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Heat transfer in the flow of a viscoelastic fluid over a stretching surface

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Abstract An analysis is made of heat transfer in the boundary layer of a viscoelastic fluid flowing over a stretching surface. The velocity of the surface varies linearly with the distance x from a fixed point and the surface is held at a uniform temperature T_w higher than the temperature T_∞ of the ambient fluid. An exact analytical solution for the temperature distribution is found by solving the energy equation after taking into account strain energy stored in the fluid (due to its elastic property) and viscous dissipation. It is shown that the temperature profiles are nonsimilar in marked contrast with the case when these profiles are found to be similar in the absence of viscous dissipation and strain energy. It is also found that temperature at a point increases due to the combined influence of these two effects in comparison with its corresponding value in the absence of these two effects. A novel result of this analysis is that for small values of x, heat flows from the surface to the fluid while for moderate and large values of x, heat flows from the fluid to the surface even when $T_w > T_\infty$. Temperature distribution and the surface heat flux are determined for various values of the Prandtl number P, the elastic parameter K_1 and the viscous dissipation parameter a. Numerical solutions are also obtained through a fourth-order accurate compact finite difference scheme.

1 Introduction

Flow of an incompressible fluid over a stretching surface has important applications in polymer industry. For instance a number of technical processes concerning polymers involves the cooling of continuous strips (or filaments) extruded from a die by drawing them through a quiescent fluid with controlled cooling system and in the process of drawing, these strips are sometimes stretched. Further glass blowing, continuous casting of metals and spinning of fibres involve the flow due to a stretching surface. In all these cases, the quality of the final product depends on the rate of heat transfer at the stretching surface. Crane [1] studied the steady two-dimensional incompressible boundary layer flow caused by the stretching of an elastic flat sheet which moves in its plane with a velocity varying linearly with distance from a fixed point due to the application of a uniform stress. The heat transfer in the flow over a stretching surface was investigated by Gupta and Gupta [2] in the case when the surface held at constant temperature is subject to suction or blowing. Dutta, Roy and Gupta [3] determined the temperature distribution in the flow over a stretching surface subject to uniform heat flux. Crane [4] studied heat transfer in the flow over a surface which is stretched in its own plane with a velocity varying linearly with distance from a fixed point in the case when the temperature difference between the surface and the ambient fluid is proportional to a power of distance from the fixed point.

All the above investigations are, however, confined to flows of Newtonian fluids. In recent years non-Newtonian fluids have become more important industrially. Specifically in certain polymer processing applications, one deals with flow of a non-Newtonian fluid over a stretching surface. Similarity solutions for the velocity distributions for the non-Newtonian flow of a power law fluid past a stretching sheet were given by Andersson and Dandapat [5]. The same flow was examined by Siddappa and Khapate [6] for a special class of non-Newtonian fluids known as second-order fluids, which are viscoelastic in nature. Bujurke, Biradar and Hiremath [7] and Dandapat and Gupta [8] examined the temperature distribution in the steady boundary layer flow of a second-order fluid past a stretching surface. Chen, Char and Cleaver [9] studied the flow and heat transfer in the boundary layer of a visco-elastic fluid of Walters' liquid B model over a stretching surface subject to either constant temperature or uniform heat flux. Interestingly the boundary layer equations for the steady two-dimensional flow studied in [8] and [9] are identical. However the heat transfer analyses in Refs. [7]–[9] suffer from the serious defect that neither the deformation energy stored in the fluid (owing to its elastic property) nor the viscous dissipation of energy were taken into account while solving the energy equation for determining temperature distribution. The aim of the present paper is to remove this defect and determine the temperature distribution in the steady boundary layer flow of a second-order fluid by taking the above two important effects into account. In the sequel it is shown that due to these effects there is a drastic change in the heat transfer characteristics and the temperature profiles become non-similar in marked contrast with the corresponding results in [8]–[9] where the temperature profiles were found to be similar.

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2 Heat transfer analysis

Consider the steady flow of an incompressible second-order fluid past a stretching surface coinciding with the plane $y = 0$ (Fig. 1). Two equal and opposite forces are applied along the x axis so that the surface is stretched with the origin fixed. Using the postulate of gradually fading memory, Coleman and Noll [10] derived the constitutive equation for the fluid as

$$\tau_{ij} = -P\delta_{ij} + \mu A_{(1)ij} + \alpha_1 A_{(1)ik} A_{(1)kj} + \alpha_2 A_{(2)ij} \quad (1)$$

where τ_{ij} is the stress tensor, P is an indeterminate pressure and μ, α_1 and α_2 are material constants. The rate-of-strain tensor $A_{(1)ij}$ and the acceleration tensor $A_{(2)ij}$ are defined by

$$A_{(1)ij} = v_{i,j} + v_{j.i} \quad (2)$$

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v_{m,i}v_{m,j} \quad (3)$$

where v_i's are the velocity components and a_i's are the acceleration components given by $v_j v_{i,j}$. It is worth pointing out that such a fluid exhibits normal stress effects in shear flows and (1) is applicable to flow of some dilute polymer solutions at low shear rates, α_2 being negative from thermodynamic considerations.

Using (1) – (3), the steady two dimensional boundary layer equations for the fluid (see [8]) are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - K\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right] \quad (5)$$

where (u, v) are the velocity components and

$$\nu = \mu/\rho, \quad K = -\alpha_2/\rho \quad (6)$$

In deriving these equations it is tacitly assumed that in addition to the usual boundary layer approximations, the contribution due to the normal stress is of the same order of magnitude as that due to shear stresses. Hence both ν and K are $O(\delta^2)$, δ being the boundary layer thickness. Note that $K > 0$.

The boundary conditions are

$$u = Cx, \quad v = 0 \quad \text{at } y = 0 \quad (7)$$

$$u \to 0 \quad \text{as } y \to \infty \quad (8)$$

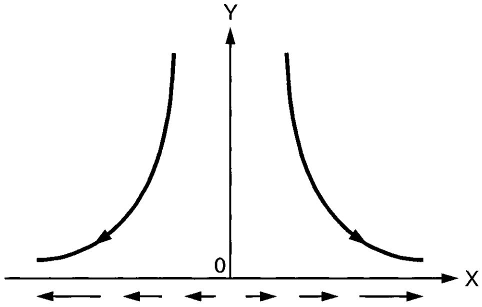

Fig. 1. A sketch of the physical problem

where C is a positive constant. A little inspection shows that (4) and (5) admit of a similarity solution

$$u = Cxf'(\eta), \quad v = -(\nu C)^{1/2} f(\eta) \quad (9)$$

where

$$\eta = (C/\nu)^{1/2} y \ . \quad (10)$$

With u and v given by (9), we find that (4) is identically satisfied while (5) gives

$$f'^2 - ff'' = f''' - K_1\left[2f'f''' - (f'')^2 - ff^{iv}\right] \quad (11)$$

where a prime denotes differentiation with respect to η and

$$K_1 = KC/\nu \ . \quad (12)$$

The boundary conditions (7) and (8) become

$$f'(0) = 1, \quad f(0) = 0, \quad f'(\infty) = 0 \ . \quad (13)$$

An exact solution of (11) satisfying (13) was given by Dandapat and Gupta [8] as

$$f(\eta) = (1 - K_1)^{1/2}\left[1 - e^{-\eta(1-K_1)^{-1/2}}\right] \ . \quad (14)$$

This solution is, of course valid for $K_1 < 1$. Such an assumption is justified for flow of a second-order fluid which displays short memory viscoelastic properties. Indeed $K_1 \ll 1$ for flow of dilute polymer solutions.

Let us now consider the heat transfer equation in the flow of a viscoelastic fluid. In this context it is necessary to establish the energy balance for a fluid element in motion and to consider it in conjunction with the equations of motion. It is important to remember that during the motion of a viscoelastic fluid, a certain amount of energy is stored up in the fluid as strain energy and some energy is lost due to viscous dissipation. Thus for an incompressible viscoelastic fluid the energy balance is determined by the internal energy, the conduction of heat, the convection of heat with the flow, the generation of heat through viscous dissipation (or friction) and the strain (or deformation) energy stored in the fluid due to its elastic properties. Following Schlichting [11], the transfer of heat in the steady flow of an incompressible second-order fluid can be expressed in the form of the energy equation given by

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \lambda\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\alpha_2}{\rho C_p}\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \quad (15)$$

where T, λ and C_p denote the temperature, thermal diffusivity and the specific heat of the fluid, respectively. In deriving (15) we use boundary layer approximations along with (1). Note that the second and the third term on the right hand side of (15) denote the terms due to viscous dissipation and strain energy, respectively.

The boundary conditions are

$$T = T_w \ \text{at } y = 0, \quad T \to T_\infty \ \text{as } y \to \infty \quad (16)$$

where T_w and T_∞ are constants with $T_w > T_\infty$.

Introducing the dimensionless temperature θ as

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{17}$$

and using (9), we find from (15)

$$xf' \frac{\partial\theta}{\partial x} - f \frac{\partial\theta}{\partial\eta} = \frac{\lambda}{\nu}\frac{\partial^2\theta}{\partial\eta^2} + x^2\left[\frac{C^2 f''^2}{C_p(T_w - T_\infty)} + \frac{\alpha_2 C^3 (f'f''^2 - ff''f''')}{\rho C_p \nu (T_w - T_\infty)}\right] \tag{18}$$

Setting

$$\theta(x,\eta) = \theta_0(\eta) \ + \ \frac{Cx^2}{\nu}\theta_1(\eta) \tag{19}$$

in (18) and equating the coefficients of x^0 and x^2, we get upon using (14)

$$\theta_0'' \ + \ \frac{\sigma}{r}(1 - e^{-r\eta})\theta_0' = 0 \tag{20}$$

$$\theta_1'' + \ \frac{\sigma}{r}(1 - e^{-r\eta})\theta_1' - 2\sigma e^{-r\eta}\theta_1 = -\sigma a e^{-2r\eta} \tag{21}$$

where

$$r = (1 - K_1)^{-1/2}, \quad \sigma = \frac{\nu}{\lambda}, \quad a = \frac{\nu C}{C_p(T_w - T_\infty)} \ . \tag{22}$$

It is clear from above that the temperature distribution $\theta(x,\eta)$ depends on three dimensionless parameters: (i) the viscoelastic parameter K_1, (ii) the Prandtl number σ and (iii) the Eckert number a (which characterizes viscous dissipation in the flow). The boundary conditions for $\theta_0(\eta)$ and $\theta_1(\eta)$ are obtained from (16), (17) and (19) as

$$\theta_0(0) = 1, \quad \theta_0(\infty) = 0 \tag{23a}$$

$$\theta_1(0) = 0, \quad \theta_1(\infty) = 0 \tag{23b}$$

The solution of (20) satisfying (23a) (see [8]) is

$$\theta_0(\eta) = 1 - \left[\int_{(\sigma/r^2)e^{-r\eta}}^{\sigma/r^2} z^{\left(\frac{\sigma}{r^2}\right)-1} e^{-z}\,dz \Big/ \int_0^{\sigma/r^2} z^{\left(\frac{\sigma}{r^2}\right)-1} e^{-z}\,dz\right] \ . \tag{24}$$

When $\sigma = r^2$, the solution for $\theta_0(\eta)$ can be obtained in a closed form as

$$\theta_0(\eta) = 1 - \frac{[\exp(-e^{-r\eta}) - e^{-1}]}{1 - e^{-1}} \tag{25}$$

To solve (21), we first note that it has a particular solution given by

$$(\theta_1)_p = \frac{a(r^2 - \sigma)}{2\sigma} + a e^{-rn} \tag{26}$$

Since (21) is linear, its complete solution is obtained by combining the solution (26) with the solution of the homogeneous equation corresponding to (21). Setting

$$t = -\frac{\sigma}{r^2} e^{-r\eta} \tag{27}$$

in this homogeneous equation, we get

$$t\frac{d^2\theta_1}{dt^2} + \left(1 - \frac{\sigma}{r^2} - t\right)\frac{d\theta_1}{dt} + 2\theta_1 = 0 \ . \tag{28}$$

This is a confluent hypergeometric equation whose general solution is

$$(\theta_1)_c = B_1 F\left(-2, 1 - \frac{\sigma}{r^2}, t\right) + B_2 t^{\sigma/r^2} F\left(\frac{\sigma}{r^2} - 2, 1 + \frac{\sigma}{r^2}, t\right) \tag{29}$$

where B_1 and B_2 are constants and F is the confluent hypergeometric function given by

$$F(a,b,x) = 1 + \frac{ax}{b} + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{x^3}{3!} + \cdots \ . \tag{30}$$

Hence the complete solution of (21) is given by

$$\theta_1 = (\theta_1)_p + (\theta_1)_c \tag{31}$$

Using (26), (27), (29) and the boundary conditions (23b) in (31), we obtain $\theta_1(\eta)$ as

$$\theta_1(\eta) = \frac{a}{2}\left(1 - \frac{r^2}{\sigma}\right)\left[F\left(-2, 1 - \frac{\sigma}{r^2}, -\frac{\sigma}{r^2}, e^{-r\eta}\right) - 1\right] + a e^{-r\eta} + \left\{\frac{\frac{a}{2}e^{-(\sigma\eta/r)}\left[\left(\frac{r^2}{\sigma} - 1\right)F\left(-2, 1 - \frac{\sigma}{r^2}, -\frac{\sigma}{r^2}\right) - \left(\frac{r^2}{\sigma} + 1\right)\right]}{F\left(\frac{\sigma}{r^2} - 2, 1 + \frac{\sigma}{r^2}, -\frac{\sigma}{r^2}\right)}\right\} \times F\left(\frac{\sigma}{r^2} - 2, 1 + \frac{\sigma}{r^2}, -\frac{\sigma}{r^2} e^{-r\eta}\right) \tag{32}$$

Thus the exact analytical solution for the temperature distribution in a closed form is obtained from (19) with $\theta_0(\eta)$ and $\theta_1(\eta)$ given by (24) and (32), respectively. It can be clearly seen from (19) that the temperature distribution is nonsimilar in marked contrast with the similarity solution for temperature distribution obtained in [8]. Using Kummer's transformation (Abramowitz and Stegun [12]) for the confluent hypergeometric function given by

$$F(a,b,z) = e^z F(b - a, b, -z) \ , \tag{33}$$

$\theta_1(\eta)$ is computed from (32).

Numerical solutions of the Eqs. (20)–(21) are obtained by using a fourth-order accurate compact Hermitian finite difference scheme. This method is described in great details by Peyret and Taylor [13] and Adam [14]. To test the accuracy of our numerical method, we have compared the values of $\theta_1(\eta)$ using this method with the corresponding values obtained from the exact analytical solution (32). These results are presented in Table 1 where the values derived from (32) are shown within parentheses.

It is clear that there is an excellent agreement between the numerical and analytical solutions.

3
Discussion

Figures 2 and 3 show the variation of $\theta_0(\eta)$ and $\theta_1(\eta)$ with η for several values of σ with $K_1 = 0.09$ and $a = 4$. It can be seen that at a given point in the flow, $\theta_0(\eta)$ decreases with increase in σ. But $\theta_1(\eta)$ increases with increase in σ

upto a certain value of η and thereafter θ_1 decreases with increase in σ. Since $\theta_1(\eta) > 0$ everywhere, it follows from (19) that temperature at a point increases due to the combined influence of viscous dissipation and stored strain energy in the flow in comparison with its value θ_0 (see [8]) in the absence of these two effects. Figure 4 shows the variation of $\theta_1(\eta)$ with η for several values of a with K_1 = 0.09 and σ = 10. It can be seen from this figure and (19) that for fixed K_1 and σ, the temperature at a point increases with increase in the viscous dissipation parameter a. This is plausible on physical grounds since viscous dissipation tends to raise the temperature of the fluid. From (19), the dimensionless heat transfer coefficient $-\theta'(0)$ is evaluated as

$$-\theta'(0) = -\theta_0'(0) - \frac{Cx^2}{\nu}\theta_1'(0) \tag{34}$$

Table 2 gives the values of $-\theta_0'(0)$ for several values of the elastic parameter K_1 when $\sigma = 10$.

Figure 5 shows the variation of $-\theta_1'(0)$ with a for several values of K_1 when $\sigma = 10$. It follows from this figure that for a fixed value of K_1, the variation of $-\theta_1'(0)$ is linear with a. This variation is, of course, evident from the analytical solution for $\theta_1(\eta)$ given by Eq. (32) which clearly shows that the parameter a can be factored out. It can be clearly seen from this figure, Table 2 and equation (34) that for given values of σ, K_1 and a, heat flows from the stretching surface to the fluid for small enough values of $x(C/\nu)^{1/2}$. But at a certain value X of $x(C/\nu)^{1/2}$, the surface heat flux vanishes and when $x(C/\nu)^{1/2}$ exceeds X, heat flows from the fluid to the stretching surface. This interesting result admits of a physical interpretation. For small enough values of $x(C/\nu)^{1/2}$, both viscous dissipation and strain energy in the flow are small (see equation (18)) and hence no significant heat is generated inside the flow. Thus for small values of $x(C/\nu)^{1/2}$, heat will flow from the surface to the fluid since $T_w > T_\infty$. On the other hand for large values of $x(C/\nu)^{1/2}$, sufficient heat is generated inside the boundary layer due to the combined influence of vis-

Table 1. Variation of $\theta_1(\eta)$ with η for σ = 10 and a = 4

	$\eta = 0.2$	$\eta = 0.4$	$\eta = 1$	$\eta = 1.2$
$K_1 = 0.09$	0.756674 (0.756576)	0.808960 (0.808923)	0.309685 (0.309683)	0.205941 (0.205940)
$K_1 = 0.25$	0.719458 (0.719621)	0.753481 (0.753521)	0.264105 (0.264105)	0.169078 (0.169078)
$K_1 = 0.33$	0.697409 (0.697595)	0.721247 (0.721299)	0.239745 (0.239745)	0.149966 (0.149966)

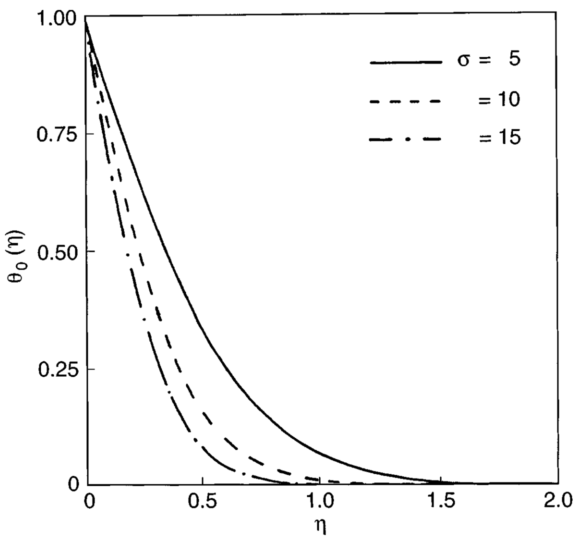

Fig. 2. Variation of $\theta_0(\eta)$ with η for several values of σ with $K_1 = 0.09$ and $a = 4$

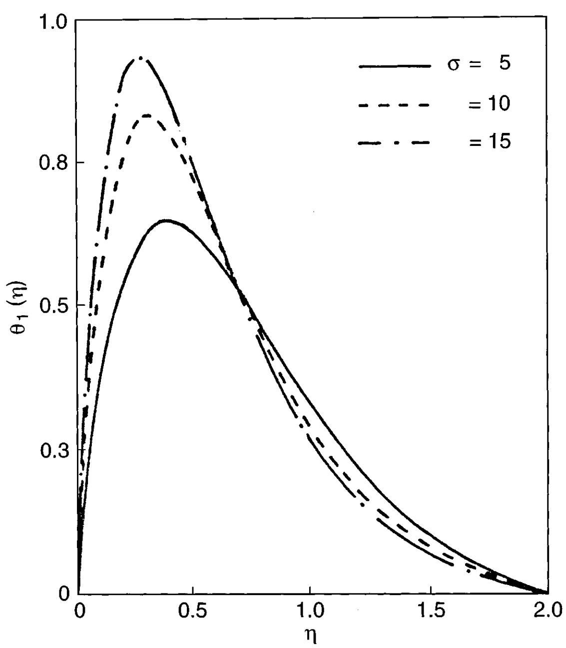

Fig. 3. Variation of $\theta_1(\eta)$ with η for several values of σ with $K_1 = 0.09$ and $a = 4$

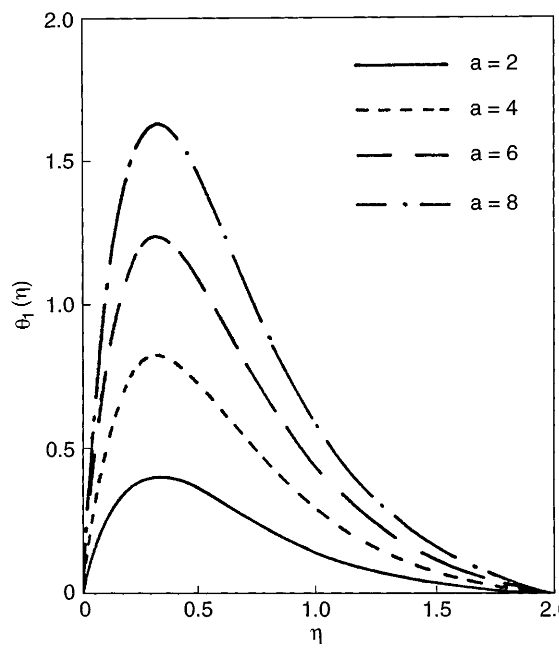

Fig. 4. Variation of $\theta_1(\eta)$ with η for several values of a with $K_1 = 0.09$ and $\sigma = 10$

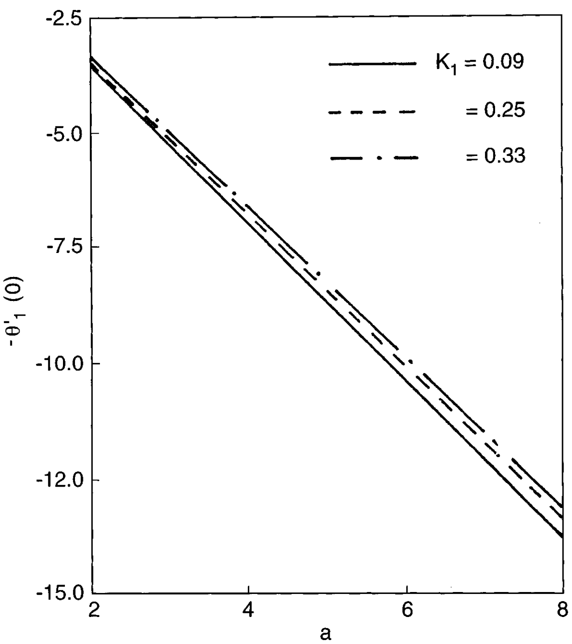

Fig. 5. Variation of $-\theta_1'(0)$ with a for several values of K_1 when $\sigma = 10$

Table 2. Values of $-\theta'_0(0)$ with $\sigma = 10$

K_1	$-\theta'_0(0)$
0.09	2.297451
0.17	2.286709
0.25	2.274215
0.33	2.259658

cous dissipation and stored deformation energy. Under such circumstances temperature very near the surface exceeds the surface temperature T_w and heat then flows from the fluid to the surface even when $T_w > T_\infty$. However in the absence of viscous dissipation and strain energy, heat always flows from the surface to the fluid as long as $T_w > T_\infty$. This can be seen from Table 2. Thus we find that there is a drastic change in the heat transfer characteristics of the flow when the above two effects are taken into account.

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