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Magneto-thermoelastic problem in non-homogeneous isotropic cylinder

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Abstract The present paper concerns the investigation of the stress, temperature and magnetic fields in an isotropic elastic cylinder in a primary magenetic field when the curved surface of the cylinder subject to certain boundary conditions.The system of fundamental equations is solved by means of a finite difference method and the numerical calculations are carried out for the temperature, the components of displacement and the components of stresses with time and through the thickness of the cylinder. The results indicate that the effects of inhomogeneity and magnetic field are very pronounced.

1 Introduction

The dynamical problem of magneto-thermoelasticity has received much attention in the literature during the past decade. In recent years the theory of magneto-thermo-elasticity which deals with the interactions among strain, temperature and electromagnetic fields has drawn the attention of many researchers because of its extensive uses in diverse field, such as geophysics for understanding the effect of the Earth's magnetic field on seismic waves, damping of acoustic waves in a magnetic field, emissions of electromagnetic radiations from nuclear devices, development of a highly sensitive superconducting magnetometer, electrical power engineering, optics etc. The thermal stress problem in a finite circular cylinder has attracted the attention of numerous investigators [1–3]. Stress functions method of plane stress thermoelastic problem in a multiply connected region of variable thickness, has been investigated by Sugano [4]. The rotation of non-homogeneous composite infinite cylinder was investigated by El-Nagaar, et al. [5]. Abd-Alla, et al. [6] have investigated the thermal stress in an infinite circular cylinder of orthotropic material. El-Naggar, et al. [7] studied the thermal stresses in a rotating non-homogeneous orthotropic hollow cylinder. Janele, et al. [8] studied the finite amplitude spherically symmetric wave propagation in a compressible hyperelastic solid. Knopoff [9] and Nowacki [10] adressed these types of problems at the begining. Kaliski [11] investigated the wave equations of thermo-electric-magneto-elasticity. Suhbi [12] studied magneto-thermo-viscoelastic interactions in a body having cylindrical geometry. Mukhopadhyay and Roychoudhurj [13] discussed magneto-thermo-elastic interactions in an infinite isotropic elastic cylinder subjected to a periodic loading.

In the present paper, we have investigated the generation of stress, temperature and magnetic field in an infinite isotropic elastic cylinder placed in a constant primary magnetic field. The governing equations for the non-homogeneous in an isotropic elastic solid are obtained in conservation form. These equations are solved using a numerical method which uses relation from the characteristics theory of finite difference scheme. This scheme is easier to implement than the method of characteristic discussed by Haddow and Mioduchowski [14, 15]. Numerical results are presented for the variation of temperature, displacement and stresses with the time t and through the thickness of the cylinder. The effects of inhomogeneity and the magnetic field are very pronounced.

2 Formulation of the problem

Let us consider an infinite isotropic elastic solid cylinder with internal radius a and external radius b. (r, θ, z) are taken as the cylindrical coordinates with z-axis as the axis of the cylinder. The cylinder is placed in a constant primary magnetic field H_O, acting in the direction of the z-axis. Assuming the medium to be non-ferromagnetic and ferroelectric and ignoring the Thompson effect, the simplified Maxwell's equations of electro-dynamics for perfectly conducting elastic medium are:

$$\nabla \times \vec{h} = \vec{j}, \tag{2.1a}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{h}}{\partial t'}, \tag{2.1b}$$

$$\nabla \cdot \vec{h} = 0 \tag{2.1c}$$

$$\nabla \cdot \vec{E} = 0 \tag{2.1d}$$

$$\vec{E} = -\mu \left(\frac{\partial \vec{u}}{\partial t'} \times \vec{H} \right) \tag{2.1e}$$

$$\vec{h} = \nabla \times (\vec{u} \times \vec{H}), \tag{2.1f}$$

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where $\vec{H}$ is the magnetic field, $\vec{E}$ is the electric field, $\vec{j}$ is the current density, $\vec{u}$ is the mechanical displacement, and $\vec{h}$ is the perturbed magnetic,

$$\vec{H} = \overrightarrow{H_o} + \vec{h}.$$

The corresponding equations for the adjoining free space are

$$\nabla \times \overrightarrow{h^o} = \varepsilon^o \frac{\partial \overrightarrow{E^o}}{\partial t'}, \tag{2.2a}$$

$$\nabla \times \overrightarrow{E^o} = -\mu^o \frac{\partial \overrightarrow{h^o}}{\partial t'}, \tag{2.2b}$$

$$\nabla \cdot \overrightarrow{h^o} = 0 \tag{2.2c}$$

$$\nabla \cdot \overrightarrow{E^o} = 0 \tag{2.2d}$$

where superscript ^{o} refers to values for the free space. The stress equations of motion in the absence of body forces are:

$$\sigma_{ij,j} + \tau_{ij,j} = \rho \ddot{u}_i, \tag{2.3}$$

where Maxwell's electro-magnetic stress tensor τ_{ij} is given by

$$\tau_{ij} = \mu(h_i H_j + h_j H_i - h_k H_k \delta_{ij}) \tag{2.4}$$

and the mechanical stress tensor σ_{ij} is given by

$$\sigma_{ij} = \lambda \Delta \delta_{ij} + 2\mu' e_{ij} - \gamma T' \delta_{ij}, \tag{2.5}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

$$e_{ii} = \Delta, \quad i,j = 1,2,3\,.$$

Considering radial vibrations of the medium, the only non-zero displacement is $u_r = u(r, t')$, so that

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = 0. \tag{2.6}$$

The field components in the medium and in the contacting free space are then obtained from equations (2.1), (2.2) and (2.4) as:

$$\vec{E} = -\mu\left[0, -H_o \frac{\partial u}{\partial t'}, 0\right], \tag{2.7a}$$

$$\vec{j} = \left[0, -\frac{\partial h_z}{\partial r}, 0\right], \tag{2.7b}$$

$$\vec{h} = \left[0, 0, -H_o\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\right], \tag{2.7c}$$

$$\overrightarrow{E^o} = \left[0, E_2^o, 0\right], \tag{2.7d}$$

$$\overrightarrow{h^o} = \left[0, 0, h_z^o\right]. \tag{2.7e}$$

The stress equations of motion (2.3) then reduce to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{\partial \tau_{rr}}{\partial r} = \rho \frac{\partial^2 u}{\partial t'^2} \tag{2.8}$$

where

$$\sigma_{rr} = \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2\mu' \frac{\partial u}{\partial r} - \gamma T', \tag{2.9a}$$

$$\sigma_{\theta\theta} = \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2\mu' \frac{u}{r} - \gamma T', \tag{2.9b}$$

$$\tau_{rr} = \mu H_o^2\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right), \tag{2.9c}$$

where u is the component of displacement in the radial direction, e_{ij} are the strain components, T′ is the absolute temperature, λ and μ' are Lame's constants, $\gamma = (3\lambda + 2\mu')\alpha_t$, α_t is the coefficient of linear thermal expansian, μ is the magnetic permeability and t′ is the time. The heat conduction equation in the presence of heat sources can be written in the following form

$$\frac{\partial^2 T'}{\partial r^2} + \frac{1}{r}\frac{\partial T'}{\partial r} - \frac{1}{k_1}\frac{\partial T'}{\partial t'} = -\frac{Q}{k_2}, \tag{2.10}$$

where k_1 and k_2 are the thermal diffusivity and thermal conductivity respectively, Q is the intensity of applied heat source.

The elastic constants λ and μ', magnetic permeability μ and density ρ are taken as a power functions of the radial coordinate.

We characterize the non-homogeneity of the material by

$$\lambda = Lr^{2m} \quad \mu' = \nu' r^{2m} \quad \mu = \nu r^{2m} \quad \rho = \rho_o r^{2m}, \tag{2.11}$$

where L, ν', ν and ρ_o are constants (the values of λ, μ', μ and ρ in homogeneous matter) and m is a rational number. Substituting from equations (2.11) into equations (2.9) we obtain the stress-displacement relations are

$$\sigma_{rr} = r^{2m}\left[L\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2\nu'\left(\frac{\partial u}{\partial r}\right) - \gamma' T'\right], \tag{2.12a}$$

$$\sigma_{\theta\theta} = r^{2m}\left[L\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2\nu'\left(\frac{u}{r}\right) - \gamma' T'\right], \tag{2.12b}$$

$$\tau_{rr} = r^{2m} \nu H_o^2\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) \tag{2.12c}$$

where $\gamma' = (3L + 2\nu')$.

Using (2.12) we have from (2.8) the displacement formulation of the equation of motion;

$$\frac{\partial^2 u}{\partial r^2} + \frac{\alpha_1}{r}\frac{\partial u}{\partial r} - \alpha_2 \frac{u}{r^2} - \frac{\gamma'}{\eta}\left(\frac{2m}{r}T' + \frac{\partial T'}{\partial r}\right) = \frac{\rho_o}{\eta}\frac{\partial^2 u}{\partial t'^2}. \tag{2.13}$$

It is convenient to introduce the following non-dimensionalization scheme

$$b(U, R) = (u, r),$$

$$t' = \frac{b}{v}t,$$

$$T = \frac{T'}{T_o}. \tag{2.14}$$

where T_o is a reference temperature and v is the dimensionless velocity. In terms of these non-dimensional variables, equations (2.10) and (2.13) can be rewritten in more convenient form as.

$$\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R} + \alpha\frac{Q}{k_2} = \alpha'\frac{\partial T}{\partial t}, \tag{2.15}$$

$$\frac{\partial^2 U}{\partial R^2} + \frac{\alpha_1}{R}\frac{\partial U}{\partial R} - \alpha_2\frac{U}{R^2} - \alpha_3\left(\frac{2m}{R}T + \frac{\partial T}{\partial R}\right) = \alpha_4\frac{\partial^2 U}{\partial t^2}. \tag{2.16}$$

where

$$\alpha = \frac{b^2}{T_o}, \quad \alpha' = \frac{\nu b}{k_1}, \quad \alpha_1 = 2m+1,$$
$$\alpha_2 = 1 - \frac{2m(L+\nu H_o^2)}{\eta},$$
$$\alpha_3 = \frac{\gamma' T_o}{\eta}, \quad \alpha_4 = \frac{\rho_o v^2}{\eta}, \quad \eta = L + 2\nu' + \nu H_o^2.$$

The stress components induced by the temperature T are related to displacement component U by

$$\sigma_{RR} = (bR)^{2m}\left[L\left(\frac{\partial U}{\partial R} + \frac{U}{R}\right) + 2\nu'\left(\frac{\partial U}{\partial R}\right) - \gamma' T_o T\right], \tag{2.17}$$

$$\sigma_{\theta\theta} = (bR)^{2m}\left[L\left(\frac{\partial U}{\partial R} + \frac{U}{R}\right) + 2\nu'\left(\frac{U}{R}\right) - \gamma' T_o T\right], \tag{2.18}$$

Assume that the intensity of the applied heat source is taken to be in the following form

$$Q = \frac{\varepsilon t \exp(-\beta R)}{R}, \tag{2.19}$$

where β being a nonnegative constant, t is the time and ε a constant.

From preceding description, the initial condition may be expressed as

$$at\ t = 0 \quad T = 0, \quad U = \frac{\partial U}{\partial t} = 0. \tag{2.20}$$

The boundary condition may be expressed as

$$at\ R = \frac{a}{b} \quad T = 0, \quad U = 0, \tag{2.21a}$$

$$at\ R = 1 \quad \frac{\partial T}{\partial R} = 0, \quad U = 0 \tag{2.21b}$$

3
Numerical Scheme

A finite difference scheme which is a modification of MacCormack's scheme is described by wachtman, et al. [16]. Where it is used to obtain solutions to problem of thermal stress emanating from cylindrical cavity in a bounded medium. This scheme is a forwared-backwared predictor corrector scheme. We take the finite difference grids with spatial intervals h in the direction R and k as the time step, and use the subscripts i and n to denote the ith discrete points in the R direction and the nth discrete time. A mesh is defined by

$$R_i = a_o + ih, \quad i = 0, 1, 2, 3, \ldots, j-1$$

$$t^n = kn, \quad n = 0, 1, 2, 3, \ldots, k \textit{ being the time step},$$

where $a_o = \frac{a}{b}$.

The functions T(R,t), U(R,t) and Q(R,t) may be at any nodal location

$$T(R_i, t^n) = T_i^n,$$
$$U(R_i, t^n) = U_i^n$$
$$Q(R_i, t_i^n) = Q_i^n.$$

Thus the heat conduction equation (2.15) may be expressed in the finite difference as follows:

$$T_i^{n+1} = T_i^n + \frac{\rho}{a}\left[T_{i+1}^n - 2T_i^n + T_{i-1}^n + \left(\frac{h}{\frac{a}{b}+h}\right)(T_{i+1}^n - T_i^n) + \frac{\delta kn}{\frac{a}{b}+h}\exp\left(-\beta\left(\frac{a}{b}+h\right)\right)\right] \tag{3.1}$$

Also, the equation of motion (2.16) may be expressed in the finite difference as follows

$$U_i^{n+1} = 2U_i^n - U_i^{n-1} + \frac{\rho_1}{\alpha_4} \times \left[U_{i+1}^n - 2U_i^n + U_{i-1}^n + \frac{\alpha_1 h(U_{i+1}^n - U_i^n)}{\frac{a}{b}+ih} - \alpha_2\left(\frac{h}{\frac{a}{b}+ih}\right)^2 U_i^n - \alpha_3 h\left(T_{i+1}^n - T_i^n + \frac{2mh}{\frac{a}{b}+ih}T_i^n\right)\right] \tag{3.2}$$

where

$$\rho = \frac{k}{h^2}, \quad \delta = \frac{h^2\alpha\varepsilon}{k_2} \quad and \quad \rho_1 = \left(\frac{k}{h}\right)^2$$

4
Numerical results and discussion

For computational work, we take cooper as the example, for which the material constants at T_o = 27°C are as follows:

$L = 1.387\times10^{12}\,dyne/cm^2,\ \nu' = 0.448\times10^{12}\,dyne/cm^2,$

$\varepsilon = 0.25,\ \beta = 0.12$

$\rho = 8.93\,g/cm^3,\ k_1 = 1.14\,cm^2/s,\ k_2 = 0.918\,cal/s^\circ C\,cm$

$H_o = 1000.000\,Oersted,\ \alpha_t = 1.67\times10^{-8}/^\circ C,$

$\nu = 0.5\,gauss/Oersted$

Results are presented for cylinder with $a = 0.1,\ b = 1.0.$

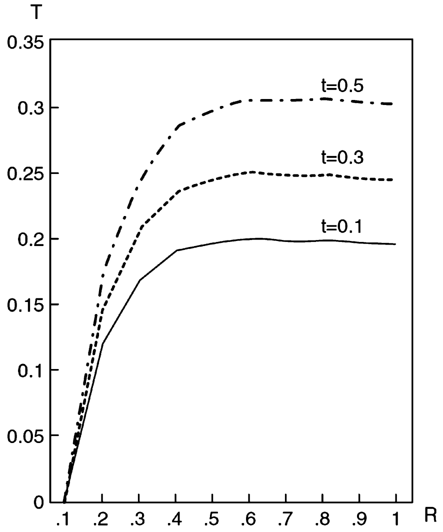

Fig. 1. Temperature distribution at different times

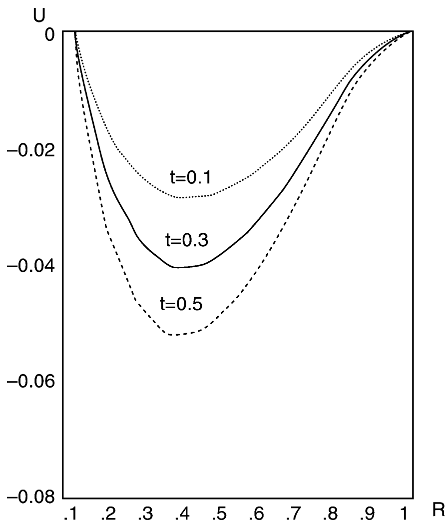

Fig. 2. Radial displacement distribution (m=0.5, ν=0.5)

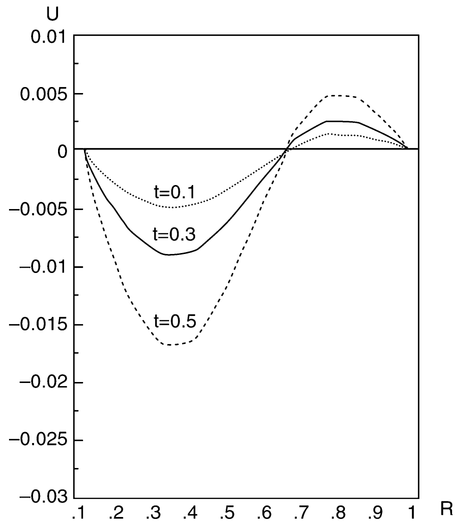

Fig. 3. Radial displacement distribution (m=0, ν=0.5)

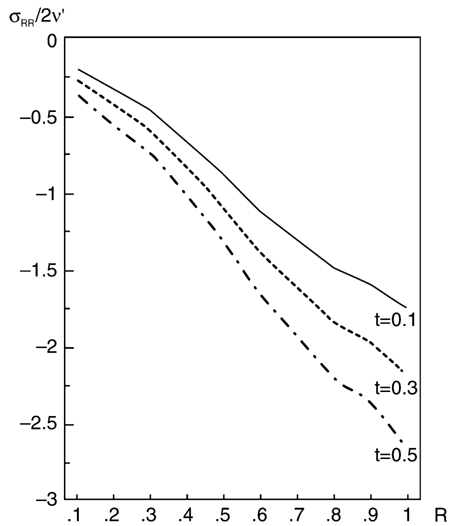

Fig. 4. Radial stress distribution (m=0, ν=0.5)

To study the non-homogeneous case, we assume that m = 0.5 and for the homogeneous case we assume that m=0.0. We represented the numerical results graphically.

Figure 1 shows the temperature variation for various non-dimensional time t. It is noticed that the temperature increases with the increasing of R in all the contexts of all three modes and satisfied the boundary conditions.

Figures 2 and 3 show the radial displacement along the radial direction R at various dimensionless t. From these figures the radial displacement U decreases and it starts to increase at the value R=0.4 for the non-homogeneous case and homogeneous case. It is noticed that the displacement component decreases with the increase of t under the effect of the magnetic field.

Figures 4, 5, 6 and 7 show the radial stress δ_{RR} and tangential stress $\delta_{\theta\theta}$ along the radial direction R at various times t. Also, they show the influence of the non-homogeneity of the material constants and the magnetic field on the stresses δ_{RR} and $\delta_{\theta\theta}$. It will be observed from those graphs that the radial stress δ_{RR} and tangential stresses $\delta_{\theta\theta}$ decrease with the increase of R for the non homogeneous case and homogeneous case. It is noticed that they decrease with the increase of t.

The variation of stresses and displacement δ_{RR}, $\delta_{\theta\theta}$ and U are due to the effect of inertia and magnetic field. Also, the influence of the non-homogeneity on displacement and stresses is very pronounced.

5
Conclusions

Some interesting conclusions can be drawn from the analysis presented here. The material is elastic and has an inhomogeneity in the direction perpendicular to the boundary for the cylinder. A finite difference predictor-corrector scheme using a relation from characteristic theory at the inner and outer radii is used to obtain solutions for the non-homogeneous infinite cylinder. Compared with the homogeneous case the inhomogeneities in which Lame's constants increase and decrease with the distance measured from the boundary have respectively

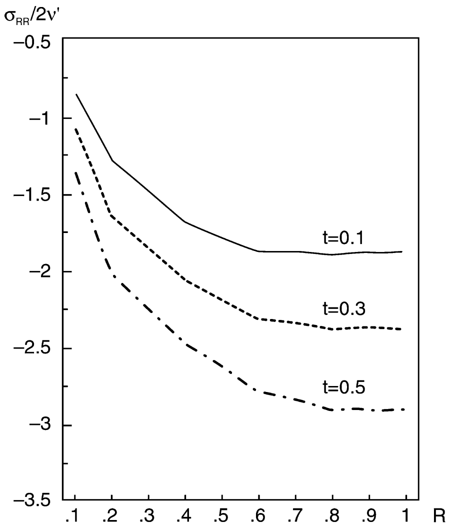

Fig. 5. Radial stress distribution (m=0, v=0.5)

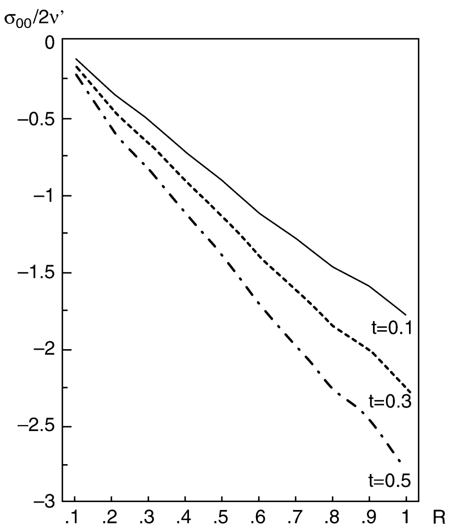

Fig. 6. Tangential stress distribution (m=0.5, v=0.5)

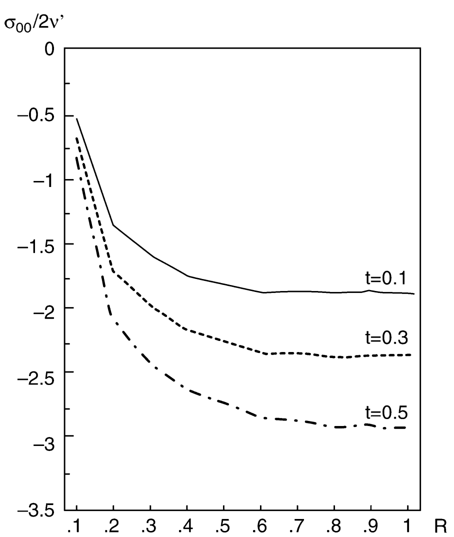

Fig. 7. Tangential stress distribution (m=0, v=0.5)

amplifying and attenuating effects on both the stress and displacement. The results are specific for the example considered, but other examples may have different trends because of the dependence of the results on the mechanical and thermal constants of the material.

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