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Counting curves on rational surfaces

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Abstract. In [CH3], Caporaso and Harris derive recursive formulas counting nodal plane curves of degree d and geometric genus g in the plane (through the appropriate number of fixed general points). We rephrase their arguments in the language of maps, and extend them to other rational surfaces, and other specified intersections with a divisor. As applications, (i) we count irreducible curves on Hirzebruch surfaces in a fixed divisor class and of fixed geometric genus, (ii) we compute the higher-genus Gromov–Witten invariants of (or equivalently, counting curves of any genus and divisor class on) del Pezzo surfaces of degree at least 3. In the case of the cubic surface in (ii), we first use a result of Graber to enumeratively interpret higher-genus Gromov–Witten invariants of certain K-nef surfaces, and then apply this to a degeneration of a cubic surface.

1. Introduction

In [CH3] Caporaso and Harris used degeneration methods and subvarieties of the Hilbert scheme to give recursions for the number of degree d geometric genus g plane curves through $3d + g - 1$ general points (the *Severi degrees* of the plane). We recast their methods in the language of stable maps, and generalize to different surfaces and multiple point conditions on a divisor.

The first application is counting curves on any Hirzebruch surface (i.e. rational ruled surface) of any genus and in any divisor class, i.e. computing Severi degrees for these surfaces.

The second application is computing genus g Gromov–Witten invariants of (all but two) del Pezzo surfaces. These invariants are of recent interest because of the Virasoro conjecture ([EHX], a generalization of Witten's conjecture, see [Ge2] and [LiuT] for more information) giving relations among them, yet surprisingly almost no higher genus invariants of any variety are known. (By [GP], all invariants of $\mathbb{P}^n$ are known. By the methods of [Ge1], resp. [BP], there is some hope of computing genus 1, resp. genus 2, invariants of some other varieties.) These invariants are known to be enumerative on Fano surfaces (Sect. 4), so once again the problem is one of counting curves, in this case, plane curves with fixed multiple points.

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In order to count curves on a cubic surface, we need a result possibly of independent interest, enumeratively interpreting invariants on certain non-Fano surfaces, where K is still numerically effective. The key technical step here is due to Graber.

More speculatively, there should be a good algebraic definition of relative Gromov–Witten invariants, loosely counting curves with prescribed intersection with a fixed divisor, meeting various homology classes (in the same way as regular Gromov–Witten invariants loosely count curves meeting various homology classes). See [Ru] for a discussion in the symplectic category (and [LiRu], especially p. 11, for more detail; also [IPa]). The numbers given here and in [CH3] (as well as the genus 0 numbers of [V1] and [Ga]) should be examples, although relative invariants in general shouldn't be enumerative).

As far as possible, we rely on analogous results of [CH3]. An example of the recursion in action is given in Sect. 8.3.

There has been a great deal of earlier work on such problems, and a brief catalogue of some of the highlights is given in Sect. 10.

1.1. Maps vs. Hilbert scheme

There seems to be an advantage in phrasing the argument in terms of maps. Many of the proofs of [CH3] essentially involve maps, and the one exception is the multiplicity calculation for "Type II components", which in any case can also be proved using maps (see Sect. 6.2). The arguments seem shorter as a result, although the content is largely the same. The disadvantage is that one needs more machinery (the compactification of the space of stable maps, Deligne–Mumford stacks), and one must worry about other components of the moduli space, parametrizing maps not of interest. The arguments here could certainly be phrased in terms of Hilbert schemes, and in the end it is probably a matter of personal taste.

1.2. Publication history

This article is a completely rewritten version of two preprints (including some extensions, most notably Sect. 9.2), math.AG/9709003 ("Counting curves of any genus on rational ruled surfaces") and math.AG/9709004 ("Genus g Gromov–Witten invariants of Del Pezzo surfaces: Counting plane curves with fixed multiple points"). They were also Mittag-Leffler preprints (Reports No. 28 and 27 of 1996/7 respectively).

2. Definitions and preliminary results

2.1. Conventions

2.1.1. Combinatorial conventions. We follow the combinatorial conventions of [CH3]. For any sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of nonnegative integers with all but finitely many α_i zero, set

$$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \dots,$$

$$I\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots,$$
$$I^\alpha = 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \dots,$$

and

$$\alpha! = \alpha_1! \alpha_2! \alpha_3! \dots.$$

The zero sequence will be denoted 0.

We denote by e_k the sequence that is zero except for a 1 in the k^{th} term (so that any sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ is expressible as $\alpha = \sum \alpha_k e_k$). By the inequality $\alpha \geq \alpha'$ we mean $\alpha_k \geq \alpha'_k$ for all k; for such a pair of sequences we set

$$\binom{\alpha}{\alpha'} = \frac{\alpha!}{\alpha'!(\alpha - \alpha')!} = \binom{\alpha_1}{\alpha'_1}\binom{\alpha_2}{\alpha'_2}\binom{\alpha_3}{\alpha'_3} \dots.$$

2.1.2. Geometric conventions. We work over the complex numbers. By *scheme*, we mean scheme of finite type over $\mathbb{C}$. By *variety*, we mean a separated integral scheme. By *stack* we mean Deligne-Mumford stack. All morphisms of schemes are assumed to be defined over $\mathbb{C}$, and fibre products are over $\mathbb{C}$ unless otherwise specified. If $f : C \to X$ is a morphism of stacks and Y is a closed substack of X, then define $f^{-1}(Y)$ as $C \times_X Y$; $f^{-1}Y$ is a closed substack of C.

For $n \geq 0$, let $\mathbb{F}_n$ be the Hirzebruch surface, or rational ruled surface, $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$. Recall that the Picard group of $\mathbb{F}_n$ is isomorphic to $\mathbb{Z}^2$, with generators corresponding to the fiber of the projective bundle F and a section E of self-intersection $-n$; E is unique if $n > 0$. Let S be the class $E + nF$. (This class is usually denoted C, but we use nonstandard notation to prevent confusion with the source of a map (C, π).) The canonical divisor $K_{\mathbb{F}_n}$ is equivalent to $-(S+E+2F)$.

A *family of n-pointed nodal curves over a base scheme or stack S* (or an *n-pointed nodal curve over S*) is a proper flat morphism $\pi : C \to S$ whose geometric fibers are reduced and pure dimension 1, with at worst ordinary double points as singularities, along with n sections $p_i : S \to C$ whose images are disjoint and lie in the smooth locus of π. (There is no connectedness condition.) If X is a scheme, then a *family of maps of pointed nodal curves to X over S* (or a *map of a pointed nodal curve to X over S*) is a morphism $\rho : C \to X \times S$ over S, where $\pi : C \to S$ is a family of pointed nodal curves over S. A *pointed nodal curve* (with no base specified) is a pointed nodal curve over $\operatorname{Spec} \mathbb{C}$, and a *map of a pointed nodal curve to X* is a map over $\operatorname{Spec} \mathbb{C}$.

2.2. Stable maps and Gromov–Witten invariants

A *stable map* to a smooth projective variety X is a map π from a connected pointed nodal curve to X such that π has finite automorphism group. The *arithmetic genus* of a stable map is defined to be the arithmetic genus of the nodal curve C. If $[C] \in H_2(C)$ is the fundamental class of C, then we say $\pi_*[C] \in H_2(X)$ is the *class* of the stable map.

A *family of stable maps* is a family of maps of pointed nodal curves to X whose fibers over maximal points are stable maps. Let $\overline{\mathcal{M}}_{g,n}(X, \zeta)$ be the stack whose category of sections of a scheme S is the category of families of n-pointed stable maps to X over S of class ζ and arithmetic genus g, with n marked points. For definitions and basic results, see [FP]. It is a fine moduli stack of Deligne-Mumford type. There is a "universal map" over $\overline{\mathcal{M}}_{g,n}(X,\zeta)$ that is a family of maps of nodal curves. There is an open substack $\mathcal{M}_{g,n}(X,\zeta)$ that is a fine moduli stack of maps of *smooth* curves. There are "evaluation" morphisms

$$ev_1,\ldots,ev_n : \overline{\mathcal{M}}_{g,n}(X,\zeta) \to X;$$

ev_i takes the point $(C, p_1, \ldots, p_n, \mu) \in \overline{\mathcal{M}}_{g,n}(X,\zeta)$ to the point $\mu(p_i)$ in X.

(Genus g) Gromov–Witten invariants were defined by Kontsevich and Manin ([KM] Sect. 2). We recall their definition, closely following the discussion in [FP] Sect. 7 of the genus 0 case. There is a *virtual fundamental class* ([LiT], [BF], [B])

$$[\overline{\mathcal{M}}_{g,n}(X,\zeta)]^{\mathrm{vir}} \in A^{\int_\zeta c_1(T_X)+(\dim X-3)(1-g)+n} X.$$

For given arbitrary classes $\gamma_1, \ldots, \gamma_n \in H^*X$,

$$I_{g,D}(\gamma_1\cdots\gamma_n) = \int_{\overline{\mathcal{M}}_{g,n}(X,D)} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \cup [\overline{\mathcal{M}}_{g,n}(X,\zeta)]^{\mathrm{vir}} \quad (1)$$

is called a *genus g Gromov–Witten invariant*. They are deformation-invariant. If the classes γ_i are homogeneous, this will be nonzero only if the sum of their codimensions is the "expected dimension" of $\overline{\mathcal{M}}_{g,n}(X,\zeta)$.

By variations of the arguments in [FP] (p. 79):

(I) If $D=0$, $I_{g,D}(\gamma_1\cdots\gamma_n)$ is non-zero only if
  i) $g=0$ and $n=3$, in which case it is $\int_X \gamma_1\cup\gamma_2\cup\gamma_3$, or
  ii) $g=1$, $n=1$, and γ_1 is a divisor class, in which case it is $(\gamma_1 \cdot c_{\dim X-1}(TX))/24$. (The author is grateful to T. Graber for pointing out this fact, which follows from a straightforward obstruction calculation. This second case is the only part of the argument that is not essentially identical to the genus 0 presentation in [FP].)

(II) If $\gamma_1 = 1 \in A^0X$, $I_{g,D}(\gamma_1\cdots\gamma_n)$ is zero unless $D=0$, $g=0$, $n=3$, in which case it is $\int_X \gamma_2\cup\gamma_3$.

(III) If $\gamma_1 \in A^1X$ and $D \neq 0$, then by the divisorial axiom ([KM] 2.2.4 or [FP] p. 79), $I_{g,D}(\gamma_1\cdots\gamma_n) = \left(\int_D \gamma_1\right)\cdot I_{g,D}(\gamma_2\cdots\gamma_n)$.

In light of these three observations, in order to compute the genus g Gromov–Witten invariants for a surface, we need only compute $I_{g,D}(\gamma^n)$ where γ is the class of a point. It is immediate that if D is the class of an exceptional curve, $I_{g,D}(\emptyset) = \delta_{g,0}$.

2.2.1. Quasi-stable maps Define *quasi-stable maps* in the same way as stable maps, except the source curve is not required to be connected. For X, ζ, g, n as before, there is a fine moduli stack $\overline{\mathcal{M}}_{g,n}(X,\zeta)'$ (of finite type) parametrizing genus g, n-pointed quasi-stable maps to X, with image in class ζ; $\overline{\mathcal{M}}_{g,n}(X,\zeta)$ is a union of connected components of $\overline{\mathcal{M}}_{g,n}(X,\zeta)'$. All constructions for stable maps carry through for quasi-stable maps.

More precisely, let $\overline{\mathcal{M}}(X)'$ be the stack whose category of sections of a scheme S is the category of families of quasi-stable maps to X over S. More conveniently, it is the choice of a finite étale cover $T \to S$ and a family of *stable* maps to X over T. This stack naturally splits into $\overline{\mathcal{M}}_{g,n}(X,\zeta)'$, which are unions of connected components of $\overline{\mathcal{M}}(X)'$. For example, $\overline{\mathcal{M}}_3(pt,0)'$ has two components, both of dimension 6. One is isomorphic to $\overline{\mathcal{M}}_3(pt,0)$, and the other is isomorphic to $(\overline{\mathcal{M}}_2(pt,0) \times \overline{\mathcal{M}}_2(pt,0))/(\mathbb{Z}/2\mathbb{Z})$.

2.3. *The surface in question*

Throughout this paper, assume:

P1. X is a smooth rational surface and $E \cong \mathbb{P}^1$ is a divisor on X.
P2. The surface $X \setminus E$ is minimal, i.e. contains no (-1)-curves.
P3. The divisor class $K_X + E$ is negative on every curve on X.
P4. If D is an effective divisor such that $-(K_X+E)\cdot D = 1$, then D is smooth.

Note that there is a natural identification $A^1_{\mathbb{Q}} X \cong H^2(X,\mathbb{Q})$ as for such a surface $h^{1,0}(X) = h^{2,0}(X) = 0$. Properties **P3.** and **P4.** would hold if $-(K_X+E)$ were very ample, which is true in all cases of interest here. Useful examples of such (X,E) include when $X=\mathbb{P}^2$ and E is a line (Sect. 7) or a conic (Sect. 9), or when X is a Hirzebruch surface and E is the "section at infinity" (Sect. 8). Property **P2.** could be removed and other properties could be weakened by modifying the results very slightly, but there seems to be no benefit to doing so.

2.4. *The stacks $V^{D,g}(\Omega,\beta)$ and $V^{D,g}(\Omega,\beta)^{\mathrm{con}}$*

Suppose S is a finite index set, $\Omega = \{(q_s, m_s)\}_{s\in S}$ is a set of ordered pairs with $q_s \in E$ and m_s a positive integer, and β is a sequence of non-negative integers with all but finitely many β_i zero. Let D be a divisor class on X and g be an integer.

Let $V^{D,g}(\Omega,\beta)$ be the (stack-theoretic) closure in $\overline{\mathcal{M}}_{g,|S|}(X,D)'$ (where the marked points are labelled p_s, $s\in S$) of quasi-stable maps $\pi : (C,\{p_s\}) \to X$ such that each component of C maps birationally to its image in X and $\pi^{-1}(E)$ consists of distinct smooth points (of C): $\{p_s\}_{s\in S}$ and $\{r_{i,j}\}_{1\le j\le\beta_i}$, with

$$\pi^*(E) = \sum_s m_s p_s + \sum_{i,j} i r_{i,j},$$

and $\pi(p_s) = q_s$. Clearly $V^{D,g}(\Omega,\beta)$ is empty if $\sum m_s + I\beta \neq D\cdot E$.

Define $V^{D,g}(\Omega,\beta)^{\mathrm{con}}$ similarly, except C is required to be connected (so the closure can be taken in $\overline{\mathcal{M}}_{g,|S|}(X,D)$). Then $V^{D,g}(\Omega,\beta)^{\mathrm{con}}$ is a union of connected components of $V^{D,g}(\Omega,\beta)$.

Let $\alpha(\Omega)$ be the sequence $\sum m_s e_s$. Example: If $X=\mathbb{P}^2$, E is a line, D is d times the class of a line, and $\{q_s\}$ are general points of E, then $V^{D,g}(\Omega,\beta)$ is a map-theoretic analogue of the Severi variety $V^{d,g}(\alpha(\Omega),\beta)(\Omega)$ of [CH3].

For convenience, let

$$\Upsilon = \Upsilon^{D,g}(\beta) := -(K_X + E) \cdot D + |\beta| + g - 1.$$

We will see that $V^{D,g}(\Omega, \beta)$ is a stack of pure dimension Υ (Theorem 3.1).

2.4.1. Enumerative relevance and transversality. Suppose

$$\begin{array}{ccc} C & \longrightarrow & X \times B \\ & \pi \searrow \quad \swarrow & \\ & B & \end{array}$$

is a family of maps of nodal curves, with B irreducible. Let the *intersection dimension* of the family (denoted idim B) be the largest number n of points on X such that for a general set S of n points of X there is a map in the family whose image contains S. Note that idim $B \leq \dim B$. If equality holds, we say the family is *enumeratively relevant*; otherwise it is *enumeratively irrelevant*.

Let $\eta : C \to X$ be the induced map. Then if q is a general point of X, $\eta^{-1}q$ is pure dimension $\dim B - 1$, and by Sard's theorem $\eta^{-1}q$ is reduced. Define the Weil divisor $H_q := \pi_* \eta^{-1} q$. As X is rational, the Weil divisor class is independent of q (so long as q is chosen so that $\eta^{-1}q$ is pure dimension $\dim B - 1$); denote this class H.

This divisor can be intersected with any irreducible substack B' of B: repeat the same construction, using the universal family over B. Hence we have a "moving lemma", and H is naturally an element of $A^1 B$ (in the operational Chow ring of B, see [V2] Section 3.10 for more complete arguments). Note that if B is enumeratively irrelevant, then $H^{\dim B}[B] = 0$.

We say the family has property (†) if for the map (C, π) corresponding to a general point of B, $\pi_* C$ has no multiple components. If the family has property (†), each component of H_q appears with multiplicity 1, and the map corresponding to a general point of each component of H_q has property (†) (not difficult, see [V2] Section 3.10). Hence by induction, if the family satisfies (†), the degree of $H^{\dim B}[B]$ is the number of maps whose image passes through $\dim B$ general points of X.

2.4.2. Enumerative invariants. Define

$$N^{D,g}(\Omega, \beta) = \begin{cases} 0 & \text{if } V^{D,g}(\Omega, \beta) = \emptyset, \\ \deg H^{\dim V^{D,g}(\Omega,\beta)}[V^{D,g}(\Omega, \beta)] & \text{otherwise.} \end{cases}$$

If $V^{D,g}(\Omega, \beta)$ has property (†), then $N^{D,g}(\Omega, \beta)$ counts maps in $V^{D,g}(\Omega, \beta)$ whose image passes through Υ general points of X. Define $N^{D,g}(\Omega, \beta)^{\text{con}}$ similarly.

For a given set $\{m_s\}_{s \in S}$, $N^{D,g}(\Omega, \beta)$ is constant for generally chosen $\{q_s\}_{s \in S}$; let this number be $N^{D,g}(\alpha, \beta)$, where $\alpha = \alpha(\Omega) = \sum e_{m_s}$. Define $N^{D,g}(\alpha, \beta)^{\text{con}}$ similarly.

The recursions of this paper for $N^{D,g}(\Omega, \beta)$ and $N^{D,g}(\Omega, \beta)^{\text{con}}$ (Theorems 6.7 and 6.8 respectively) correspond to specializing one of the divisors H_q by specializing q to be a point of E.

3. Dimension counts

In this section, we prove a key dimension count (Theorem 3.1) and a bound on intersection dimension (Proposition 3.4).

Theorem 3.1. *Let V be a component of $V^{D,g}(\Omega,\beta)$, and suppose $(C,\{p_s\},\pi)$ is the map corresponding to a general point of V.*

(a) $\dim V \le \Upsilon = \Upsilon^{D,g}(\beta) = -(K_X+E)\cdot D + |\beta| + g - 1$.
(b) *If* $\dim V = \Upsilon$*, then C is smooth, and the map π is unramified.*
(c) *Conversely, if C is smooth and π is unramified, then* $\dim V = \Upsilon$.
(d) *If* $\dim V = \Upsilon$ *and $q_s \neq q_{s'}$ for each (s,s') such that p_s and $p_{s'}$ lie on different components of C, then V has property (†).*

Note that the hypotheses of (d) are satisfied if V is a component of $V^{D,g}(\Omega,\beta)^{\mathrm{con}}$ or the $\{q_s\}$ are distinct (and $\dim V = \Upsilon$). Hence $N^{D,g}(\Omega,\beta)^{\mathrm{con}}$ is enumerative, and by (b) counts irreducible (not just connected) curves.

We use the following lemma, which appears (in a different guise) in Section 2.2 of [CH3]: (a) is contained in Corollary 2.4 and part (b) is Lemma 2.6. Part (a) was proven earlier by E. Arbarello and M. Cornalba in [AC], Sect. 6.

Lemma 3.2 (Arbarello–Cornalba, Caporaso–Harris). *Let V be an irreducible substack of $\overline{\mathcal{M}}_g(Y,\beta)'$ where Y is smooth, such that if (C,π) corresponds to a general point of V then C is smooth and π maps C birationally onto its image. Let $N = \operatorname{coker}(T_C \to \pi^*T_Y)$, and let N_{tors} be the torsion subsheaf of N. Then:*

(a) *If (C,π) corresponds to a general point of V then* $\dim V \le h^0(C, N/N_{\mathrm{tors}})$.
(b) *Assume further that Y is a surface. Fix a smooth curve G in Y and points $\{q_{i,j}\}\subset G$, and assume that*

$$\pi^*G = \sum_{i,j} i p_{i,j} + \sum_{i,j} i r_{i,j}$$

with $\pi(p_{i,j}) = q_{i,j}$ if (C,π) corresponds to a general point of V. Then

$$\begin{aligned}\dim V &\le h^0(C, N/N_{\mathrm{tors}}(-\sum_{i,j} i p_{i,j} - \sum_{i,j}(i-1) r_{i,j}))\\ &= h^0(C, N/N_{\mathrm{tors}}(-\pi^*G + \sum_{i,j} r_{i,j})).\end{aligned}$$

Lemma 3.3. *Let V be a component of $V^{D,g}(\Omega,\beta)$ whose general point corresponds to a map $\pi : C \to X$ where C is a smooth curve (not necessarily irreducible). Then* $\dim V \le \Upsilon$. *If π is not unramified then the inequality is strict.*

Proof. Note that by the definition of $V^{D,g}(\Omega,\beta)$, π is a birational map from C to its image in X, so we may invoke Lemma 3.2. The map $T_C \to \pi^*T_X$ is injective (as it is generically injective, and there are no nontrivial torsion subsheaves of invertible sheaves). Define the normal sheaf N (of π) and N_{tors} as in Lemma 3.2. The map π is unramified if and only if $N_{\mathrm{tors}} = 0$. By property **P3.**, the divisor

$-\pi^*(K_X+E)+\sum r_{i,j}$ is positive on each component of C, so by Kodaira vanishing or Serre duality

$$H^1(C, \mathcal{O}_C(K_C - \pi^*(K_X + E) + \sum r_{i,j})) = 0.$$

As N/N_{tors} is a subsheaf of the invertible sheaf $\mathcal{O}_C(-\pi^* K_X + K_C)$,

$$\begin{aligned} & h^0(C, N/N_{\text{tors}}(-\pi^* E + \sum r_{i,j})) \\ & \quad \leq h^0(C, \mathcal{O}_C(-\pi^* K_X + K_C - \pi^* E + \sum r_{i,j})) \\ & \quad = \chi(C, \mathcal{O}_C(-\pi^* K_X + K_C - \pi^* E + \sum r_{i,j})) \\ & \quad = -(K_X + E) \cdot D + |\beta| + g - 1 \\ & \quad = \Upsilon. \end{aligned} \tag{2}$$

If C' is a component of C with $-\pi^*(K_X + E) \cdot C' = 1$, then $\pi : C' \to X$ is an immersion (hence unramified) by property **P4**. Thus if $N_{\text{tors}} \neq 0$, then it is non-zero when restricted to some component C'' for which $-\pi^*(K_X + E) \cdot C'' \geq 2$. Let p be a point on C'' in the support of N_{tors}. Then $-\pi^*(K_X + E) + \sum r_{i,j} - p$ is positive on each component of C, so by the same argument as above, N/N_{tors} is a subsheaf of $\mathcal{O}_C(-\pi^* K_X + K_C - p)$, so

$$\begin{aligned} & h^0(C, N/N_{\text{tors}}(-\pi^* E + \sum r_{i,j})) \\ & \quad \leq h^0(C, \mathcal{O}_C(-\pi^* K_X + K_C - \pi^* E + \sum r_{i,j} - p)) \\ & \quad = \Upsilon - 1. \end{aligned}$$

Therefore, equality holds at (2) only if $N_{\text{tors}} = 0$, i.e. π is an immersion. By Lemma 3.2(a), the result follows. $\square$

Proof of Theorem 3.1. Suppose $\dim V \geq \Upsilon$. Let the normalizations of the components of C be $C(1), C(2), \ldots, C(s)$, so $p_a(\coprod_k C(k)) \leq p_a(C)$ with equality if and only if C is smooth. Let $\beta = \sum_{k=0}^s \beta(k)$ be the partition of β induced by $C = \cup_{k=1}^s C(k)$, let $g(k)$ be the arithmetic genus of $C(k)$, and let

$$\Upsilon(k) = (K_X + E) \cdot \pi_*[C(k)] + |\beta(k)| + g(k) - 1.$$

By the definition of $V^{D,g}(\Omega, \beta)$, π maps $C(k)$ birationally onto its image.

By Lemma 3.3, C moves in a family of dimension at most

$$\begin{aligned} \sum_{k=1}^s \Upsilon(k) &= \sum_{k=1}^s (-(K_X + E) \cdot \pi_*[C(k)] + |\beta(k)| + g(k) - 1) \\ &= -(K_X + E) \cdot D + |\beta| + p_a\left(\coprod C(k)\right) - 1 \\ &\leq -(K_X + E) \cdot D + |\beta| + p_a(C) - 1 \qquad (3) \\ &= \Upsilon. \end{aligned}$$

This proves part (a).

If $\dim V = \Upsilon$, then equality must hold in (3), so C is smooth, and by Lemma 3.3, π is unramified, proving (b).

For part (c), let N be the normal sheaf to the map π (which is invertible as π is unramified). As $\pi^* E$ contains no components of C,

$$N(-K_C) = \mathcal{O}_C(-\pi^*(K_X + E)) \otimes \mathcal{O}_C(\pi^* E).$$

is positive on every component of C by property **P3.**, so N is nonspecial. By Riemann-Roch,

$$h^0(N) = -K_X \cdot D + \deg K_C - g + 1 = -K_X \cdot D + g - 1.$$

Requiring the curve to remain m_s-fold tangent to E at the point p_s of C (where $\pi(p_s)$ is required to be the fixed point q_s) imposes at most m_s independent conditions. Requiring the curve to remain i-fold tangent to E at the point $r_{i,j}$ of C imposes at most $(i-1)$ independent conditions. Thus

$$\begin{aligned} \dim V &\geq -K_X \cdot D + g - 1 - \sum m_s - I\beta + |\beta| \\ &= -(K_X + E) \cdot D + |\beta| + g - 1 \end{aligned}$$

as $\sum m_s + I\beta = D \cdot E$, completing the proof of (c).

For (d), we need only prove that two components of C have distinct images in X. It suffices to show that if $(C(i), \pi(i))$ is the general map in a component of $V^{D,g(i)}(\Omega(i), \beta(i))^{\text{con}}$ of dimension $\Upsilon(i) := \Upsilon^{D,g(i)}(\Omega(i), \beta(i))$ $(i = 1, 2)$, and $\{q(1)_s\}_{s \in S(1)} \cap \{q(2)_s\}_{s \in S(2)} = \emptyset$, then $\pi(1)(C(1)) \neq \pi(2)(C(2))$ as sets. If $\Upsilon(i) > 0$ for $i = 1$ or 2 (i.e. one of the images "moves") then the result is clear. Otherwise, for $i = 1, 2$, $-(K_X + E) \cdot D \geq 1$ (by property **P3.**), $|\beta(i)| \geq 0$, and $g(i) - 1 \geq -1$. As $\Upsilon(i) = 0$ is the sum of these three terms, equality must hold in each case, so D is smooth (property **P4.**) and rational ($g(i) = 0$). Also, D meets E (or else $D^2 = -2 - (K_X + E) \cdot D = -1$, violating property **P2.**), so (as $|\beta(i)| = 0$) $\Omega(i)$ is non-empty. Hence $\pi(1)(C(1))$ meets E at different points than $\pi(2)(C(2))$, proving (d). □

Fix an index set S. Let V be an irreducible substack of $\overline{\mathcal{M}}_{g,|S|}(X, D)'$, and let $\pi : C \to X$ be the map corresponding to a general point of a component of V. Assume that $\pi^* E = \sum m_s p_s + \sum i r_{i,j}$ where $\pi(p_s)$ is required to be a fixed point q_s of E as C varies. (In particular, no component C is mapped to E.) Define β by $\beta_i = \#\{r_{i,j}\}_j$ and $\Omega = \{(q_s, m_s)\}_{s \in S}$.

Proposition 3.4. *The intersection dimension of V is at most $\Upsilon^{D,g}(\Omega, \beta)$. If equality holds then V is a component of $V^{D,g}(\Omega, \beta)$.*

The main obstacle to proving this result is that the map π may not map components of C birationally onto their image: the map π may collapse components or map them multiply onto their image.

Proof. If necessary, pass to a dominant generically finite cover of V that will allow us to distinguish components of C. (Otherwise, monodromy on V may induce a nontrivial permutation of the components of C.)

For convenience, first assume that C has no contracted rational or elliptic components. We may replace C by its normalization; this will only make the bound worse. (The map from the normalization of C is also a quasi-stable map.) We may further assume that C is irreducible, as $-(K_X + E) \cdot D + |\beta| + g - 1$ is additive.

Suppose C maps with degree m to the irreducible curve $D_0 \subset X$. Then the map $\pi : C \to D_0$ factors through the normalization $\tilde{D}$ of D_0. Let r be the total ramification index of the morphism $C \to \tilde{D}$. By Theorem 3.1(a),

$$\begin{aligned}
\operatorname{idim} V &\leq \dim V \\
&\leq -(K_X + E) \cdot D_0 + |\beta| + g(\tilde{D}) - 1 \\
&= -\frac{1}{m}(K_X + E) \cdot \pi_*[C] + |\beta| + \frac{1}{m}(g(C) - 1 - r/2) \\
&\leq -(K_X + E) \cdot \pi_*[C] + |\beta| + g(C) - 1
\end{aligned}$$

where we use the Riemann–Hurwitz formula for the map $C \to \tilde{D}$ and the fact (property **P3**.) that $-(K_X + E) \cdot D_0 > 0$. Equality holds only if $m = 1$, so by Theorem 3.1, equality holds only if V is a component of $V^{D,g}(\Omega, \beta)$ for some g, Ω, β.

If C has contracted rational or elliptic components, replace C with those components of its normalization that are not contracted elliptic or rational components (which reduces the genus of C) and follow the same argument. $\square$

4. Enumerative interpretation of higher-genus Gromov–Witten invariants of K-nef surfaces

In this section, we interpret higher-genus Gromov–Witten invariants of some K-nef surfaces. By Sect. 2.2, we need only concern ourselves with point conditions. For convenience, we make a definition.

4.1. Almost Fano surfaces

Suppose X is a smooth surface, and K_X is positive on all curves of X except for a rational $G \subset X$ with $G^2 = -2$ (so $K_X \cdot G = 0$). Then we say (X, G) is *almost Fano*.

One example is if X is the blow-up of a Fano surface at a point lying on exactly one (-1)-curve E, and G is the proper transform of E, e.g. $X = \mathbb{P}^2$ blown up at 6 distinct points on a smooth conic C, G the proper transform of E (see Sect. 9.2).

By essentially the same argument as that of Proposition 3.4, one shows:

Proposition 4.1. *Let X be a smooth rational surface with nonzero effective divisor D. Suppose M is an irreducible component of $\overline{\mathcal{M}}_g(X, D)$ with general map (C, π). If*

(a) *X is Fano, or*
(b) *(X, G) is almost Fano, and no component of C is mapped with positive degree to G, then*

$$\operatorname{idim} M \leq -K_X \cdot D + g - 1. \tag{4}$$

If equality holds, then π is unramified, C is smooth, and π maps C birationally onto its image.

4.2. *Fano surfaces*

It is known that the higher-genus Gromov–Witten invariants of a Fano surface X are enumerative. The author is unaware of a reference in the literature, so for completeness the argument is sketched here.

By the preceding Proposition, the only components of $\overline{\mathcal{M}}_g(X, D)$ whose images can meet

$$n := c_1(T_X) \cdot D + (\dim X - 3)(1 - g) = -K_X \cdot D + g - 1$$

general points of X is the closure $\overline{\mathcal{M}}$ of the locus of unramified maps from smooth curves. Let $\overline{\mathcal{M}}' := \overline{\mathcal{M}} \times_{\overline{\mathcal{M}}_g(X,D)} \overline{\mathcal{M}}_{g,n}(X, D)$ be the "n^{th} universal curve over $\overline{\mathcal{M}}$," a component of $\overline{\mathcal{M}}_{g,n}(X, D)$ of dimension $2n$. Then by Proposition 4.1, (1) reduces to an integral over this component $\overline{\mathcal{M}}'$, where $[\overline{\mathcal{M}}']^{\text{vir}} = [\overline{\mathcal{M}}']$. By Sard's theorem, if $q_1, \ldots, q_n$ are general points of X, $\cap_i ev^{-1} q_i$ is a reduced scheme of dimension 0, and the intersection lies in the open set corresponding to unramified maps. Hence if γ is the class of a point, the genus g Gromov–Witten invariant $I_{g,D}(\gamma^n)$ counts unramified genus g maps to X.

4.3.

In [K1], p. 22–23, Kleiman gives an enumerative interpretation for a particular genus 0 Gromov–Witten invariant of $\mathbb{F}_2$, due to Abramovich and Bertram. See Sect. 8.2 for their formula (AB0) and a generalization. This interpretation suggests the following result.

Theorem 4.2. *Suppose (X, G) is almost Fano, D is an effective divisor on X (not a multiple of G), and γ is the class of a point. Let $n := -K_X \cdot D + g - 1$. Then the Gromov–Witten invariant $I_{g,D}(\gamma^n)$ is the number of stable maps $\pi : C \to X$ with $\pi_*[C] = D$, where*

(i) *the union C'' of components of C not mapping to E is connected, and*
(ii) *any other component C_0 of C maps isomorphically to E, and C_0 intersects $\overline{C \setminus C_0}$ at one point, which is contained in C''.*

Proof. Suppose $\mathcal{N}$ is a component of $\overline{\mathcal{M}}_g(X, D)$ with $\operatorname{idim} \mathcal{N} \geq n$ (the only components that could contribute to the Gromov–Witten integral (1)). Restrict to an open subset of $\mathcal{N}$ so that the universal curve $\mathcal{C}$ can be written $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$, where $\mathcal{C}'$ corresponds to the union of components (of the general curve) mapping with positive degree to G, and $\mathcal{C}''$ corresponds to the union of the remaining components. Suppose the curve C' corresponding to a general (closed) point in $\mathcal{C}'$ has t connected components, maps to G with total degree k, and meets C'', the curve corresponding to a general point in $\mathcal{C}''$, at s points. Then $p_a(C') \geq 1 - t$ and (as C is connected) $s \geq t$, so

$$g = p_a(C') + p_a(C'') - 1 + s \geq p_a(C'') + s - t \geq p_a(C'').$$

Let $\mathcal{M}$ be the component of $\overline{\mathcal{M}}_{p_a(C'')}(X, D - kG)'$ induced by $(\mathcal{C}'', \pi|_{\mathcal{C}''})$. Then by Proposition 4.1(b),

$$\begin{aligned} \operatorname{idim} \mathcal{N} &= \operatorname{idim} \mathcal{M} \\ &\leq -K_X \cdot (D - kG) + p_a(C'') - 1 \\ &\leq -K_X \cdot D + g - 1 \\ &= n. \end{aligned}$$

As equality holds, if M is any component of $\overline{\mathcal{M}}_g(X, D)$ with $\operatorname{idim} M \geq n$, then $\operatorname{idim} M = n$, and the general map is as described in the statement of the theorem except that in (ii), all we know is that C_0 is rational and must map to G with degree at least 1.

But by a result of Graber ([G] Sects. 3.2 and 3.3, and Proposition 3.5) if any C_0 maps with degree greater than 1, this component does not contribute to the invariant; the integral (1) is 0. $\square$

Remark 4.3. The image of any map in $\overline{\mathcal{M}}_g(X, kG)$ $(k > 0)$ must lie in G. The construction of $[\overline{\mathcal{M}}_g(X, kG)]^{\mathrm{vir}}$ depends only on the first-order neighborhood of G, so we can compute $I_{g,kG}(\cdot)$ when $X = \mathbb{F}_2$ and $G = E$. When $\mathbb{F}_2$ is deformed to $\mathbb{F}_0$, the class kE deforms to a non-effective class, so $I_{g,kG}(\cdot) = 0$.

Remark 4.4. Section 2.2, Theorem 4.2, and the Remark above give an enumerative interpretation of all genus g Gromov–Witten invariants on an almost Fano surface.

5. Identifying potential components

Fix D, g, Ω, β, and let q be any point of E not in $\{q_s\}$. Let H_q be the Weil divisor on $V^{D,g}(\Omega, \beta)$ corresponding to maps whose image contain q. In this section, we will derive a list of subvarieties (call them *potential components*) in which each component of H_q of intersection dimension $\Upsilon - 1$ appears. We will see in the next section that each potential component actually appears in H_q.

The potential components come in two classes. First, one of the "moving tangencies" $r_{i,j}$ could map to q. Call such components *Type I potential components*.

Second, the image could degenerate to contain E as a component. Call such components *Type II potential components*. For any sequence $\gamma \geq 0$ and subset

$S'' \subset S$ (inducing $\Omega'' \subset \Omega$), let $g'' = g - |\gamma| + 1$. Define the Type II component $K(\Omega'' \subset \Omega, \beta, \gamma)$ as the closure in $\overline{\mathcal{M}}_{g,|S|}(X, D)'$ of points representing maps $\pi : C' \cup C'' \to X$ where

K1. the curve C' maps isomorphically to E, and contains the points $\{p_{s'}\}_{s' \in S \setminus S''}$,
K2. the curve C'' is smooth, each component maps birationally onto its image, and there exist distinct $r_{i,j} \in C''$ $(1 \le j \le \beta_i)$ and $t_{i,j} \in C''$ $(1 \le j \le \gamma_i)$ such that

$$\pi(p_s) = q_s \quad \text{and} \quad (\pi|_{C''})^*(E) = \sum_{s'' \in S''} m_{s''} p_{s''} + \sum i r_{i,j} + \sum i t_{i,j},$$

K3. the intersection of the curves C' and C'' is $\{t_{i,j}\}_{i,j}$, and
K4. the points $q_{s'}$ $(s' \in S \setminus S'')$ are distinct.

The stack $K(\Omega'' \subset \Omega, \beta, \gamma)$ is empty unless $\sum_{s'' \in S''} m_{s''} + I(\beta + \gamma) = (D - E) \cdot E$. The genus of C'' is g'', and there is a degree $\binom{\beta+\gamma}{\beta}$ rational map

$$K(\Omega'' \subset \Omega, \beta, \gamma) \dashrightarrow V^{D-E,g''}(\Omega'', \beta + \gamma)$$

corresponding to "forgetting the curve C'".

Theorem 5.1. *Let K be an irreducible component of H_q with intersection dimension $\Upsilon - 1$. Then*

I. *K is a component of $V^{D,g}(\Omega', \beta - e_k)$, where $\Omega' = \Omega \cup \{(q, k)\}$, or*
II. *K is a component of $K(\Omega'' \subset \Omega, \beta, \gamma)$ for some γ and Ω''.*

This is the analogue of [CH3] Theorem 1.2. The approach is the same.

Proof. Let $(C_0, \{p_s\}, \pi_0)$ be the map corresponding to a general point of K.

Let

$$\begin{array}{ccc} \Pi : (\mathcal{C}, \{p_s\}) & \longrightarrow & X \times B \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

be a smooth irreducible one-parameter family of pointed quasi-stable maps (with total space $\mathcal{C}$) with point $0 \in B$ and an isomorphism of the fiber over 0 with $(C_0, \{p_s\}, \pi_0)$, such that the image of the induced map $B \to \overline{\mathcal{M}}_{g,|S|}(X, D)'$ lies in $V^{D,g}(\Omega, \beta)$, but not in K. The total space of the family $\mathcal{C}$ is a surface, so the pullback of the divisor E to this family has pure dimension 1. The components of $\Pi^{-1}E$ not contained in a fiber $\mathcal{C}_t$ $(t \in B)$ must intersect the general fiber and thus be the sections p_s or multisections coming from the $r_{i,j}$. Therefore $\pi_0^{-1}E$ consists of components of C and points that are limits of the p_s or $r_{i,j}$. In particular:

(*) The number of zero-dimensional components of $\pi_0^{-1}E$ not mapped to any q_s is at most $|\beta|$, and

(**) If there are exactly $|\beta|$ such components, the multiplicities of $\pi_0^* E$ at these points must be given by the sequence β.

Case I. If C contains no components mapping to E, then

$$\pi_0^* E = \sum i a_{i,j} + \sum i b_{i,j}$$

where $\{a_{i,j}\}_{i,j}$ are the points mapped to $\{p_s\}_{s \in S} \cup \{q\}$, $\{b_{i,j}\}$ are the rest, and the second sum is over all i, $1 \le j \le \beta'_i$ for some sequence β'. By (*), $|\beta'| \le |\beta| - 1$. Then by Proposition 3.4,

$$\begin{aligned} \operatorname{idim} K &\le -(K_X + E) \cdot D + |\beta'| + g - 1 \\ &\le -(K_X + E) \cdot D + |\beta| - 1 + g - 1 \\ &= \Upsilon - 1. \end{aligned}$$

Equality must hold, so $|\beta'| = |\beta| - 1$ and K is of the form $V^{D,g}(\Omega', \beta - e_k)$ for some Ω', k. The set $\pi_0^{-1} E$ consists of $|S| + |\beta|$ points ($\{p_s\}$, the preimage of q, and $\{b_{i,j}\}$). This is also true of $\pi^{-1} E$ for a general map (C, π) in $V^{D,g}(\Omega, \beta)$, so the multiplicities at these points must be the same as for the general map (i.e. $\pi_0^* E$ has multiplicity m_s at p_s, etc.) so K must be as described in I.

Case II. If otherwise a component of C maps to E, assume first that no components of C are contracted to a point of E. Say $C = C' \cup C''$ where C' is the union of irreducible components of C mapping to E and C'' is the union of the remaining components. Define m by $\pi_{0*}[C'] = mE$, so $\pi_{0*}[C''] = D - mE$. Let $s = \#(C' \cap C'')$.

Then $p_a(C') \ge 1 - m$, so

$$p_a(C'') = g - p_a(C') + 1 - s \le g + m - s.$$

Assume $(\pi_0|_{C''})^* E = \sum i a_{i,j} + \sum i b_{i,j}$ where $\pi(a_{i,j})$ are fixed points of E as C'' varies (as (C, π) varies in K), and the second sum is over all i and $1 \le j \le \beta''_i$ for some sequence β''. By (*), $|\beta''| \le |\beta| + s$.

There is an open substack $U \subset K$ (containing 0) such that the universal map over U may be written

$$\begin{tikzcd} \Pi : (\mathcal{C}' \cup \mathcal{C}'', \{p_s\}) \arrow[r] \arrow[dr] & X \times U \arrow[d] \\ & U \end{tikzcd}$$

where for all $t \in U$, $\Pi_t(\mathcal{C}'_t) \subset E$, and $\Pi_t(\mathcal{C}''_t)$ has no component mapping to E, and the fiber over 0 has the given isomorphism with $(C_0, \{p_s\}, \pi_0)$. Let K' be the family $(\mathcal{C}'', \Pi|_{\mathcal{C}''})$. By Proposition 3.4 (applied to the family K'):

$$\begin{aligned} \operatorname{idim} K &= \operatorname{idim} K' \\ &\le -(K_X + E) \cdot (D - mE) + |\beta''| + p_a(C'') - 1 \\ &\le (-(K_X + E) \cdot D - 2m) + (|\beta| + s) + (g + m - s) - 1 \\ &= (-(K_X + E) \cdot D + |\beta| + g - 1) - 1 - (m - 1) \\ &= \Upsilon - 1 - (m - 1) \\ &\le \Upsilon - 1. \end{aligned} \tag{5}$$

In the third line, we used property **P1.**: E is rational, so $(K_X + E) \cdot E = -2$.

Equality must hold, so $m = 1$ and $|\beta''| = |\beta| + s$. By (**), the multiplicity of $\pi_0^* E$ at the $|\beta|$ points of C'' not in $C' \cup \{\pi_0^{-1} q_s\}$ is given by the sequence β. Let γ be the sequence given by the multiplicities of $(\pi_0|_{C''})^* E$ at the s points $C' \cap C''$. Define $S'' \subset S$ by $S'' = \{s'' \in S | p_{s''} \in C''\}$ (inducing $\Omega'' \subset \Omega$). The remaining zero-dimensional components of $\pi_0^{-1} E$ (aside from the $|\beta|$ points $r_{i,j}$) must be $\{p_{s''}\}_{s'' \in S''}$, and the multiplicity of $\pi_0^* E$ at $p_{s''}$ must be $m_{s''}$.

Suppose the point $r \in E$ appears in Ω exactly n times, say $q_s = r$ for $1 \leq s \leq n$. For a general map (C, π) in $V^{D,g}(\Omega, \beta)$, $\pi^{-1}(z)$ is a length n subscheme, supported at $\{p_s\}_{1 \leq s \leq n}$. In this limit, the map $\pi_0|_{C'} \to E$ is an immersion, so $(\pi_0|_{C'})^{-1}(r)$ has length 1. Thus at most one of the $\{p_s\}_{1 \leq s \leq n}$ can lie on C'. Thus K is a component of $K(\Omega'' \subset \Omega, \beta, \gamma)$ for some Ω'' and γ.

Finally, if a component of C is contracted to a point of E, follow the same argument but discard the contracted components (so the new source curve has arithmetic genus $g' < g$, and β_i', defined to be the number of $r_{i,j}$ on the new source curve, is at most β_i). Then at (5), we have

$$\operatorname{idim} K \leq \Upsilon^{D,g'}(\beta') - 1 < \Upsilon^{D,g}(\beta) - 1.$$

Hence such K are enumeratively irrelevant. $\square$

There are other (enumeratively irrelevant) components of the divisor H_q not counted in Theorem 5.1. For example, suppose $X = \mathbb{P}^2$ and E is a line L, and q_1 and q_2 are distinct points of E. If $D = 2L$, $g = 0$, $\Omega = \{(q_1, 1), (q_2, 1)\}$, $\beta = 0$, then $V^{D,g}(\Omega, \beta)$ is a three-dimensional family (generically) parametrizing conics through 2 fixed points q_1, q_2 of L. One component of H_q (generically) parametrizes a line union L; this is a Type II potential component. The other (generically) parametrizes degree 2 maps from $\mathbb{P}^1$ to L; it has intersection dimension 0.

Remark 5.2. Theorem 5.1 also describes components of H_q on $V^{D,g}(\Omega, \beta)^{\text{con}}$.

6. Multiplicities and recursions

We next compute the multiplicities of H_q along each component K described in Theorem 5.1.

6.1. Type I components

Proposition 6.1. *The component $K = V^{D,g}(\Omega', \beta - e_k)$ appears with multiplicity k.*

The proof is essentially that of the analogous proposition in [CH3] (Theorem 1.3a, Proposition 4.5, Subsect. 4.3). Only one minor change is necessary: consider the natural map

$$\sigma : V^{D,g}(\Omega, \beta) \dashrightarrow \left| \mathcal{O}_E \left(\sum i \beta_i \right) \right|$$

that is a morphism where the divisors $R_i = \{r_{i,j}\}_{1 \leq j \leq \beta_i}$ are defined (with $r_{i,j}$ as in the definition of $V^{D,g}(\Omega, \beta)$), given by

$$(C, \pi) \to \sum_i i\pi(R_i).$$

Then [CH3] Lemma 4.6 should be replaced by:

Lemma 6.2. *The differential $d\sigma$ is surjective at a general point of K.*

The proof is essentially the same.

6.2. Type II components

6.2.1. Versal deformation spaces of tacnodes. We first recall facts about versal deformation spaces from tacnodes, following [CH1] and [CH3] Section 4. Let (C, p) be an m^{th} order tacnode, that is, a curve singularity equivalent to the origin in the plane curve given by the equation $y(y + x^m) = 0$.

The miniversal deformation space of (C, p) is an étale neighborhood of the origin in $\mathbb{A}^{2m-1}$ with co-ordinates $a_0, \dots, a_{m-2}$, and $b_0, \dots, b_{m-1}$, and the "universal curve" $\pi : \mathcal{S} \to \Delta$ is given by

$$y^2 + yx^m + a_0 y + a_1 xy + \cdots + a_{m-2}x^{m-2}y + b_0 + b_1 x + \cdots + b_{m-1}x^{m-1} = 0.$$

There are two loci in Δ of interest to us. Let $\Delta_m \subset \Delta$ be the closure of the locus representing a curve with m nodes. It is smooth of dimension $m-1$, and corresponds to locally reducible curves. Let $\Delta_{m-1} \subset \Delta$ be the closure of the locus representing a curve with $m-1$ nodes. It is irreducible of dimension m, smooth away from Δ_m, with m sheets of Δ_{m-1} crossing transversely at a general point of Δ_m.

Let $m_1, m_2, \dots$ be any finite sequence of positive integers, and let (C_j, p_j) be an $(m_j)^{\text{th}}$ order tacnode. Denote the versal deformation space of (C_j, p_j) by Δ_j, and let $a_{j,m_j-2}, \dots, a_{j,0}, b_{j,m_j-1}, \dots, b_{j,0}$ be coordinates on Δ_j as above. For each j, let Δ_{j,m_j} and $\Delta_{j,m_j-1} \subset \Delta_j$ be as above the closures of loci in Δ_j over which the fibers of π_j have m_j and $m_j - 1$ nodes respectively. Set

$$\Delta = \Delta_1 \times \Delta_2 \times \dots,$$

$$\Delta_m = \Delta_{1,m_1} \times \Delta_{2,m_2} \times \dots,$$

$$\Delta_{m-1} = \Delta_{1,m_1-1} \times \Delta_{2,m_2-1} \times \dots.$$

Note that Δ, Δ_m and Δ_{m-1} have dimensions $\sum(2m_j - 1)$, $\sum(m_j - 1)$ and $\sum m_j$ respectively.

Let $W \subset \Delta$ be a smooth subvariety of dimension $\sum(m_j - 1) + 1$, containing the linear space Δ_m. Suppose that the tangent plane to W is not contained in the union of hyperplanes $\cup_j \{b_{j,0} = 0\} \subset \Delta$. Let $\kappa := \prod m_j / \operatorname{lcm}(m_j)$. Then:

Lemma 6.3. *With the hypotheses above, in an étale neighborhood of the origin in* Δ,

$$W \cap \Delta_{m-1} = \Delta_m \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_\kappa$$

where $\Gamma_1, \dots, \Gamma_\kappa \subset W$ *are distinct reduced unibranch curves having intersection multiplicity exactly* $\operatorname{lcm}(m_j)$ *with* Δ_m *at the origin.*

This lemma arose in conversations with J. Harris, and appears (with proof) as part of [CH3] Lemma 4.3. Results of a similar flavor appear in [V1] Sect. 1 and [V3] Sect. 2.5, although the proofs are different.

6.2.2. Calculating the multiplicity. Suppose $K = K(\Omega'' \subset \Omega, \beta, \gamma)$ is a Type II component of H_q (on $V^{D,g}(\Omega, \beta)$). Let $m_1, \dots, m_{|\gamma|}$ be a set of positive integers with j appearing γ_j times ($j = 1, 2, \dots$), so $\sum m_i = I\gamma$.

Proposition 6.4. *The multiplicity of* H_q *along* K *is* $m_1 \dots m_{|\gamma|} = I^\gamma$.

The proof of this proposition will occupy us until Sect. 6.3.

Fix general points $s_1, \dots, s_{\Upsilon-1}$ on X, and let H_i be the divisor on $V^{D,g}(\Omega, \beta)$ corresponding to requiring the image curve to pass through s_i. By Sard's Theorem, the intersection of $V^{D,g}(\Omega, \beta)$ with $\cap_i H_i$ is a curve V and the intersection of K with $\cap_i H_i$ is a finite set of points (non-empty as K has intersection dimension $\Upsilon - 1$). Choose a point (C, π) of $K \cap H_1 \cap \cdots \cap H_{\Upsilon-1}$. The multiplicity of H_q along K on $V^{D,g}(\Omega, \beta)$ is the multiplicity of H_q at the point (C, π) on the curve V.

For such (C, π) in $K(\Omega'' \subset \Omega, \beta, \gamma)$ there are unique choices of points $\{r_{i,j}\}$ on C (up to permutations of $\{r_{i,j}\}$ for fixed i).

Define the map $(\tilde{C}, \tilde{\pi})$ as follows: $C \stackrel{\pi}{\to} X$ factors through

$$C \stackrel{\nu}{\to} \tilde{C} \stackrel{\tilde{\pi}}{\to} X$$

where ν is a homeomorphism (a seminormalization) and $\tilde{\pi}$ is locally an immersion. Each node of C is mapped to a tacnode (of some order) of $\tilde{C}$, and $\nu : C \to \tilde{C}$ is a partial normalization. Then $\tilde{C}$ has arithmetic genus $\tilde{g} := g + \sum(m_i - 1)$.

Let $\operatorname{Def}(\tilde{C}, \tilde{\pi})$ be the deformations of $(\tilde{C}, \tilde{\pi})$ preserving the incidences to s_1, $\dots, s_{\Upsilon-1}$ and the tangencies to E ($\tilde{\pi}^* E = \sum m_s p_s + \sum i r_{i,j}, \tilde{\pi}(p_s) = q_s$). For convenience, let $N := N_{\tilde{C}/X}(-\sum m_s p_s - \sum(i-1) r_{i,j})$.

Lemma 6.5. *The space* $\operatorname{Def}(\tilde{C}, \tilde{\pi})$ *is smooth of dimension* $\sum(m_j - 1) + 1$.

Proof. We will show the equivalent result: the vector space of first-order deformations of $(\tilde{C}, \tilde{\pi})$ preserving the tangency conditions (but not necessarily the incidence conditions $s_1, \dots, s_{\Upsilon-1}$) has dimension $\Upsilon + \sum(m_i - 1)$, and is unobstructed.

As $(\tilde{C}, \tilde{\pi})$ is an immersion, let $N_{\tilde{C}/X} = \mathcal{O}_{\tilde{C}}(-\tilde{\pi}^* K_X + K_{\tilde{C}})$ be the normal bundle to $\tilde{\pi}$. By property **P3.**, as $\tilde{\pi}^*(K_X + E - \sum r_{i,j})$ is negative on every component of $\tilde{C}$, $h^1(\tilde{C}, N) = 0$ so

$$\begin{aligned} h^0(\tilde{C}, N) &= \chi\left(\tilde{C}, N_{\tilde{C}/X}\left(-\sum m_s p_s - \sum (i-1) r_{i,j}\right)\right) \\ &= \deg(\tilde{\pi}^*(-K_X - E + \sum r_{i,j})) + \deg K_{\tilde{C}} - \tilde{g} + 1 \\ &= -(K_X + E) \cdot D + |\beta| + \tilde{g} - 1 \\ &= -(K_X + E) \cdot D + |\beta| + g + \sum (m_i - 1) - 1 \\ &= \Upsilon + \sum (m_i - 1). \end{aligned}$$

Thus there are $\Upsilon + \sum(m_i - 1)$ first-order deformations, and as $h^1(\tilde{C}, N) = 0$ they are unobstructed. □

By the proof of the above lemma, $H^0(\tilde{C}, N)$ is naturally the tangent space to $\mathrm{Def}(\tilde{C}, \tilde{\pi})$. Now $-K_X$ restricted to C' has degree $K_X \cdot E = 2 + E^2$; $K_{\tilde{C}}$ restricted to C' has degree $I\gamma - 2$, which is ($\deg K_{C'}$) plus the length of the scheme-theoretic intersection of C' and C''. Therefore

$$\begin{aligned} \deg N|_{C'} &= 2 + E^2 + I\gamma - 2 - \sum_{s' \in S \setminus S''} m_{s'} \\ &= D \cdot E - (D - E) \cdot E + I\gamma - \sum_{s' \in S \setminus S''} m_{s'} \\ &= \left(\sum_{s \in S} m_s + I\beta\right) - \left(\sum_{s'' \in S''} m_{s''} + I\beta + I\gamma\right) + I\gamma - \sum_{s' \in S \setminus S''} m_{s'} \\ &= 0 \end{aligned}$$

so the restriction of N to C' is the trivial line bundle.

Also, if p is a general point on C' then $h^0(\tilde{C}, N(-p)) = h^0(\tilde{C}, N) - 1$. (*Proof:* From above, $h^1(\tilde{C}, N) = 0$. By the same argument, as $\deg(K_X + E)|_E = -2$, $\tilde{\pi}^*(K_X + E - \sum r_{i,j} + p)$ is negative on every component of $\tilde{C}$, so $h^1(\tilde{C}, N(-p)) = 0$. Thus $h^0(\tilde{C}, N(-p)) - h^0(\tilde{C}, N) = \chi(\tilde{C}, N(-p)) - \chi(\tilde{C}, N) = -1$.) Thus there is a section of N that is nonzero on C'.

Let $\mathcal{J}$ be the Jacobian ideal of $\tilde{C}$. In an étale neighborhood of the (C, π), there are natural maps

$$V \xrightarrow{\rho} \mathrm{Def}(\tilde{C}, \tilde{\pi}) \xrightarrow{\sigma} \Delta$$

where the differential of σ is given by the natural map

$$H^0(\tilde{C}, N) \to H^0(\tilde{C}, N \otimes (\mathcal{O}_{\tilde{C}}/\mathcal{J})). \tag{6}$$

Lemma 6.6. *In a neighborhood of the origin, the morphism*

$$\sigma : \mathrm{Def}(\tilde{C}, \tilde{\pi}) \to \Delta$$

is an immersion, and the tangent space to $\sigma(\mathrm{Def}(\tilde{C}, \tilde{\pi}))$ *contains* Δ_m *and is not contained in the union of hyperplanes* $\cup_j \{b_{j,0} = 0\}$.

Proof. From (6), the Zariski tangent space to the divisor $\sigma^*(b_{j,0}=0)$ is a subspace Z of $H^0(\tilde{C},N)$ vanishing at a point of C' (the j^{th} tacnode). But $N|_{C'}$ is a trivial bundle, so this subspace of sections Z must vanish on all of C'. As there is a section of N that is non-zero on C', Z has dimension at most $h^0(\tilde{C},N)-1 = \dim \operatorname{Def}(\tilde{C},\pi)-1$. This proves that σ is an immersion, and that the tangent space to $\sigma(\operatorname{Def}(\tilde{C},\tilde{\pi}))$ is not contained in $\{b_{j,0}=0\}$.

Finally, if S is the divisor (on $\operatorname{Def}(\tilde{C},\tilde{\pi})$) corresponding to requiring the image curve to pass through a fixed general point of E, then $\sigma(S)\subset\Delta_m$, as the image curve must be reducible. As σ is an immersion,

$$\begin{aligned}\sum(m_i-1) &= \dim \operatorname{Def}(\tilde{C},\tilde{\pi})-1\\ &= \dim S\\ &= \dim \sigma(S)\\ &\leq \dim \Delta_m \qquad (7)\\ &= \sum(m_i-1)\end{aligned}$$

so we must have equality at (7), and the linear space $\Delta_m=\sigma(S)$ is contained in $\sigma(\operatorname{Def}(\tilde{C},\tilde{\pi}))$, and thus in the tangent space to $\sigma(\operatorname{Def}(\tilde{C},\tilde{\pi}))$. $\square$

Therefore the image $\sigma(\operatorname{Def}(\tilde{C},\tilde{\pi}))$ satisfies the hypotheses of Lemma 6.3, so the closure of the inverse image $\sigma^{-1}(\Delta_{m-1}\setminus\Delta_m)$ will have $\prod m_i/\operatorname{lcm}(m_i)$ reduced branches, each having intersection multiplicity $\operatorname{lcm}(m_i)$ with $\sigma^{-1}(\Delta_m)$ and hence with the hyperplane H_q. Since in a neighborhood of (C,π) the variety V is a curve birational with $\rho(V)=\overline{\sigma^{-1}(\Delta_{m-1}\setminus\Delta_m)}$, we conclude that the divisor H_q contains $K(\Omega''\subset\Omega,\beta,\gamma)$ with multiplicity $m_1\cdots m_{|\gamma|}=I^\gamma$.

This completes the proof of Proposition 6.4. As an added benefit, we see that $V^{D,g}(\Omega,\beta)$ has $I^\gamma/\operatorname{lcm}(\gamma)$ branches at a general point of $K(\Omega''\subset\Omega,\beta,\gamma)$, where $\operatorname{lcm}(\gamma)$ is the least common multiple of the set $\#\{i\,|\,\gamma_i\neq 0\}$.

6.3. Recursions

Theorem 5.1 and Propositions 6.1 and 6.4 give a rational equivalence between H_q and a linear combination of boundary components. Intersecting this equivalence with $H^{\Upsilon-1}$ yields the following recursion (the generalization of [CH3] Theorem 1.1).

Theorem 6.7. *If* $\Upsilon=\dim V^{D,g}(\Omega,\beta)>0$,

$$N^{D,g}(\Omega,\beta)=\sum_k kN^{D,g}(\Omega',\beta-e_k)+\sum I^\gamma\binom{\beta+\gamma}{\beta}N^{D-E,g''}(\Omega'',\beta+\gamma)$$

where

- *in the first sum,* $\Omega'=\Omega\cup\{(q,k)\}$, *and*
- *the second sum runs over choices* $S''\subset S$ *such that the points* $\{q_{s'}\}_{s'\in S\setminus S''}$ *are distinct, and* $\gamma\geq 0$*; also,* $g'':=g-|\gamma|+1$.

By considering divisors only on $V^{D,g}(\Omega,\beta)^{\mathrm{con}}$, we get a recursion for irreducible curves. The proof is identical, except that rather than considering all maps, we just consider maps from connected curves. The Type I components that can appear are analogous. The Type II components consist of maps from curves $C = C(0) \cup \cdots \cup C(l)$ where $C(0)$ maps isomorphically to E, and $C(i)$ intersects $C(j)$ if and only if $ij = 0$. (In the general case Theorem 6.7, $C(i)$ intersected $C(j)$ only if $ij = 0$.)

Theorem 6.8. *If $\Upsilon = \dim V^{D,g}(\Omega,\beta)^{\mathrm{con}} > 0$, then*

$$\begin{aligned} N^{D,g}(\Omega,\beta)^{\mathrm{con}} = &\sum_k k N^{D,g}(\Omega',\beta - e_k)^{\mathrm{con}} \\ &+ \sum \frac{1}{\sigma} \binom{\Upsilon^{D,g}(\beta) - 1}{\Upsilon^{D^1,g^1}(\beta^1), \dots, \Upsilon^{D^l,g^l}(\beta^l)} \\ &\cdot \prod_{i=1}^{l} \binom{\beta^i + \gamma^i}{\beta^i} I^{\beta^i - \gamma^i} N^{D^i,g^i}(\Omega^i, \beta^i + \gamma^i)^{\mathrm{con}} \end{aligned}$$

(cf. [CH3] Sect. 1.4) where

- *in the first sum, $\Omega' = \Omega \cup \{(q,k)\}$, and*
- *the second sum runs over choices of $D^i, g^i, \Omega^i, \beta^i, \gamma^i$ $(1 \le i \le l)$ where*
  - *D^i is a divisor class (with $\sum D^i = D - E$),*
  - *g^i is a non-negative integer,*
  - *β^i and γ^i are sequences of non-negative integers (with $\sum_i \beta^i = \beta$, $\gamma^i \neq 0$),*
  - *$\coprod \Omega^i \subset \Omega$ (with $\Omega \setminus \coprod \Omega^i$ consisting of distinct points $\{q_s\}$), and*
  - *σ is the order of the symmetry group of the set $\{(D^i, g^i, \Omega^i, \beta^i, \gamma^i)\}_{1 \le i \le l}$.*

6.4. Theorems 6.7 and 6.8 as differential equations, following Getzler

Assemble the enumerative invariants (in the case where $\{q_s\}$ are general) in a generating function

$$G = \sum_{D,g,\alpha,\beta} N^{D,g}(\alpha,\beta) v^D w^{g-1} \left(\frac{x^\alpha}{\alpha!}\right) y^\beta \left(\frac{z^\Upsilon}{\Upsilon!}\right)$$

(where w and z are variables, $x = (x_1, x_2, \dots)$, $\quad y = (y_1, y_2, \dots)$, and

$$\{v^D\}_{D \text{ effective}, D \neq E}$$

generates a semigroup algebra, the Novikov ring). Then Theorem 6.7 is equivalent to the differential equation

$$\frac{\partial G}{\partial z} = \left(\sum k y_k \frac{\partial}{\partial x_k} + \frac{v^E}{w} \operatorname{res}_{t=0} e^{\sum (t^{-k} x_k + k w t^k \frac{\partial}{\partial y_k})} \right) G. \tag{8}$$

The corresponding observation for the plane is due to Getzler ([Ge1] Sect. 5.3), and nothing essentially new is involved here, although the notation is slightly different from Getzler's.

Define the generating function

$$G_{\mathrm{irr}} = \sum_{D,g,\alpha,\beta} N^{D,g}(\alpha,\beta)^{\mathrm{con}} v^D w^{g-1} \left(\frac{x^\alpha}{\alpha!}\right) y^\beta \left(\frac{z^\Upsilon}{\Upsilon!}\right).$$

Then by a simple combinatorial argument (see e.g. [W] Chapter 3), $G = e^{G_{\mathrm{irr}}}$. Substituting this into (8) yields a differential equation satisfied by G_{irr}:

$$\frac{\partial G_{\mathrm{irr}}}{\partial z} = \sum k y_k \frac{\partial}{\partial x_k} G_{\mathrm{irr}} + \frac{v^E}{w} \operatorname{res}_{t=0} e^{\sum (t^{-k} x_k + G_{\mathrm{irr}}|_{y_k \mapsto y_k + kwt^k}) - G_{\mathrm{irr}}} \tag{9}$$

where $G_{\mathrm{irr}}\ |_{y_k \mapsto y_k + kwt^k}$ is the same as G_{irr} except y_k has been replaced by $(y_k + kw)$. (Once again, this should be compared with Getzler's formula [Ge1], p. 993.)

7. Application: Caporaso–Harris revisited

If $X = \mathbb{P}^2$ and E is a line, Theorem 6.7 applied when the $\{q_s\}$ are distinct points is the recursion of Caporaso and Harris ([CH3] Sect. 1.4). A minor additional observation: by induction, $N^{D,g}(\Omega, \beta)$ is independent of the points $\{q_s\}$ (so long as they are distinct). This is true of the applications in the next two sections as well.

Computationally, it is simpler to apply Theorem 6.7 when the $\{q_s\}$ are distinct. It is always possible to reduce a more complicated enumerative problem to this case. For example,

Lemma 7.1. *Suppose $s_i \in E$, $s_i \notin \{q_s\}$ $(i = 1, 2, 3)$, s_i distinct. Then*

$$\begin{aligned} N^{D,g}(\Omega \cup \{(s_1,1),(s_1,1)\},\beta) &= N^{D,g}(\Omega \cup \{(s_1,1),(s_2,1)\},\beta) \\ &\quad - N^{D,g}(\Omega \cup \{(s_1,2)\},\beta), \\ N^{D,g}(\Omega \cup \{(s_1,1),(s_1,1),(s_1,1)\},\beta) &= N^{D,g}(\Omega \cup \{(s_1,1),(s_2,1),(s_3,1)\},\beta) \\ &\quad -3N^{D,g}(\Omega \cup \{(s_1,1),(s_2,2)\},\beta) \\ &\quad +2N^{D,g}(\Omega \cup \{(s_1,3)\},\beta). \end{aligned}$$

This tells us how to reduce the conditions of a double or triple point on E to tangency conditions. (*Warning:* the left side counts curves with multiple points with *labelled* branches at the n-fold point; to forget the labelling, one must divide by $n!$.) There are analogous expressions for all other cases where a point s_1 on E appears multiply in $\{q_s\}$. The result still holds when $N^{D,g}(\cdot,\cdot)$ is replaced by $N^{D,g}(\cdot,\cdot)^{\mathrm{con}}$. The lemma can be proved by induction (on Ω and β) using Theorem 6.7.

8. Application: Counting curves on Hirzebruch surfaces

Theorems 6.7 and 6.8 count curves of any genus in any divisor class on $\mathbb{F}_n$. The "seed data" necessary are the cases where $\Upsilon^{D,g}(\Omega,\beta) = \dim V^{D,g}(\Omega,\beta) = 0$. It can be easily checked (using Theorem 3.1) that the only non-empty $V^{D,g}(\Omega,\beta)^{\mathrm{con}}$ where $\Upsilon = 0$ has $D = F$, $g = 0$, $\Omega = \{(pt, 1)\}$, $\beta = 0$; in this case, $N^{D,g}(\Omega,\beta)^{\mathrm{con}} = 1$.

Hence the only non-empty $V^{D,g}(\Omega,\beta)$ where $\Upsilon = 0$ has (for some integer $k > 0$) $D = kF$, $g = 1-k$, $|S| = k$, $m_s = 1$ for all $s \in S$, and $|\beta| = 0$. In this case, if σ is the order of the symmetry group of the set $\{q_s\}$,

$$N^{D,g}(\Omega,\beta) = \deg[V^{D,g}(\Omega,\beta)] = \frac{1}{\sigma}.$$

The following proposition shows that if the points $\{q_s\}$ are distinct, $N^{D,g}(\Omega,\beta)$ counts nodal curves.

Proposition 8.1. *If $X = \mathbb{F}_n$, the $\{p_s\}$ are distinct, and (C,π) is a general curve in a component of $V^{D,g}(\Omega,\beta)$, then $\pi(C)$ has at most nodes as singularities.*

The proof is easily adapted from that of [CH3] Prop. 2.2 a), and is omitted.

Warning: The proof requires more than properties **P1.–P4.**. The following example shows that the result does not hold for every (X,E) satisfying properties **P1.–P4.**. Let $X = \mathbb{P}^2$ and E be a smooth conic (see the next Section). Choose six distinct points $a,\ldots,f$ on E such that the lines ab, cd, and ef meet at a point. Then if L is the class of a line,

$$V^{D=3L,g=-2}(\Omega = \{(a,1),\ldots,(f,1)\}, \beta = 0)$$

consists of a finite number of maps, one of which is the map sending three disjoint $\mathbb{P}^1$'s to the lines ab, cd, and ef.

8.1. Higher genus Gromov–Witten invariants of Hirzebruch surfaces

Suppose X is $\mathbb{F}_0$ or $\mathbb{F}_1$. As X is Fano, the higher genus Gromov–Witten invariants are enumerative (Section 4), so for fixed g, $D \neq 0$, if γ is the class of a point, then invariant

$$I_{g,D}(\gamma^{-K_X\cdot D+g-1}) = \begin{cases} \delta_{g,0} & \text{if } (X,D) = (\mathbb{F}_1,E) \\ N^{D,g}(\emptyset,(D\cdot E)e_1)^{\text{con}} & \text{otherwise} \end{cases}$$

can be recursively calculated by Theorem 6.8. As $\mathbb{F}_n$ is deformation-equivalent to $\mathbb{F}_0$ if n is even, or $\mathbb{F}_1$ if n is odd ([N] p. 9–10), this computes the invariants of *all* $\mathbb{F}_n$.

8.2. Curves in $\mathbb{F}_2$ in terms of curves in $\mathbb{F}_0$

Let $N^g_{\mathbb{F}_n}(aS+bF)$ be the number of irreducible genus g curves in class $aS+bF$ through the appropriate number of points. Abramovich and Bertram have proved

$$N^0_{\mathbb{F}_0}(aS+(a+b)F) = \sum_{i=0}^{a-1}\binom{b+2i}{i} N^0_{\mathbb{F}_2}(aS+bF-iE). \tag{AB0}$$

by degenerating $\mathbb{F}_2$ to $\mathbb{F}_0$ (so the class $aS+(a+b)F$ on $\mathbb{F}_0$ degenerates to $aS+bF$ on $\mathbb{F}_2$, [AB1]). Graber has given another proof ([G] Sect. 3.5). From Sect. 4, computing

Table 1. Number of genus g curves in class $2S + kF$ on $\mathbb{F}_n$

	$\mathbb{F}_0$	$\mathbb{F}_1$	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{F}_4$
$2S$		0: 1	1: 1 0: 10	2: 1 1: 17 0: 69	3: 1 2: 24 1: 177 0: 406
$2S + F$	0: 1	1: 1 0: 12	2: 1 1: 20 0: 102 (93)	3: 1 2: 28 1: 246 (234) 0: 781 (594)	
$2S + 2F$	1: 1 0: 12	2: 1 1: 20 0: 105 (96)	3: 1 2: 28 1: 252 (240) 0: 856 (636)		
$2S + 3F$	2: 1 1: 20 0: 105 (96)	3: 1 2: 28 1: 252 (240) 0: 860 (640)			
$2S + 4F$	3: 1 2: 28 1: 252 (240) 0: 860 (640)				

the invariants of $\mathbb{F}_2$ in two ways (by deforming to $\mathbb{F}_0$, and by Theorem 4.2), this formula generalizes to higher genus:

$$N^g_{\mathbb{F}_0}(aS + (a+b)F) = \sum_{i=0}^{a-1} \binom{b+2i}{i} N^g_{\mathbb{F}_2}(aS + bF - iE).$$

8.3. Examples

Table 1 gives the number of genus g curves in certain classes on certain $\mathbb{F}_n$. The number preceding the colons in the table is the genus g. Where the number of irreducible curves is different, it is given in parentheses. Tables 2 and 3 give more examples; only the total number is given, although the number of irreducible curves could also be easily computed (using Theorem 6.8). Many of these numbers were computed by a maple program written by L. Göttsche to implement the algorithm of Theorem 6.7.

As an example of the algorithm in action, we calculate $N^{4S,1}(\emptyset, 0) = 225$ on $\mathbb{F}_1$. (This is also the number of two-nodal elliptic plane quartics through 11 fixed general points.) There are a finite number of such elliptic curves through 11 fixed

Table 2. Number of (possibly reducible) genus g curves in various classes on $\mathbb{F}_1$

Class	Genus	Number	Class	Genus	Number	Class	Genus	Number
$3S$	-2	15	$3S+2F$	0	22647	$3S+3F$	0	642434
	-1	21		1	14204		1	577430
	0	12		2	4249		2	291612
	1	1		3	615		3	83057
$3S+F$	0	675		4	41		4	13405
	1	225		5	1		5	1200
	2	27					6	55
	3	1					7	1

Table 3. Number of (possibly reducible) genus g curves in class $3S$ on $\mathbb{F}_2$

Genus	-2	-1	0	1	2	3	4
Number	280	1200	2397	1440	340	32	1

general points on $\mathbb{F}_1$. We calculate the number by specializing the fixed points to lie on E one at a time, and following what happens to the finite number of curves.

The divisor E is represented by the horizontal dotted line, and fixed points on E are represented by fat dots. Part of the figure, the calculation that $N^{2S+2F,0}(\emptyset, 2e_1) = 105$, has been omitted.

After the first specialization, the curve must contain E. (Reason: As $4S \cdot E = 0$, any representative of $4S$ containing a point of E must contain all of E.) The residual curve is in class $3S + F$. Theorem 6.7 gives

$$N^{4S,1}(\emptyset, 0) = N^{3S+F,1}(\emptyset, e_1).$$

After specializing a second point q to lie on E, two things could happen to the elliptic curve. First, the limit curve could remain smooth, and pass through the fixed point q of E. This will happen $N^{3S+F,1}(\{(q, 1)\}, 0)$ times. Second, the curve could contain E. Then the residual curve C' is in class $2S + 2F$, and is a nodal curve intersecting E at two distinct points. Of the two nodes of the original curve C, one goes to the node of C', and the other tends to one of the intersection of C' with E. The choice of the two possible limits of the node gives a multiplicity of 2 (indicated by the "$\times 2$" in the figure). Theorem 6.7 gives

$$N^{3S+F,1}(\emptyset, e_1) = N^{3S+F,1}(\{(q, 1)\}, 0) + 2N^{2S+2F,1}(\emptyset, 2e_1).$$

The rest of the derivation is similar.

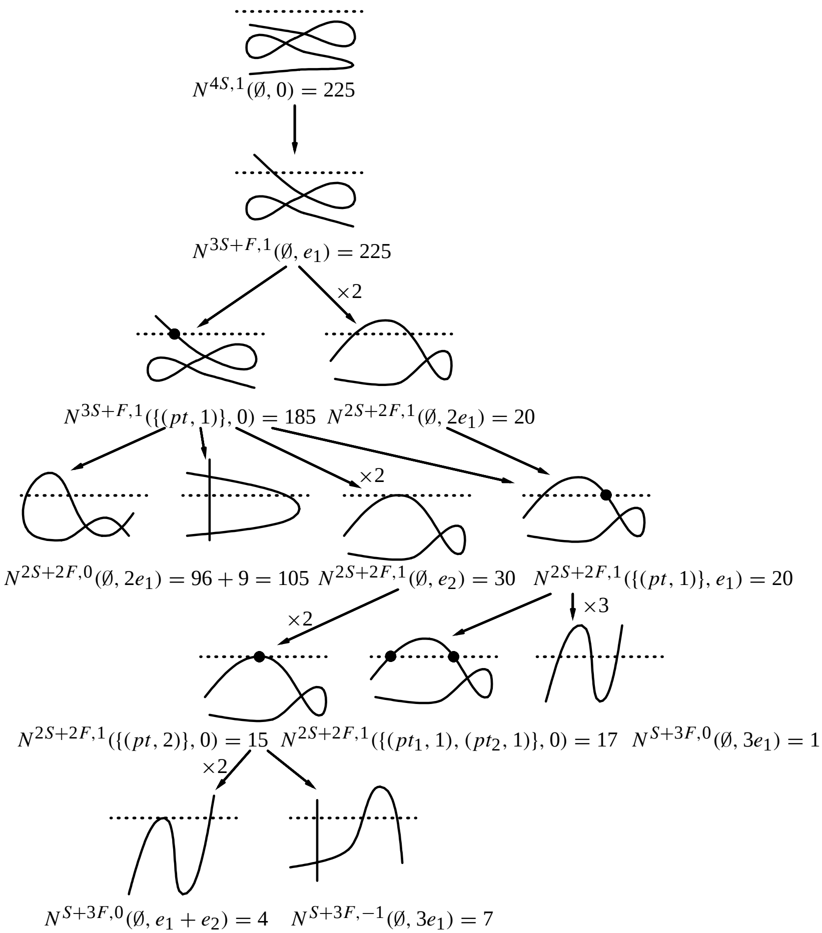

Fig. 1. Calculating $N^{4S,1}(\emptyset, 0) = 225$.

9. Application: Higher-genus Gromov–Witten invariants of del Pezzo surfaces

In this section, we compute the higher-genus Gromov–Witten invariants of $\mathbb{P}^2$ blown up at $s \leq 6$ points. By Section 4, it suffices to count maps through various numbers of points, i.e. compute $I_{g,D}(\gamma^n)$ where γ is the class of a point. If D is an exceptional curve, the invariant is $\delta_{g,0}$.

9.1. The case $s \leq 5$

If $D \neq E$ is not an exceptional curve, then by blowing down the s exceptional divisors, the invariants count maps with "s multiple points".

More precisely, let Y be the del Pezzo surface that is $\mathbb{P}^2$ blown up at s points $q_1, \ldots, q_s$ (no 3 collinear). Let $X = \mathbb{P}^2$, H the class of a line, and E the smooth

Table 4. Numbers of irreducible plane curves with fixed multiple points, or invariants of Fano surfaces

$N^{1,0}$	$N^{2,0}$	$N^{3,1}$	$N^{3,0}$	$N_2^{3,0}$	$N^{4,3}$	$N^{4,2}$	$N^{4,1}$	$N^{4,0}$	$N_2^{4,2}$	$N_2^{4,1}$	$N_2^{4,0}$
1	1	1	12	1	1	27	225	620	1	20	96

$N_{2^2}^{4,1}$	$N_{2^2}^{4,0}$	$N_{2^3}^{4,0}$	$N_3^{4,0}$	$N^{5,6}$	$N^{5,5}$	$N^{5,4}$	$N^{5,3}$	$N^{5,2}$	$N^{5,1}$	$N^{5,0}$
1	12	1	1	1	48	882	7915	36855	87192	87304

$N_2^{5,5}$	$N_2^{5,4}$	$N_2^{5,3}$	$N_2^{5,2}$	$N_2^{5,1}$	$N_2^{5,0}$
1	41	615	4235	13775	18132

$N_{2^2}^{5,4}$	$N_{2^2}^{5,3}$	$N_{2^2}^{5,2}$	$N_{2^2}^{5,1}$	$N_{2^2}^{5,0}$	$N_{2^3}^{5,3}$	$N_{2^3}^{5,2}$	$N_{2^3}^{5,1}$	$N_{2^3}^{5,0}$
1	34	396	1887	3510	1	27	225	620

$N_{2^4}^{5,2}$	$N_{2^4}^{5,1}$	$N_{2^4}^{5,0}$	$N_{2^5}^{5,1}$	$N_{2^5}^{5,0}$	$N_3^{5,3}$	$N_3^{5,2}$	$N_3^{5,1}$	$N_3^{5,0}$
1	20	96	1	12	1	28	240	640

$N_{3,2}^{5,2}$	$N_{3,2}^{5,1}$	$N_{3,2}^{5,0}$	$N_{3,2^2}^{5,1}$	$N_{3,2^2}^{5,0}$	$N_{3,2^3}^{5,0}$	$N_4^{5,0}$
1	20	96	1	12	1	1

conic through $q_1, \dots, q_s$. Then if $dH - \sum f_i E_i \neq E_j$,

$$I_{g,dH-\sum f_i E_i}(\gamma^n) = N^{dH,g}(\Omega, (2d - \sum f_i)e_1)$$

where Ω consists of f_i copies of $(q_i, 1)$ $(1 \leq i \leq s)$ and $n = \operatorname{idim} V^{dH,g}(\Omega, (2d - \sum f_i)e_1)$ is the appropriate number of point conditions.

Theorem 6.8 calculates these numbers recursively, given "seed data" of the cases when $\Upsilon = 0$. It can be easily checked (using Theorem 3.1) that the only non-empty $V^{D,g}(\Omega, \beta)^{\text{con}}$ where $\Upsilon = 0$ is the case $D = H$, $g = 0$, $\Omega = \{(pt_1, 1), (pt_2, 1)\}$ or $\{(pt, 2)\}$, $\beta = 0$, in which case $N^{D,g}(\Omega, \beta)^{\text{con}} = 1$ (there is only one line through 2 distinct fixed points of a conic, and only one line tangent to a conic at a fixed point). Theorem 6.7 counts maps from reducible curves, of course. Lemma 7.1 applies here as well, and can be used to simplify calculations.

9.1.1. Examples. If f is the sequence $f_1, \dots, f_s$, let $N_f^{d,g}$ be the genus g invariant for class $dH - \sum f_i E_i$. For convenience, we indicate repetitions of f_i with exponents, e.g. $N_{2,2,2}^{d,g} = N_{2^3}^{d,g}$. Then Table 4 gives values of $N_f^{d,g}$ for $d \leq 5$.

It is computationally more convenient to count maps of possibly disconnected curves, and the results in Table 4 were obtained by first counting such maps and

Table 5. Number of degree 6 maps (from possibly reducible curves) to Fano surfaces

$R_2^{6,9}$	$R_2^{6,8}$	$R_2^{6,7}$	$R_2^{6,6}$	$R_2^{6,5}$	$R_2^{6,4}$	$R_2^{6,3}$	$R_2^{6,2}$	$R_2^{6,1}$	$R_2^{6,0}$	$R_{2^2}^{6,8}$	$R_{2^2}^{6,7}$	$R_{2^2}^{6,6}$	$R_{2^2}^{6,5}$
1	68	1914	29091	261945	1448496	4960863	10430567	13170063	9753850	1	61	1507	19581

$R_{2^2}^{6,4}$	$R_{2^2}^{6,3}$	$R_{2^2}^{6,2}$	$R_{2^2}^{6,1}$	$R_{2^2}^{6,0}$	$R_{2^3}^{6,7}$	$R_{2^3}^{6,6}$	$R_{2^3}^{6,5}$	$R_{2^3}^{6,4}$	$R_{2^3}^{6,3}$	$R_{2^3}^{6,2}$	$R_{2^3}^{6,1}$	$R_{2^3}^{6,0}$
146070	645102	1691553	5188493/2	4559365/2	1	54	1149	12437	743170	251295	479472	1016067/2

$R_{2^4}^{6,6}$	$R_{2^4}^{6,5}$	$R_{2^4}^{6,4}$	$R_{2^4}^{6,3}$	$R_{2^4}^{6,2}$	$R_{2^4}^{6,1}$	$R_{2^4}^{6,0}$	$R_{2^5}^{6,5}$	$R_{2^5}^{6,4}$	$R_{2^5}^{6,3}$	$R_{2^5}^{6,2}$	$R_{2^5}^{6,1}$	$R_{2^5}^{6,0}$	$R_3^{6,7}$	$R_3^{6,6}$	$R_3^{6,5}$
1	47	840	7316	33484	82128	107214	1	40	580	3874	12785	21029	1	55	1200

$R_3^{6,4}$	$R_3^{6,3}$	$R_3^{6,2}$	$R_3^{6,1}$	$R_3^{6,0}$	$R_{3,2}^{6,6}$	$R_{3,2}^{6,5}$	$R_{3,2}^{6,4}$	$R_{3,2}^{6,3}$	$R_{3,2}^{6,2}$	$R_{3,2}^{6,1}$	$R_{3,2}^{6,0}$	$R_{3,2^2}^{6,5}$	$R_{3,2^2}^{6,4}$
13405	83057	291612	577430	642434	1	48	884	7996	38170	195473/2	269719/2	1	41

$R_{3,2^2}^{6,3}$	$R_{3,2^2}^{6,2}$	$R_{3,2^2}^{6,1}$	$R_{3,2^2}^{6,0}$	$R_{3,2^3}^{6,4}$	$R_{3,2^3}^{6,3}$	$R_{3,2^3}^{6,2}$	$R_{3,2^3}^{6,1}$	$R_{3,2^3}^{6,0}$	$R_{3,2^4}^{6,3}$	$R_{3,2^4}^{6,2}$	$R_{3,2^4}^{6,1}$	$R_{3,2^4}^{6,0}$	$R_{3^2}^{6,4}$	$R_{3^2}^{6,3}$	$R_{3^2}^{6,2}$
617	4316	14970	26337	1	34	399	2022	4735	1	27	229	751	1	35	430

$R_{3^2}^{6,1}$	$R_{3^2}^{6,0}$	$R_{3^2,2}^{6,3}$	$R_{3^2,2}^{6,2}$	$R_{3^2,2}^{6,1}$	$R_{3^2,2}^{6,0}$	$R_{3^2,2^2}^{6,2}$	$R_{3^2,2^2}^{6,1}$	$R_{3^2,2^2}^{6,0}$	$R_{3^2,2^3}^{6,1}$	$R_{3^2,2^3}^{6,0}$	$R_{3^3}^{6,1}$	$R_{3^3}^{6,0}$	$R_{3^3,2}^{6,0}$
4681/2	12141/2	1	28	254	1883/2	1	21	127	1	13	1	15	1

$R_4^{6,4}$	$R_4^{6,3}$	$R_4^{6,2}$	$R_4^{6,1}$	$R_4^{6,0}$	$R_{4,2}^{6,3}$	$R_{4,2}^{6,2}$	$R_{4,2}^{6,1}$	$R_{4,2}^{6,0}$	$R_{4,2^2}^{6,2}$	$R_{4,2^2}^{6,1}$	$R_{4,2^2}^{6,0}$	$R_{4,2^3}^{6,1}$	$R_{4,2^3}^{6,0}$	$R_{4,2^4}^{6,0}$
1	36	463	2632	7038	1	29	280	2223/2	1	22	146	1	15	1

$R_{4,3}^{6,1}$	$R_{4,3}^{6,0}$	$R_{4,3,2}^{6,0}$	$R_5^{6,0}$
1	41/2	1	1

then inductively subtracting the maps from reducible curves. A short Maple program computing these numbers (based on one by Göttsche) is available from the author. Table 5 gives all numbers of maps from possibly disconnected curves for degree 6. We use "R" rather than "N" to remind the reader that the source may be reducible. (The generating function for such numbers is the exponential of the Gromov–Witten potential, see Sect. 6.4.) Note that these values need not be integral, as some such maps have nontrivial automorphisms.

9.2. The cubic surface, $s = 6$

By deformation-invariance of Gromov–Witten invariants, we can compute the invariants on the surface X that is $\mathbb{P}^2$ blown up along 6 distinct points $q_1, \dots, q_6$ on a smooth conic C. If G is the proper transform of C, (X, G) is almost Fano, and we can use Theorem 4.2 to compute the invariants of X by counting curves. This is the same as counting irreducible curves in $\mathbb{P}^2$ with fixed multiple points at $q_1, \dots,$ q_6 and through an appropriate number of other fixed general points, which we can do using Theorem 6.8 applied to $(\mathbb{P}^2, C)$.

As an example, we compute the number of rational sextic curves in the plane with six nodes at fixed points $q_1, \dots, q_6$, and passing through five other fixed points $p_1, \dots, p_5$, where all the points are in general position. (This is the Gromov–Witten invariant $N^{6,0}_{2^6}$ of the cubic surface, see Sect. 9.1.1 for notation.) [DI] p. 119 gives this number as 2376, while [GöP] p. 25 gives the number as 3240. Göttsche and Pandharipande checked their number using different recursive strategies.

By Theorem 4.2, this invariant is the sum of three contributions.

1. Those (irreducible) rational sextics with six fixed nodes $q_1, \dots, q_6$ lying on a conic, passing through $p_1, \dots, p_5$. By Theorem 6.8 (and some computation), this number is 2002.
2. A stable map $\pi : C \to \mathbb{P}^2$ where C has two irreducible rational components C_0 and C_1 joined at one point, π maps C_1 isomorphically to E, and π maps C_0 to an irreducible rational quartic through $q_1, \dots, q_6$ (which lie on a conic) and $p_1, \dots, p_5$. The image of the node $C_0 \cap C_1$ is one of the two points $\pi(C_0) \cap E \setminus \{q_1, \dots, q_6\}$. By Theorem 6.8, there are 616 such quartics. There are two choices for the image of the node $C_0 \cap C_1$, so the contribution is 1232.
3. A stable map $\pi : C \to \mathbb{P}^2$ where C has three irreducible rational components C_0, C_1, C_2, where C_1 and C_2 intersect C_0, π maps C_1 and C_2 isomorphically to E, and π maps C_0 isomorphically to the conic through $p_1, \dots, p_5$. There are 12 choices of pairs of images of the nodes $C_0 \cap C_1$ and $C_0 \cap C_2$, and we must divide by 2 as exchanging C_1 and C_2 preserves the stable map. This contribution is 6.

Therefore $N^0_{6,2^6} = 2002 + 1232 + 6 = 3240$, in agreement with [GöP].

9.3. An approach for the two remaining del Pezzo surfaces

To count curves on $\mathbb{P}^2$ blown up at $s = 7$ or 8 general points $q_1, \dots, q_s$, one might want to degenerate point conditions to lie on a fixed smooth cubic E through the s points. Although the surface $(\mathbb{P}^2, E)$ does not satisfy **P1.–P4.**, many of the arguments carry through without change. Probably the most significant problem is the calculation of "seed data", i.e. counting maps when $\Upsilon = 0$. One can check that this corresponds to counting maps from degree d rational curves to $\mathbb{P}^2$ with intersection with E specified by (Ω, β) with $\beta = e_k$ (there is one "loose" tangency r, although its position on E is actually specified up to a finite number of choices by the location of the points q_s, as $\mathcal{O}_E(\sum m_s q_s + kr) \cong \mathcal{O}_E(d)$). This can be loosely thought of as "counting rational curves on a log K3 surface", and hence potentially related to [YZ].

10. Earlier results

There has been a great deal of earlier work on counting curves on Hirzebruch or Fano surfaces, and this is only a partial, brief sketch. Undoubtedly some important work has been missed.

10.1. The surfaces $\mathbb{P}^2$, $\mathbb{F}_0$ and $\mathbb{F}_1$ are convex, so the ideas of [KM] allow one to count (irreducible) rational curves in all divisor classes on these surfaces (see [DI] for further discussion). Di Francesco and Itzykson calculated the genus 0 Gromov–Witten invariants of the plane blown up at up to six points in [DI], Sect. 3.3. Kleiman gave recursions for all del Pezzo surfaces, and for $\mathbb{F}_0$ and $\mathbb{F}_1$ ([K1] Sect. 6). Ruan and Tian gave recursive formulas for the genus 0 Gromov–Witten invariants of Fano surfaces, and indicated their enumerative significance ([RuT] Sect. 10). Göttsche and Pandharipande later derived recursive formulas for the genus 0 Gromov–Witten invariants of the plane blown up at any number of points ([GöP]).

10.2. The algorithms [CH3] and [R1] count degree d genus g plane curves, and hence also count $N^g_{\mathbb{F}_1}(dS) = N^g_{\mathbb{F}_1}((d-1)S + F)$ (as defined in Sect. 8.2).

10.3. Recursions for curves of any genus in $X = \mathbb{F}_1$ were given in [R1]. The case of $\mathbb{F}_0$ is similar and was worked out by Ran's student Y. Choi (manuscript in preparation). As $\mathbb{F}_n$ may be degenerated to a union of $\mathbb{F}_{n-1}$ and $\mathbb{F}_1$ meeting along a fiber, arguments similar to those in [R1] should count curves on any $\mathbb{F}_n$ ([R2]).

10.4. Abramovich and Bertram have proved several (unpublished) formulas counting irreducible rational curves in certain classes on $\mathbb{F}_n$ ([AB2]):

$$N^0_{\mathbb{F}_n}(2S+bF) = N^0_{\mathbb{F}_{n-2}}(2S+(b+2)F) - \sum_{l=1}^{n-1} \binom{2(n+b)+3}{n-l-1}\left(l^2(b+2)+\binom{l}{2}\right), \tag{AB1}$$

$$N^0_{\mathbb{F}_n}(2S) = 2^{2n}(n+3) - (2n+3)\binom{2n+1}{n}, \tag{AB2}$$

$$N^0_{\mathbb{F}_n}(2S+bF) = N^0_{\mathbb{F}_{n-1}}(2S+(b+1)F) - \sum_{l=1}^{n-1}\binom{2(n+b)+2}{n-l-1} l^2(b+2). \tag{AB3}$$

Their method for (AB0) (in Section 8.2) and (AB1) is to deform the surface $\mathbb{F}_n$ to $\mathbb{F}_{n-2}$. For (AB2) and (AB3), they relate curves on $\mathbb{F}_n$ to curves on $\mathbb{F}_{n-1}$.

The author has obtained the formula

$$\begin{aligned} N^g_{\mathbb{F}_n}(2S+bF) = N^g_{\mathbb{F}_{n-1}}&(2S+(b+1)F) \\ &- \sum_{f=0}^{n-g-1} \sum \binom{\alpha_1}{|\alpha|-g-1} \binom{|\alpha|}{\alpha_1,\dots,\alpha_n} \\ &\times \binom{2(n+b)+2+g}{f} I^{2\alpha} \end{aligned}$$

where the second sum is over all integers f and sequences α such that $I\alpha = n+b-f$, $|\alpha| = b+2+g$, $b < \alpha_1$. This generalizes (AB3) above. The author's method is to specialize a single point condition to E, then perform an elementary transformation to turn $\mathbb{F}_n$ into $\mathbb{F}_{n-1}$.

10.5. Caporaso and Harris (in [CH1] and [CH2]) obtained recursive formulas for $N^0_{\mathbb{F}_n}(aS+bF)$ when $n \leq 3$, and the remarkable result that $N^0_{\mathbb{F}_n}(2S)$ is the coefficient of t^n in $(1+t)^{2n+3}/(1-t)^3$. Coventry has generalized the "rational fibration method" of [CH2] and found a recursive formula for the number of rational curves in *any* class in $\mathbb{F}_n$ ([Co]). E. Kussell has recovered the Gromov–Witten invariants of $\mathbb{P}^2$ blown up at 2 points by the rational fibration method ([Ku]).

10.6. Kleiman and Piene have examined systems with a fixed number δ of nodes ([KP]). The postulated number of δ-nodal curves is given (conjecturally) by a polynomial. Vainsencher determined the entire polynomial for $\delta \leq 6$ ([Va]). Kleiman and Piene extended his work to $\delta \leq 8$, and gave new techniques to determine explicit conditions on the line bundle for the formulas to be enumerative.

10.7. Graber and Pandharipande's powerful technique of virtual localization ([GP]) can also be used to compute the Gromov–Witten invariants of *any* rational surface. (Deform the rational surface so that it is a toric variety.) However, the graph-theoretic sums involved are extremely cumbersome to calculate in practice, even in simple cases with the aid of a computer.

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