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ORIGINAL

Forced convection heat transfer of Giesekus viscoelastic fluid in pipes and channels

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Abstract A theoretical solution is presented for the convective heat transfer of Giesekus viscoelastic fluid in pipes and channels, under fully developed thermal and hydrodynamic flow conditions, for an imposed constant heat flux at the wall. The fluid properties are taken as constant and axial conduction is negligible. The effect of Weissenberg number (We), mobility parameter (α) and Brinkman number (Br) on the temperature profile and Nusselt number are investigated. The results emphasize the significant effect of viscous dissipation and fluid elasticity on the Nusselt number in all circumstances. For wall cooling and the Brinkman number exceeds a critical value (Br_1), the heat generated by viscous dissipation overcomes the heat removed at the wall and fluid heats up longitudinally. Fluid elasticity shifts this critical Brinkman number to higher values.

List of symbols

Br	Brinkman number
C	Integration constant in Eq. 11
c_p	Heat capacity (J/K)
D/Dt	Substantial derivative
G	Dimensionless pressure
h	Heat transfer coefficient (watt/m^2 k)
H	Half-width of channel and pipe radius (m)
k	Thermal conductivity (watt/m k)
K_1, K_2	First and second integration constants in Eq. 26
n	Plane ($n = 0$) or axisymmetric ($n = 1$) conditions
Nu	Nusselt number (2hH/k)
P	Pressure (Pa)
q	Heat flux (w/m^2)
t	Time (s)
T	Temperature (K)
u	Velocity (m/s)
We	Weissenberg number $\left(\frac{\lambda \bar{u}}{H}\right)$
x	Axial coordinate (m)
y	Lateral coordinate (m)

Greek symbols

α	Mobility parameter of Giesekus
$\dot{\gamma}$	Shear rate tensor (s^{-1})
η	Zero-shear viscosity (Pa s)
θ	Dimensionless temperature
λ	Zero-shear relaxation time (s)
ρ	Density (kg/m^3)
τ	Stress tensor (Pa)
Φ	Viscous dissipation function

Subscripts

b	Refers to bulk value
w	Refers to wall value

Superscripts

*	Refers to dimensionless quantities
–	Refers to the average value
T	Transpose of tensor

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1 Introduction

In a broad variety of chemical and industrial processes, non-Newtonian fluids have to be heated or cooled.

Examples related to the heat transfer characteristics of non-Newtonian fluid flows in pipes appear in double pipes and shell and tube heat exchangers; for instance in the polymer and food industries where, it is possible to generate well-defined heat transfer rates in these geometries, which have a strong influence to control extrusion processes. Sometimes, it may also be necessary to know the rate at which heat is removed from a pipe system or physical configurations, such as screw conveyors.

Knowledge of temperature distribution as well as heat transfer coefficient in the hot molten polymers that flow in the pipes and channels prior to enter the extrusion is valuable information in the polymer industry. This is so, because due to the low heat conduction of molten polymers, the temperature variation is quite nonuniform which may cause hot spots and thermal instability in the temperature distribution; which can make the temperature control quite complicated. In the polymer industries, the quality of the final products strongly depends on the ability to control the heat transfer. Therefore, the knowledge of temperature distribution as well as heat transfer coefficient is very important.

The majority of research work about heat transfers in non-Newtonian fluids considers the Power Law model to describe the rheological properties of fluid. Various conditions have been considered in solving the energy equation. Also different solution techniques have been devised. Some solutions ignore the viscous dissipation term [1, 2], while others include it [3, 4]. Similar treatment applies to the axial heat conduction term [5]. In [6], the energy equation is solved by including viscous dissipation term and neglecting axial conduction term and assuming constant thermo physical properties. The viscoelastic fluid is assumed to follow the basic SPTT model. The solution is offered for pipe and channel with constant heat flux boundary condition. Reference [7] is similar to the preceding one but with constant wall temperature boundary condition. Thermo-physical properties of fluids are constant in some works, while they vary with temperature in other works [8].

In the present study, Giesekus model has been employed to describe the hydrodynamic behavior of viscoelastic fluid in pipe and channel. To avoid the severe complexity arise from non-linear velocity profile equation; a suitable approximation using power series is used to solve the momentum equation. Subsequently, by using the aforementioned approximate velocity and shear stress equation in the energy equation; the temperature distribution and heat transfer coefficient is obtained. The flow under consideration is assumed to be thermally and hydrodynamically fully developed and the axial heat conduction is neglected. Details of work are presented in the following sections.

2 Mathematical formulation

The Giesekus model which is used as our rheological model is as follows:

$$\tau + \frac{\alpha\lambda}{\eta}(\tau \cdot \tau) + \lambda\tau_{(1)} = \eta\dot{\gamma} \tag{1}$$

where

$$\dot{\gamma} = \left[\nabla u + (\nabla u)^{\mathrm{T}}\right] \tag{2}$$

$$\tau_{(1)} = \frac{\mathrm{D}\tau}{\mathrm{D}t} - \left\{\tau \cdot \nabla u + (\nabla u)^{T} \cdot \tau\right\} \tag{3}$$

$$\frac{\mathrm{D}\tau}{\mathrm{D}t} = \frac{\partial\tau}{\partial t} + (u \cdot \nabla)\tau \tag{4}$$

τ and t are the stress tensor and the time, respectively. u is the velocity vector. η and λ are model parameters representing zero-shear viscosity and zero-shear relaxation time, respectively [9]. The parameter α in Eq. 1 is a model parameter and the term containing α in the constitutive equation is attributed to the anisotropic Brownian motion and/or anisotropic hydrodynamic drag on the constituent polymer molecules [10], and it is required that $0 \le \alpha \le 1$ as discussed by Giesekus [11]. Setting $\alpha = 0$ reduces the model to the upper convected Maxwell (UCM). The solution of continuity and momentum equations as well as the Giesekus equation results in Eq. 5 for shear rate. Equation 5, in fact is similar to the Eq. 17 of work done by Yoo and Choi [12]:

$$\frac{\mathrm{d}u^*}{\mathrm{d}y^*} = 2\alpha\tau^* \frac{1 \pm (2\alpha - 1)\sqrt{1 - 4\alpha^2 We^2 \tau^{*2}}}{\left(2\alpha - 1 \pm \sqrt{1 - 4\alpha^2 We^2 \tau^{*2}}\right)^2} \tag{5}$$

The starred quantities are used for the dimensionless parameters.The dimensionless stress term is define as below:

$$\tau^* = \frac{Gy^*}{2^n} \tag{6}$$

Weissenberg number and the dimensionless terms are defined as below:

$$We = \frac{\lambda\bar{u}}{H}, \quad y^* = \frac{y}{H}, \quad \tau^* = \frac{\tau H}{\eta\bar{u}}, \quad u^* = \frac{u}{\bar{u}}, \quad G = \frac{H^2}{\eta\bar{u}}\left(\frac{\mathrm{d}P}{\mathrm{d}x}\right) \tag{7}$$

H is tube radius or channel half-height. P is the pressure. x and y are axial coordinate and lateral coordinate, respectively.

For pipe flow $n = 1$ and for channel flow as flow between parallel plates $n = 0$. The classical positive and negative solutions of Eq. 5 have been discussed by Schleiniger and Weinacht [13] using linear stability analysis and the requirements arising from configuration tensor.

They concluded that for the case of no-solvent viscosity, there is only one stable, physically relevant solution and that is the positive solution with the following restrictions:

$$|\tau^*| < \frac{1}{We}\sqrt{\frac{1}{\alpha} - 1} \quad 0 < \alpha \le \frac{1}{2} \tag{8}$$

$$|\tau^*| < \frac{1}{2\alpha We} \quad \frac{1}{2} < \alpha \le 1 \tag{9}$$

To reduce complexity in the heat transfer calculations, we have decided to use an approximate solution of Eq. 5. To do so, the term $\sqrt{1 - 4\alpha^2 We^2 \tau^{*2}}$ in Eq. 5 can be expressed in a power series, using the binominal expansion:

$$\sqrt{1 - 4\alpha^2 We^2 \tau^{*2}} \cong 1 - 2\alpha^2 We^2 \tau^{*2} \tag{10}$$

where all terms of higher order have been ignored compared to the leading term, in the approximation, which is valid for small values of $4\alpha^2 We^2 \tau^{*2}$. A similar approach has been considered by other researchers [14–16]. The truncation error is less than 6% when $4\alpha^2 We^2 \tau^{*2}$ is less than $\frac{1}{2}$ (6% relative to the exact value of, $\sqrt{1 - 4\alpha^2 We^2 \tau^{*2}}$). Hence, if $4\alpha^2 We^2 \tau^{*2} < \frac{1}{2}$ the accuracy of this approximation is more than 94%. Because in this inequality τ^* is a function of We and α, therefore this inequality implicitly indicates the conditions for having acceptable approximation errors. Therefore the revelant constitutive stability condition i.e. Eqs. 8 or 9 and the approximation validity condition i.e. $|\tau^*| < \frac{1}{2\sqrt{2}\alpha We}$ should be satisfied simultaneously.

By substituting Eq. 10 in 5, it is possible to integrate Eq. 5 to obtain the following analytical expression for the velocity profile:

$$u^* = \frac{1}{2^{1-n}\alpha G We^2}\left[\frac{2^{1+2n}(\alpha - 1)}{\alpha We^2 G^2 y^{*2} - 4^n} + (1 - 2\alpha) \times \mathrm{Ln}\left(\left|\alpha We^2 G^2 y^{*2} - 4^n\right|\right)\right] + C \tag{11}$$

By using the following boundary condition, the integration constant (i.e. C) can be obtained.

$$u^* = 0 \quad \text{at } y^* = 1 \tag{12}$$

The average velocity is obtained as below:

$$\bar{u} = \frac{\int_0^H 2\pi y^n u \, dy}{\int_0^H 2\pi y^n \, dy} \tag{13}$$

Dimensionless form of Eq. 13 is as below:

$$\int_0^1 y^{*n} u^* dy^* = \frac{1}{n+1} \tag{14}$$

For each prescribed α and We, the solution of Eq. 14 will determine the value of G. Having G known, the velocity profile as well as shear stress will be known.

The governing energy equation, with the assumption of significant viscous dissipation and negligible axial heat conduction can be represented by the following equation:

$$\rho c_p u \frac{\partial T}{\partial x} = k \frac{1}{y^n} \frac{\partial}{\partial y}\left(y^n \frac{\partial^2 T}{\partial y^2}\right) + \Phi \tag{15}$$

where κ, ρ and c_p stand for thermal conductivity, density and specific heat, respectively. T is the local fluid temperature. The last term on the right-hand side in Eq. 15 represents viscous dissipation, which is defined as:

$$\Phi = \tau \frac{du}{dy} \tag{16}$$

The thermal boundary conditions for the case of constant wall heat flux are as follow:

$$k \frac{\partial T}{\partial y} = q_w \quad \text{at } y = H \tag{17}$$

$$\frac{\partial T}{\partial y} = 0 \quad \text{at } y = 0 \tag{18}$$

where q_w is positive for the case of wall heating and is negative for the case of wall cooling.

3 Solution procedure

The flow is considered to be fully developed both thermally and hydrodynamically. It is also assumed that the flow is steady and laminar. Fluid properties and model parameters are considered independent of temperature. Fluids found in polymer processing (polymer melts and concentrated solutions) are usually very viscous and industrial flows frequently involve large velocity gradients, thus viscous dissipation effects can be important and therefore will be taken into account. However, as mentioned above, we assume that the temperature variations will not be high enough to impose significant changes in fluid properties.

When the temperature profile is fully developed, one can write (Bejan [17]):

$$\frac{\partial}{\partial x}\left(\frac{T_w - T}{T_w - T_b}\right) = 0 \tag{19}$$

For the special case of constant wall heat flux, Eq. 19 reduces to:

$$\frac{\partial T}{\partial x} = \frac{\partial T_b}{\partial x} = \text{const.} \tag{20}$$

where T_b is the bulk temperature of the fluid. Performing an energy balance over an infinitesimal element dx of fluid, the following expression can be written:

$$\frac{\partial T_b}{\partial x}=\frac{2^n}{\rho c_p \bar{u} H}\left(q_w+\frac{1}{H^n}\int_0^H y^n \tau \frac{du}{dy}dy\right) \tag{21}$$

Substitution of Eq. 21 into 15 and using dimensionless terms results in the following equation:

$$\frac{1}{y^{*n}}\frac{\partial}{\partial y^*}\left(y^{*n}\frac{\partial^2\theta}{\partial y^{*2}}\right)=\left(1+2^n Br\int_0^1 y^{*n}\tau^*\frac{du^*}{dy^*}\,dy^*\right)u^* - Br\Phi^* \tag{22}$$

Φ*, Brinkman number and dimensionless temperature are defined as below:

$$\Phi^*=\tau^*\frac{du^*}{dy^*},\quad Br=\frac{\eta\bar{u}^2}{2Hq_w},\quad \theta=\frac{k(T-T_b)}{2Hq_w} \tag{23}$$

The dimensionless boundary conditions become:

$$\frac{\partial\theta}{\partial y^*}=1/2 \quad \text{at } y^*=1 \tag{24}$$

$$\frac{\partial\theta}{\partial y^*}=0 \quad \text{at } y^*=0 \tag{25}$$

Integrating Eq. 22 twice yields:

$$\theta=\left(1+2^n Br\int_0^1 y^{*n}\tau^*\frac{du^*}{dy^*}\,dy^*\right)U_{\text{II}}-Br\Phi_{\text{II}} + K_1\int\frac{1}{y^{*n}}dy^*+K_2 \tag{26}$$

$$U_{\text{II}}=\int\frac{1}{y^{*n}}\int y^{*n}u^*dy^*dy^* \tag{27}$$

$$\Phi_{\text{II}}=\int\frac{1}{y^{*n}}\int y^{*n}\Phi^*dy^*dy^* \tag{28}$$

Mathematical expression for U_{II} and Φ_{II} are presented in the Appendix 1. K_1 and K_2 are the first and second integration constants, respectively, which can be obtained using the boundary conditions. However, instead of determining the value of K_2, it is more appropriate to use wall dimensionless temperature θ_w and subtract it from dimensionless temperature θ. Doing so, will result in the elimination of K_2 from the equation as is shown below.

$$\theta-\theta_w=\left(1+2^n Br\int_0^1 y^{*n}\tau^*\frac{du^*}{dy^*}\,dy^*\right)\left(U_{\text{II}}-U_{\text{II}}|_{y^*=1}\right) - Br\left(\Phi_{\text{II}}-\Phi_{\text{II}}|_{y^*=1}\right)+K_1\int\frac{1}{y^{*n}}dy^* \tag{29}$$

Using mean temperature definition $\left(T_b=\frac{\int_0^H 2\pi y^n uT\,dy}{\int_0^H 2\pi y^n u\,dy}\right)$ and dimensionless temperature definition (Eq. 23) combined with some simple mathematical manipulation will result in the following expression:

$$\theta_w=(n+1)\int_0^1 y^{*n}u^*(\theta_w-\theta)dy^* \tag{30}$$

By inserting Eq. 29 into 30, θ_w may be evaluated. However, due to the complexity of the integration process it was decided to use numerical technique. Gaussian method was employed and the dimensionless temperature distribution θ was obtained. Using the definition of heat transfer coefficient $\left(h=\frac{q_w}{T_w-T_b}\right)$, the Nusselt number $\left(Nu=\frac{2hH}{k}\right)$ can be obtained as below:

$$Nu=\frac{1}{\theta_w} \tag{31}$$

where the Nusselt number can be calculated by substitution of θ_w from Eq. 30 into 31.

4 Results and discussion

Figure 1 compares the velocity profiles obtained for small α values of approximate solution with the exact solution of Scheiniger and Weinacht [13]. The agreement is very good and therefore, makes it suitable to extend this approximation to the energy equation as well.

The presence of viscous dissipation term causes variation in the viscoelastic fluid behavior during heating and cooling processes. Taking this into consideration in the coming section, the influence of various parameters on the

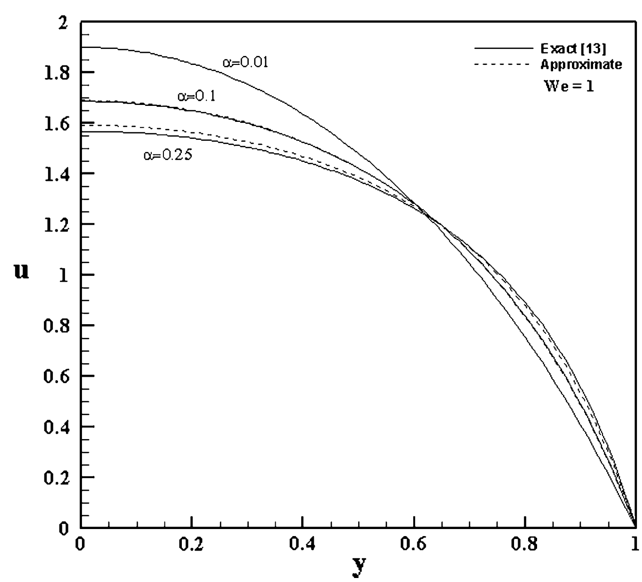

Fig. 1 Comparison of exact [13] and approximate velocity profile for different values of α

heat transfer will be discussed for the cases of wall heating and wall cooling, respectively.

4.1 Wall heating

According to Eq. 17, a positive value of wall heat flux ($q_w > 0$) imply that heat is being supplied across the wall into the fluid and Eq. 23 requires that $Br > 0$ in this case Eq. 21 implies a positive longitudinal gradient of temperature $\left(\frac{\partial T}{\partial x} > 0\right)$, i.e. fluid is being heated. Figure 2 shows the Nu number variations with We with Br number and α as parameters. As it can be seen, Nu number increases with fluid elasticity (i.e. We number). Fluid viscosity decreases as fluid elasticity increases due to the shear thinning behavior of fluid. As viscosity decreases; the heat generation inside the fluid decreases and therefore the temperature difference between wall and bulk decreases. Therefore Nusselt number $\left(Nu = \frac{2Hq_w}{k(T_w - T_b)}\right)$ will increase. This can be verified from Fig. 3 as well; for instance compare the curve for $We = 0.1$ and $We = 5$ at $Br = 1$.

According to Fig. 2 for a given Br number, when We number approaches zero Nu number becomes independent of α value. This is expected because viscoelastic fluids behave as a Newtonian fluid when extensional parameter i.e. α losses its effect.

It is also apparent from Fig. 2 that for the case of no viscous dissipation i.e. $Br = 0$ and elasticity (i.e. $We = 0$); Nu number approaches the value 4.364, which is in agreement with the reported value of Nusselt number of Newtonian fluids in pipes with constant heat flux boundary as reported by Holman [18]. Figure 1 highlights that velocity profile is being more flat by α increasing, it is

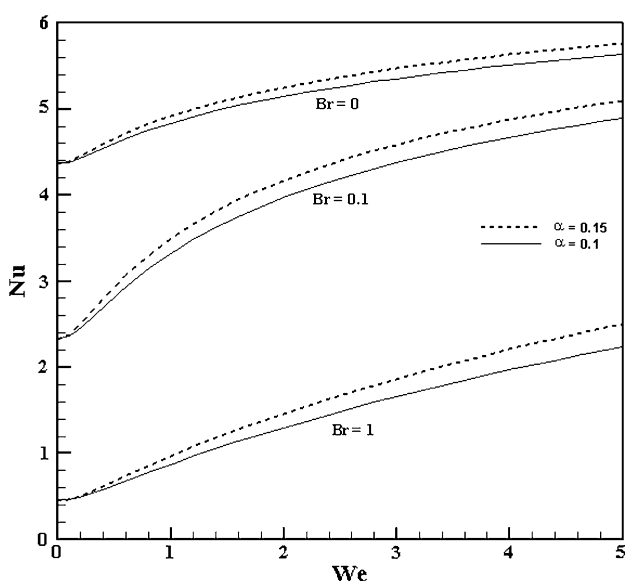

Fig. 2 Variation of Nu number with We number

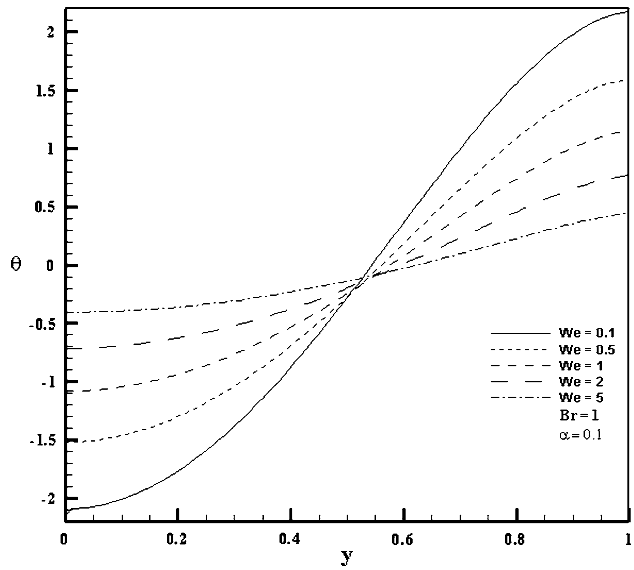

Fig. 3 Dimensionless temperature profile for different values of We number

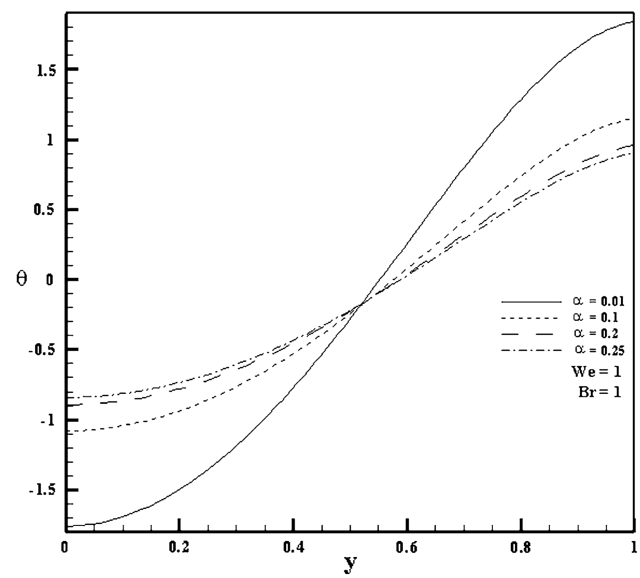

Fig. 4 Dimensionless temperature profile for different values of α

evident that the elasticity increases by α increasing, according to above explanations about elasticity anyone can find that the effect of α on heat transfer is similar to the effect of We number. (Also, it can been concluded from Figs. 2 and 4.)

Figures 2 and 5 show that viscous dissipation term has quite strong effect on the Nu number. For instance compare the curves for $Br = 0.1$ and $Br = 0.5$, it is evident that by increasing Br number, Nu number decreases. With reference to the definition of Br number, one should notice that as Br number increases, the inside heat generation by viscous dissipation increases and therefore the temperature

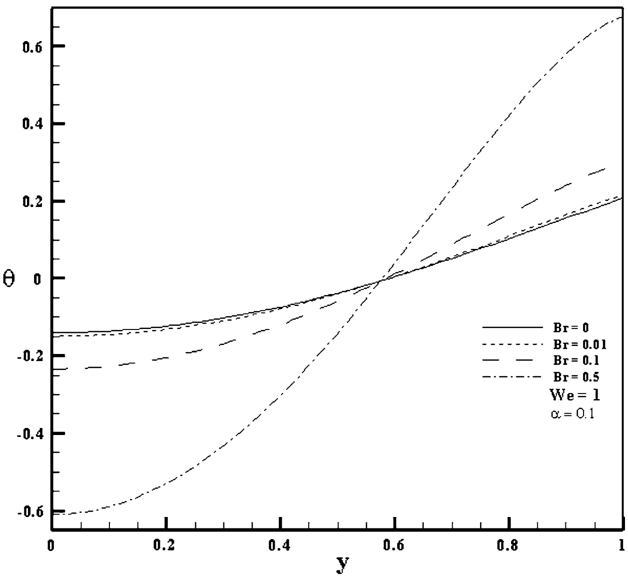

Fig. 5 Dimensionless temperature profile for different values of Brinkman number ($Br > 0$)

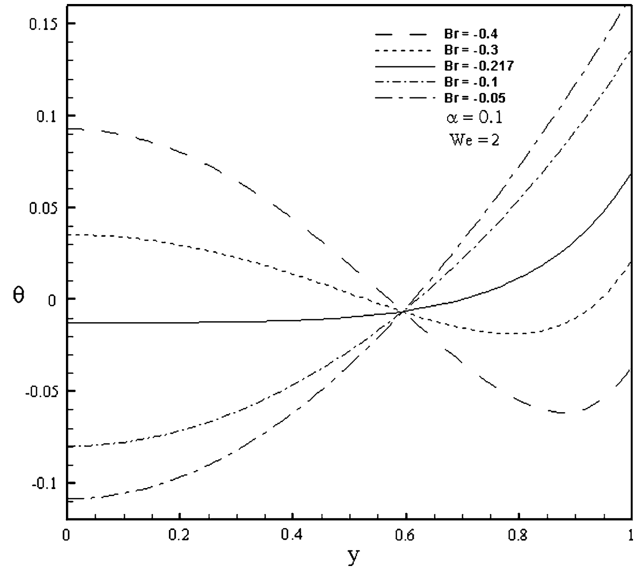

Fig. 6 Dimensionless temperature profile for different values of Brinkman number ($Br < 0$)

difference between wall and bulk increases and as a result the *Nu* number decreases.

4.2 Wall cooling

Wall cooling ($q_w < 0$) is applied to reduce the bulk temperature of fluid. However, in this process, the value of viscous dissipation term might change the overall heat balance.

While the absolute value of *Br* number is small (negligible viscous dissipation), the fluid temperature along the pipe will continuously decreases $\left(\frac{dT}{dx}<0\right)$. However, when $|Br|$ becomes larger than a critical value; the inside heat generation by viscous dissipation will overcome the cooling process and the fluid will start to warm up itself $\left(\frac{dT}{dx}>0\right)$. The aforementioned critical value of *Br* number can be obtained by equating the temperature gradient of Eq. 21 to zero, which yield the following equation.

$$Br_1 = \frac{-1}{2^n \int_0^1 y^{*n} \tau^* \frac{du^*}{dy^*} dy^*} \tag{32}$$

Solution of the integral term of Eq. 32 results in different mathematical expressions for the pipe and channel which can be found in Appendix 2. According to Fig. 6 for $|Br| < |Br_1|$ fluid is cooled and for $|Br| > |Br_1|$ fluid is heated. It is evident from Fig. 7 that by increasing the fluid elasticity (i.e. increasing *We* number) the absolute value of Br_1 increases. This means that, by increasing the fluid elasticity the cooling range of fluid is extended. This effect is again related to the shear thinning behavior of fluid, where by increasing the fluid elasticity the viscosity of fluid decreases and the internal heat generated by viscous dissipation decreases and therefore the cooling range of fluid is extended.

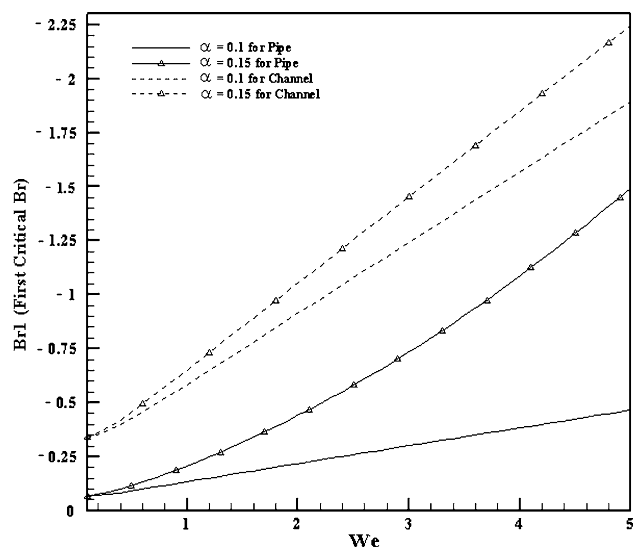

Fig. 7 Variation of the critical Brinkman number (*Br*) with the Weissenberg number (*We*)

As *Br* number becomes larger than Br_1; the internal heat generation by viscous dissipation increases rapidly and overcomes the wall cooling effect. In this situation, the fluid heats up $\left(\frac{\partial T}{\partial x} > 0\right)$. Figure 8 represents the effect of Brinkman number and fluid elasticity on the Nusselt number for the case of wall cooling. As can be seen from this figure, there is a second critical Brinkman number

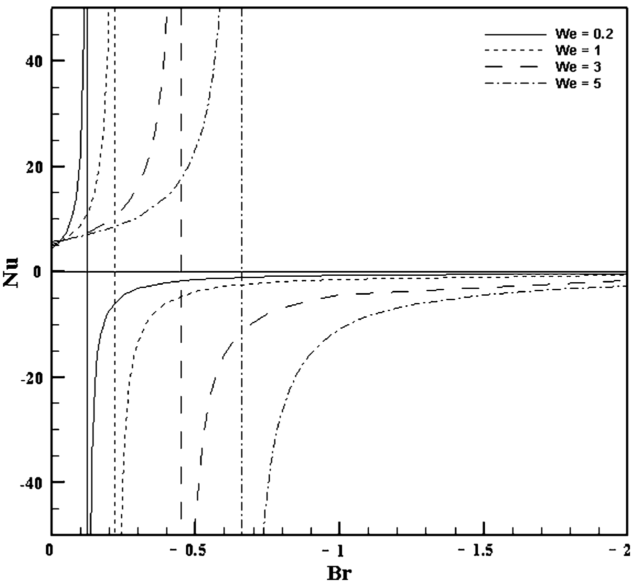

Fig. 8 Variation of Nusselt number (Nu) with Brinkman number ($Br < 0$)

(Br_2) in which the Nusselt number approaches infinity. At this Brinkman number, there is a change in the sign of temperature difference leading to singularities in the Nusselt number. It was found from this figure that for low absolute values of Brinkman number up to Br_2, the difference between wall and bulk temperature decreases and then the Nusselt number increases. Since the fluid elasticity reduces the effect of viscous dissipation, therefore by increasing the fluid elasticity when Brinkman umber is kept below Br_2, the difference between wall and bulk temperatures increases and consequently the Nusselt number decreases. At $Br = Br_2$ the sing of temperature difference will change and the Nusselt number approaches infinity. For $|Br| > |Br_2|$, the increase of viscous dissipation increases the temperature difference and therefore reduces the Nusselt number (in the absolute term).the beheviors of discussed above are all related to the variation of Brinkman number. Notethat, opposit trends are expected is we investigate the variation of fluid elasticity.

5 Conclusion

In this study, the velocity profile derived from approximate hydrodynamics solution was employed to obtain the temperature distribution and the Nu number for viscoelastic fluid flow in pipes and channels. The viscoelastic fluid chosen was assumed to obey one-mode Giesekus model. It was assumed that thermo-physical properties of fluid are constant, the axial heat conduction was negligible and the fluid was assumed to be hydro-dynamically fully developed. Effect of parameters such as generation of internal heat due to viscous dissipation (Br), fluid elasticity (We) and mobility factor (α) on temperature distribution and Nu number was studied. Results were discussed for fluid heating as well as for fluid cooling. Results showed that viscous dissipation as well as elasticity have strong effects on the heat transfer. Also in the cooling process, a critical Brinkman number (Br_1) is obtained such that for Br numbers above Br_1, the fluid in the cooling process starts to warming up. This retrograd range increases with increasing fluid elasticity.

Appendix 1

$$
\begin{aligned}
U_{\mathrm{II}} = {} & \frac{1}{\alpha^2 We^4 G^3 (\alpha We^2 G^2 - 4)} \\
& \times \Big(\big(32\alpha - 24 + 2(3 - 4\alpha)\alpha We^2 G^2\big) \\
& \times \mathrm{Ln}[r^*]\mathrm{Ln}\left[1 + \frac{r^*\sqrt{\alpha} WeG}{2}\right] \\
& + \big(16\alpha - 12 + 4\alpha We^2 G^2\big)\mathrm{diLog}\left[1 - \frac{\alpha We^2 G^2 r^{*2}}{4}\right] \\
& + (1 - \alpha)\big(16 - 4\alpha We^2 G^2\big)\mathrm{Ln}[r^*]\mathrm{Ln}\left[\alpha We^2 G^2 r^{*2} - 4\right] \\
& + \big(2(1 - 2\alpha)\alpha We^2 G^2 + 32\alpha - 24\big)\mathrm{Ln}[r^*] \\
& + (8 - 16\alpha + 2(2\alpha - 2))\mathrm{Ln}[r^*]\mathrm{Ln}\left[\frac{\alpha We^2 G^2 r^{*2} - 4}{\alpha We^2 G^2 - 4}\right] \\
& \times \big(4 - 8\alpha + (2\alpha - 1)\alpha We^2 G^2\big)\mathrm{Ln}\left[\frac{\alpha We^2 G^2 r^{*2} - 4}{\alpha We^2 G^2 - 4}\right] \\
& + (1 - 2\alpha)\alpha We^2 G^2 + \big(-24 + 32\alpha + (6 - 8\alpha)\alpha We^2 G^2\big) \\
& \times \mathrm{Ln}[r^*]\mathrm{Ln}\left[1 - \frac{r^*\sqrt{\alpha} WeG}{2}\right] + 4 + 8\alpha \Big) \\
& + \left(\alpha - \frac{1}{2}\right) G r^{*2} + \left(\frac{1}{4} - \frac{1}{2}\alpha\right) G r^{*2}\mathrm{Ln}\left[\frac{\alpha We^2 G^2 r^{*2} - 4}{\alpha We^2 G^2 - 4}\right] \\
& + (4 - 6\alpha)\frac{r^*}{\alpha We^2 G} + \frac{(2\alpha - 1)}{\alpha We^2 G} r^{*2}\mathrm{Ln}\left[\frac{\alpha We^2 G^2 r^{*2} - 4}{\alpha We^2 G^2 - 4}\right]
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{\mathrm{II}} = {} & \frac{1}{\alpha^2 We^2 G^2}(\mathrm{Ln}[r^*] + (16\alpha - 12) \\
& \times \mathrm{Ln}\left[r^{*2}\right]\mathrm{Ln}\left[1 - \frac{r^*\sqrt{\alpha} WeG}{2}\right] \\
& + (4\alpha - 4)\mathrm{Ln}\left[\alpha We^2 G^2 r^{*2} - 4\right] \\
& + \frac{1}{2}\alpha We^2 G^2 (1 - 2\alpha) r^{*2} + (8 - 8\alpha)\mathrm{Ln}[r^*] \\
& + (8\alpha - 6)\mathrm{diLog}\left[1 - \frac{\alpha We^2 G^2 r^{*2}}{4}\right]\Big)
\end{aligned}
$$

where

$$
\mathrm{diLog}[x] = \int_1^x \frac{\mathrm{Ln}[t]}{1 - t}\mathrm{d}t
$$

Appendix 2

First critical Br number for channel:

$$Br_1 = \left(\frac{2(-1+2\alpha)}{\alpha We^2} - \frac{8(-1+\alpha)}{\alpha We^2(-4+\alpha We^2G^2)} - \frac{1}{\alpha^{\frac{3}{2}}We^3G}\left(4(-2+3\alpha)Arc\tanh\left(\frac{1}{2}\sqrt{\alpha}WeG\right)\right)\right)^{-1}$$

First critical Br number for pipe:

$$Br_1 = \frac{\alpha^2We^4G^2}{-2}\left(\frac{32(\alpha-1)}{-4+\alpha We^2G^2} - (2\alpha-1)\left(-4+\alpha We^2G^2\right) - 4(4\alpha-3)\mathrm{Ln}\left(-4+\alpha We^2G^2\right)\right)^{-1}$$

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