manuscripta math. 106, 213 – 217 (2001)

M. Fernández-López · E. García-Río · D.N. Kupeli · B. Ünal

A curvature condition for a twisted product to be a warped product

Received: 8 March 2001 / Revised version: 25 July 2001

Abstract. It is shown that a mixed Ricci-flat twisted product semi-Riemannian manifold can be expressed as a warped product semi-Riemannian manifold. As a consequence, any Einstein twisted product semi-Riemannian manifold is in fact, a warped product semi-Riemannian manifold.

1. Introduction

Twisted and warped product structures are widely used in geometry to construct new examples of semi-Riemannian manifolds with interesting curvature properties. (See, for example, [1,3,7]). Twisted product metric tensors, as a generalization of warped product metric tensors, have also been useful in the study of several aspects of submanifold theory, namely, in hypersurfaces of complex space forms [5], in Lagrangian submanifolds [2] and in curvature netted hypersurfaces [4], etc. In [8], the relations between the twisted and warped product structures in semi-Riemannian geometry are studied. There, essentially the following is proven (cf. [8, Proposition 3]):

Let g be a semi-Riemannian metric tensor on the manifold $M = B \times F$ and assume that the canonical foliations L_B and L_F intersect perpendicularly everywhere. Then g is the metric tensor of

(i) a doubly twisted product ${}_fB \times {}_bF$ if and only if L_B and L_F are totally umbilic foliations,
(ii) a twisted product $B \times_b F$ if and only if L_B is a totally geodesic foliation and L_F is a totally umbilic foliation,
(iii) a warped product $B \times_b F$ if and only if L_B is a totally geodesic foliation and L_F is a spheric foliation,
(iv) a usual product of semi-Riemannian manifolds if and only if L_B and L_F are totally geodesic foliations.

M. Fernández-López, E. García-Río: Department of Geometry and Topology, Faculty of Mathematics, Univ. Santiago, 15706 Santiago de Compostela, Spain. e-mail: xtedugr@usc.es

D.N. Kupeli, B. Ünal: Department of Mathematics, Atılım University, İncek, 06836 Ankara, Turkey

Mathematics Subject Classification (2000): 53C50, 53C20

In this note, we state a curvature condition, called the *mixed Ricci flatness,* relating (ii) and (iii) above. Note that the condition relating (ii) and (iii) in the above is the normal parallelism of the mean curvature vector field of the totally umbilic foliation L_F. Here we use the mixed Ricci flatness condition, that is, $\mathrm{Ric}(X, V) = 0$ for every $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$ to relate the cases (ii) and (iii) of the above. In order to achieve this result, we give the expressions for the Levi–Civita connection, curvature tensor, Ricci curvature and scalar curvature of a doubly twisted product of two semi-Riemannian manifolds without giving the straightforward but lengthly computations of these connection and curvatures. (The first two of them are also given in [8] in a slightly different form). Then by using the expression of the Ricci tensor of a doubly twisted product of two semi-Riemannian manifolds, we obtain the main result mentioned above.

Roughly speaking, our main result says that when we write physical models in general relativity, replacing a warped product by a twisted product is not a good way to make a generalization in order to incorporate some further physical features because at that time the spacetime is no longer mixed Ricci-flat.

2. Geometry of twisted products

Here we give some notation and terminology used throughout this note.

Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds of dimensions r and s, respectively, and let, $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ be the canonical projections. Also let $b : B \times F \to (0, \infty)$ and $f : B \times F \to (0, \infty)$ be smooth functions. Then the *doubly twisted product* of (B, g_B) and (F, g_F) with twisting functions b and f is defined to be the product manifold $M = B \times F$ with metric tensor $g = f^2 g_B \oplus b^2 g_F$ given by $g = f^2 \pi^* g_B + b^2 \sigma^* g_F$. For brevity in notation, we denote this semi-Riemannian manifold (M, g) by ${}_f B \times {}_b F$. In particular, if $f = 1$ then ${}_1 B \times {}_b F = B \times {}_b F$ is called the *twisted product* of (B, g_B) and (F, g_F) with twisting function b. Moreover, if b only depends on the points of B then $B \times {}_b F$ is called the *warped product* of (B, g_B) and (F, g_F) with warping function b. As a generalization of the warped product of two semi-Riemannian manifolds, ${}_f B \times {}_b F$ is called the *doubly warped product* of semi-Riemannian manifolds (B, g_B) and (F, g_F) with warping functions b and f if b and f only depend on the points of B and F, respectively.

Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds with Levi–Civita connections ∇^B and ∇^F, respectively, and let ∇ both denote the Levi–Civita connection and the gradient of the doubly twisted product ${}_f B \times {}_b F$ of (B, g_B) and (F, g_F) with twisting functions b and f. Also, let $k = \log(b)$ and $l = \log(f)$. Now if we denote the set of lifts of vector fields on B and F to $B \times F$ by $\mathfrak{L}(B)$ and $\mathfrak{L}(F)$, respectively and use the same notation for a vector field and for its lift for our convenience (see [7, p. 205]) then we have the following relations:

Proposition 1. *If $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$, then*

$$\nabla_X Y = \nabla^B_X Y + X(l)Y + Y(l)X - g(X, Y)\nabla l, \tag{2.1}$$

$$\nabla_X V = V(l)X + X(k)V. \tag{2.2}$$

Note that, in a doubly twisted product ${}_fB \times {}_bF$, the roles of the base manifold (B, g_B) and the fiber manifold (F, g_F) can be interchanged. In the above proposition, we stated the expression of the Levi–Civita connection for vector fields $X, Y \in \mathfrak{L}(B)$ in (2.1). Clearly, by making corresponding changes, one can easily see the expression of Levi–Civita connection for vector fields $V, W \in \mathfrak{L}(F)$ as in (2.1). In the rest of the paper, we only state the expressions of the geometric objects for vector fields in $\mathfrak{L}(B)$. Obviously, the analogue expressions hold for vector fields in $\mathfrak{L}(F)$ by making corresponding changes.

Now, for $X, Y \in \mathfrak{L}(B)$, define $\mathrm{h}_B^k(X, Y) = XY(k) - (\nabla_X^B Y)(k)$. First note that, if $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$, we have $XV(k) = VX(k)$ and the Hessian form h^k of k on ${}_fB \times {}_bF$ satisfies

$$\mathrm{h}^k(X, V) = XV(k) - X(k)V(l) - X(k)V(k),$$
$$h^k(X, Y) = \mathrm{h}_B^k(X, Y) - X(l)Y(k) - X(k)Y(l) + g(X, Y)g(\nabla k, \nabla l).$$

Let R_B and R_F be the curvature tensors of (B, g_B) and (F, g_F), respectively, and let R be the curvature tensor of ${}_fB \times {}_bF$. Then we have the following relations:

Proposition 2. *Let $X, Y, Z \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$. Then*

$$\begin{aligned} R(X, Y)Z = {} & R_B(X, Y)Z - g(Y, Z)(H^l(X) + X(l)\nabla l) \\ & + g(X, Z)(H^l(Y) + Y(l)\nabla l) \\ & + (\mathrm{h}_B^l(X, Z) - X(l)Z(l))Y - (\mathrm{h}_B^l(Y, Z) - Y(l)Z(l))X \qquad (2.3) \end{aligned}$$

$$R(X, Y)V = \mathrm{h}^l(X, V)Y - \mathrm{h}^l(Y, V)X + V(l)X(l)Y - V(l)Y(l)X \qquad (2.4)$$

$$\begin{aligned} R(X, V)Y = {} & \big(\mathrm{h}_B^k(X, Y) + X(k)Y(k) - X(l)Y(k) - X(k)Y(l)\big)V \\ & + \big(Y(k)V(l) - VY(l)\big)X + \big(V(l)\nabla l + H^l(V)\big)g(X, Y), \qquad (2.5) \end{aligned}$$

where H^l is the Hessian tensor of l on ${}_fB \times {}_bF$, i.e. $h^l(X, Y) = g(H^l(X), Y)$.

Now let Ric_B and Ric_F be the Ricci tensors of (B, g_B) and (F, g_F), respectively, and let Ric be the Ricci tensor of ${}_fB \times {}_bF$. Then we have the following relation:

Proposition 3. *Let $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$. Then*

$$\begin{aligned} \mathrm{Ric}(X, Y) = {} & \underset{B}{\mathrm{Ric}}(X, Y) + h^l(X, Y) + (1 - r)h_B^l(X, Y) + rX(l)Y(l) \\ & - g(X, Y)[\Delta l + g(\nabla l, \nabla l)] \\ & - s[h_B^k(X, Y) + X(k)Y(k) - X(l)Y(k) - X(k)Y(l)], \qquad (2.6) \end{aligned}$$

where $\Delta(l)$ is the Laplacian of l.

$$\mathrm{Ric}(X, V) = (1 - r)VX(l) + (1 - s)XV(k) + (n - 2)X(k)V(l). \qquad (2.7)$$

Next we use the following construction to express the scalar curvature of ${}_fB \times {}_bF$. Let $\{X_1, \cdots, X_r\}$ and $\{V_1, \cdots, V_s\}$ be orthonormal frames on open sets $U_B \subseteq$

B and $U_F \subseteq F$ of (B, g_B) and (F, g_F), respectively. Then $\{Z_1, \cdots, Z_n\}$ is an orthonormal frame on the open set $U_B \times U_F \subseteq B \times F$, where

$$Z_i = \begin{cases} \dfrac{X_i}{f} & \text{if } i \in \{1, \cdots, r\} \\ \dfrac{V_{i-r}}{b} & \text{if } i \in \{r+1, \cdots, n\}\,. \end{cases}$$

Here, as mentioned before, X_i and V_j denote the lifts of X_i and V_j to $B \times F$, respectively, and $n = r + s$.

Hence, if we define $\widetilde{\Delta}_B(k) = \sum_{i=1}^{r} g(Z_i, Z_i)\mathrm{h}^k(Z_i, Z_i)$ and also likewise, $\widetilde{\Delta}_F(k) = \sum_{j=r+1}^{n} g(Z_j, Z_j)\mathrm{h}^k(Z_j, Z_j)$, we can express the Laplacian $\Delta(k)$ of k on ${}_fB \times {}_bF$ as

$$\Delta(k) = \widetilde{\Delta}_B(k) + \widetilde{\Delta}_F(k)$$

Finally, for the scalar curvatures we have the following relation:

Proposition 4. *Let τ_B and τ_F be the scalar curvatures of (B, g_B) and (F, g_F), respectively. Then the scalar curvature of ${}_fB \times {}_bF$ satisfies:*

$$\begin{aligned} \tau = {} & \frac{\tau_B}{f^2} + \frac{\tau_F}{b^2} - r\Delta l - s\Delta k \\ & + (2-r)\widetilde{\Delta}_B l + (2-s)\widetilde{\Delta}_F k - s\widetilde{\Delta}_B k - r\widetilde{\Delta}_F l \\ & + r(r-2)g(\nabla l, \nabla l) + s(s-2)g(\nabla k, \nabla k) + 2rsg(\nabla l, \nabla k) \\ & + (2-r)f^2 g_B(\pi_*\nabla l, \pi_*\nabla l) + (2-s)b^2 g_F(\sigma_*\nabla k, \sigma_*\nabla k) \\ & - sf^2 g_B(\pi_*\nabla k, \pi_*\nabla k) - rb^2 g_F(\sigma_*\nabla l, \sigma_*\nabla l), \end{aligned} \tag{2.8}$$

where $\widetilde{\Delta}_F(l)$ and $\widetilde{\Delta}_B(l)$ are defined as in the above by replacing k with l.

3. Mixed Ricci flat twisted products

Next, let $B \times {}_bF$ be the twisted product of (B, g_B) and (F, g_F) with twisting function b. Recall that $B \times {}_bF$ *is called mixed Ricci-flat if* $\mathrm{Ric}(X, V) = 0$ *for all* $X \in \mathfrak{L}(B)$ *and* $V \in \mathfrak{L}(F)$.

Theorem 1. *Let $B \times {}_bF$ be a twisted product of (B, g_B) and (F, g_F) with twisting function b and* $\dim F > 1$. *Then,* $\mathrm{Ric}(X, V) = 0$ *for all $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$ if and only if $B \times {}_bF$ can be expressed as a warped product, $B \times {}_\Phi F$ of (B, g_B) and $(F, \widetilde{g_F})$ with a warping function Φ, where $\widetilde{g_F}$ is a conformal metric tensor to g_F.*

Proof. Note that, if $M = B \times {}_bF$ is a semi-Riemannian twisted product of (B, g_B) and (F, g_F) with twisting function b, then by using Proposition 3 and the properties of the Hessian forms of 1 and B on ${}_1B \times_b F$, we obtain $\mathrm{Ric}(X, V) = (1-s)XV(k)$, where $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$.

Hence, if $\mathrm{Ric}(X, V) = 0$ for all $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$ then it follows that $XV(k) = 0$ and $VX(k) = 0$. Now, $XV(k) = 0$ implies that $V(k)$ only depends on the points of F, and likewise, $VX(k) = 0$ implies that $X(k)$ only depends on the points of B. Thus k can be expressed as a sum of two functions ϕ and ψ which are defined on B and F, respectively, that is, $k(p,q) = \phi(p) + \psi(q)$ for any $(p,q) \in B \times F$. Hence $b = \exp(\phi)\exp(\psi)$, that is, $b(p,q) = \Phi(p)\Psi(q)$, where $\Phi = \exp(\phi)$ and $\Psi = \exp(\psi)$ for any $(p,q) \in B \times F$. Thus we can write, $g = g_B \oplus \Phi^2 \widetilde{g_F}$, where $\widetilde{g_F} = \Psi^2 g_F$, that is, the twisted product $B \times_b F$ can be expressed as a warped product $B \times_\Phi F$, where the metric tensor of F is $\widetilde{g}_F$ given above. The converse is obvious from Proposition 3. $\square$

Corollary 1. *Let $B \times_b F$ be a twisted product of (B, g_B) and (F, g_F) with twisting function b and* $\dim F > 1$. *If $B \times_b F$ is Einstein, then $B \times_b F$ can be expressed as a warped product as in the statement of Theorem 1.*

As an immediate application of Corollary 1, the Einstein generalized Taub-NUT metrics discussed in [6], indeed, can be expressed as warped product metric tensors.

Remark 1. Note that, both necessary and sufficient conditions of Theorem 1 fail in the case of doubly twisted and doubly warped products as shown in Proposition 3.

Remark 2. Most physically realistic spacetimes, such as Robertson–Walker and Kruskal spacetimes, satisfy the mixed Ricci flatness condition with respect to their product manifold structure. (See [7]). In fact, this is a physical requirement induced by the stress-energy tensor on these spacetimes. Note that these spacetimes are warped products. Now, in the view of Theorem 1, clearly this is the maximal generality since these spacetimes cannot be further generalized to twisted products in a nontrivial way.

Acknowledgements. M. Fernández-López E. García-Río were partially supported by projects DGESIC PB97-0504-C02-01 and PGIDT99 PXI 20703B (Spain).

References

[1] Beem, J.K., Ehrlich, P.E. and Easley, K.L.: Global Lorentzian Geometry. Marcel Dekker, Second Edition, New York, 1996

[2] Chen, B.Y., Dillen, F., Verstraelen, L. and Vranken, L.: Lagrangian isometric immersions of a real-space-form $M^n(c)$ into a complex-space-form $\tilde{M}^n(4c)$. Math. Proc. Cambridge Philos. Soc. **124**, 107–125 (1998)

[3] Ganchev, G. and Mihova, V.: Riemannian manifolds of quasi-constant sectional curvatures. J. Reine Angew. Math. **522**, 119–141 (2000)

[4] Koike, N.: The decomposition of curvature netted hypersurfaces. Geom. Dedicata **54**, 1–11 (1995)

[5] Lohnherr, M. and Reckziegel, H.: On ruled real hypersurfaces in complex space forms. Geom. Dedicata **74**, 267–286 (1999)

[6] Miyake, Y.: Self-dual generalized Taub-NUT metrics. Osaka J. Math. **32**, 659–675 (1995)

[7] O'Neill, B.: Semi-Riemannian Geometry With Applications to Relativity. New York: Academic Press, 1983

[8] Ponge, R. and Reckziegel, H.: Twisted Products in Pseudo-Riemannian Geometry. Geom. Dedicata **48**, 15–25 (1993)