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Exact formulae for determination of the mean temperature and wear during braking

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Abstract This paper deals with the one-dimensional transient heat conductivity contact problem of a sliding two semi-spaces, which induces effects of friction, heat generation and water during braking. In the present temperature analysis the capacity of the frictional source on the contact plane dependent on the time of braking. The problem solved exactly using the Laplace transform technique. Numerical results for the temperature are obtained for the different values of the input parameter, which characterise the duration of the increase of the contact pressure during braking from zero to the maximum value.

An analytical formulae for the abrasive wear of the contact plane is obtained in the assumption, that the wear coefficient is the linear function of the contact temperature.

List of symbols

z	axial coordinate
T	temperature
V	sliding speed
V_0	initial sliding speed
p	pressure
p_0	maximum pressure
q	capacity of frictional heat source per unit contact area
K	thermal conductivity
k	thermal diffusivity
f_0	friction coefficient
α_1	the coefficient of the thermal expansion of the frictional pad
ν_1	Poisson's coefficient of the pad
m	coefficient of wear
t	time
t_m	duration of the increase of the loading from zero to maximum value
t_s^0	duration of braking in the case of constant load
t_s	duration of the stop
t_{max}	time at which the maximum contact temperature is reached
W	initial kinetic energy per unit contact area
Λ	parameter given by formula (12) and taking the dimension of the temperature
$\mathrm{erf}(\cdot)$	Gauss error function
$\mathrm{erfc}(\cdot) = 1 - \mathrm{erf}(\cdot)$	complementary error function

Dimensionless parameters

$\tau = t/t_s^0$

$\tau_m = t_m/t_s^0$

$\tau_{max} = t_{max}/t_s^0$

$\tau_s = t_s/t_s^0$

$\tau^* = t/t_m$

$\zeta_i = |z|/\left(2\sqrt{k_i t_i^o}\right), \quad i = 1,\ 2$

$T^* = T/\Lambda$

$T_i^* = T_i/\Lambda, \quad i = 1,\ 2$

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1 Introduction

A mathematical model which permits to study the contact temperature of various types of braking systems has been presented by Chichinadze [1], Chichinadze et al. [2]. It is assumed that the maximum temperature rise on the contact surface was represented as the sum of the mean temperature of the nominal contact area and the flash temperature. The following one-dimensional heat conductivity boundary-value problem is used for determination of the mean temperature:

$$\frac{\partial^2 T_i(z,t)}{\partial z^2} = \frac{1}{k_i}\frac{\partial T_i(z,t)}{\partial t}, \quad z > 0 \quad \text{for } i = 1, \quad z < 0 \quad \text{for } i = 2; 0 \le t \le t_s\ , \tag{1}$$

$$T_i(z,0) = 0, \quad i = 1,\ 2 \tag{2}$$

$$T_1(z,0) = T_2(z,0) \equiv T(t), \quad 0 \le t \le t_s \ , \tag{3}$$

$$K_1 \frac{\partial T_1(z,t)}{\partial z}\bigg|_{z=0+} - K_2 \frac{\partial T_2(z,t)}{\partial z}\bigg|_{z=0-} = q(t),\ 0 \le t \le t_s\ , \tag{4}$$

$$T_i \to 0, \quad i = 1,\ 2 \quad \text{for } |z| \to \infty,\ 0 \le t \le t_s \ . \tag{5}$$

Here and further the indices $i = 1, 2$ denote the upper and the lower half-spaces respectively. The rate of frictional heat generation throughout the contact plane $z = 0$ is equal

$$q(t) = f(t)p(t)V(t), \quad 0 \le t \le t_s \ , \tag{6}$$

The exact solution of the contact problem with frictional heating (1)–(5) can be obtained for three cases of the function $q(t)$ [3]: (1) $q(t) = q_0 = \text{const.}$; (2) $q(t) = q_0\sqrt{t}$; (3) $q(t) = q_0 t$. However, really the contact pressure rises from zero to a value, when the motion comes to a still, in the form [2];

$$p(t) = p_0 p^*(t/t_m), \quad p^*(t) = 1 - \exp(-t) \ . \tag{7}$$

At the known pressure (7) and the constant friction coefficient $f(t) = f_0$ we obtain the speed changing law during braking [5]:

$$V(t) = V_0 V^*(\tau),\ V^*(\tau) = 1 - \tau + \tau_m p^*(\tau^*),\ 0 \le t \le t_s\ , \tag{8}$$

where $\tau = t/t_s^0,\ \tau_m = t_m/t_s^0, \tau^* = \tau/\tau_m,\ t_s^0 = 2W/(f_0 p_0 V_0)$.

Using the condition $V(t_s) = 0$, from expression (8) we find the equation for the dimensionless braking time

$$\tau_s = t_s/t_s^0 \ge 1$$

$$\tau_s - \tau_m p^*(\tau_s/\tau_m) = 1 \ . \tag{9}$$

The numerical solution of the equation (9) is shown in Fig. 1. These results can be obtained from the approximate linear function:

$$\tau_s = 1 + 0.9975\tau_m \ . \tag{10}$$

The behaviour of the nondimensional contact pressure p^* (7) and sliding speed V^* (8) is presented in Fig. 2. We see that in the case when the contact pressure is constant ($t_m = 0$) the sliding speed decreases from V_0 to zero linearly with time so that the deceleration is constant (uniform braking).

In this paper the exact solution of the transient one-dimensional heat conductivity problem (1)–(5) with the rate of the frictional heating q Eq. (6), contact pressure p Eq. (7) and sliding speed V Eq. (8) is obtained.

2
Temperature

Employing the convolution theorem for the Laplace transform with respect to time t [5], the solution of the boundary-value heat conductivity problem (1)–(5) we find in the form

$$T_i(z,t) = \Lambda \int_0^{\tau} p^*[(\tau - \tau_0)/\tau_m] V^*(\tau - \tau_0)\tau_0^{-1/2} \times \exp(-\zeta_i^2/\tau_0)\,\mathrm{d}\tau_0, \quad 0 \le t \le t_s \tag{11}$$

where

$$\Lambda = \frac{f_0 p_0 V_0}{(1+k_\varepsilon)K_1}\sqrt{\frac{k_1 t_s^0}{\pi}}, \quad k_\varepsilon = \frac{K_2}{K_1}\sqrt{\frac{k_1}{k_2}},$$

$$\zeta_i = \frac{|z|}{2\sqrt{k_i t_s^0}}, \quad i = 1,\ 2 \ . \tag{12}$$

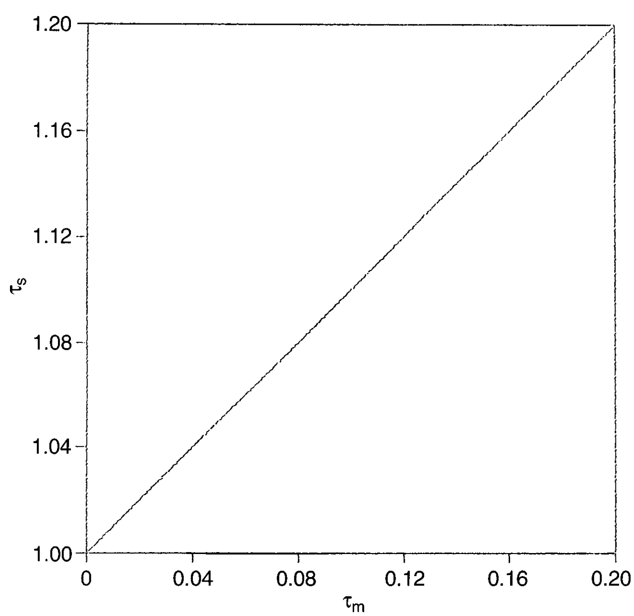

Fig. 1. Dependence of the dimensionless time τ_s on the dimensionless parameter τ_m

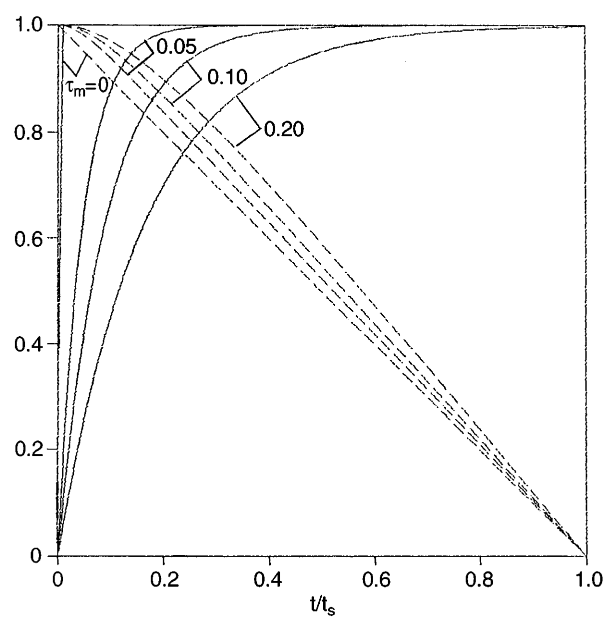

Fig. 2. Change of the dimensionless pressure p^* (continuous curve) and dimensionless speed V^* (dotted curves) during braking with different dimensionless parameters τ_m

Having an analytical expressions for the contact pressure $p^*(\tau)$ Eq. (7) and sliding speed $V^*(\tau)$ Eq. (8), from Eq. (11) we obtain the temperature of the braking elements:

$$T_i(z,t) = \Lambda\big[(1+\tau_m-\tau)N^-(\zeta_i,\tau) - (1+2\tau_m-\tau) \times L^-(\zeta_i,\tau,\tau_m) + \tau_m L^-(\zeta_i,\tau,\tau_m/2) + N^+(\zeta_i,\tau) + L^+(\zeta_i,\tau,\tau_m) - \tau_m\sqrt{\tau}\exp(-\eta_i^2)\big], \quad 0 \le t \le t_s, \ i = 1,\ 2 \ , \tag{13}$$

where

$$\left\{\begin{matrix} N^\pm(\zeta_i,\tau) \\ L^\pm(\zeta_i,\tau,\tau_m) \end{matrix}\right\} = \int_0^\tau \tau_0^{\pm 1/2}\exp(-\zeta_i^2/\tau_0) \times \left\{\begin{matrix} 1 \\ \exp[-(\tau-\tau_0)/\tau_m] \end{matrix}\right\} d\tau_0, \quad i = 1,\ 2 \tag{14}$$

We evaluate the integrals $N^\pm(\zeta_i,\tau)$ appearing in Eq. (14) as shown below. After integrating by parts in (14) we have

$$N^-(\zeta_i,\tau) = 2\sqrt{\tau}\exp(-\eta_i^2) - 2\zeta_i^2\int_0^\tau \tau_0^{-3/2}\exp(-\zeta_i^2/\tau_0)\,d\tau_0 \ , \tag{15}$$

$$N^+(\zeta_i,\tau) = \frac{2}{3}\tau\sqrt{\tau}\exp(-\eta_i^2) - \frac{2}{3}\zeta_i^2 N^-(\zeta_i,\tau) \ , \tag{16}$$

where

$$\eta_i = \zeta_i/\sqrt{\tau} \ .$$

Using the integral[6]

$$\int_x^\infty \exp(-y^2)\,dy = 1/2\sqrt{\pi}\ \mathrm{erfc}(x) \ ,$$

from the Eq. (15) we obtain

$$N^-(\zeta_i,\tau) = 2\sqrt{\tau}\exp(-\eta_i^2) - 2\zeta_i\sqrt{\pi}\ \mathrm{erfc}(\eta_i) \ . \tag{17}$$

By methods [4], the integrals $L^\pm(\zeta_i,\tau,\tau_m)$ Eq. (14) can be rewritten as:

$$L^-(\zeta_i,\tau,\tau_m) = \sqrt{\pi\tau_m}\exp(-\tau^*)\{[F_c(\zeta_i,\tau,\tau_m)-1]\sin(2\eta_{im}) + C(\zeta_i,\tau,\tau_m)\cos(2\eta_{im})\}, \quad i = 1,\ 2 \tag{18}$$

$$L^-(\zeta_i,\tau,\tau_m) = \frac{1}{2}\tau_m\sqrt{\pi\tau_m}\exp(-\tau^*)\{[F_s(\zeta_i,\tau,\tau_m) + 2\eta_{im}C(\zeta_i,\tau,\tau_m) - 1]\sin(2\eta_{im}) + [C(\zeta_i,\tau,\tau_m) - 2\eta_{im}F_s(\zeta_i,\tau,\tau_m) + 2\eta_{im}]\cos(2\eta_{im})\}, \quad i = 1, 2 \tag{19}$$

$$\left\{\begin{matrix} F_s(\zeta_i,\tau,\tau_m) \\ F_c(\zeta_i,\tau,\tau_m) \end{matrix}\right\} = \mathrm{erf}(2\eta_i) + \left\{\begin{matrix} S(\zeta_i,\tau,\tau_m) \\ C(\zeta_i,\tau,\tau_m) \end{matrix}\right\} \ , \tag{20}$$

$$\left\{\begin{matrix} S(\zeta_i,\tau,\tau_m) \\ C(\zeta_i,\tau,\tau_m) \end{matrix}\right\} = \frac{2}{\sqrt{\pi}}\exp(-\eta_i^2) \int_0^{\sqrt{\tau^*}} \exp(x^2)\left\{\begin{matrix} \sin(2\eta_i x) \\ \cos(2\eta_i x) \end{matrix}\right\} dx, \tag{21}$$

where

$$\eta_m = \zeta_i/\sqrt{\tau_m} \ .$$

Substituting integrals $N^\pm(\zeta_i,\tau)$ Eq. (16), (17) and $L^\pm(\zeta_i,\tau,\tau_m)$ Eqs. (18)–(21) into Eqs. (13) we determined the temperature field of the working elements (the frictional pad and the disc) during braking.

The contact temperature we find from Eqs. (13) at $z = 0$ in the form

$$T(t) = \Lambda\big[(2+\tau_m - \frac{4}{3}\tau)\sqrt{\tau} - (1+\frac{3}{2}\tau_m - \tau)2\sqrt{\tau_m}\,F\Big(\sqrt{\tau^*}\Big) + \tau_m\sqrt{2\tau_m}F(\sqrt{2\tau^*})\big], \ 0 \le t \le t_s \ . \tag{22}$$

where $F(\tau) = \exp(-\tau^2)\int_0^\tau \exp(x^2)\,dx$ is Douson's integral [7]. To count $F(\tau)$ we use formulae [8]:

$$F(\tau) = \sum_{i=0}^{\infty}\frac{(-2\tau^2)^i}{(2i+1)!!}, \quad 0 \le \tau \le 3 \ ,$$

$$F(\tau) = \sum_{i=0}^{n}\frac{(2i-1)!!}{(2\tau^2)^{i+1}}, \quad \tau > 3$$

where $(-1)!! = 1$.

At $t_m = 0\,(t_s = t_s^0)$ from Eq. (22) we obtain the known result [9] for the contact temperature in the case of uniform braking

$$T(t) = 2\Lambda\left(1 - \frac{2}{3}\frac{t}{t_s}\right)\sqrt{\frac{t}{t_s}}, \quad 0 \le t \le t_s \ .$$

3
The temperature-dependence wear

We assume the Archard's law of wear [10] in which the rate of material removal is proportional to pressure and speed of sliding:

$$I(t) = \int_0^t m(T)q(t_0)\,dt_0, \quad 0 \le t \le t_s \ . \tag{23}$$

In according to [11] the wear coefficient $m(T)$ at the small gradients of the contact temperature takes the form

$$m(T) = m_0 + m_1\beta T(t), \quad \beta = \alpha_1(1+\nu_1)/(1-\nu_1) \ . \tag{24}$$

Substitutes Eqs. (22), (24) and (6)–(8) into Eq. (23) after integrating we find:

$$I(t) = m_0^* I_0(t) + m_1^*\beta\Lambda_0 I_1(t), \quad 0 \le t \le t_s, \tag{25}$$

where

$$m_k^* = m_k f_0 V_0 p_0 t_s^0, \quad k = 0,\ 1 \ ,$$

$$I_0(t) = \tau - \tau^2/2 + \tau_m(\tau - \tau_m - 1)p^*(\tau^*) + \tau_m^2 p^*(2\tau^*)/2 \ , \tag{26}$$

$$I_1(t) = I_1^{(1)}(t) + I_1^{(2)}(t) + I_1^{(3)}(t) \ , \tag{27}$$

$$\begin{aligned} I_1^{(1)}(t) = {} & \frac{2}{3}(1+\tau_{\mathrm{m}})(2+\tau_{\mathrm{m}})\tau\sqrt{\tau} - \frac{2}{15}(10+7\tau_{\mathrm{m}})\tau^2\sqrt{\tau} \\ & + \frac{8}{21}\tau^3\sqrt{\tau} - (1+2\tau_{\mathrm{m}})(2+\tau_{\mathrm{m}})\tau_{\mathrm{m}}\sqrt{\tau_{\mathrm{m}}} \\ & \times \left[\frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(\sqrt{\tau^*}) - \tau^*\exp(-\tau^*)\right] \\ & + \frac{1}{3}(10+11\tau_{\mathrm{m}})\tau_{\mathrm{m}}^2\sqrt{\tau_{\mathrm{m}}} \\ & \times \left[\frac{3}{4}\sqrt{\pi}\,\mathrm{erf}(\sqrt{\tau^*}) - \sqrt{\tau^*}\left(\frac{3}{2}+\tau^*\right)\exp(-\tau^*)\right] \\ & - \frac{4}{3}\tau_{\mathrm{m}}^3\sqrt{\tau_{\mathrm{m}}}\left[\frac{15}{8}\sqrt{\pi}\,\mathrm{erf}(\sqrt{\tau^*}) \right. \\ & \left. - \sqrt{\tau^*}\left(\frac{15}{4}+\frac{5}{2}\tau^* + \tau^{*2}\right)\exp(-\tau^*)\right] \\ & + \frac{1}{2}(2+\tau_{\mathrm{m}})\tau_{\mathrm{m}}^2\sqrt{\tau_{\mathrm{m}}} \\ & \times \left[\frac{1}{2}\sqrt{\frac{\pi}{2}}\mathrm{erf}(\sqrt{2\tau^*}) - \sqrt{\tau}^*\exp(-\tau^*)\right] - \frac{2}{3}\tau_{\mathrm{m}}^3\sqrt{\tau_{\mathrm{m}}} \\ & \times \left[\frac{3}{8}\sqrt{\frac{\pi}{2}}\mathrm{erf}(\sqrt{2\tau^*}) - \sqrt{\tau^*}\left(\frac{3}{8}+\tau^*\right)\exp(-\tau^*)\right], \end{aligned}$$

$$\begin{aligned} I_1^{(2)}(t) = 2\sqrt{\tau_{\mathrm{m}}}\Bigg[& \left(2+\frac{5}{2}\tau_{\mathrm{m}}\right)M_{101}(\tau) - (1+\tau_{\mathrm{m}}) \\ & \times \left(1+\frac{3}{2}\tau_{\mathrm{m}}\right)M_{001}(\tau) - M_{201}(\tau) \\ & + M_{211}(\tau) - \left(2+\frac{7}{2}\tau_{\mathrm{m}}\right)M_{111}(\tau) \\ & + (1+2\tau_{\mathrm{m}})\left(1+\frac{3}{2}\tau_{\mathrm{m}}\right)M_{011}(\tau) \\ & + \tau_{\mathrm{m}}M_{121}(\tau) - \tau_{\mathrm{m}}\left(1+\frac{3}{2}\tau_{\mathrm{m}}\right)M_{021}(\tau)\Bigg] \ , \end{aligned}$$

$$\begin{aligned} I_1^{(3)}(t) = \tau_{\mathrm{m}}\sqrt{2\tau_{\mathrm{m}}}\big[& (1+\tau_{\mathrm{m}})M_{002}(\tau) - M_{102}(\tau) \\ & + M_{112}(\tau) - (1+2\tau_{\mathrm{m}})M_{012}(\tau) \\ & + \tau_{\mathrm{m}}M_{022}(\tau)\big] \ , \end{aligned}$$

$$\begin{aligned} M_{kjl}(\tau) = {} & \int_0^{\tau} \tau_0^k \exp(-j\tau_0)/\tau_{\mathrm{m}})F\left(\sqrt{l\tau_0/\tau_{\mathrm{m}}}\right)\mathrm{d}\tau_0, \\ & k, j = 0,\ 1,\ 2; \quad l = 1,\ 2\ . \end{aligned} \tag{28}$$

Taking into account the values of the integrals:

$$\int xF(x)\,\mathrm{d}x = \frac{1}{2}[x - F(x)] \ ,$$

$$\int x^3F(x)\,\mathrm{d}x = \frac{1}{2}\left[x + \frac{1}{3}x^3 - (1+x^2)F(x)\right] \ ,$$

$$\int x^5F(x)\,\mathrm{d}x = \frac{1}{2}\left[2x + \frac{2}{x}x^3 + \frac{1}{5}x^5 - (2+2x^2+x^4)F(x)\right] \ ,$$

$$\int x\exp(-x^2)F(x)\,\mathrm{d}x = \frac{1}{4}\left[\int \exp(-x^2)\,\mathrm{d}x - \exp(-x^2)F(x)\right] \ ,$$

$$\begin{aligned} \int x^3\exp(-x^2)F(x)\,\mathrm{d}x = {} & \frac{1}{4}\left[\int \exp(-x^2)\,\mathrm{d}x - \frac{1}{2}\,x\exp(-x^2) \right. \\ & \left. - \left(\frac{1}{2}+x^2\right)\exp(-x^2)F(x)\right] \ , \end{aligned}$$

$$\begin{aligned} \int x^5\exp(-x^2)F(x)\,\mathrm{d}x = {} & \frac{1}{4}\left[\frac{7}{4}\int \exp(-x^2)\,\mathrm{d}x - \frac{1}{2}x\left(\frac{5}{2}+x^2\right)\exp(-x^2) \right. \\ & \left. - x^4\exp(-x^2)F(x) - \left(\frac{1}{2}+x^2\right)\exp(-x^2)F(x)\right] \ , \end{aligned}$$

$$\int x\ \exp(-2x^2)F(x)\,\mathrm{d}x = \frac{1}{2}\left[\int \exp(-2x^2)\,\mathrm{d}x - \exp(-2x^2)F(x)\right] \ ,$$

$$\begin{aligned} \int x^3\exp(-2x^2)F(x)\,\mathrm{d}x = {} & \frac{1}{6}\left[\frac{7}{12}\int \exp(-2x^2)\,\mathrm{d}x - \frac{1}{4}x\exp(-2x^2) \right. \\ & \left. - \left(\frac{1}{3}+x^2\right)\exp(-2x^2)F(x)\right] \ , \end{aligned}$$

$$\int xF\left(\sqrt{2}x\right)\,\mathrm{d}x = \frac{1}{2}\left[\frac{1}{2\sqrt{2}}x - \frac{1}{2}F\left(\sqrt{2}x\right)\right] \ ,$$

$$\begin{aligned} \int x^3F\left(\sqrt{2}x\right)\,\mathrm{d}x = {} & \frac{1}{2}\left[\frac{1}{2\sqrt{2}}x + \frac{1}{3\sqrt{2}}x^3 - \frac{1}{2} \right. \\ & \left. \times \left(\frac{1}{2}+x^2\right)F\left(\sqrt{2}x\right)\right] \ , \end{aligned}$$

$$\int x\exp(-x^2)F\left(\sqrt{2}x\right)\,\mathrm{d}x = \frac{1}{3}\left[\frac{1}{\sqrt{2}}\int \exp(-x^2)\,\mathrm{d}x - \frac{1}{2}\exp(-x^2)F\left(\sqrt{2}x\right)\right],$$

$$\begin{aligned} \int x^3\exp(-x^2)F(\sqrt{2}x)\,\mathrm{d}x = {} & \frac{1}{6}\left[\frac{5}{3\sqrt{2}}\int \exp(-x^2)\,\mathrm{d}x - \frac{1}{\sqrt{2}}x\exp(-x^2) \right. \\ & \left. - \left(\frac{1}{3}+x^2\right)\exp(-x^2)F(\sqrt{2}x)\right] \ , \end{aligned}$$

$$\int x \exp(-2x^2) F(\sqrt{2}x)\,\mathrm{d}x = \frac{1}{4}\left[\frac{1}{\sqrt{2}}\int \exp(-2x^2)\,\mathrm{d}x - \frac{1}{\sqrt{2}}\exp(-2x^2)F(\sqrt{2}x)\right] .$$

for the function M_{kjl} (28) we find:

$$M_{001}(\tau) = \tau_\mathrm{m}\left[\sqrt{\tau^*} - F(\sqrt{\tau^*})\right] ,$$

$$M_{101}(\tau) = \tau_\mathrm{m}^2\left[\sqrt{\tau^*} + \frac{1}{3}\tau^*\sqrt{\tau^*} - (1+\tau^*)F(\sqrt{\tau^*})\right] ,$$

$$M_{201}(\tau) = \tau_\mathrm{m}^3\left[2\sqrt{\tau^*} + \frac{2}{3}\tau^*\sqrt{\tau^*} + \frac{1}{5}\tau^{*2}\sqrt{\tau^*} - (2 + 2\tau^* + \tau^{*2})F(\sqrt{\tau^*})\right] ,$$

$$M_{011}(\tau) = \frac{\tau_\mathrm{m}}{2}\left[\mathrm{erf}(\sqrt{\tau^*}) - \exp(-\tau^*)F(\sqrt{\tau^*})\right] ,$$

$$M_{111}(\tau) = \frac{\tau_\mathrm{m}^2}{2}\left[\mathrm{erf}(\sqrt{\tau^*}) - \frac{1}{2}\sqrt{\tau^*}\exp(-\tau^*) - \left(\frac{1}{2}+\tau^*\right)\exp(-\tau^*)F(\sqrt{\tau^*})\right] ,$$

$$M_{211}(\tau) = \frac{\tau_\mathrm{m}^3}{2}\left[\frac{7}{4}\mathrm{erf}(\sqrt{\tau^*}) - \frac{1}{2}\sqrt{\tau^*}\left(\frac{5}{2}+\tau^*\right) \times \exp(-\tau^*) - \tau^{*2}\exp(-\tau^*)F(\sqrt{\tau^*}) - \left(\frac{1}{2}+\tau^*\right)\exp(-\tau^*)F(\sqrt{\tau^*})\right] ,$$

$$M_{021}(\tau) = \frac{\tau_\mathrm{m}}{3}\left[\frac{1}{2}\sqrt{\frac{\pi}{2}}\mathrm{erf}(\sqrt{2\tau^*}) - \exp(-2\tau^*)F(\sqrt{\tau^*})\right] ,$$

$$M_{121}(\tau) = \frac{\tau_\mathrm{m}^2}{2}\left[\frac{1}{2}\sqrt{\frac{\pi}{2}}\mathrm{erf}(\sqrt{2\tau^*}) - \frac{1}{4}(\sqrt{\tau^*})\exp(-2\tau^*) - \left(\frac{1}{3}+\tau^*\right)\exp(-2\tau^*)F(\sqrt{\tau^*})\right] ,$$

$$M_{002}(\tau) = \tau_\mathrm{m}\left[\sqrt{\frac{\tau^*}{2}} - \frac{1}{2}F(\sqrt{2\tau^*})\right] ,$$

$$M_{102}(\tau) = \tau_\mathrm{m}^2\left[\frac{1}{2}\sqrt{\frac{\tau^*}{2}} + \frac{\tau^*}{3}\sqrt{\frac{\tau^*}{2}} - \frac{1}{2}\left(\frac{1}{2}+\tau^*\right)F(\sqrt{2\tau^*})\right] ,$$

$$M_{112}(\tau) = \tau_\mathrm{m}^2\left[\frac{5}{9\sqrt{2}}\mathrm{erf}(\sqrt{\tau^*}) - \frac{1}{3}\sqrt{\frac{\tau^*}{2}}\exp(-\tau^*) - \frac{1}{3}\left(\frac{1}{3}+\tau^*\right)\exp(-\tau^*)F(\sqrt{2\tau^*})\right] ,$$

$$M_{012}(\tau) = \tau_\mathrm{m}\left[\frac{2}{3\sqrt{2}}\mathrm{erf}(\sqrt{\tau^*}) - \frac{1}{3}\exp(-\tau^*)F(\sqrt{2\tau^*})\right] ,$$

$$M_{022}(\tau) = \tau_\mathrm{m}\left[\frac{\sqrt{\pi}}{8}\mathrm{erf}(\sqrt{2\tau^*}) - \frac{1}{4}\exp(-2\tau^*)F(\sqrt{2\tau^*})\right] ,$$

where we using the known integral [6]

$$\int_0^\tau \exp(-x^2)\,\mathrm{d}x = 1/2\sqrt{\pi}\ \mathrm{erf}(x) .$$

4 Numerical results

The input parameters of problem are two dimensionless quantities: $0 \le \tau_m \le 0.2$ – the duration of the application of load p Eq. (7) from zero to the maximum value p_0 [2] and $0 \le \zeta_i < \infty$ Eq. (12) – the axial coordinate.

The effect of τ_m during braking on the variation of the dimensionless contact temperature $T^* = T/\Lambda$, where T is given by formula (22), is shown in Fig. 3. We see that the largest value of the contact temperature is reached during braking with the uniform retardation ($\tau_\mathrm{m} = 0$). In this case the maximum temperature occurs at the middle-point ($t \approx 0,5t_\mathrm{s}$) of the stop. For $\tau_\mathrm{m} = 0.2$ this maximum is attained at $t \approx 0.62t_\mathrm{s}$.

The maximum dimensionless contact temperature

$$T^*_\mathrm{max} = \max_{0 \le t \le t_\mathrm{s}} T^*(t)$$

falls with increasing τ_m (Fig. 4). Thus, the maximum contact temperature in the case of uniform braking is always larger than at the non-uniform stopping. The corresponding dependence for dimensionless time $\tau_\mathrm{max} = t_\mathrm{max}/t_\mathrm{s}^0$ at which T^*_max is attained, is shown in Fig. 5. In additing, we obtain the following engineering expressions for T^*_max and τ_max:

$$T^*_\mathrm{max} = \sum_{k=0}^{3} a_k \tau_\mathrm{m}^k, \quad \tau_\mathrm{max} = 0.501 + 0.53\tau_\mathrm{m} , \tag{29}$$

where $a_0 = 0.9426$, $a_1 = 0.0201$, $a_2 = -1.2409$, $a_3 = 1.6024$.

Distribution in axial direction of the maximum dimensionless temperature

$$T^*_{t,\mathrm{max}}(\zeta_i) = \max_{0 \le t \le t_\mathrm{s}} T_i(t,z)/\Lambda, \quad i = 1,\ 2 ,$$

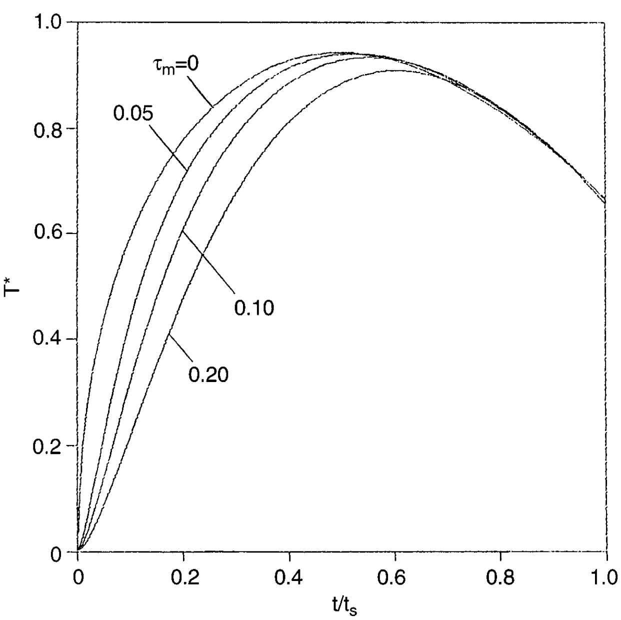

Fig. 3. Change of dimensionless contact temperature $T^* = T/\Lambda$ during braking with different dimensionless parameters τ_m

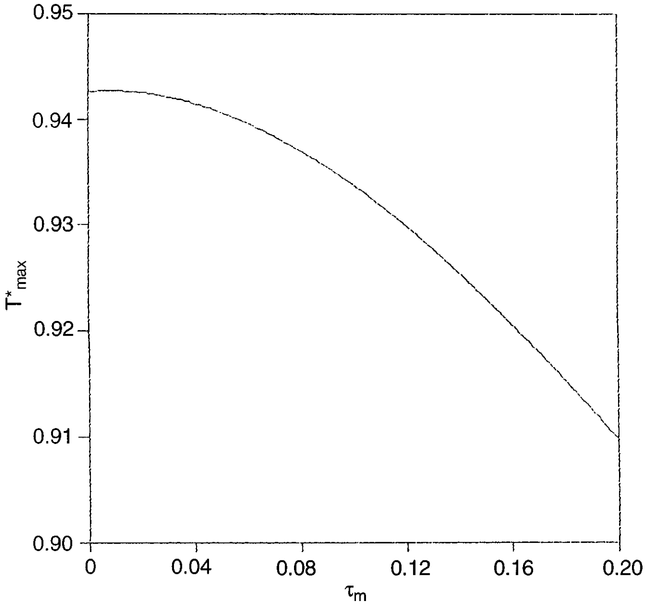

Fig. 4. Dependence of the dimensionless maximum contact temperature $T^*_{\max}$ on the dimensionless parameter τ_m

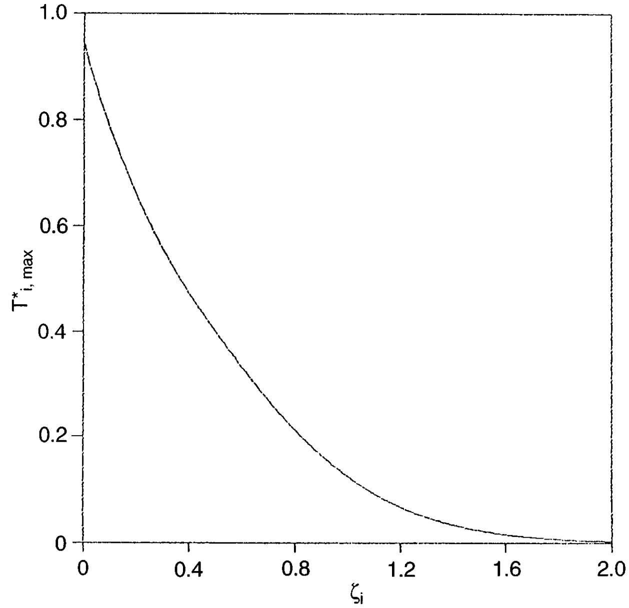

Fig. 6. Distribution of the dimensionless maximum temperature $T^*_{i,\max}$ in axial direction during uniform braking ($\tau_m = 0$)

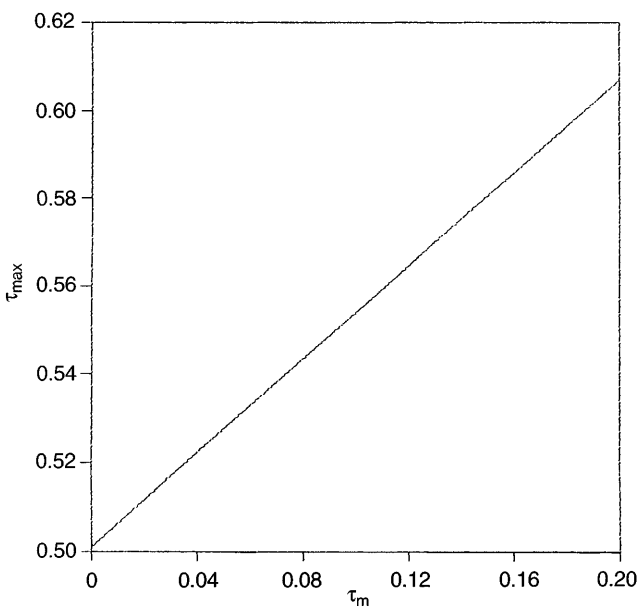

Fig. 5. Dependence on the dimensionless time $\tau_{\max}$, when the maximum contact temperature is attained, on the dimensionless parameter t_m

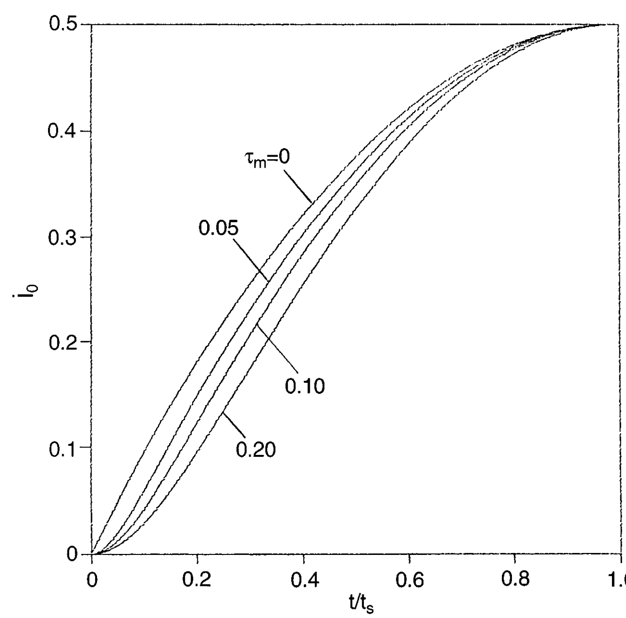

Fig. 7. Change of the dimensionless function I_0 during braking with different dimensionless parameters τ_m

where T_i is given by expression (13), is shown in Fig. 6 for $\tau_m = 0$ (the curves for different values of $0 \le \tau_m \le 0.2$ almost coincides). We see that the temperature field is strongly localized and has a sharp gradient in axial direction. According to Chichinadze et al. [2], the effective depth of the frictional heating during braking characterises the depth when the following condition takes place

$$T_i^*(\zeta_i)/T^*_{\max} \cdot 100\% \le 5\% \ .$$

It is seen that the dimensionless effective depth is equal to $\zeta_i^{\text{eff}} = 2$ at uniform braking. A numerical results presented in Fig. 6 has permitted us to construct such approximate formula

$$T^*_{i\max}(\zeta_i) = \sum_{k=0}^{4} b_k \zeta_i^k, \quad i = 1,\ 2 \quad \text{for } \tau_m = 0 \ , \qquad (30)$$

where $b_0 = 0.9426$, $b_1 = -1.6711$, $b_2 = 1.6947$, $b_3 = -1.3668$, $b_4 = 0.6551$.

Figures 7, 8 shown, respectively, the distribution of the dimensionless function $I_0(t)$ (26) and $I_1(t)$ (27) during braking. The maximum values of these functions are reached in the end of stopping time at $t = t_s$. Thus, the

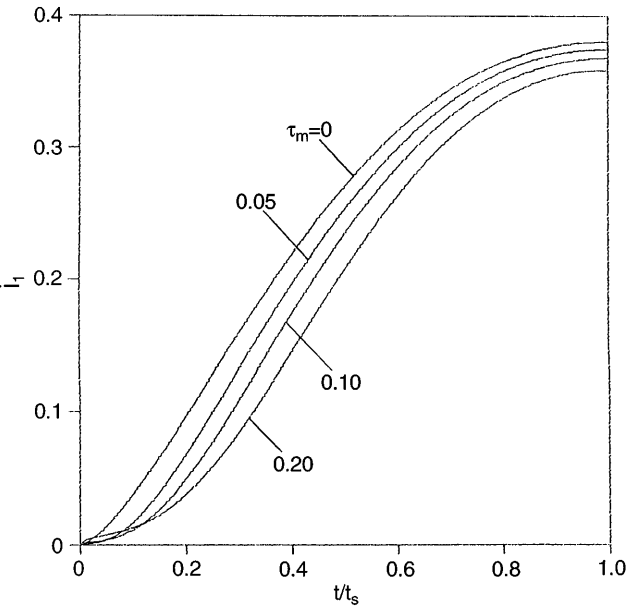

Fig. 8. Change of the dimensionless function I_1 during braking with different dimensionless parameters τ_m

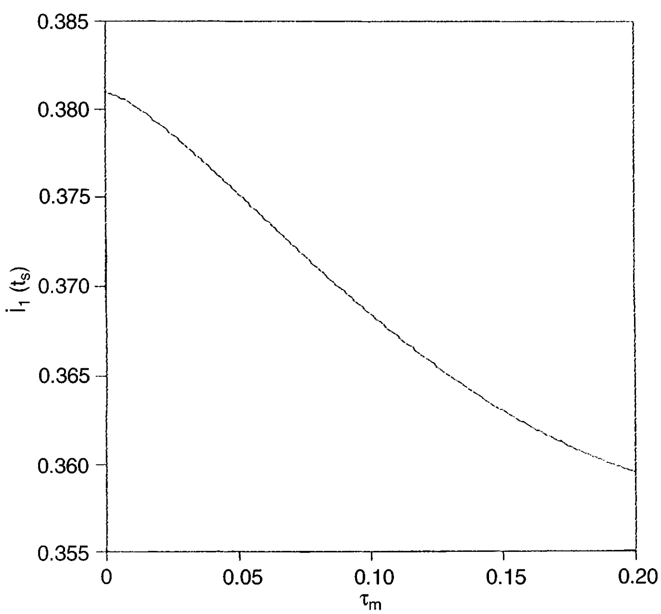

Fig. 9. Dependence of the dimensionless function I_1 on the dimensionless parameter τ_m in the stop time moment $t = t_s$

wear during braking is largest in the stop time moment. We observe also that the function $I_0(t)$, which characterise of the wear in the absence of frictional heating, nearly not-dependence from the parameter τ_m at $t = t_s$. Another picture is observed for the function $I_1(t)$, which falls with increasing τ_m (Fig. 9). Thus, the maximum value of wear is attained during braking with uniform retardation. The least square method were applied to approximate of the function $I_1(t_s)$ as polynomial of τ_m:

$$I_1(t_s) = \sum_{k=0}^{4} c_k \tau_m^k \qquad (31)$$

where $c_0 = 0.3810$, $c_1 = 0.0837$, $c_2 = -0.9369$, $c_3 = 6.3810$, $c_4 = -11.4620$.

We note that the absolute error of the approximate formulae (10), (29)–(31) is at most 0.5%.

5 Conclusions

By method of the Laplace transform the exact solution of the one-dimensional transient heat conductivity problem with frictional heating during braking is obtained in the general case of contact pressure distribution. We assumed that the coefficient of friction is constant and coefficient of wear is linearly dependence on the contact temperature.

It is established that the contact temperature and wear essentially depends on the one input parameter: the time when the maximum value of the pressure is reached. If this parameter increasing then the maximum contact temperature and wear falls. The temperature fields proves to be strongly localized and possesses a sharp gradient in axial directions.

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