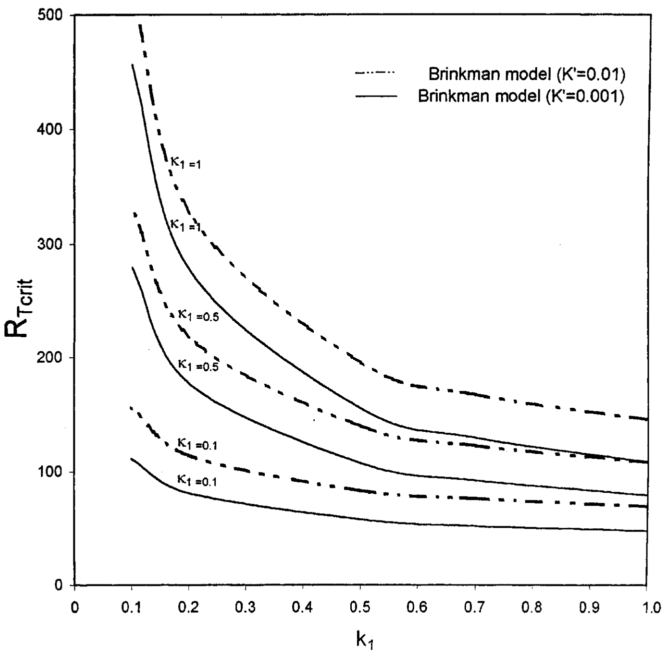

Fig. 4. The effect of anisotropy in thermal diffusivity on the critical Rayleigh no. for the Brinkman models ($K' = 0.01$ and $K' = 0.001$)

Further, the tables reveal that a decrease in the values of the anisotropy parameters, causes the critical wave number to increase, implying a shrinkage in the cell sizes.

It can be observed that the critical Rayleigh number obtained for the isotropic case ($k_1 = 1, \kappa_1 = 1$), coincides with that obtained by Vadasz [1].

References

1. **Vadasz P** (1994) Stability of free convection in a narrow porous layer subject to rotation. Int Comm Heat and Mass Transfer 21: 881–890
2. **Vadasz P** (1993) Fluid flow through heterogeneous porous media in a rotating square channel. Transport in Porous Media 12: 43–54
3. **Nielson DG; Incropera FP** (1993) Effect of rotation on fluid motion and channel formation during unidirectional solidification of a binary alloy. Int J Heat Mass Transfer 36: 489–505
4. **Jou JJ; Kung KY; Hsu CH** (1996) Thermal stability of horizontally superposed porous and fluid layers in a rotating system. Int J Heat Mass Transfer 39: 1847–1857
5. **Zhao M; Robillard L; Prudhomme M** (1996) Effect of weak rotation on natural convection in a horizontal porous cylinder. Heat and Mass Transfer Warme 31: 403–409
6. **Robillard L; Torrance KE** (1990) Convective heat transfer inhibition in an annular porous layer rotating at weak angular velocity. Int J Heat Mass Transfer 33: 953–963
7. **Patil PR; Parvathy CP; Venkatakrishnan KS** (1990) Effect of rotation on the stability of a doubly diffusive fluid layer in a porous medium. Int J Heat Mass Transfer 33: 1073–1080
8. **Soong CY; Hwang GJ** (1990) Laminar mixed convection in a radially rotating semiporous channel. Int J Heat Mass Transfer 33: 1805–1816
9. **Sacheti NC; Singh AK** (1992) Free convection through a vertical channel in a rotating porous medium. Int Comm Heat Mass Transfer 19: 423–433
10. **Vadasz P** (1993) Three-dimensional free convection in a long rotating porous box: Analytical solution. ASME J Heat Transfer 115: 639–644
11. **Vadasz P** (1995) Coriolis effect on free convection in a long rotating porous box subject to uniform heat generation. Int J Heat Mass Transfer 38: 2011–2018
12. **Vadasz P** (1998) Coriolis effect on gravity-driven convection in a rotating porous layer heated from below. J Fluid Mech 376: 351–375
13. **Vadasz P** (1992) Natural convection in porous media induced by the centrifugal body force: The solution for small aspect ratio. ASME J Energy Res Tech 114: 250–254
14. **Vadasz P** (1994) Centrifugally generated free convection in a rotating porous box. Int J Heat Mass Transfer 37: 2399–2404
15. **Vadasz P** (1996) Stability of free convection in a rotating porous layer distant from the axis of rotation. Transport in Porous Media 23: 153–173
16. **Vadasz P** (1996) Convection and stability in a rotating porous layer with alternating direction of the centrifugal body force. Int J Heat Mass Transfer 39: 1639–1647
17. **Vadasz P; Olek S** (1998) Transitions and chaos for free convection in a rotating porous layer. Int J Heat Mass Transfer 41: 1417–1435
18. **Vadasz P; Govender S** (1998) Two-dimensional convection induced by gravity and centrifugal forces in a rotating porous layer for away from the axis of rotation. Int J Rot Mach 4: 73–90
19. **Vadasz P; Heerah A** (1998) Experimental confirmation and analytical results of centrifugally-driven free convection in a rotating porous media. J Porous Media 1: 261–272
20. **Walker K; Homsy GM** (1977) A note on convective instability of Boussinesq fluids in porous media. ASME J Heat Transfer 99: 338–339
21. **Finlayson BA** (1972) The Method of Weighted Residuals and Variational Principles. New York: Academic Press
22. **Neale G** (1977) Degrees of anisotropy for fluid flow and diffusion (electrical conduction) through anisotropic porous media. AIChE J 23: 55–63