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Periodic heat conduction with relaxation time in cylindrical geometry

E. Zanchini, B. Pulvirenti

Abstract Steady-periodic heat conduction with relaxation time in an infinitely long hollow cylinder is considered. Four boundary value problems, with boundary conditions of the first and of the second kind, are solved analytically. The solution for a solid cylinder with a sinusoidally varying surface temperature is obtained as a special case of a solution found for the hollow cylinder. The effects of the relaxation time on the steady-periodic temperature field are analysed, in details, for a solid cylinder with a sinusoidally varying surface temperature and for a hollow cylinder with a sinusoidally varying heat flux at the inner surface and with a constant temperature at the outer surface. The results show that thermal resonances may occur and suggest that accurate measurements of the relaxation time could be obtained by means of experiments on steady-periodic heat conduction in cylindrical geometry.

List of symbols

Arg	argument of a complex number
A_1	amplitude of the temperature fluctuations at the inner surface [K]
A_2	amplitude of the temperature fluctuations at the outer surface [K]
a_1	$= (i\omega_1 - \tau\omega_1^2)/\alpha$, complex constant [$m^{-2}$]
a_2	$= (i\omega_2 - \tau\omega_2^2)/\alpha$, complex constant [$m^{-2}$]
B_1	amplitude of the heat-flux fluctuations at the inner surface [W/m^2]
B_2	amplitude of the heat-flux fluctuations at the outer surface [W/m^2]
c	specific heat at constant density [J/(kgK)]
$C_0(r)$	complex function defined by Eq. (5) [K]
$C_1(r)$	complex function defined by Eq. (5) [K]
$C_2(r)$	complex function defined by Eq. (5) [K]
$D_0(r)$	complex function defined by Eq. (12) [W/m^2]
$D_1(r)$	complex function defined by Eq. (12) [W/m^2]
$D_2(r)$	complex function defined by Eq. (12) [W/m^2]
i	$= \sqrt{-1}$, imaginary unit
I_n	modified Bessel function of first kind and order n
k	thermal conductivity [W/(mK)]
K_n	modified Bessel function of second kind and order n
m_0, m_1, m_2	complex coefficients which appear in Eqs. (9)–(11) [K]
n_0, n_1, n_2	complex coefficients which appear in Eqs. (9)–(11) [K]
$\mathbf{q}$	heat-flux density vector [W/m^2]
q	heat-flux density in the radial direction [W/m^2]
Q_0	time average of the heat flux for unit length [W/m]
r	radial coordinate [m]
r_1	inner radius of a hollow cylinder [m]
r_2	outer radius of a hollow cylinder, radius of a solid cylinder [m]
Re	real part of a complex number
t	time [s]
T	temperature, complex temperature [K]
T_1	time average of the temperature of the inner surface [K]
T_2	time average of the temperature of the outer surface [K]
u	internal energy per unit mass [J/kg]
w	$= \sqrt{i\Phi - \Phi^2\Gamma}$, dimensionless complex constant
$\lvert\ \rvert$	modulus of a complex number

Greek symbols

α	thermal diffusivity [m^2/s]
Γ	$= \alpha\tau/r_1^2$, dimensionless parameter defined in Eq. (48)
ζ	$= r/r_1$, dimensionless radial coordinate defined in Eq. (48)
η	$= r/r_2$, dimensionless radial coordinate defined in Eq. (46)
ϑ	dimensionless temperature defined in Eq. (46)
Λ	$= \alpha\tau/r_2^2$, dimensionless parameter defined in Eq. (46)
ξ	$= t/\tau$, dimensionless time defined in Eq. (46)
ρ	mass density [Kg/m^3]
τ	thermal relaxation time [s]

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v	dimensionless temperature defined in Eq. (48)
Φ	$= (\omega_1 r_1^2)/\alpha$, dimensionless angular frequency defined in Eq. (48)
Ω	$= (\omega_2 r_2^2)/\alpha$, dimensionless angular frequency defined in Eq. (46)
ω_1	angular frequency of temperature and heat flux at the inner surface [rad/s]
ω_2	angular frequency of temperature and heat flux at the outer surface [rad/s]

1 Introduction

In the analysis of transient heat conduction with quick temperature changes, the classical Fourier's diffusion theory may be inaccurate. In order to obtain more reliable previsions one can replace Fourier's law with Cattaneo-Vernotte's constitutive equation for the heat flux density vector $\mathbf{q}$, namely [1–3]

$$\mathbf{q} + \tau \frac{\partial \mathbf{q}}{\partial t} = -k\nabla T \ , \tag{1}$$

where t is time, T is temperature, τ is the relaxation time and k is the thermal conductivity of the substance. Let us consider a medium such that k, τ, the mass density ρ and the specific heat at constant density c can be considered as constants. For this medium, if the differential of the internal energy per unit mass can be expressed as $\mathrm{d}u = c\mathrm{d}T$, Eq. (1) and the energy balance equation $\nabla \bullet \mathbf{q} = -\rho c(\partial T/\partial t)$ yield the telegraph equation for the temperature field

$$\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} = \alpha \nabla^2 T \ , \tag{2}$$

where $\alpha = k/(\rho c)$ is the thermal diffusivity.

Many solutions of unsteady heat conduction problems based on Eqs. (1) and (2) have been presented in the literature; most of them are reviewed in [4–6]. Almost all the available solutions concern the transient behaviour of the temperature field due to a sudden change of the boundary conditions. Very few solutions of steady-periodic heat conduction problems based on Eqs. (1) and (2) have been presented. Glass, Özisik and Vick [7] deal with a semi-infinite medium bounded by a plane surface and subjected to a periodic on-off type heat flux at the surface. At the initial instant, the medium is in thermodynamic equilibrium. Two solutions, with and without heat radiation from the surface to an external ambient, are presented. Clearly, in the absence of heat radiation the temperature field cannot reach a true steady-periodic regime. Yuen and Lee [8] consider the time evolution, starting from thermodynamic equilibrium, of a semi-infinite medium bounded by a plane surface. For $t > 0$, the surface is exposed to a sinusoidal heat flux with zero mean value. The steady-periodic temperature field is obtained by considering the limit of the solution for $t \to \infty$. Tang and Araki [9] analyse a plane slab whose front surface is exposed to a sinusoidal heat flux with zero mean value, while the rear surface is insulated. The medium is initially in thermodynamic equilibrium. The time evolution of the temperature at both surfaces is evaluated. Novikov [10] studies the steady-periodic heat conduction in a plane slab such that the temperature of one surface is a sinusoidal function of time while the other surface is insulated. Tzou [11] considers a non-uniform heat generation which is a sinusoidal function of time within a plane slab with insulated surfaces. The author points out that resonance phenomena may occur and determines the values of the resonance frequencies. The heat source considered in Ref. [11] is not easily obtained experimentally. Barletta and Zanchini [12] deal with an infinitely long solid cylinder with an internal heat generation produced by an alternating current. The power generated per unit volume is non-uniform and steady-periodic. The surface of the cylinder is assumed to exchange heat by convection with an external fluid. The authors determine the steady-periodic temperature field within the cylinder and the thermal-resonance frequencies.

Thus, while some attention has already been devoted to steady-periodic heat conduction with relaxation time and no heat generation in plane geometry [8, 10], no analysis of hyperbolic heat conduction in cylindrical geometry, in the absence of internal heat generation and in steady-periodic regime, is available in the literature.

The aim of this paper is to analyse the effects of the relaxation time on the temperature field for steady-periodic heat conduction in an infinitely long hollow cylinder. The following boundary conditions are considered: sinusoidally varying temperature at each surface; sinusoidally varying heat flux at each surface; sinusoidally varying temperature at the inner surface and sinusoidally varying heat flux at the outer surface; sinusoidally varying heat flux at the inner surface and sinusoidally varying temperature at the outer surface. The solution for a solid cylinder with a sinusoidally-varying temperature at the surface is obtained as a particular case of the last boundary condition.

The results show that non-Fourier effects are relevant and that thermal resonances may occur even for rather low values of the relaxation time τ. In particular, for a solid cylinder whose radius is 2 mm and whose thermal diffusivity is $10^{-6}\ \mathrm{m^2/s}$, for $\tau = 1$ s a thermal resonance occurs at the axis if the angular frequency of the sinusoidal temperature prescribed at the surface is $1.183\ \mathrm{rad\,s^{-1}}$. Moreover, the amplitude and the phase of the temperature fluctuations at the axis of the cylinder are appreciably different from those predicted by Fourier's theory even for $\tau = 0.2$ s. These results suggest that careful measurements of the relaxation time could be performed by measuring the amplitude and the phase of the temperature field in steady-periodic heat conduction in cylindrical geometry.

2 Steady-periodic solutions of the telegraph equation

In this section, the telegraph equation for the temperature field in cylindrical geometry is recalled. Then, the general solution of this equation in steady-periodic regime is obtained for an infinitely long hollow cylinder with either a periodically varying temperature or a periodically varying heat flux at each boundary surface. Finally, the corresponding heat flux distribution in steady-periodic regime is determined.

Let us consider an infinitely long hollow cylinder, with internal radius r_1 and external radius r_2. Let us assume that the density, the thermal conductivity, the thermal diffusivity and the thermal relaxation time of the annulus are constants and that the temperature of the annulus depends only on time and on the radial coordinate r. With this assumptions, Eq. (2) holds and can be expressed as

$$\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau}{\alpha}\frac{\partial^2 T}{\partial t^2} \quad . \tag{4}$$

If either a periodically varying temperature or a periodically varying heat flux is prescribed at each boundary surface, the steady-periodic solutions of Eq. (4) are of the kind

$$T(r,t)=C_0(r)+C_1(r)\mathrm{e}^{i\omega_1 t}+C_2(r)\mathrm{e}^{i\omega_2 t} \quad , \tag{5}$$

where T is the complex temperature, ω_1 and ω_2 are the angular frequencies at r_1 and at r_2, $C_0(r)$, $C_1(r)$ and $C_2(r)$ are undetermined complex functions. By substituting Eq. (5) in Eq. (4), the following conditions on $C_0(r)$, $C_1(r)$ and $C_2(r)$ are obtained:

$$\frac{1}{r}\frac{\mathrm{d}C_0}{\mathrm{d}r}+\frac{\mathrm{d}^2 C_0}{\mathrm{d}r^2}=0 \quad , \tag{6}$$

$$\frac{1}{r}\frac{\mathrm{d}C_1}{\mathrm{d}r}+\frac{\mathrm{d}^2 C_1}{\mathrm{d}r^2}=\frac{1}{\alpha}(i\omega_1-\tau\omega_1^2)C_1(r) \quad , \tag{7}$$

$$\frac{1}{r}\frac{\mathrm{d}C_2}{\mathrm{d}r}+\frac{\mathrm{d}^2 C_2}{\mathrm{d}r^2}=\frac{1}{\alpha}(i\omega_2-\tau\omega_2^2)C_2(r) \quad . \tag{8}$$

Equations (6), (7) and (8) yield

$$C_0(r)=m_0\ln r+n_0 \quad , \tag{9}$$

$$C_1(r)=m_1 I_0(\sqrt{a_1}r)+n_1 K_0(\sqrt{a_1}r) \quad , \tag{10}$$

$$C_2(r)=m_2 I_0(\sqrt{a_2}r)+n_2 K_0(\sqrt{a_2}r) \quad , \tag{11}$$

where $a_1=(i\omega_1-\tau\omega_1^2)/\alpha$, $a_2=(i\omega_2-\tau\omega_2^2)/\alpha$. The coefficients m_i and n_i, $(i=0,\ 1,\ 2)$ are determined by the boundary conditions.

Moreover, if either a periodically varying temperature or a periodically varying heat flux is prescribed at each boundary surface, the steady-periodic heat flux distributions are of the kind

$$q(r,t)=D_0(r)+D_1(r)\mathrm{e}^{i\omega_1 t}+D_2(r)\mathrm{e}^{i\omega_2 t} \quad . \tag{12}$$

From Eq. (12), Eq. (5) and Eq. (1), the following conditions on $D_0(r)$, $D_1(r)$ and $D_2(r)$ are obtained:

$$D_0(r)=-k\frac{\mathrm{d}C_0}{\mathrm{d}r} \quad , \tag{13}$$

$$D_1(r)(1+i\omega_1\tau)=-k\frac{\mathrm{d}C_1}{\mathrm{d}r} \quad , \tag{14}$$

$$D_2(r)(1+i\omega_2\tau)=-k\frac{\mathrm{d}C_2}{\mathrm{d}r} \quad . \tag{15}$$

By substituting Eqs. (9)–(11) in Eqs. (13)–(15) and by applying the properties of Bessel functions [13], one obtains

$$D_0(r)=-k\frac{m_0}{r} \quad , \tag{16}$$

$$D_1(r)=-\frac{k\sqrt{a_1}}{(1+i\omega_1\tau)}[m_1 I_1(\sqrt{a_1}r)-n_1 K_1(\sqrt{a_1}r)] \quad , \tag{17}$$

$$D_2(r)=-\frac{k\sqrt{\alpha_2}}{(1+i\omega_2\tau)}[m_2 I_1(\sqrt{a_2}r)-n_2 K_1(\sqrt{a_2}r)] \quad . \tag{18}$$

3
Boundary value problems

In this section, four steady-periodic boundary conditions for an infinitely long hollow cylinder are selected and the corresponding steady-periodic distributions of temperature and heat flux are determined.

Let us consider the following steady-periodic boundary conditions.

$$\begin{cases} T(r_1,t)=T_1+A_1\mathrm{e}^{i\omega_1 t} \\ T(r_2,t)=T_2+A_2\mathrm{e}^{i\omega_2 t} \quad , \end{cases} \tag{19}$$

$$\begin{cases} q(r_1,t)=\dfrac{Q_0}{2\pi r_1}+B_1\mathrm{e}^{i\omega_1 t} \\ q(r_2,t)=\dfrac{Q_0}{2\pi r_2}+B_2\mathrm{e}^{i\omega_2 t} \quad , \end{cases} \tag{20}$$

$$\begin{cases} T(r_1,t)=T_1+A_1\mathrm{e}^{i\omega_1 t} \\ q(r_2,t)=\dfrac{Q_0}{2\pi r_2}+B_2\mathrm{e}^{i\omega_2 t} \quad , \end{cases} \tag{21}$$

$$\begin{cases} q(r_1,t)=\dfrac{Q_0}{2\pi r_1}+B_1\mathrm{e}^{i\omega_1 t} \\ T(r_2,t)=T_2+A_2\mathrm{e}^{i\omega_2 t} \quad . \end{cases} \tag{22}$$

In Eqs. (19)–(22), T is the complex temperature, while T_1, T_2, A_1, A_2, B_1, B_2 and Q_0 are real quantities. Thus, for instance, Eq. (19) represents the real boundary condition $\mathrm{Re}[T(r_1,t)]=T_1+A_1\cos(\omega_1 t)$ and $\mathrm{Re}[T(r_2,t)]=T_2+A_2\cos(\omega_2 t)$. In Eqs. (20)–(22), Q_0 is the time average of the heat flux per unit length which crosses the annulus.

The case of a solid cylinder with a surface temperature $T(r_2,t)=T_2+A_2\mathrm{e}^{i\omega_2 t}$ can be obtained from Eq. (22), as follows: r_1 tends to zero, as well as $q(r_1,t)$. The case of an infinite solid medium which surrounds a cylindrical surface with a temperature $T(r_1,t)=T_1+A_1\mathrm{e}^{i\omega_1 t}$ can be obtained from Eq. (21) as follows: r_2 tends to infinity, $q(r_2,t)$ tends to zero and $Q_0=0$.

By employing Eq. (1), Eqs. (20)–(22) can be rewritten as follows:

$$\begin{cases} \left.\dfrac{\partial T}{\partial r}\right|_{r_1,t}=-\dfrac{Q_0}{2\pi k r_1}-\dfrac{B_1}{k}(1+i\omega_1\tau)\mathrm{e}^{i\omega_1 t} \\ \left.\dfrac{\partial T}{\partial r}\right|_{r_2,t}=-\dfrac{Q_0}{2\pi k r_2}-\dfrac{B_2}{k}(1+i\omega_2\tau)\mathrm{e}^{i\omega_2 t} \quad , \end{cases} \tag{23}$$

$$\begin{cases} T(r_1,t)=T_1+A_1\mathrm{e}^{i\omega_1 t} \\ \left.\dfrac{\partial T}{\partial r}\right|_{r_2,t}=-\dfrac{Q_0}{2\pi k r_2}-\dfrac{B_2}{k}(1+i\omega_2\tau)\mathrm{e}^{i\omega_2 t} \quad , \end{cases} \tag{24}$$

$$\begin{cases} \left.\dfrac{\partial T}{\partial r}\right|_{r_1,t}=-\dfrac{Q_0}{2\pi k r_1}-\dfrac{B_1}{k}(1+i\omega_1\tau)\mathrm{e}^{i\omega_1 t} \\ T(r_2,t)=T_2+A_2\mathrm{e}^{i\omega_2 t} \quad . \end{cases} \tag{25}$$

Let us first consider the boundary condition given by Eq. (19).
On account of Eq. (5), Eq. (19) can be rewritten as

$$\begin{cases} C_0(r_1) + C_1(r_1)\mathrm{e}^{i\omega_1 t} + C_2(r_1)\mathrm{e}^{i\omega_2 t} = T_1 + A_1\mathrm{e}^{i\omega_1 t} \\ C_0(r_2) + C_1(r_2)\mathrm{e}^{i\omega_1 t} + C_2(r_2)\mathrm{e}^{i\omega_2 t} = T_2 + A_2\mathrm{e}^{i\omega_2 t} \end{cases}. \tag{26}$$

Equation (26) implies

$$\begin{cases} C_0(r_1) = T_1 \\ C_0(r_2) = T_2 \end{cases}, \tag{27}$$

$$\begin{cases} C_1(r_1) = A_1 \\ C_1(r_2) = 0 \end{cases}, \tag{28}$$

$$\begin{cases} C_2(r_1) = 0 \\ C_2(r_2) = A_2 \end{cases}. \tag{29}$$

Equations (9)–(11) and (27)–(29) yield

$$\begin{cases} m_0 = -\dfrac{T_1 - T_2}{\ln(r_2/r_1)} \\ n_0 = \dfrac{T_1 \ln r_2 - T_2 \ln r_1}{\ln(r_2/r_1)} \end{cases}, \tag{30}$$

$$\begin{cases} m_1 = \dfrac{A_1 K_0(\sqrt{a_1} r_2)}{I_0(\sqrt{a_1} r_1)K_0(\sqrt{a_1} r_2) - I_0(\sqrt{a_1} r_2)K_0(\sqrt{a_1} r_1)} \\ n_1 = \dfrac{-A_1 I_0(\sqrt{a_1} r_2)}{I_0(\sqrt{a_1} r_1)K_0(\sqrt{a_1} r_2) - I_0(\sqrt{a_1} r_2)K_0(\sqrt{a_1} r_1)} \end{cases}, \tag{31}$$

$$\begin{cases} m_2 = \dfrac{-A_2 K_0(\sqrt{a_2} r_1)}{I_0(\sqrt{a_2} r_1)K_0(\sqrt{a_2} r_2) - I_0(\sqrt{a_2} r_2)K_0(\sqrt{a_2} r_1)} \\ n_2 = \dfrac{A_2 I_0(\sqrt{a_2} r_1)}{I_0(\sqrt{a_2} r_1)K_0(\sqrt{a_2} r_2) - I_0(\sqrt{a_2} r_2)K_0(\sqrt{a_2} r_1)} \end{cases}. \tag{32}$$

By substituting Eqs. (30)–(32) in Eqs. (9)–(11), one obtains the following steady-periodic temperature distribution in the hollow cylinder:

$$\begin{aligned} T(r,t) = {} & A_1 \frac{K_0(\sqrt{a_1} r_2)I_0(\sqrt{a_1} r) - I_0(\sqrt{a_1} r_2)K_0(\sqrt{a_1} r)}{I_0(\sqrt{a_1} r_1)K_0(\sqrt{a_1} r_2) - I_0(\sqrt{a_1} r_2)K_0(\sqrt{a_1} r_1)} \\ & \times \mathrm{e}^{i\omega_1 t} \\ & + A_2 \frac{I_0(\sqrt{a_2} r_1)K_0(\sqrt{a_2} r) - K_0(\sqrt{a_2} r_1)I_0(\sqrt{a_2} r)}{I_0(\sqrt{a_2} r_1)K_0(\sqrt{a_2} r_2) - K_0(\sqrt{a_2} r_1)I_0(\sqrt{a_2} r_2)} \\ & \times \mathrm{e}^{i\omega_2 t} \\ & + T_1 - \frac{T_1 - T_2}{\ln(r_2/r_1)}\ln(r/r_1). \end{aligned} \tag{33}$$

By substituting Eqs. (30)–(32) in Eqs. (16)–(18), one obtains the following steady-periodic heat-flux distribution in the hollow cylinder:

$$\begin{aligned} & q(r,t) = \\ & -\frac{kA_1\sqrt{a_1}}{1+i\omega_1\tau}\frac{K_0(\sqrt{a_1} r_2)I_1(\sqrt{a_1} r) + I_0(\sqrt{a_1} r_2)K_1(\sqrt{a_1} r)}{I_0(\sqrt{a_1} r_1)K_0(\sqrt{a_1} r_2) - I_0(\sqrt{a_1} r_2)K_0(\sqrt{a_1} r_1)}\mathrm{e}^{i\omega_1 t} \\ & +\frac{kA_2\sqrt{a_2}}{1+i\omega_2\tau}\frac{K_0\left(\sqrt{a_1} r_1\right)I_1\left(\sqrt{a_2} r\right) + I_0\left(\sqrt{a_2} r_1\right)K_1\left(\sqrt{a_2} r\right)}{I_0\left(\sqrt{a_2} r_1\right)K_0\left(\sqrt{a_2} r_2\right) - I_0\left(\sqrt{a_2} r_2\right)K_0\left(\sqrt{a_2} r_1\right)}\mathrm{e}^{i\omega_2 t} \\ & + k\frac{T_1 - T_2}{r\ln(r_2/r_1)}. \end{aligned} \tag{34}$$

Let us now consider the boundary condition given by Eq. (23).
On account of Eq. (5), Eq. (23) can be rewritten as

$$\begin{cases} \left.\dfrac{\mathrm{d}C_1}{\mathrm{d}r}\right|_{r_1}\mathrm{e}^{i\omega_1 t} + \left.\dfrac{\mathrm{d}C_2}{\mathrm{d}r}\right|_{r_1}\mathrm{e}^{i\omega_2 t} + \left.\dfrac{\mathrm{d}C_0}{\mathrm{d}r}\right|_{r_1} \\ \quad = -\dfrac{Q_0}{2\pi k r_1} - \dfrac{B_1}{k}(1+i\omega_1\tau)\mathrm{e}^{i\omega_1 t} \\ \left.\dfrac{\mathrm{d}C_1}{\mathrm{d}r}\right|_{r_2}\mathrm{e}^{i\omega_1 t} + \left.\dfrac{\mathrm{d}C_2}{\mathrm{d}r}\right|_{r_2}\mathrm{e}^{i\omega_2 t} + \left.\dfrac{\mathrm{d}C_0}{\mathrm{d}r}\right|_{r_2} \\ \quad = -\dfrac{Q_0}{2\pi k r_2} - \dfrac{B_2}{k}(1+i\omega_2\tau)\mathrm{e}^{i\omega_2 t} \end{cases}. \tag{35}$$

By means of the same procedure as that employed to deduce Eq. (33), from Eqs. (9)-(11) and Eq. (35) one obtains

$$\begin{aligned} & T(r,t) = \\ & -\frac{B_1(1+i\omega_1\tau)[K_1(\sqrt{a_1} r_2)I_0(\sqrt{a_1} r) + I_1(\sqrt{a_1} r_2)K_0(\sqrt{a_1} r)]}{\sqrt{a_1}k[I_1(\sqrt{a_1} r_1)K_1(\sqrt{a_1} r_2) - I_1(\sqrt{a_1} r_2)K_1(\sqrt{a_1} r_1)]}\mathrm{e}^{i\omega_1 t} \\ & -\frac{B_2(1+i\omega_2\tau)[K_1(\sqrt{a_2} r_1)I_0(\sqrt{a_2} r) + I_1(\sqrt{a_2} r_1)K_0(\sqrt{a_2} r)]}{\sqrt{a_2}k[I_1(\sqrt{a_2} r_2)K_1(\sqrt{a_2} r_1) - I_1(\sqrt{a_2} r_1)K_1(\sqrt{a_2} r_2)]}\mathrm{e}^{i\omega_2 t} \\ & + T_2 - \frac{Q_0}{2\pi k}\ln(r/r_2). \end{aligned} \tag{36}$$

By the same method as that used to deduce Eq. (34), one obtains the following steady-periodic heat-flux distribution in the hollow cylinder:

$$\begin{aligned} & q(r,t) = \\ & B_1\frac{K_1(\sqrt{a_1} r_2)I_1(\sqrt{a_1} r) - I_1(\sqrt{a_1} r_2)K_1(\sqrt{a_1} r)}{I_1(\sqrt{a_1} r_1)K_1(\sqrt{a_1} r_2) - I_1(\sqrt{a_1} r_2)K_1(\sqrt{a_1} r_1)}\mathrm{e}^{i\omega_1 t} \\ & + B_2\frac{K_1(\sqrt{a_2} r_1)I_1(\sqrt{a_2} r) - I_1(\sqrt{a_2} r_1)K_1(\sqrt{a_2} r)}{I_1(\sqrt{a_2} r_2)K_1(\sqrt{a_2} r_1) - I_1(\sqrt{a_2} r_1)K_1(\sqrt{a_2} r_2)}\mathrm{e}^{i\omega_2 t} \\ & + \frac{Q_0}{2\pi r}. \end{aligned} \tag{37}$$

Let us consider the boundary condition given by Eq. (24).
On account of Eq. (5), Eq. (24) can be rewritten as

$$\begin{cases} C_0(r_1) + C_1(r_1)\mathrm{e}^{i\omega_1 t} + C_2(r_1)\mathrm{e}^{i\omega_2 t} \\ \quad = T_1 + A_1\mathrm{e}^{i\omega_1 t} \\ \left.\dfrac{\mathrm{d}C_0}{\mathrm{d}r}\right|_{r_2} + \left.\dfrac{\mathrm{d}C_1}{\mathrm{d}r}\right|_{r_2}\mathrm{e}^{i\omega_1 t} + \left.\dfrac{\mathrm{d}C_2}{\mathrm{d}r}\right|_{r_2}\mathrm{e}^{i\omega_2 t} \\ \quad = \dfrac{-Q_0}{2\pi k r_2} - \dfrac{B_2}{k}(1+i\omega_2\tau)\mathrm{e}^{i\omega_2 t} \end{cases}. \tag{38}$$

By means of the same procedure as that employed to deduce Eq. (33), from Eqs. (9)-(11) and Eq. (38) one obtains

$$T(r,t) = A_1 \frac{K_1(\sqrt{a_1}r_2)I_0(\sqrt{a_1}r) + I_1(\sqrt{a_1}r_2)K_0(\sqrt{a_1}r)}{I_0(\sqrt{a_1}r_1)K_1(\sqrt{a_1}r_2) + I_1(\sqrt{a_1}r_2)K_0(\sqrt{a_1}r_1)} e^{i\omega_1 t} + \frac{B_2(1+i\omega_2\tau)[I_0(\sqrt{a_2}r_1)K_0(\sqrt{a_2}r) - K_0(\sqrt{a_2}r_1)I_0(\sqrt{a_2}r)]}{\sqrt{a_2}k[I_1(\sqrt{a_2}r_2)K_0(\sqrt{a_2}r_1) + I_0(\sqrt{a_2}r_1)K_1(\sqrt{a_2}r_2)]} e^{i\omega_2 t} + T_1 - \frac{Q_0}{2\pi k}\ln(r/r_1). \tag{39}$$

By the same method as that used to deduce Eq. (34), one obtains the following steady-periodic heat-flux distribution in the hollow cylinder:

$$q(r,t) = -\frac{kA_1\sqrt{a_1}}{1+i\omega_1\tau} \frac{K_1(\sqrt{a_1}r_2)I_1(\sqrt{a_1}r) - I_1(\sqrt{a_1}r_2)K_1(\sqrt{a_1}r)}{I_0(\sqrt{a_1}r_1)K_1(\sqrt{a_1}r_2) + I_1(\sqrt{a_1}r_2)K_0(\sqrt{a_1}r_1)} e^{i\omega_1 t} + B_2 \frac{K_0(\sqrt{a_2}r_1)I_1(\sqrt{a_2}r_1 + I_0(\sqrt{a_2}r_1)K_1(\sqrt{a_2}r)}{I_1(\sqrt{a_2}r_2)K_0(\sqrt{a_2}r_1) + I_0(\sqrt{a_2}r_1)K_1(\sqrt{a_2}r_2)} e^{i\omega_2 t} + \frac{Q_0}{2\pi r}. \tag{40}$$

Finally, let us consider the boundary condition given by Eq. (25).
On account of Eq. (5), Eq. (25) can be rewritten as

$$\begin{cases} \left.\frac{dC_0}{dr}\right|_{r_1} + \left.\frac{dC_1}{dr}\right|_{r_1} e^{i\omega_1 t} + \left.\frac{dC_2}{dr}\right|_{r_1} e^{i\omega_2 t} = \\ -\frac{Q_0}{2\pi k r_1} - \frac{B_1}{k}(1+i\omega_1\tau)e^{i\omega_1 t} . \\ C_0(r_2) + C_1(r_2)e^{i\omega_1 t} + C_2(r_2)e^{i\omega_2 t} = \\ T_2 + A_2 e^{i\omega_2 t} \end{cases} \tag{41}$$

By means of the same procedure as that employed to deduce Eq. (33), from Eqs. (9)-(11) and Eq. (41) one obtains

$$T(r,t) = \frac{B_1(1+i\omega_1\tau)[I_0(\sqrt{a_1}r_2)K_0(\sqrt{a_1}r) - K_0(\sqrt{a_1}r_2)I_0(\sqrt{a_1}r)]}{\sqrt{a_1}k[I_1(\sqrt{a_1}r_1)K_0(\sqrt{a_1}r_2) + I_0(\sqrt{a_1}r_2)K_1(\sqrt{a_1}r_1)]} \times e^{i\omega_1 t} + A_2 \frac{K_1(\sqrt{a_2}r_1)I_0(\sqrt{a_2}r) + I_1(\sqrt{a_2}r_1)K_0(\sqrt{a_2}r)}{I_0(\sqrt{a_2}r_2)K_1(\sqrt{a_2}r_1) + I_1(\sqrt{a_2}r_1)K_0(\sqrt{a_2}r_2)} \times e^{i\omega_2 t} + T_2 - \frac{Q_0}{2\pi k}\ln(r/r_2). \tag{42}$$

By the same method as that used to deduce Eq. (34), one obtains the following steady-periodic heat-flux distribution in the hollow cylinder:

$$q(r,t) = B_1 \frac{K_0(\sqrt{a_1}r_2)I_1(\sqrt{a_1}r) + I_0(\sqrt{a_1}r_2) + K_1(\sqrt{a_1}r)}{I_1(\sqrt{a_1}r_1)K_0(\sqrt{a_1}r_2) + I_0(\sqrt{a_1}r_2)K_1(\sqrt{a_1}r_1)} \times e^{i\omega_1 t} - \frac{kA_2\sqrt{a_2}}{1+i\omega_2\tau} \frac{K_1(\sqrt{a_2}r_1)I_1(\sqrt{a_2}r) - I_1(\sqrt{a_2}r_1)K_1(\sqrt{a_2}r)}{I_0(\sqrt{a_2}r_2)K_1(\sqrt{a_2}r_1) + I_1(\sqrt{a_2}r_1)K_0(\sqrt{a_2}r_2)} \times e^{i\omega_2 t} + \frac{Q_0}{2\pi r}. \tag{43}$$

4 Examples

In this section, two special cases of the boundary value problem given by Eq. (25) are illustrated. The first case is a solid cylinder whose surface temperature is a sinusoidal function of time. The second is a hollow cylinder whose inner surface is exposed to a heat flux which varies in time with a sinusoidal law and whose outer surface is kept at constant temperature.

Let us first consider an infinitely long solid cylinder, with radius r_2, with the steady-periodic surface temperature

$$T(r_2,t) = T_2 + A_2 e^{i\omega_2 t}. \tag{44}$$

Since for $r_1 \to 0$ the condition $q(r_1,t) = 0$ must be fulfilled, Eq. (22) yields $Q_0 = 0$ and $B_1 = 0$. By considering the limit for $r_1 \to 0$ of Eq. (42), with these values of Q_0 and B_1, one obtains

$$T(r,t) = T_2 + A_2 \frac{I_0(\sqrt{a_2}r)}{I_0(\sqrt{a_2}r_2)} e^{i\omega_2 t}. \tag{45}$$

By employing the dimensionless quantities

$$\vartheta = \frac{T - T_2}{A_2}, \qquad \eta = r/r_2, \qquad \xi = t/\tau, \qquad \Omega = \frac{\omega_2 r_2^2}{\alpha}, \quad \Lambda = \frac{\alpha\tau}{r_2^2}, \tag{46}$$

Eq. 45 can be rewritten as

$$\vartheta(\eta,\xi) = \frac{I_0(\sqrt{i\Omega - \Omega^2\Lambda}\ \eta)}{I_0(\sqrt{i\Omega - \Omega^2\Lambda})} e^{i\Omega\Lambda\xi}. \tag{47}$$

The amplitude of ϑ versus η, for $\Omega = 8$, is plotted in Fig. 1 for $\Lambda = 1$, $\Lambda = 0.25$ and $\Lambda = 0$. The figure shows that, both for $\Lambda = 1$ and for $\Lambda = 0.25$, the dimensionless temperature field is quite different from that predicted by Fourier's law ($\Lambda = 0$). Indeed, for $\Lambda = 0$ the amplitude of ϑ decreases monotonically from the surface to the axis of the cylinder. For $\Lambda = 1$ the amplitude of ϑ presents three maxima, with the greatest in $\eta = 0$. For $\Lambda = 0.25$ the amplitude of ϑ presents a maximum in $\eta = 0$. All maxima are greater than 1.

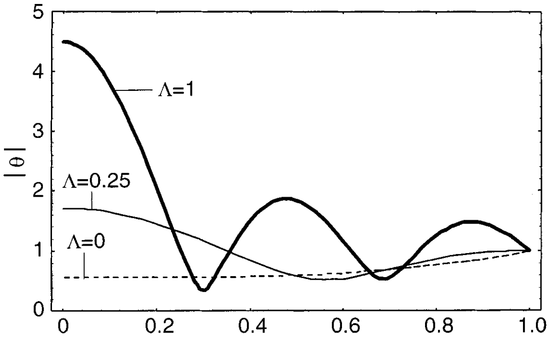

Fig. 1. Solid cylinder: amplitude of ϑ versus η for $\Omega = 8$ and $\Lambda = 1$, $\Lambda = 0.25$, $\Lambda = 0$

The amplitude of ϑ versus Ω, for $\eta = 0$, is plotted in Fig. 2, for $\Lambda = 1$, $\Lambda = 0.25$ and $\Lambda = 0$. For $\Lambda = 1$ several maxima, whose values increase with Ω, are present in the range $0 < \Omega \leq 20$. These maxima occur for the following values of Ω : 2.4030, 5.5215, 8.6548, 11.792, 14.932, 18.072. For $\Lambda = 0.25$ three maxima, whose values increase with Ω, are present in the same range and occur for $\Omega = 4.7314, \;\Omega = 11.082, \;\Omega = 17.344$. The values of Ω which maximize the amplitude of ϑ correspond to thermal-resonance frequencies. Thermal resonances do not occur for $\Lambda = 0$: in this case the amplitude of ϑ is a decreasing function of Ω. The phase of ϑ for $\xi = 0$ and $\eta = 0$ is plotted versus Ω in Fig. 3, for $\Lambda = 1$, $\Lambda = 0.25$ and $\Lambda = 0$. The figure shows that, for the values of Λ considered, also the phase of ϑ is quite different from that predicted by Fourier's law. Indeed, for $\Lambda = 0$ the phase of ϑ decreases from 0 to -2.748 when Ω increases from 0 to 20. For $\Lambda = 1$, the phase of ϑ decreases from 0 to $-\pi$ and from $+\pi$ to 0 when Ω increases from 0 to 6.992. The same changes of the phase of ϑ occur in the intervals $6.992 \leq \Omega \leq 13.311$ and $13.311 \leq \Omega \leq 19.607$. For $\Lambda = 0.25$, the phase of ϑ decreases from 0 to $-\pi$ and from $+\pi$ to 0 in the interval $0 \leq \Omega \leq 13.847$. The discontinuities of $\mathrm{Arg}(\vartheta)$ which appear in Fig. 3 have no physical relevance, since $\mathrm{Arg}(\vartheta)$ is defined up to integer multiples of 2π.

The amplitude of ϑ versus Ω for $\eta = 0$ and the phase of ϑ for $\xi = 0$ and $\eta = 0$ are plotted in Fig. 4 and Fig. 5 respectively, for $\Lambda = 0.1$, $\Lambda = 0.05$ and $\Lambda = 0$. Figure 4 shows that for $\Lambda = 0.1$ the amplitude of ϑ presents two maxima in the range $0 < \Omega \leq 20$, for $\Omega = 6.2104$ and for $\Omega = 17.863$. For $\Lambda = 0.05$ the amplitude of ϑ is a decreasing function of Ω, but has values higher than those predicted by Fourier's law. Figure 5 shows that, if $\Omega > 10$, the values of the phase of ϑ are appreciably different from those predicted by Fourier's law, both for $\Lambda = 0.1$ and for $\Lambda = 0.05$. Therefore, Figs. 4 and 5 reveal that even for $\Lambda = 0.05$ the effects of the relaxation time should be easily observed experimentally. For a medium with $\alpha \cong 10^{-6}\,\mathrm{m}^2/\mathrm{s}$ and a solid cylinder with radius $r_2 \cong 2\,\mathrm{mm}$, this value of Λ corresponds to a relaxation time $\tau \cong 0.2\,\mathrm{s}$.

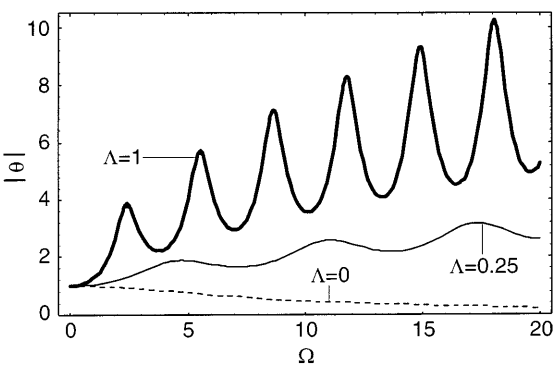

Fig. 2. Solid cylinder: amplitude of ϑ versus Ω for $\eta = 0$ and $\Lambda = 1$, $\Lambda = 0.25$, $\Lambda = 0$

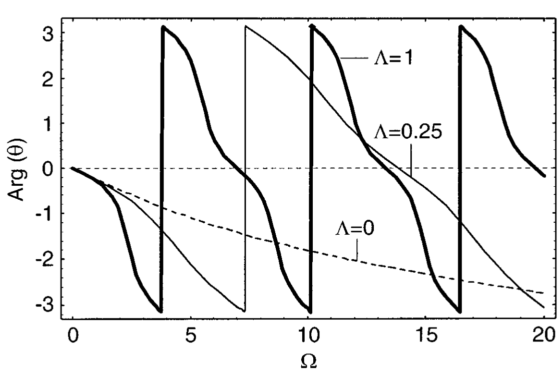

Fig. 3. Solid cylinder: phase of ϑ versus Ω for $\eta = 0$ and $\Lambda = 1$, $\Lambda = 0.25$, $\Lambda = 0$

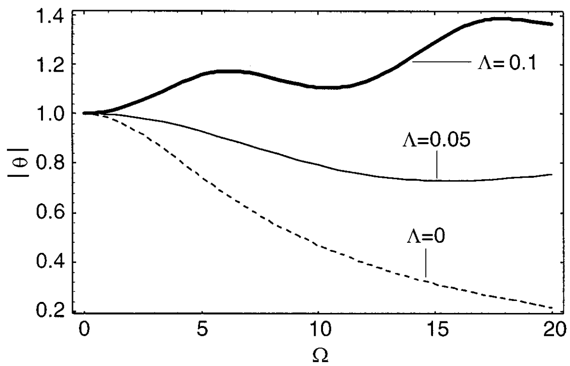

Fig. 4. Solid cylinder: amplitude of ϑ versus Ω for $\eta = 0$ and $\Lambda = 0.1$, $\Lambda = 0.05$, $\Lambda = 0$

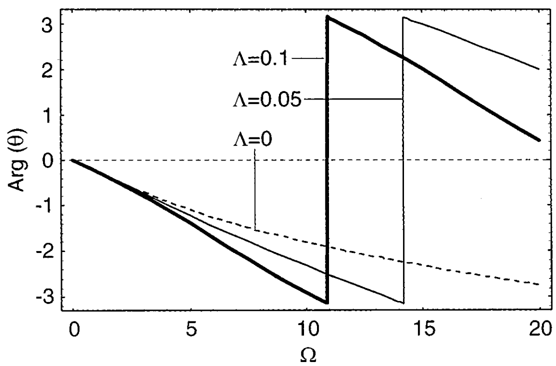

Fig. 5. Solid cylinder: phase of ϑ versus Ω for $\eta = 0$ and $\Lambda = 0.1$, $\Lambda = 0.05$, $\Lambda = 0$

Let us now consider an infinitely long hollow cylinder with the boundary conditions given by Eq. (22), in the special case $A_2 = 0$.

By employing the dimensionless quantities

$$\upsilon = \frac{\sqrt{a_1}k}{B_1}\left(T - T_2 - \frac{Q_0}{2k\pi}\ln\frac{r_2}{r}\right), \quad \zeta = r/r_1 \;,$$
$$\xi = t/\tau \;, \quad \Phi = \frac{\omega_1 r_1^2}{\alpha} \;, \quad \Gamma = \frac{\alpha\tau}{r_1^2} \;, \tag{48}$$

from Eq. (42), with $A_2 = 0$, one obtains

$$\upsilon(\zeta, \xi) = \frac{(1 + i\Phi\Gamma)[I_0(w\zeta_2)K_0(w\zeta) - K_0(w\zeta_2)I_0(w\zeta)]}{I_1(w)K_0(w\zeta_2) + I_0(w\zeta_2)K_1(w)} e^{i\Phi\Gamma\xi} \;, \tag{49}$$

where $w = \sqrt{i\Phi - \Phi^2\Gamma}$.

The amplitude of υ versus ζ, for $\Phi = 8$, is plotted in Fig. 6 for $\Gamma = 1$, $\Gamma = 0.25$ and $\Gamma = 0$. Both for $\Gamma = 1$ and for $\Gamma = 0.25$, the dimensionless temperature field is quite different from that predicted by Fourier's law ($\Gamma = 0$). In particular, for $\Gamma = 1$ the amplitude of υ presents three maxima, with the greatest in $\zeta = 1$. All maxima are greater than 10. On the other hand, for $\Gamma = 0$ the amplitude of υ is less than 1 for every value of ζ and decreases monotonically from $\zeta = 1$ to $\zeta = 2$.

The amplitude of υ versus Φ, in $\zeta = 1$, is plotted in Fig. 7, for $\Gamma = 1$, $\Gamma = 0.25$ and $\Gamma = 0$. For $\Gamma = 1$, three maxima are present in the range $0 < \Phi \leq 10$, for $\Phi = 1.7940$, $\Phi = 4.8046$ and $\Phi = 7.9104$. For $\Gamma = 0.25$, two maxima are present in the same range and occur for $\Phi = 3.3755$ and $\Phi = 9.6501$. The phase of υ for $\xi = 0$ and $\zeta = 1$ is plotted versus Φ in Fig. 8, for $\Gamma = 1$, $\Gamma = 0.25$ and $\Gamma = 0$. While the phase of υ predicted by Fourier's law is a decreasing function of Φ, both for $\Gamma = 1$ and for $\Gamma = 0.25$ the phase of υ is an oscillating function of Φ.

The amplitude of υ for $\zeta = 1$ and the phase of υ for $\xi = 0$ and $\zeta = 1$ are plotted versus Φ in Fig. 9 and in Fig. 10 respectively, for $\Gamma = 0.1$, $\Gamma = 0.05$ and $\Gamma = 0$. As it is shown by Fig. 9, for $\Gamma = 0.1$ the amplitude of υ presents two maxima in the range $0 < \Phi \leq 20$, namely for $\Phi = 4.4109$ and for $\Phi = 15.807$; moreover, for $\Phi > 2$ it has values higher than 1. The same figure reveals that for $\Gamma = 0.05$ the amplitude of υ has still values higher than those predicted by Fourier's law. Finally, Fig. 10 shows that the values of the phase of υ are appreciably different from those predicted by Fourier's law even for $\Gamma = 0.05$.

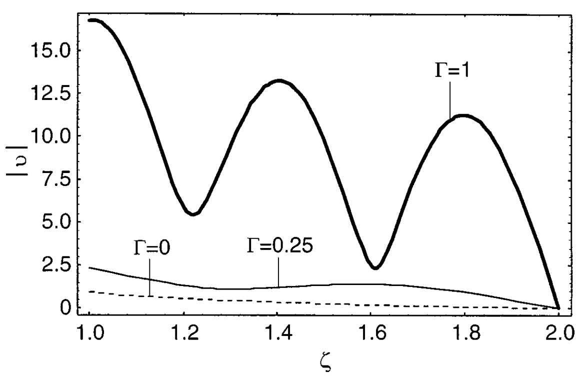

Fig. 6. Hollow cylinder with $1 \leq \zeta \leq 2$: amplitude of υ versus ζ for $\Phi = 8$ and $\Gamma = 1$, $\Gamma = 0.25$, $\Gamma = 0$

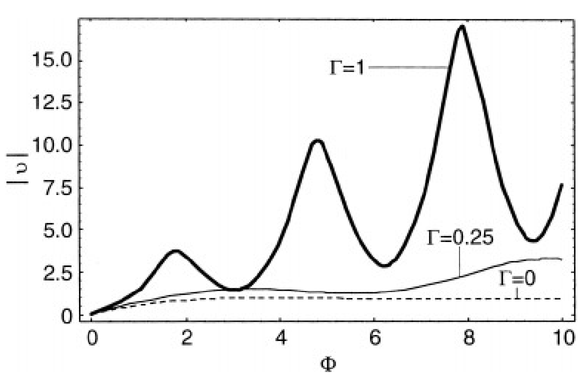

Fig. 7. Hollow cylinder with $1 \leq \zeta \leq 2$: amplitude of υ versus Φ for $\zeta = 1$ and $\Gamma = 1$, $\Gamma = 0.25$, $\Gamma = 0$

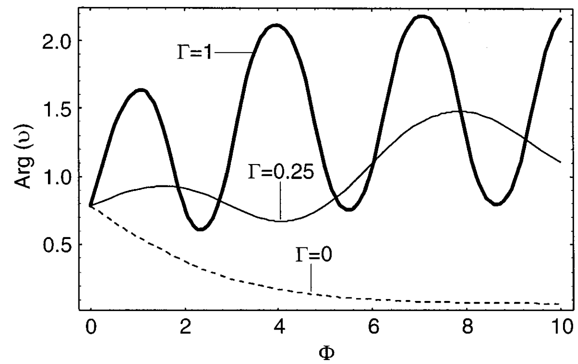

Fig. 8. Hollow cylinder with $1 \leq \zeta \leq 2$: phase of υ versus Φ for $\zeta = 1$ and $\Gamma = 1$, $\Gamma = 0.25$, $\Gamma = 0$

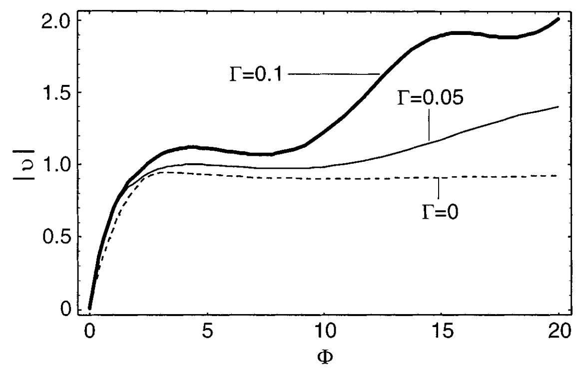

Fig. 9. Hollow cylinder with $1 \leq \zeta \leq 2$: amplitude of υ versus Φ for $\zeta = 1$ and $\Gamma = 0.1$, $\Gamma = 0.05$, $\Gamma = 0$

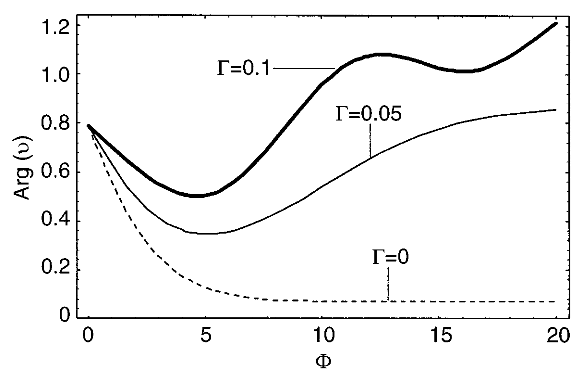

Fig. 10. Hollow cylinder with $1 \leq \zeta \leq 2$: phase of υ (see caption of Fig. 10) versus Φ for $\zeta = 1$ and $\Gamma = 0.1$, $\Gamma = 0.05$, $\Gamma = 0$

5
Conclusions

The differential equation of hyperbolic heat conduction has been solved analytically, in steady periodic regime, for an infinitely long hollow cylinder exposed to the following boundary conditions: sinusoidally varying temperature at each surface; sinusoidally varying heat flux at each surface; sinusoidally varying temperature at the inner surface and sinusoidally varying heat flux at the outer surface, and vice versa. The solution for a solid cylinder with a sinusoidally varying surface temperature has been obtained as a particular case of the last boundary condition. The effects of the relaxation time on the steady-periodic temperature field have been analysed, in dimensionless form, for a solid cylinder with a sinusoidally varying surface temperature and for a hollow cylinder with a sinusoidally varying heat flux at the inner surface and with a constant temperature at the outer surface. Results show that, for a solid cylinder with a 2 mm radius and a thermal diffusivity of about $10^{-6}\,\mathrm{m^2 s^{-1}}$, thermal resonances occur for a relaxation time $\tau \geq 1$ s and the temperature field is appreciably different from that predicted by Fourier's equation for $\tau \geq 0.2$ s. Similar results hold for a hollow cylinder with an internal radius of 2 mm, an external radius of 4 mm and a thermal diffusivity of about $10^{-6}\,\mathrm{m^2 s^{-1}}$. Therefore, measurements of the relaxation time by means of experiments on steady-periodic heat conduction in cylindrical geometry could be very accurate.

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