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The multiplicity of solutions in non-homogeneous boundary value problems

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Abstract. We use a method recently devised by Bolle to establish the existence of an infinite number of solutions for various non-homogeneous boundary value problems. In particular, we consider second order systems, Hamiltonian systems as well as semi-linear partial differential equations. The non-homogeneity can originate in the equation but also from the boundary conditions. The results are more satisfactory than those obtained by the standard "*Perturbation from Symmetry*" method that was developed – in various forms – in the early eighties by Bahri–Berestycki, Struwe and Rabinowitz.

1. Introduction

Equivariant variational methods often yield multiple solutions for partial differential equations and Hamiltonian systems that are invariant under certain group actions. However, there are no satisfactory general answers yet to the cases where the group symmetry is broken by some non-equivariant – and even linear – perturbations. A partially successful method to deal with such problems was devised in the early eighties by Bahri–Berestycki [Ba-Be1,2] and Struwe [S1,2]. The variational principle underlying these results was later formulated by Rabinowitz [R]. The main idea being to think of the non-symmetric functional I under study as a perturbation of its symmetric part I_0 and then to estimate how the growth rate of the critical levels of I_0 is affected by the perturbation from symmetry $I - I_0$.

This method has been somewhat successful in dealing with certain "lower order" perturbations like the one resulting from non-homogeneous Hamiltonian systems and second order systems [Ba-Be2]. More recently, the authors of [E-G-T] had to deal with a new type of perturbation from symmetry which appeared in their study of a second order system with non-homogeneous boundary conditions. Because of the high order of the perturbation term, the method described above (to which we shall refer thereafter as the "standard method") did not yield a satisfactory result. To remedy that, P. Bolle introduced in [B] a more refined version that succeeded in improving the result in [E-G-T] on the Bolza problem.

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The new approach deals with I as the end-point of a continuous path of functionals $(I_\theta)_{\theta\in[0,1]}$ which starts at the symmetric functional I_0. Bolle's abstract result roughly says that the preservation of the min-max critical levels along the path (I_θ) depends only on the variations $\frac{\partial}{\partial\theta}I_\theta(u)$ *at the critical points* u *of* I_θ. As we shall see in this paper, better estimates can be obtained at such points since they often obey certain conservation laws as they are solutions of the corresponding Euler–Lagrange equations.

In this paper, we try this approach on three problems: a second order system, a semi-linear pde and a Hamiltonian system. We show how it can improve old results and prove new ones about the existence of multiple solutions for ordinary or partial differential equations in the absence of symmetry. We also include a fourth example (a non-linear wave equation) which illustrates how the failure to exploit conservation laws to obtain estimates, can lead to unsatisfactory results.

We now state the main results of this paper.

Theorem 1.1 (Non-homogeneous Bolza problem**).** *Suppose* $V \in C^2(\mathbf{R}^n, \mathbf{R})$ *is even and that there exists* $p > 2$ *such that:*

$$0 < pV(x) \le \langle \nabla V(x), x\rangle \text{ for all } |x| \text{ large.}$$

Then for any $f \in C([0,T], \mathbf{R}^n)$*, the Bolza problem*

$$\begin{cases} \ddot{x} + \nabla V(x) = f(t) & x \in \mathbf{R}^n \\ x(0) = x_0, & x(T) = x_1. \end{cases} \tag{P1}$$

has infinitely many solutions.

As noted in [E-G-T], the "standard method" yields the above result only when $2 < p < 4$. The case where $p < 2$ is different and has been dealt with by Clarke–Ekeland [C-E].

Theorem 1.2 (Non-homogeneous semi-linear equations**).** *Let* Ω *be an open bounded subset of* $\mathbf{R}^n$ *(of class* C^2*) and* $u_0 \in C^2(\partial\Omega, \mathbf{R})$. *Assume* $1 < p < \frac{n+1}{n-1}$*, then for any* $f \in C(\bar{\Omega}, \mathbf{R})$*, the problem*

$$\begin{cases} \Delta u + |u|^{p-1}u = f & \text{in } \Omega \\ \qquad u = u_0 & \text{on } \partial\Omega \end{cases} \tag{P2}$$

has infinitely many solutions.

Again, the "standard method" yields the above result for $1 < p < \frac{n+2}{n}$. We note that if $u_0 = 0$, Bolle's method yields the same result as the standard one which was last used by Bahri-Lions [Ba-L] to improve the range of p up to $\frac{n}{n-2}$. It is still an open problem whether problem (P_2) (even when $u_0 = 0$) has an infinite number of solutions for all p all the way up to $\frac{n+2}{n-2}$.

Theorem 1.3 (Non-homogeneous Hamiltonian systems**).** *Let* H *be a function in* $C^2(\mathbf{R}^{2n}, \mathbf{R})$ *satisfying*

(h1) *H is even,*
(h2) $\exists \mu > 2$ *such that* $H'(x) \cdot x \geq \mu H(x) > 0$ *for* $|x|$ *large,*
(h3) *There are r and s such that* $1 < r \leq s < 2r + 1$ *and*

$$A|x|^{r+1} - B \leq H(x) \leq C|x|^{s+1} + D,$$

where A, B, C, D are positive constants.

Let q_0 and q_1 be two given vectors in $\mathbf{R}^n$, then for any two functions f_1, f_2 in $C^1([0,1]; \mathbf{R}^n)$, the following problem

$$\begin{cases} \frac{dq}{dt} = \frac{\partial H}{\partial p}(q, p) + f_1(t) \\ \frac{dp}{dt} = -\frac{\partial H}{\partial q}(q, p) + f_2(t) \\ q(0) = q_0 \text{ and } q(1) = q_1. \end{cases} \tag{P3}$$

has an infinite number of solutions.

Assumptions (h2) and (h3) were used by Bahri and Berestycki in [Ba-Be2] to prove the existence of infinitely many periodic solutions to the systems above, where f_1 and f_2 are 1-periodic. Here we deal with the existence of orbits which connect the subspaces $\{q = q_0\}$ and $\{q = q_1\}$ in a given time period. Note that in terms of symplectic geometry, $\{q = q_0\}$ and $\{q = q_1\}$ are two Lagrangian submanifolds of $\mathbf{R}^{2n}$ (endowed with the canonical symplectic structure $\sum_1^n dq_i \wedge dp_i$), and we could raise a more general question, namely the existence of orbits connecting two given Lagrangian submanifolds of $\mathbf{R}^{2n}$ in given time.

Finally, we describe the following highly unsatisfactory result which we obtain by only using the “standard method”.

Theorem 1.4 (Non-homogeneous semi-linear wave equations**).** *For any given x_0 and x_1 in $\mathbf{R}$ and any continuous f, the following equation:*

$$\begin{cases} u_{tt} - u_{xx} + |u|^{p-1}u = f(x,t) \\ u(0,t) = x_0, \quad u(\pi, t) = x_1 \\ u(x, t + 2\pi) = u(x,t). \end{cases} \tag{P4}$$

has an infinite number of solutions provided $1 < p < 2$.

Note that for the homogeneous case ($x_0 = x_1 = 0$), the multiplicity holds for any $p > 1$ (Tanaka [T2]). We were led here to this restriction on p because we had to use the estimates of the “standard method” which do not take advantage of the fact that the energy estimates are only needed for the critical points of the associated functional.

2. Preservation of critical levels along a path of functionals

In this section we recall Bolle's method for dealing with problems with broken symmetry. Let E be a Hilbert space and consider a C^2 functional $I = [0,1] \times E \to \mathbf{R}$. We denote by $\langle , \rangle$ and $\|.\|$ the scalar product in E and the associated norm. For $\theta \in [0,1]$ we shall use the abbreviation I_θ for $I(\theta, .)$. We make the following hypotheses:

(H1) I satisfies the Palais–Smale condition, which means here that for every sequence $((\theta_n, x_n))$ (with $\theta_n \in [0,1]$, $x_n \in E$) such that $\|I'_{\theta_n}(x_n)\| \to 0$ as $n \to +\infty$ and $I(\theta_n, x_n)$ is bounded, there is a subsequence converging in $[0,1] \times E$. (The limit (θ, x) then satisfies $I'_\theta(x) = 0$).

(H2) For all $b > 0$ there is a constant $C_1(b)$ such that:

$$|I_\theta(x)| < b \ \text{ implies } \ |\tfrac{\partial}{\partial \theta} I(\theta, x)| \le C_1(b)(\|I'_\theta(x)\| + 1)(\|x\| + 1).$$

(H3) There exist two continuous functions f_1 and $f_2 : [0,1] \times \mathbf{R} \to \mathbf{R}$ with $f_1 \le f_2$, that are Lipschitz-continuous relative to the second variable and such that, for all critical points x of I_θ,

$$f_1(\theta, I_\theta(x)) \le \frac{\partial}{\partial \theta} I(\theta, x) \le f_2(\theta, I_\theta(x)).$$

(H4) There are two closed subsets of E, A and $B \subset A$, such that:
(i) I_0 has an upper-bound on A and $\lim\limits_{\substack{|x| \to +\infty \\ x \in A}} (\sup\limits_{\theta \in [0,1]} I_\theta(x)) = -\infty$.
(ii) $c_{A,B} > c_B$ where $c_B = \sup_B I_0$ and $c_{A,B} = \inf\limits_{g \in \mathcal{D}_B} \sup\limits_{g(A)} I_0$ where for some $R > 0$, $\mathcal{D}_B = \{g \in C^0(E, E);\ g(x) = x$ for $x \in B$ and for $x \in E$ with $\|x\| \ge R\}$.

Denote by ψ_i ($i = 1, 2$) the functions defined on $[0,1] \times \mathbf{R}$ by

$$\begin{cases} \psi_i(0, s) = s \\ \frac{\partial}{\partial \theta} \psi_i(\theta, s) = f_i\,(\theta, \psi_i(\theta, s))\,. \end{cases}$$

Note that ψ_1 and ψ_2 are continuous and that for all $\theta \in [0,1]$, $\psi_1(\theta, \cdot)$ and $\psi_2(\theta, \cdot)$ are non-decreasing on $\mathbf{R}$. Moreover, since $f_1 \le f_2$, we have $\psi_1 \le \psi_2$.

In the sequel, we set $\bar{f}_i(s) = \sup_{\theta \in [0,1]} |f_i(\theta, s)|$, $i = 1, 2$. Here is the result of Bolle [B].

Theorem 2.1 (Bolle)**.** *Assume that* $I = [0,1] \times E \to \mathbf{R}$ *is* C^2 *and satisfies* (H1), (H2), (H3) *and* (H4). *If* $\psi_2(1, c_B) < \psi_1(1, c_{A,B})$, *then* I_1 *has a critical point at a level* $\bar{c}$ *such that:* $\psi_1(1, c_{A,B}) \le \bar{c} \le \psi_2(1, c_{A,B})$.

Assume now $E = E_- \oplus E_+$ and let $(E_n)_n$ be an increasing sequence of subspaces of E such that $E_0 = E_-$ and $E_{n+1} = E_n \oplus \mathbf{R}e_{n+1}$. If E_- is finite dimensional, set:

$$\mathcal{G} = \{g \in C(E; E);\ \ g \text{ is odd and } g(x) = x \text{ for } x \in E \text{ and } \|x\| \text{ large}\}.$$

and $c_k = \inf_{g \in \mathcal{G}} \sup_{g(E_k)} I_0$.

Theorem 2.2. *Assume I satisfies hypothesis (H1),(H2),(H3). In addition, we suppose*

(H4') *I_0 is even and for any finite dimensional subspace W of E, we have:* $\sup_{\theta\in[0,1]} I(\theta, y) \to -\infty$ *as* $y \in W$ *and* $\|y\| \to \infty$.

Then, there is $C > 0$ such that for every k:

(1) *Either I_1 has a critical level $\bar{c}_k$ with $\psi_2(1, c_k) < \psi_1(1, c_{k+1}) \le \bar{c}_k$.*
(2) *Or $c_{k+1} - c_k \le C((\bar{f}_1(c_{k+1}) + \bar{f}_2(c_k) + 1)$.*

Proof. Suppose that $\psi_2(1, c_k) < \psi_1(1, c_{k+1})$, we shall show that we are then in the context of Theorem 2.1 above and that I_1 has a critical level $\ge \psi_1(1, c_{k+1})$. Indeed, find $\epsilon > 0$ such that $\psi_2(1, c_k + \epsilon) < \psi_1(1, c_{k+1})$. Fix $g \in \mathcal{G}$ such that $\sup_{g(E_k)} I_0 < c_k + \epsilon$. Let $E^+_{k+1} = E_k \oplus \mathbf{R}^+ e_{k+1}$ and set $A_k = g(E^+_{k+1})$ and $B_k = g(E_k)$.

We only need to verify (H4)(ii), that is:

$$\psi_1(1, c_{A_k,B_k}) > \psi_2(1, \sup_{B_k} I_0)$$

where $c_{A_k,B_k} = \inf_{\ell\in\mathcal{D}_{B_k}} \sup_{\ell(A_k)} I_0$ and $\mathcal{D}_{B_k}$ is defined as in Theorem 2.1.

Indeed if $\ell \in \mathcal{D}_{B_k}$, the function $m = \ell \circ g_{|E^+_{k+1}}$ is odd on E_k and therefore it trivially extends to an odd function $\bar{m}$ on E_{k+1} (hence to an odd function on the whole space satisfying $m(x) = x$ for large $|x|$, by the Tietze theorem). We now have since I_0 is even and $\bar{m}$ is odd:

$$\sup_{\ell(A_k)} I_0 = \sup_{m(E^+_{k+1})} I_0 = \sup_{\bar{m}(E^+_{k+1})} I_0 = \sup_{\bar{m}(E_{k+1})} I_0 \ge c_{k+1}.$$

It follows that $c_{A_k,B_k} \ge c_{k+1}$ and therefore

$$\psi_1(1, c_{A_k,B_k}) \ge \psi_1(1, c_{k+1}) > \psi_2(1, c_k + \epsilon) \ge \psi_2(1, \sup_{B_k} I_0).$$

Theorem 2.1 then applies to yield a critical level for I_1 at level $\bar{c}_k$ such that $\psi_2(1, c_k) < \psi_1(1, c_{k+1}) \le \bar{c}_k \le \psi_2(1, c_{A_k,B_k})$, which is the first alternative.

Otherwise, $\psi_2(1, c_k) \ge \psi_1(1, c_{k+1})$. Now, by the Lipschitz continuity of f_i w.r.t. the second argument, $|\psi_i(1, s) - s| \le C_i \bar{f}_i(s) + K_i$ for $s \ge 0$ and $i = 1, 2$, where C_i, K_i are positive constants. Hence we get that:

$$\begin{aligned} c_{k+1} - c_k &\le \psi_1(1, c_{k+1}) + C_1 \bar{f}_1(c_{k+1}) + K_1 - \psi_2(1, c_k) + C_2 \bar{f}_2(c_k) + K_2 \\ &\le C_1 \bar{f}_1(c_{k+1}) + C_2 \bar{f}_2(c_k) + K_1 + K_2. \end{aligned}$$

which completes the proof of Theorem 2.2. □

Remark 2.1. The class $\mathcal{D}_{A,B}$ in Theorem 2.1 may be replaced by a less general class of deformations, containing the (possibly truncated) gradient flows of the functionals $I(\theta, \cdot)$. If the gradient of I_θ is of the form $U + K$ with U an invertible linear operator and K compact, then the subspaces $(E_k)_k$ of Theorem 2.2 need not be finite dimensional as long as the class $\mathcal{G}$ is suitably defined. We shall see an example in Section 5.

3. The nonhomogeneous Bolza problem

We first revisit the following non-homogeneous second order system:

$$\begin{cases} \ddot{x} + \nabla V(x) = f(t) & x \in \mathbf{R}^n \\ x(0) = x_0, & x(T) = x_1. \end{cases} \tag{P1}$$

As was done in [E-G-T], we introduce the change of variable: $x = z + u$, where $z : [0, T] \to \mathbf{R}^n$, $z(t) = x_0 + \frac{t}{T}(x_1 - x_0)$ in such a way that u satisfies:

$$\begin{cases} \ddot{u} + \nabla V(u + z(t)) = f(t) \\ u(0) = u(T) = 0 \end{cases} \tag{P'1}$$

In order to use Bolle's method we introduce the function: $I : [0,1] \times H \to \mathbf{R}$ defined by:

$$I(\theta, u) = I_\theta(u) = \int_0^T \left(\frac{1}{2}|\dot{u}|^2 - V(u + \theta z) + \theta f u\right) dt$$

where: $H = H_0^1(0, T)$. We shall denote $J_\theta(u) := \frac{\partial}{\partial\theta} I_\theta(u)$.

In the next lemmas, we show that I satisfies the hypothesis (H_1), (H_2) and (H_3).

Lemma 3.1. *There exists positive constants $(C_i)_{i=1}^5$ such that for any $y \in H$:*

(i) $\|y\|^2 \le C_1 \left(pI_\theta(y) - \langle I'_\theta(y), y\rangle - \theta J_\theta(y) - \theta(p-2)\int_0^T f(t) y\, dt\right)$.

(ii) *For all $a > 0$, there exists $K(a) > 0$ such that*

$$|J_\theta(y)| \le \frac{C_2}{a}\left(\|y\|^2 + \|I'_\theta(y)\|\|y\|\right) + C_3\|I'_\theta(y)\| + K(a).$$

(iii) $\|y\|^2 \le C_4(\|I'_\theta(y)\|^2 + 1) + C_5|I_\theta(y)|$.

Proof. The proof of this lemma follows the same lines as Lemma 3.1 in [B]. We repeat the argument here for completeness. We have

$$J_\theta(y) = -\int_0^T \nabla V(y + \theta z) z + \int_0^T f(t) y\, dt \quad \text{and,}$$

$$\begin{aligned} \langle I'_\theta(y), y\rangle &= \int_0^T |\dot{y}|^2 - \int_0^T \nabla V(y + \theta z) y + \theta \int_0^T f(t) y\, dt \\ &= \int_0^T |\dot{y}|^2 - \int_0^T \nabla V(y + \theta z)(y + \theta z) \\ &\quad + \theta \int_0^T \nabla V(y + \theta z) z + \int_0^T f(t) y\, dt \\ &\le \int_0^T |\dot{y}|^2 - p\int_0^T V(y + \theta z) - \theta J_\theta(y) + 2\theta \int_0^T f(t) y\, dt \\ &\le pI_\theta(y) - (\frac{p}{2} - 1)\int_0^T |\dot{y}|^2 - \theta(p-2)\int_0^T f(t) y\, dt. \end{aligned}$$

Since $\frac{p}{2} - 1 > 0$, we get estimate (i).

To get (ii) set $r = \frac{a^2 T}{||y||^2 + 2a^2}$. Note that

$$t \in [0, r) \cup (T - r, T] \Rightarrow |y(t)| \leq \sqrt{r}||y|| \leq a\sqrt{T} \tag{3.1}$$

since $y(0) = y(T) = 0$.

Let $h \in H$ be defined by $h(t) = z(r)\frac{t}{r}$ in $[0, r]$, $h(t) = z(t)$ in $[r, T - r]$, and $h(t) = \frac{T-t}{r}$ for $t \in [T - r, T]$. We have

$$\begin{aligned} J_\theta(y) = &-\int_0^T \langle \dot{y}, \dot{h} \rangle \, dt + \langle I'_\theta(y), h \rangle - \theta \int_0^T f(t)h - \int_0^r \nabla V(y + \theta z)(z - h) \\ &- \int_{T-r}^T \nabla V(y + \theta z)(z - h). \end{aligned}$$

Since V is C^1, from (3.1) we get

$$\left| \int_0^r \nabla V(y + \theta z)(z - h) \right| + \left| \int_{T-r}^T \nabla V(y + \theta z)(z - h) \right| \leq C(a)$$

Hence

$$\| J_\theta(y) \| \leq ||y|| ||h|| + ||I'_\theta(y)|| ||h|| + K||h||$$

Now;

$$||h|| \leq \frac{K}{\sqrt{r}} + K \leq \frac{C}{a}||y|| + K$$

From which (ii) follows. Finally from (i) and (ii) we get

$$\begin{aligned} ||y||^2 \leq \ &I_\theta(y) - \langle I'_\theta(y), y \rangle + \frac{C}{a}||y||^2 + \frac{C}{a}||I'_\theta|| ||y|| \\ &+ c||I'_\theta|| + K(a) + C||y|| \end{aligned}$$

From which (iii) follows. Now we use this result to prove:

Lemma 3.2. *There exist constants $a, b > 0$ such that whenever u is a critical point of I_θ, we have:*

$$-a\Big(|I_\theta(u)|^2 + 1\Big)^{1/4} \leq J_\theta(u) = \frac{\partial}{\partial \theta} I_\theta(u) \leq b\Big(|I_\theta(u)|^2 + 1\Big)^{1/4}.$$

Proof.

$$J_\theta(u) = \int_0^T (-\nabla V(u + \theta z)z + f(t)u) \, dt.$$

We need to estimate $J_\theta(u)$ at a point u, such that $I'_\theta(u) = 0$, i.e.

$$\ddot{u} + \nabla V(u + \theta z) = \theta f(t) \qquad t \in [0, T].$$

$$\begin{aligned} J_\theta(u) &= \int_0^T \ddot{u}(t)z(t) - \theta f(t)z(t) + f(t)u(t) \, dt \\ &= \dot{u}(T)z(T) - \dot{u}(0)z(0) + \int_0^T f(t)u \, dt - \int_0^T \theta f(t)z \, dt \end{aligned}$$

which implies:

$$|J_\theta(u)| \le c\,(|\dot u(T)| + |\dot u(0)| + \|u\| + 1)$$

Let $E(t) = \frac{1}{2}|\dot u + \theta \dot z|^2 + V(u+\theta z)$. Then $\dot E(t) = (\ddot u + \nabla V(u+\theta z))\,(\dot u + \theta \dot z) = \theta(\dot u + \theta \dot z) f(t)$. Since V is bounded below on $\mathbf{R}^n$:

$$\begin{cases} |\dot u(T)|^2 \le E(T) - V\,(u(T) + \theta z(T)) \text{ and} \\ |\dot u(0)|^2 \le E(0) - V\,(u(0) + \theta z(0)) \text{ imply} \\ |\dot u(T)| + |\dot u(0)| \le C\left(\sqrt{\max\limits_{0\le t\le 1} |E(t)|} + 1\right) \end{cases}$$

On the other hand, for $t_1, t_2 \in [0,T]$, the formula $E(t_2) - E(t_1) = \int_{t_1}^{t_2} E'(s)\,ds$ implies that

$$T E(t_2) - \int_0^T E(t)\,dt = \int_0^T \int_{t_1}^{t_2} E'(s)\,ds\,dt_1$$

which in turn gives:

$$E(t) = \frac{1}{T}\int_0^T E(s)\,ds + \frac{1}{T}\int_0^T \int_r^t E'(s)\,ds\,dr \qquad \forall t \in [0,T].$$

From this last inequality we derive:

$$\begin{aligned} |E(t)| &\le \left|\frac{1}{T}\int_0^T E(s)\,ds\right| + \int_0^T |E'(s)|\,ds \\ &\le \left|\frac{1}{T}\int_0^T \left(\frac{1}{2}|\dot u + \theta \dot z|^2 + V(u+\theta z)\right) dt\right| + \theta \int_0^T |\dot u + \theta \dot z|.|f(t)|\,dt \\ &\le \left|\frac{1}{T}\left(-I_\theta(u) + \theta\int_0^T f(t) u\,dt + \frac{1}{2}\int_0^T |\dot u + \theta \dot z|^2\,dt + \frac{1}{2}\int_0^T |\dot u|^2\,dt\right)\right| \\ &\quad + \|u\| + C \\ &\le \frac{1}{T}|I_\theta(u)| + C_1\|u\|^2 + C_2. \end{aligned}$$

Now using Lemma 3.1 (iii) and recalling that $I'_\theta(u) = 0$ we have:

$$|E(t)| \le C_3|I_\theta(u)| + C_4, \ \ \forall t \in [0,T]$$

which implies:

$$|\dot u(T)|^2 + |\dot u(0)|^2 \le C_5\,(|I_\theta(u)| + 1)$$

and we finally get:

$$|J_\theta(u)| \le C\Big(I_\theta(u)|^2 + 1\Big)^{1/4}.$$

The proof of Lemma 3.2 is complete. $\square$

End of proof of Theorem 1.1. We first prove that I satisfies the hypothesis of Theorem 2.2. It is clear that I_0 is even and that, in view of the above lemmas, it satisfies hypothesis (H1), (H2) and (H3). Moreover, the condition on V implies the existence of constants C_1 and C_2 such that $V(x) \geq C_1|x|^p - C_2$. Hence

$$\begin{aligned} I(\theta, y) &\leq \frac{1}{2}\|y\|^2 - C_1 \int_0^T |y + \theta z|^p \, dt + C_2 + \theta \|y\|_1 \|f\|_\infty \\ &\leq \frac{1}{2}\|y\|^2 - C_1 \|y\|_p^p + C_3 + C_4 \|y\|_1. \end{aligned}$$

which implies that for every finite dimensional subspace W of E the quantity $\sup_{\theta \in [0,1]} I(\theta, y)$ goes to $-\infty$ since all norms are equivalent on W.

Let now $(\mathbf{e}_i)_{1 \leq i \leq n}$ be an orthonormal basis of $\mathbf{R}^n$. For $k \in \mathbf{N}$, let $a_k \in E$ be defined by $a_k(t) = \sin \pi (q(k) + 1)t.\mathbf{e}_{r(k)+1}$ where $k = q(k)n + r(k)$, $0 \leq r(k) \leq n - 1$.

Denote by E_k the subspace of E spanned by $\{a_0, ..., a_k\}$ and let

$$\mathcal{G} = \{g \in C(E; E);\ g \text{ is odd and } g(x) = x \text{ for } |x| \text{ large}\}$$

and set $c_k = \inf_{g \in \mathcal{G}} \sup_{g(E_k)} I_0$. Theorem 2.2 applied to $f_1(\theta, s) = -a(s^2 + 1)^{1/4}$ and $f_2(\theta, s) = b(s^2 + 1)^{1/4}$ yields that if the set of critical levels of I_1 has an upper bound then:

$$|c_{k+1} - c_k| \leq K(\sqrt{c_k} + \sqrt{c_{k+1}} + 1)$$

which implies that $(\frac{c_k}{k^2})$ is bounded. On the other hand, it is shown in [E-G-T] that if V has a polynomial growth like $V(x) \leq \gamma_1 |x|^q + \gamma_2$ then there is a positive constant $L > 0$ such that $c_k \geq L k^{\frac{2q}{q-2}}$. An adaptation of the proof in [EGT] to general superquadratic potentials yields that $\frac{c_k}{k^2} \to +\infty$ as $k \to +\infty$. This contradiction completes the proof of Theorem 1.1. □

4. Semilinear elliptic PDEs with nonhomogeneous boundary conditions

Let Ω be an open bounded subset of $\mathbf{R}^n$ (of class C^2), $f \in C(\bar{\Omega}, \mathbf{R})$ and u_0 in $C^2(\bar{\Omega}, \mathbf{R})$ such that $\Delta u_0 = 0$. We consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1} u + f & \text{in } \Omega \\ \quad u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{P2}$$

We shall prove that for $1 < p < \frac{n+1}{n-1}$, (P_2) has infinitely many solutions. Again, we first reformulate the problem by setting $u = v + u_0$. (P_2) is then equivalent to

$$\begin{cases} -\Delta v = |u_0 + v|^{p-1}(u_0 + v) + f & \text{in } \Omega \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{P'2}$$

Let $E = H_0^1(\Omega; \mathbf{R})$ be endowed with the scalar product $(v_1, v_2) = \int\limits_\Omega \nabla v_1.\nabla v_2 \, dx$ and let $\| \;\|$ denote the associated norm. For $1 < p \le \frac{n+2}{n-2}$, we can define on $[0, 1] \times E$ the functional I by

$$I(\theta, v) = \int\limits_\Omega (\frac{1}{2}|\nabla v|^2 - \frac{|v + \theta u_0|^{p+1}}{p+1} - \theta f v) \, dx.$$

The solutions of (P_2') coincide with the critical points of $I_1 = I(1, \cdot)$ in E. In order to apply Theorem 2.2, one must first check that I satisfies the Palais–Smale condition (H1). That proof can be carried out in a standard way. Now we establish (H2).

Lemma 4.1. *For all $b > 0$ there is a constant $C_1(b)$ such that:*

$$\left| \frac{\partial}{\partial \theta} I(\theta, v) \right| \le C_1(b)(\|I_\theta'(v)\| + 1)(\|v\| + 1) \text{for all } (\theta, v) \text{ with } |I_\theta(v)| \le b.$$

Proof. Let $b > 0$ be given. The condition $|I_\theta(v)| \le b$ is equivalent to

$$\left| \int\limits_\Omega (\frac{1}{2}|\nabla v|^2 - \frac{1}{p+1}|v + \theta u_0|^{p+1} - \theta f v) \, dx \right| \le b \tag{4.1}$$

and

$$\frac{\partial}{\partial \theta} I(\theta, v) = \int\limits_\Omega (-|v + \theta u_0|^{p-1}(v + \theta u_0)u_0 - f v) \, dx.$$

Moreover, using (4.1), we have

$$\begin{aligned} -\langle I_\theta'(v), v\rangle &= \int_\Omega (-|\nabla v|^2 + |v + \theta u_0|^{p-1}(v + \theta u_0)v + \theta f v) \, dx \\ &\ge \left(\frac{p+1}{2} - 1 \right) \int_\Omega |\nabla v|^2 \, dx \\ &\quad - \int\limits_\Omega (|v + \theta u_0|^{p-1}(v + \theta u_0)\theta u_0 + p\theta f v) \, dx - (p+1)b. \end{aligned}$$

Set $C_0 = \frac{p+1}{2} - 1 > 0$ and note that

$$\left| \int_\Omega |v + \theta u_0|^{p-1}(v + \theta u_0)\theta u_0 \right| \le C_1 \|v + \theta u_0\|_{p+1}^p$$

and

$$\left| \int_\Omega f v \, dx \right| \le C_2 + C_3 \|v + \theta u_0\|_{p+1}^{p+1}.$$

From (4.1) we have, $\|v + \theta u_0\|_{p+1}^{p+1} \le C_4 \int_\Omega |\nabla v|^2 + C_5$. Hence

$$-\langle I'_\theta(v), v\rangle \geq C_0 \int_\Omega |\nabla v|^2 - C_6(\int_\Omega |\nabla v|^2)^{\frac{p}{p+1}} - C_7(\int_\Omega |\nabla v|^2)^{\frac{1}{p+1}} - C_8 \quad (4.2)$$

whenever $|I_\theta(v)| \leq b$.

Now (4.1) and (4.2) imply

$$\|v + \theta u_0\|_{p+1}^{p+1} \leq C_4 \int_\Omega |\nabla v|^2 + C_5 \leq C_{10}(\|I'_\theta(v)\| + 1)(\|v\| + 1).$$

Since

$$\begin{aligned} \left|\frac{\partial}{\partial\theta} I(\theta, v)\right| &\leq C_{11}\|v + \theta u_0\|_{p+1}^{p} + C_{12}\|v\|_{p+1}^{1/p+1} \\ &\leq C_{13}\left(\|v + \theta u_0\|_{p+1}^{p} + 1\right), \end{aligned}$$

the lemma is proved. □

Note that the same computations show that there exist $C, K > 0$ such that if v is a critical point of $I_\theta(v)$, then

$$C(I_\theta(v) - 1) \leq \int_\Omega |\nabla v|^2 \leq C\,(I_\theta(v) + 1) \quad (4.3)$$

and

$$K(I_\theta(v) - 1) \leq \int_\Omega |v + \theta u_0|^{p+1} \leq K\,(I_\theta(v) + 1)\,. \quad (4.4)$$

We shall need the following "Pohozaev-type" lemma.

Lemma 4.2. *There exists a constant $C > 0$ such that if v is a critical point for I_θ, then for $u = v + \theta u_0$ we have*

$$\int_{\partial\Omega} (\frac{1}{2}|\nabla u|^2 - |\frac{\partial u}{\partial n}|^2)d\sigma \leq C \int_\Omega (|\nabla u|^2 + |u|^{p+1} + 1)\,dx.$$

Proof. Let v be a critical point of I_θ. Then

$$\begin{cases} -\triangle v = |v + \theta u_0|^{p-1}(v + \theta u_0) + \theta f & \text{on } \Omega \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that, by classical regularity results we have that v and u belong to $C^2(\bar{\Omega})$.

For $x \in \bar{\Omega}$, let $\ell(x) = d(x, \partial\Omega)$ be the distance to the boundary. Since Ω is of class C^2, there is $\delta > 0$ such that ℓ is C^2 on $\bar{\Omega} \cap \{\ell < \delta\}$ and $n(x) = \nabla\ell(x)$ coincides on $\partial\Omega$ with the inner normal. Let φ denote a smooth function $\mathbf{R} \to [0, 1]$ such that $\varphi = 1$ on $(-\infty, 0]$ and $\varphi = 0$ on $[\delta, +\infty)$. Set $g(x) = \varphi\,(\ell(x))$.

Multiply now the equation

$$\begin{cases} -\Delta u = |u|^{p-1}u + \theta f & \text{on } \Omega \\ \quad u = \theta u_0 & \text{on } \partial\Omega \end{cases} \quad (4.5)$$

by $g(x)\nabla u.n(x)$ and integrate over Ω. As a first term, we get:

$$\int_{\Omega} -\Delta u.g\nabla u.n\,dx = \int_{\partial\Omega} -g\left|\frac{\partial u}{\partial n}\right|^2 d\sigma + \int_{\Omega} \nabla u\cdot\nabla(g\nabla u\cdot n))\,dx.$$

Note that $g = 1$ on the boundary, while the last term of the equation is equal to

$$\begin{aligned}
&\int_\Omega \sum_{1\le i,j\le n} u_{x_i}(gu_{x_j}n_j)_{x_i}\,dx\\
&\qquad = \int_\Omega \sum_{i,j} u_{x_i}u_{x_j}(gn_j)_{x_i}\,dx + \int_\Omega \sum_{i,j} u_{x_i}u_{x_jx_i}gn_j\,dx\\
&\qquad = O\left(\int_\Omega |\nabla u|^2\right) + \sum_{i,j}\int_\Omega \left(\frac{1}{2}|u_{x_i}|^2\right)_{x_j} gn_j\\
&\qquad = O\left(\int_\Omega |\nabla u|^2\right) + \int_{\partial\Omega}\sum_{i,j}\frac{1}{2}|u_{x_i}|^2 g|n_j|^2\,d\sigma
\end{aligned}$$

and on $\partial\Omega$, $\sum_{i,j}|u_{x_i}|^2 g|n_j|^2 = |\nabla u|^2$. Therefore,

$$\int_\Omega -\Delta ug\nabla u\cdot n\,dx = \int_{\partial\Omega}(\frac{1}{2}|\nabla u|^2 - \left|\frac{\partial u}{\partial n}\right|^2)\,d\sigma + O\left(\int_\Omega |\nabla u|^2\,dx\right). \tag{4.6}$$

In the same way, we get

$$\int_\Omega |u|^{p-1}ug(x)\nabla u\cdot n\,dx = \theta^{p+1}\int_{\partial\Omega}\frac{|u_0|^{p+1}}{p+1}\,d\sigma + O\left(\int_\Omega |u|^{p+1}\,dx\right), \tag{4.7}$$

$$\int_\Omega f(x)g(x)\nabla u\cdot n\,dx = \theta\int_{\partial\Omega} fu_0\,d\sigma + O\left(\left(\int_\Omega |u|^{p+1}\,dx\right)^{1/p+1}\right). \tag{4.8}$$

By putting together (4.5)–(4.8), one can complete the proof of Lemma 4.2. □

Lemma 4.3. *There exists a constant $C > 0$ such that, if v is a critical point of I_θ, then*

$$\left|\frac{\partial}{\partial\theta}I(\theta,v)\right| \le C\left(I(\theta,v)^2+1\right)^{1/4}.$$

Proof. Let v be a critical point of I_θ. Then

$$\begin{cases} -\Delta v = |v+\theta u_0|^{p-1}(v+\theta u_0)+\theta f & \text{on } \Omega\\ \quad v = 0 & \text{on } \partial\Omega\end{cases}$$

and therefore:

$$
\begin{aligned}
\frac{\partial}{\partial\theta} I(\theta,v) &= \int_\Omega \left(-|v+\theta u_0|^{p-1}(v+\theta u_0)u_0 - fv\right) dx \\
&= \int_\Omega \left((\Delta v + \theta f)u_0 - fv\right) dx \\
&= \int_\Omega -\nabla v \nabla u_0 \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial n} u_0 \, d\sigma + \theta \int_\Omega f u_0 - \int_\Omega f v.
\end{aligned}
$$

Note first that $\int_\Omega -\nabla v \nabla u_0 \, dx = \int_\Omega v \Delta u_0 \, dx = 0$. We also have

$$
\begin{aligned}
\left|\int_\Omega f v \, dx\right| &\le \|v\|_{p+1} \|f\|_{\frac{p+1}{p}} \le C\left(\|v+\theta u_0\|_{p+1} + 1\right) \\
&\le C'\left(|I(\theta,v)|^{1/p+1} + 1\right).
\end{aligned}
\tag{4.9}
$$

So, since $p > 1$, it is enough to check that

$$
\left|\int_{\partial\Omega} \frac{\partial v}{\partial n} \cdot u_0 d\sigma\right| \le C(|I\left(\theta,v)|^2 + 1\right)^{1/4}.
$$

We shall see that in fact

$$
\int_{\partial\Omega} \left|\frac{\partial v}{\partial n}\right|^2 d\sigma \le c\,(|I(\theta,v)| + 1) \tag{4.10}
$$

which obviously implies the desired estimate. Since $\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} - \theta \frac{\partial u_0}{\partial n}$, (4.10) is equivalent to prove the same estimate for $\int_{\partial\Omega} |\frac{\partial u}{\partial n}|^2 \, d\sigma$.

Now

$$
\int_{\partial\Omega} \left(\frac{1}{2}|\nabla u|^2 - |\frac{\partial u}{\partial n}|^2\right) d\sigma = \int_{\partial\Omega} \left(\frac{1}{2}|D_{\partial\Omega} u_0|^2 - \frac{1}{2}|\frac{\partial u}{\partial n}|^2\right) d\sigma
$$

(where $D_{\partial\Omega} u_0$ is the gradient of $u_{0|\partial\Omega}$). Since $u_0 \in C^2(\bar\Omega)$, we get from Lemma 4.2 that there is a constant C' such that

$$
\begin{aligned}
\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma &\le C'\left(\int_\Omega (|\nabla u|^2 + |u|^{p+1})\, dx + 1\right) \\
&\le C''\left(\int_\Omega (|\nabla v|^2 + |v|^{p+1})\, dx + 1\right)
\end{aligned}
$$

and by (4.3) and (4.4)

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma \le C''' \left(I(\theta, v) + 1\right),$$

which yields Lemma 4.3. □

End of Proof of Theorem 1.2. Again, we show that $I(\theta, x)$ satisfies the hypothesis of Theorem 2.2. It is clear that I_0 is even and that, in view of the above lemmas, it satisfies hypothesis (H1), (H2) and (H3). It is also clear that for every finite dimensional subspace W of E, $\sup_{\theta\in[0,1]} I(\theta, y) \to -\infty$ as $|y| \to +\infty$, $y \in W$.

For each k, denote by E_k the subspace of E spanned by the first k eigenfunctions of Δ, let

$$\mathcal{G} = \{g \in C(E; E);\ g \text{ is odd and } g(x) = x \text{ for } |x| \text{ large}\}$$

and set $c_k = \inf_{g\in\mathcal{G}} \sup_{g(E_k)} I_0$.

Theorem 2.2 applied with $f_1(\theta, s) = -a(s^2+1)^{1/4}$ and $f_2(\theta, s) = b(s^2+1)^{1/4}$ then yield that if the set of critical levels of I_1 has an upper bound then:

$$|c_{k+1} - c_k| \le K(\sqrt{c_k} + \sqrt{c_{k+1}} + 1)$$

which implies that $(\frac{c_k}{k^2})$ is bounded. On the other hand, it is shown by Tanaka [Ta] (see also Bahri–Lions [B-B]), that there is a positive constant $L > 0$ such that $c_k \ge Lk^{\frac{p+1}{p-1}\cdot\frac{2}{n}}$. This is a contradiction as long as $\frac{2}{n}\cdot\frac{p+1}{p-1} > 2$, that is when $p < \frac{n+1}{n-1}$. This completes the proof of the theorem. □

Remark 4.1. A rougher estimate for $\left|\frac{\partial}{\partial\theta} I(\theta, v)\right|$, without using that v is a critical point of I_θ, would have led to:

$$\left|\frac{\partial}{\partial\theta} I(\theta, v)\right| \le \int_\Omega |v + \theta u_0|^p \le \|v + \theta u_0\|_{p+1}^p \le C(I(\theta, v) + 1)^{\frac{p}{p+1}}.$$

The existence of solution would have been obtained in this way only for $p < \frac{n+2}{n}$.

Also note that, in order to obtain the same multiplicity result for the full range of p (up to the critical Sobolev exponent), one needs to show that for each $\epsilon > 0$,

$$\left|\frac{\partial}{\partial\theta} I(\theta, v)\right| \le C_\epsilon(|I(\theta, v)|^\epsilon + 1).$$

5. Non-homogeneous Hamiltonian systems

We now consider the following Hamiltonian system

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p}(q, p) + f_1(t) \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q}(q, p) + f_2(t) \\ q(0) = q_0, \quad q(1) = q_1 \end{cases} \tag{P3}$$

where $q, p : [0, 1] \to \mathbf{R}^n$; $f_i \in C^1\big([0, 1]; \mathbf{R}^n\big)$ and q_0, q_1 are given.

We shall use the following assumptions on H:

(h1) H is even and belongs to $C^2(\mathbf{R}^{2n}, \mathbf{R})$.
(h2) $\exists \mu > 2$ such that $H'(x) \cdot x \geq \mu H(x) > 0$ for $|x|$ large.
(h3) There are r and s such that $1 < r \leq s < 2r + 1$ and

$$A|x|^{r+1} - B \leq H(x) \leq C|x|^{s+1} + D,$$

where A, B, C, D are positive constants.

In this section we prove Theorem 1.3, which state that under assumptions (h1), (h2), (h3), problem (P3) has infinitely many solutions. To do that, we again perform a change of variable, replacing $q(t)$ by $q(t) + z(t)$, where $z(t) = tq_1 + (1 - t)q_0$. We find that (P3) is equivalent to

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p}(q + z, p) + f_1(t) - (q_1 - q_0) \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q}(q + z, p) + f_2(t) \\ q(0) = q(1) = 0. \end{cases} \tag{P'3}$$

Let a be a smooth even map $\mathbf{R} \to \mathbf{R}$, non decreasing on $\mathbf{R}_+$ and such that $|a'(x)| \leq 1$, $a(x) = x$ for $x \geq 1$. Let $\tilde{m} : \mathbf{R}^2 \to \mathbf{R}$ be defined by $\tilde{m}(x, y) = (x + y)/2 - a(x - y)/2$ ($\tilde{m}$ is a smooth approximation of “min”). It is easy to see that $\tilde{m}$ is non-decreasing with respect to each of its arguments. More precisely, $0 \leq \partial\tilde{m}/\partial x \leq 1$ and $0 \leq \partial\tilde{m}/\partial y \leq 1$.

Fix $\alpha \in (2, \min(\mu, r + 1))$ and $\alpha' \in (2, \alpha)$. For R large, define G_R by

$$G_R(x) = \tilde{m}(H(x), R|x|^\alpha + D + 1).$$

By the definition of $\tilde{m}$ and the properties of H, $G_R(x) = H(x)$ if $|x| \leq m_R := (R/C)^{1/(s+1-\alpha)}$ and, provided R is large enough, $G_R(x) = |Rx|^\alpha + D + 1$ if $|x| \geq M_R := (2R/A)^{1/(r+1-\alpha)}$.

Now define H_R by

$$H_R(q, p) = \tilde{m}(G_R(q, p), L_R(q, p)),$$

where $L_R(q, p) = 2R(2M_R)^{\alpha-2}(|p|^2 + |q|^{\alpha'} + 1)$. One could easily verify that, provided R is large enough, $H_R(x) = G_R(x)$ if $|x| \leq 2M_R$. Hence $H_R(x) =$

$R|x|^\alpha + D + 1$ if $M_R \le |x| \le 2M_R$. Moreover, for $|x|$ large, $H_R(x) = L_R(x)$. Notice also that $H_{R'}(x) \ge H_R(x)$ if $R' \ge R$. Finally, by the properties of H, there are two positive constants E and F such that, for all $R \ge 1$, for all $x \in \mathbf{R}^{2n}$, $H_R(x) \ge E(|p|^2 + |q|^{\alpha'}) - F$.

Let $E = H_0^1\big((0,1);\mathbf{R}^n\big) \times L^2\big((0,1);\mathbf{R}^n\big)$ be endowed with the scalar product

$$\langle (q,p);(q',p')\rangle = \int_0^1 \dot{q}\dot{q}' + pp'\,dt.$$

Let I^R be defined on $(0,1)\times E$ by

$$I^R(\theta,q,p) = \int_0^1 (\dot{q}\cdot p - H_R(q+\theta z,p) + \theta(q_1-q_0) \cdot p - \theta f_1\cdot p + \theta f_2 \cdot q)\ dt.$$

It is easy to see that I^R is smooth on $(0,1)\times E$ and that the critical points $x=(q,p)$ of $I^R(1,.)$ which satisfy $|q+z|^2+|p|^2 \le m_R$ are solutions to problem (P_3'). Notice also that if $R' \ge R$ then $I^{R'} \le I^R$. Moreover we have the following:

Lemma 5.1. *There is a function $\ell : \mathbf{R}_+ \to (0,+\infty)$ such that:*

(i) $\lim_{R\to+\infty} \ell(R) = +\infty$.
(ii) *If $x=(q,p)$ is a critical point of $I^R(\theta,\cdot)$ which satisfies $I^R(\theta,x) \le \ell(R)$ then $|y| \le m_R$ where $y = x + (\theta z, 0)$.*

Proof. We first prove that, if x is a critical point of $I^R(\theta,\cdot)$, then

$$|H_R(y(t)) - \int_0^1 H_R(y(s))\,ds| \le C_1 \int_0^1 (H_R^2(y(s))+1)^{1/4}\,ds + C_2(H_R^2(y(t))+1)^{1/4} \tag{5.1}$$

where $y(t) = x(t) + \theta\big(z(t),0\big)$.

In fact, let $x=(q,p)$ be a critical point of $I^R(\theta,\cdot)$. Then

$$\begin{cases} \dot{q}(t) = \frac{\partial H_R}{\partial p}(q+\theta z,p) + \theta f_1 - \theta(q_1-q_0) \\ \dot{p}(t) = -\frac{\partial H_R}{\partial q}(q+\theta z,p) + \theta f_2. \end{cases}$$

Hence

$$\begin{aligned} &\frac{d}{dt} H_R\big(q(t)+\theta z(t),p(t)\big) \\ &\quad = \frac{\partial H_R}{\partial q}(q+\theta z,p)\big(\dot{q}+\theta(q_1-q_0)\big) + \frac{\partial H_R}{\partial p}(q+\theta z,p)\dot{p} \\ &\quad = (\theta f_2 - \dot{p})\big(\dot{q}+\theta(q_1-q_0)\big) + \big(\dot{q}-\theta f_1+\theta(q_1-q_0)\big)\dot{p} \\ &\quad = \theta f_2\dot{q} + \theta^2 f_2(q_1-q_0) - \theta f_1\dot{p}. \end{aligned}$$

We get:

$$\begin{aligned}
&H_R\big(y(t_2)\big) - H_R\big(y(t_1)\big)\\
&\quad = \int_{t_1}^{t_2} \theta f_2(s)\dot q(s) + \theta^2 f_2(s)(q_1 - q_0) - \theta f_1(s)\dot p(s)\, ds\\
&\quad = \theta^2(q_1 - q_0)\cdot \int_{t_1}^{t_2} f_2(s)\, ds + \theta \int_{t_1}^{t_2} (\dot f_1(s)p(s) - \dot f_2(s)q(s))\, ds\\
&\qquad + \theta[f_2(s)q(s)]_{t_1}^{t_2} - \theta[f_1(s)p(s)]_{t_1}^{t_2}.
\end{aligned}$$

Since $f_i \in C^1\left([0,1];\mathbf{R}^n\right)$, by (h3) and the definition of H_R we get:

$$\begin{aligned}
K\left|H_R\big(y(t_2)\big) - H_R\big(y(t_1)\big)\right| &\le \Big(H_R\big(y(t_2)\big)^2 + 1\Big)^{\frac14} + \Big(H_R\big(y(t_1)\big)^2 + 1\Big)^{\frac14}\\
&\quad + \int_0^1 \Big(H_R\big(y(s)\big)^2 + 1\Big)^{\frac14} ds \qquad (5.2)
\end{aligned}$$

where $K > 0$ is independent of R. It is easy to see that (5.2) implies (5.1).

Moreover, by the properties of H_R, (5.1) implies that, if $x = (q, p)$ is a critical point of $I^R(\theta,\cdot)$ then either $|y(t)| \le 2M_R$ for all t or $|y(t)| \ge M_R$ for all t (provided R is large enough).

Now assume that there is $t \in [0,1]$ such that $|y(t)| > m_R$. Then using (h2), the definition of H_R and (5.1), one can derive that $\int_0^1 H_R\big(y(t)\big)\, dt \ge \beta_R$, where $\lim_{R\to+\infty}\beta_R = +\infty$ and $y(t) = x(t) + (\theta z(t), 0)$. □

We now prove that $I^R\big(\theta, x(t)\big) \ge \ell(R)$, where $\lim_{R\to+\infty}\ell(R) = +\infty$.

Case 1. $|y| \le 2M_R$.

On the ball $B(0, 2M_R)$, we have

$$\begin{aligned}
H_R(y) = G_R(y) &= \tilde m(H(y), R|x|^\alpha + D + 1)\\
H_R'(y) &= \frac12(1 - a'(H(y) - R|y|^\alpha - D - 1))H'(y)\\
&\quad + \frac12(1 + a'(H(y) - R|y|^\alpha - D - 1)\alpha|y|^{\alpha-2}y.
\end{aligned}$$

Choose $\mu' \in (2,\alpha)$ (so $\mu' < \mu$). Using the properties of a, we can get easily by (h2):

$$H_R'(y)\cdot y \ge \mu' H_R(y) > 0 \quad \text{on} \quad B(0, 2M_R) \quad \text{for} \quad |y| \ge C, \qquad (5.3)$$

where C is independent of R. Now x satisfies

$$\dot x(t) = J H_R'\big(y(t)\big) + \theta\big(f_1 - (q_1 - q_0), f_2\big)$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $y(t) = x(t) + \big(\theta z(t), 0\big)$.

$$
\begin{aligned}
&I^R(\theta, x)\\
&\quad = \int_0^1 \frac{1}{2}(\dot{q}p - \dot{p}q) - H_R(q+\theta z, p) + \theta(q_1 - q_0)p - \theta f_1 p + \theta f_2 \cdot q\, dt\\
&\quad = \int_0^1 [\frac{1}{2}\big(\dot{q}p - (q+\theta z)\dot{p}\big) - H_R(q+\theta z, p) + \theta(q_1 - q_0)\cdot p - \theta f_1 p + \theta f_2 q\,]dt\\
&\qquad + \frac{\theta}{2}\int_0^1 z\dot{p}\, dt\\
&\quad = \int_0^1 (-\frac{1}{2}\langle J\dot{x}, y\rangle - H_R(q+\theta z, p) + \frac{\theta}{2}(q_1 - q_0)\cdot p - \theta f_1 \cdot p + \theta f_2 q\,)dt\\
&\qquad + \frac{\theta}{2}\left[\big(q_1, p(1)\big) - \big(q_0, p(0)\big)\right]\\
&\quad = \int_0^1 \frac{1}{2}\big(H_R'(y), y\big) - H_R(y) + O\left(\int_0^1 |y|\,dt + |y(1)| + |y(0)|\right)\\
&\quad \geq \left(\frac{\mu'}{2} - 1\right)\int_0^1 H_R(y)\,dt + O\left(\int_0^1 |y|\,dt + |y(1)| + |y(0)| + 1\right).
\end{aligned}
$$

Now, by (5.1) and the fact that $H_R(y) \geq K|y|^2 - K'$ (where K and K' are independent of R), we get that for appropriate constants $K''', K'' > 0$,

$$
I^R(\theta, x) \geq K''\left(\int_0^1 H_R(y)\,dt\right) - K''' \geq K''\beta_R - K'''.
$$

Case 2. $|y| \geq M_R$.

On the set $\{|y| \geq M_R\}$, we have $H_R(y) = \tilde{m}(R|x|^\alpha + D + 1, L_R(y))$, and a straightforward computation shows that there is $\delta \in (0, 1)$ such that

$$
(1+\delta)\frac{\partial H_R}{\partial p}(y)\cdot p + (1-\delta)\frac{\partial H_R}{\partial q}(y)\cdot q \geq \nu H_R(y) - K \tag{5.4}
$$

where $\nu > 2$ and K is a constant independent of R.

Noting that $\int_0^1 \dot{q}p = \frac{(1+\delta)}{2}\int_0^1 \dot{q}p - \frac{(1-\delta)}{2}\int_0^1 \dot{p}q$ and using the equation satisfied by x, we get by the same type of estimates as in the first case;

$$
I^R(\theta, x) \geq \tilde{K}''\beta_R - \tilde{K}'''.
$$

Since $\lim_{R\to+\infty}\beta_R = +\infty$, we get the claim of Lemma 5.1, with

$$
\ell(R) = \min(K''\beta_R - K'''; \tilde{K}''\beta_R - \tilde{K}''').
$$

As a consequence of Lemma 5.1, we get that a critical point of $I^R(1, .)$ of critical value $\leq \ell(R)$ is a solution to (P_3').

Lemma 5.2. *For every R, I^R satisfies the Palais–Smale condition (i.e. if (θ_n, x_n) is a sequence in $[0, 1] \times E$ s.t. $(I^R_{\theta_n})'(x_n) \to 0$ and $I^R(\theta_n, x_n)$ is bounded, then it has a convergent subsequence).*

Proof. Note that $H_R(x) = L_R(x) + U_R(x) = 2R(2M_R)^{\alpha-2}(|p|^2 + |q|^{\alpha'} + 1) + U_R(x)$ where U_R and U'_R are bounded.

The same type of estimates as in the proof of Lemma 5.1 (second case) enables one to prove that, if (θ_n, x_n) is a PS sequence, then (x_n) is bounded in E. The convergence of a subsequence is then proved by standard arguments. □

Lemma 5.3. *For all R, for all $b > 0$, there is C_b^R such that:*

$$|I^R(\theta, x)| \leq b \Rightarrow \left|\frac{\partial}{\partial\theta} I^R(\theta, x)\right| \leq C_b^R \big(\|(I_\theta^R)'(x)\| + 1\big)(\|x\| + 1).$$

We do not give the proof, which is an easy consequence of the fact that the truncated Hamiltonian $H_R(x) = 2R(2M_R)^{\alpha-2}(|p|^2 + |q|^{\alpha'} + 1) + U_R(x)$, where U_R and U'_R are bounded.

Lemma 5.4. *If x is a critical point of $I^R(\theta, \cdot)$ of critical value $\leq \ell(R)$ then*

$$\left|\frac{\partial}{\partial\theta} I^R(\theta, x)\right| \leq C_2 \left|I^R(\theta, x)\right|^{1/r+1} + C_3,$$

where C_2 and C_3 are two constants.

Proof. Let x be a critical point of $I^R(\theta, \cdot)$ of critical value $\leq \ell(R)$. From Lemma 5.1, we have:

$$\begin{cases} \dot{q}(t) = \frac{\partial H}{\partial p}(q + \theta z, p) + \theta f_1 - \theta(q_1 - q_0) \\ \dot{p}(t) = -\frac{\partial H}{\partial q}(q + \theta z, p) + \theta f_2 \\ q(0) = q(1) = 0 \end{cases}$$

where $x = (q, p)$. We therefore have:

$$\begin{aligned} \frac{\partial}{\partial\theta} I^R(\theta, x) &= \int_0^1 -\frac{\partial H}{\partial q}(q + \theta z, p) \cdot z + (q_1 - q_0)p - f_1 \cdot p + f_2 \cdot q \\ &= \int_0^1 (\dot{p} - \theta f_2) \cdot z + O\left(\int_0^1 |p| + |q|\right) \\ &= p(1) \cdot q_1 - p(0) \cdot q_0 - \int_0^1 p \cdot (q_1 - q_0) \\ &\quad - \theta \int_0^1 f_2 \cdot z + O\left(\int_0^1 |p| + |q|\right) \\ &= p(1) \cdot q_1 - p(0) \cdot q_0 + O\left(\int_0^1 |p| + |q| + 1\right). \end{aligned}$$

By (h3) and (5.1) we get:

$$\left|\frac{\partial}{\partial\theta} I^R(\theta, x)\right| \leq C\left(\left(\int_0^1 H(q + \theta z, p)\, dt\right)^2 + 1\right)^{1/2(r+1)} \tag{5.5}$$

Now, as in the proof of Lemma 5.1 (first case), we get:

$$I^R(\theta, x) \geq K'' \int_0^1 H(q + \theta z, p)\, dt - K'''.$$

from which our claim follows. □

End of Proof of Theorem 1.3. The functional $I^R(0,\cdot) = I_0^R$ is clearly even as H_R is an even Hamiltonian. Let

$$E^- = \{(q,p) \in H_0^1 \times L^2 \mid p = -\dot q\}; \quad E^+ = \{(q,p) \in H_0^1 \times L^2 \mid p = \dot q\}$$

and

$$E^0 = \{(q,p) \in H_0^1 \times L^2 \mid q = 0 \text{ and } p = \text{ Cst.}\}$$

We have $E = E^- \oplus E^+ \oplus E^0$. For $x \in E$ we shall denote by x^-, x^+, x^0 the elements of E^-, E^+, E^0 resp. such that $x = x^- + x^+ + x^0$.

Let $\{\mathbf{e_i}, 1 \le i \le n\}$ denote the standard basis of $\mathbf{R}^n$. For $k \ge 0$, let $k = l(k)n + r(k)$, $r(k) \in \{0, \dots, n-1\}$, and define $a_k \in E^+$ by

$$a_k = \big(\sin \pi(l(k)+1)t \ \mathbf{e_{r(k)+1}}\, , (l(k)+1)\pi \cos \pi(l(k)+1)t \ \mathbf{e_{r(k)+1}}\big)$$

which are valued in $\mathbf{R}^n \times \mathbf{R}^n$. Let $E_k^+ = \text{span}\{a_0, a_1, ..., a_k\} \subset E^+$.

Set $E_k = E^- \oplus E^0 \oplus E_k^+$,

$$\mathcal{G} = \left\{ g \in \mathcal{C}(E,E);\ \begin{array}{l} g(x) = e^{\gamma^+(x)}x^+ + e^{\gamma^-(x)}x^- + K(x) \text{ where } \gamma^\pm \\ \text{are continuous, even, maps bounded} \\ \text{sets to compact subsets} \\ \text{of } \mathbf{R} \text{ and } \gamma^\pm(x) = 0 \text{ for } \|x\| \text{ large} \\ \text{while } K \text{ is compact, odd and } K(x) = x^0, \end{array} \right\}$$

and let $c_k^R = \inf_{g\in\mathcal{G}} \sup_{g(E_k)} I_0^R \in (-\infty, +\infty]$. Without loss of generality we will assume that $H(0) = 0$. Note that if $R' \ge R$ then $I^{R'} \le I^R$ and $c_k^{R'} \le c_k^R$. We may define $c_k = \lim_{R\to+\infty} c_k^R$.

Lemma 5.5. *With the above hypothesis, we have:*

(i) $0 \le c_k \le c_k^R \le \sup_{E_k} I_0^R < +\infty$.
(ii) $c_k \ge K'k^{(s+1)/(s-1)} - K$ *(where K and K' are positive constants).*

Proof. (i) For all $g \in \mathcal{G}$, $g(0) = 0$ and since $I_0^R(0) = 0$, it follows that $c_k^R \ge 0$ for all R, hence $c_k \ge 0$. Moreover, for all $R \ge 1$, $H_R(x) \ge \underline{H}(x) := E\big(|p|^2 + |q|^{\alpha'}\big) - F$. Let ψ be the functional defined by: $\psi(x) = \int_0^1 \dot q p - \underline{H}(x)\, dt$ (where $x = (q,p)$). It is enough to check that ψ has an upper bound d_k on E_k. If x belongs to E_k, it can be written: $x = (q, -\dot q + v)$ with $v \in \text{span}\{\mathbf{e_1}, \mathbf{e_2}, .., \mathbf{e_n}, \cos\pi(q(l)+1)t\mathbf{e_{r(l)+1}};\ 0 \le l \le k\}$.

$$\begin{aligned}\psi(x) &= \int_0^1 -|\dot q|^2 + \dot q v - E|v - \dot q|^2 - E|q|^{\alpha'} + F \\ &= \int_0^1 -(1+E)|\dot q|^2 + (1+2E)q\dot v - E|v|^2 - E|q|^{\alpha'} + F \\ &\le \int_0^1 \delta|\dot v|^2 - E|v|^2 + K_\delta|q|^2 - E|q|^{\alpha'} + F\end{aligned}$$

where $\delta > 0$ is arbitrary and K_δ depends on δ.

Now, since $v \in \operatorname{span}\{\mathbf{e_1}, \mathbf{e_2}, .., \mathbf{e_n}, \cos\pi(q(l)+1)t\mathbf{e_{r(l)+1}}; \quad 0 \le l \le k\}$, we can write $\int_0^1 |\dot{v}|^2 \le K(k)\int_0^1 |v|^2$ and, choosing δ small enough, we see that ψ has an upper bound on E_k.

(ii) By (h3) there are positive numbers $a_1, \ldots, a_n$, independent over $\mathbf{Q}$, such that for all x, $H(x) \le \bar{H}(x) := Q(x)^{\frac{(s+1)}{2}} + K$, where

$$Q(x) = a_1(p_1^2 + q_1^2) + \ldots + a_n(p_n^2 + q_n^2).$$

The non-zero solutions of

$$\begin{cases} \dot{q} &= \partial H/\partial p \\ \dot{p} &= -\partial H/\partial q \end{cases}$$

which satisfy $q(0) = q(1) = 0$ are

$$\left(x^{i,k} = (q^{i,k}, p^{i,k})\right)_{1\le i\le n, k\ge 1}, \quad \text{where} \quad q_j^{i,k}(t) = p_j^{i,k}(t) = 0 \quad \text{for} \quad j \ne i$$

and

$$q_i^{i,k}(t) = \frac{1}{\sqrt{a_i}}\left(\frac{k\pi}{a_i(s+1)}\right)^{\frac{1}{s-1}} \sin k\pi t;$$
$$p_i^{i,k}(t) = \frac{1}{\sqrt{a_i}}\left(\frac{k\pi}{a_i(s+1)}\right)^{\frac{1}{s-1}} \cos k\pi t.$$

The value of the functional $\int_0^1 (p\dot{q} - \bar{H}(q,p))\,dt$ at $x^{i,k}$ is equal to

$$c_{i,k} = \left(\frac{s}{2} - \frac{1}{2}\right)\left(\frac{k\pi}{(s+1)a_i}\right)^{(s+1)/(s-1)} - K$$

Note that the $c_{i,k}$ are distinct. Let us call $c_1' < c_2' < \ldots < c_k' < \ldots$ their ordered sequence.

Now we can define $\bar{H}_R$ (for R large) in the same way as H_R is defined, and we can define $\bar{c}_k^R$ in the same way as c_k^R. Let R_k be such that $\bar{\ell}(R) > c_k'$ for all $R \ge R_k$ ($\bar{\ell}(R)$, defined in lemma 5.1, is associated to $\bar{H}$ here). Then, for $R \ge R_k$, the first k nontrivial (i.e. $\ne I_0^R(0)$) critical values of $\bar{I}_0^R$ (the functional associated to $\bar{H}$) are $c_1' < c_2' < \ldots < c_k'$. Moreover it is easy to see that the infimum of I_0^R over $S_\rho^+ = \{x \in E^+ \mid ||x|| = \rho\}$ is $> I_0^R(0)$ for $\rho > 0$ small. Hence ($\mathcal{G}$ has been defined such that, for all $i \ge 1$, for all $g \in \mathcal{G}$, $g(E_i) \cap S_\rho^+ \ne \emptyset$), $\bar{c}_i^R$ is a nontrivial critical value of I_0^R. Now all the critical points of critical level $\le \bar{\ell}(R)$ are isolated. Hence by classical results on even functionals, $\bar{c}_i^R > \bar{\ell}(R)$ or $\bar{c}_{i+1}^R > \bar{c}_i^R$. We can conclude that for all $R \ge R_k$, $\quad \bar{c}_k^R \ge c_k' \ge K'k^{\frac{s+1}{s-1}} - K$. Since $\bar{H}_R \ge H_R$, $\bar{c}_k^R \le c_k^R$ and we can derive (ii) of the Lemma.

Now assume that problem (P_3') has a finite number N of solutions. Let $\bar{R}$ be such that the corresponding trajectories are included in the ball of radius $m_{\bar{R}}$. Then, for $R \ge \bar{R}$, these solutions give rise to critical points of I_1^R. Let D be an upper bound of the corresponding critical values.

By Lemma 5.1, for $R \geq \bar{R}$, $I^R(1,.)$ has no critical value $> D$ and $\leq l(R)$.
Let ψ_i denote the flow of f_i, where

$$f_1(s) = -C_2(1+s^2)^{1/2(r+1)} - C_3 \text{ and } f_2(s) = C_2(1+s^2)^{1/2(r+1)} + C_3.$$

As in the proof of Theorem 2.2, for any $\epsilon > 0$ we can find $\tilde{R}$ and construct A_k and $B_k \subset A_k$ such that $c^{\tilde{R}}_{B_k} \leq c_k + \epsilon$ and

$$\forall R \geq \tilde{R} \;\; c^{\tilde{R}}_{A_k,B_k} \geq c^R_{A_k,B_k} \geq c^R_{k+1} \geq c_{k+1},$$

where $c^{\tilde{R}}_{B_k} = \sup_{B_k} I^{\tilde{R}}_0$, $c^{\tilde{R}}_{A_k,B_k} = \inf_{g\in\mathcal{D}'_{B_k}} \sup_{g(A_k)} I^{\tilde{R}}_0$ and

$$\mathcal{D}'_{B_k} = \{g \in \mathcal{G};\, g_{|B_k} = Id_{B_k}\}.$$

Now choose $R \geq \tilde{R}$ such that $\psi_2(1, c^{\tilde{R}}_{A_k,B_k}) \leq l(R)$. Then $c^R_{B_k} \leq c^{\tilde{R}}_{B_k} \leq c_k + \epsilon$, $c^R_{A_k,B_k} \geq c_{k+1}$ and $\psi_2(1, c^R_{A_k,B_k}) \leq \psi_2(1, c^{\tilde{R}}_{A_k,B_k}) \leq l(R)$.

Using Theorem 2.1 and Remark 2.1, one can see by lemma 5.4 that if $\psi_2(1, c_k+\epsilon) < \psi_1(1, c_{k+1})$ then I^R_1 has a critical value in the interval $[\psi_1(1, c_{k+1}), \psi_2(1, c^R_{A_k,B_k})]$, which corresponds to a solution to the problem by Lemma 5.1.

Since we may choose $\epsilon > 0$ arbitrarily small, we derive that, for all k,

$$\psi_1(1, c_{k+1}) \leq D \quad \text{or} \quad \psi_1(1, c_{k+1}) \leq \psi_2(1, c_k),$$

which yields $c_{k+1} - c_k \leq K\big((c_k)^{1/(r+1)} + (c_{k+1})^{1/r+1}\big)$. It is then easy to see that this implies that $c_k \leq \alpha k^{\frac{r+1}{r}} + \beta$ where α and β are some constants, which contradicts Lemma 5.5 (ii), because $s < 2r+1$. $\square$

6. Non-homogeneous, semi-linear wave equations

In this last section, we deal with the following non-linear wave equation with non-homogeneous boundary conditions:

$$\begin{cases} u_{tt} - u_{xx} + |u|^{p-1}u = f(x,t) \\ u(x, t+2\pi) = u(x,t) \\ u(0,t) = x_0 \quad u(\pi,t) = x_1. \end{cases} \tag{P4}$$

Again, we introduce $u = v + z$ where $z(x) = x_0 + \frac{x}{\pi}(x_1 - x_0)$. Note that z is independent of t and so is obviously 2π periodic in t. We need then to solve:

$$\begin{cases} v_{tt} - v_{xx} + |v+z|^{p-1}(v+z) = f(x,t) \\ v(x, t+2\pi) = v(x,t) \\ v(0,t) = 0, \quad v(\pi,t) = 0. \end{cases} \tag{P'4}$$

As in Tanaka [T2] we consider the space of functions defined on $\Omega = (0,\pi) \times (0, 2\pi)$ satisfying the above (homogeneous) boundary conditions. Any smooth function u in this space has a Fourier expansion of the form

$$u = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{jk} \sin jx e^{ikt}, \qquad a_{j,-k} = \bar{a}_{j,k}$$

We define:

$$\langle u, v\rangle = \frac{1}{4\pi^2} \sum_{j,k} |k^2 - j^2| a_{jk} \bar{b}_{jk} \quad \text{and} \quad ||.||_E = \langle u, u\rangle$$

for $u = \sum_{j,k} a_{jk} \sin jx e^{ikt}$ and $v = \sum_{j,k} b_{jk} \sin jx e^{ikt}$. It is clear that $||.||_E$ is a norm on the set $\{u : a_{jk} = 0 \text{ if } j = |k|\}$. Next we set

$$E^+ = \left\{ u; u = \sum_{j=1}^{\infty} \sum_{|k|>j} a_{jk} \sin jx e^{ikt} : \sum_{j,k} |k^2 - j^2| |a_{jk}|^2 < \infty \right\}$$
$$E^- = \left\{ u; u = \sum_{j=1}^{\infty} \sum_{|k|<j} a_{jk} \sin jx e^{ikt} : \sum_{j,k} |k^2 - j^2| |a_{jk}|^2 < \infty \right\},$$

and $E = E^+ \oplus E^-$. Furthermore set:

$$N = L^{p+1} - \text{closure of span}\{\sin jx e^{ikt}; j = |k|, j \in \mathbf{N}\}$$

with L^{p+1}-norm $||.||_{p+1}$. E^+, E^- and N are complementary subspaces of the space of functions satisfying homogeneous boundary conditions: $v(0,t) = v(\pi, t) = 0$, $v(x, t + 2\pi) = v(x, t)$ and the wave form $\mathcal{L} = v_{tt} - v_{xx}$ is positive definite, negative definite and null respectively, on E^+, E^- and N. Next we consider the one parameter family of functionals:

$$K_\theta(u) = \int_\Omega \left\{ \frac{1}{2}(u_t^2 - u_x^2) - \frac{1}{p+1}|u + \theta z|^{p+1} + \theta f u \right\} dx dt$$

on the space $E \oplus N$. Clearly, critical points of K_1 are solutions of our problem. Note that

$$\begin{aligned} K_\theta(u + v) = &\frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 \\ &- \frac{1}{p+1}||u^+ + u^- \\ &+ v + \theta z||_{p+1}^{p+1} + \big\langle \theta f, u^+ + u^- + v + \theta z \big\rangle \end{aligned}$$

for $u = u^+ + u^- \in E = E^+ \oplus E^-$ and $v \in N$. Now the function Q_θ defined by the minimization problem

$$Q_\theta(u) = \min_{\omega \in N} \left[\frac{\|u + \theta z + \omega\|_{p+1}^{p+1}}{p+1} - \langle \theta f, u + \theta z + \omega\rangle \right] \qquad u \in E$$

is well defined and continuous –as the minimized term is strictly convex. Actually, for each u, the minimum is attained at a unique point $\omega_\theta(u)$ in such a way that:

$$Q_\theta(u) = \frac{\|u+\theta z+\omega_\theta(u)\|_{p+1}^{p+1}}{p+1} - \langle \theta f, u+\theta z+\omega_\theta(u)\rangle .$$

In particular:

$$\left\langle |u+\theta z+\omega_\theta(u)|^{p-1}(u+\theta z+\omega_\theta(u)) - \theta f, h\right\rangle = 0 \quad \forall h \in N.$$

Now it is easily seen that there is a one to one correspondence between the critical points of K_θ and those of I_θ, where $I_\theta : E \to \bar{R}$ is defined by

$$I_\theta(u) = \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2. - Q_\theta(u)$$

So we seek critical points of I_θ. One can easily see that Q_θ is of class C^1 on E and

$$\langle Q'_\theta(u), h\rangle = (|u+\theta z+\omega_\theta(u)|^{p-1}(u+\theta z+\omega_\theta(u))-\theta f, h) = 0 \quad \text{for all } u, h \in E.$$

Elementary estimates then show that:

$$\begin{aligned} \left|(Q'_\theta(u), u) - (p+1)Q_\theta(u)\right| &\le c_1\|u+\theta z+\omega\|_{p+1}^p + c_2 \\ &\le C\left(|Q_\theta(u)|^{\frac{p}{p+1}} + 1\right) \end{aligned} \tag{6.3}$$

since: $|Q_\theta(u)| \ge a|u+\theta z+\omega|_{p+1}^{p+1} - b$.

On the other hand at a point u where $I'_\theta(u) = 0$, we have

$$\begin{aligned} I_\theta(u) &= \frac{\|u^+\|^2}{2} - \frac{\|u^-\|^2}{2} - Q_\theta(u) \\ 0 = \langle I'_\theta(u), h\rangle &= \langle u^+ - u^-, h\rangle - \langle Q'_\theta(u), h\rangle \text{ for } h \in E. \end{aligned} \tag{6.4}$$

Setting $h = u$ and $h = u^+ - u^-$ in the above formula we get

$$\|u^+\|^2 - \|u^-\|^2 - \langle Q'_\theta(u), u\rangle = 0 \tag{6.5}$$

$$\|u\|^2 + \langle Q'_\theta(u), u^+ - u^-\rangle = 0 \tag{6.6}$$

So by (6.4) and (6.5)

$$\frac{1}{2}\langle Q'_\theta(u), u\rangle - Q_\theta(u) = I_\theta(u)$$

Now from (6.3) and the above equality we get

$$\left(\frac{p+1}{2} - 1\right)Q_\theta(u) - C(|Q_\theta(u)|^{\frac{p}{p+1}} + 1) \le I_\theta(u)$$

which implies

$$|Q_\theta(u)| \le C(|I_\theta(u)| + 1) \quad \text{for} \quad I'_\theta(u) = 0.$$

To apply Bolle's method we need to estimate

$$\begin{aligned}
J_\theta(u) &= \frac{\partial}{\partial\theta} I_\theta(u) \\
&= -\left[\left\langle |u+\theta z+\omega|^{p-1}(u+\theta z+\omega), z+\frac{\partial}{\partial\theta}\omega\right\rangle\right. \\
&\le -\left(\theta f, z+\frac{\partial}{\partial\theta}\omega\right) - \left.(f, u+\theta z+\omega)\right] \\
&= -\left[\langle |u+\theta z+\omega|^{p-1}(u+\theta z+\omega), z\rangle - (f, u+\theta z+\omega)\right]
\end{aligned}$$

Now if we estimate the first term of J_θ in the usual way we get

$$|J_\theta(u)| \le C\left(|I_\theta(u)|^{\frac{p}{p+1}} + 1\right).$$

By applying Theorem 2.2, we get that either (P_4) has an infinite number of solutions or I_0 has a sequence of critical levels (c_n) satisfying $c_n \le Cn^{p+1}$. On the other hand, by a result of Tanaka [T2] we have that for each $\epsilon > 0$, $c_n \ge c_\epsilon n^{\frac{p+1}{p-1}-\epsilon}$ which proves the result only for $1 < p < 2$!

Remark 6.1. In order to improve the above result, we need better estimates on J_θ and to use the full strength of Theorem 2.2, we need to exploit that the estimate is only needed at a point $I'_\theta(u) = 0$. For such points, we have

$$u_{tt} - u_{xx} + |u+\theta z|^{p-1}(u+\theta z) = \theta f$$

and $J_\theta(u) = \langle u_{tt} - u_{xx}, z\rangle + \langle f, u+\theta z+\omega\rangle$.

Now we have a better estimate on the second term as

$$\langle f, u+\theta z+\omega\rangle \le \|u+\theta z+\omega\|_{p+1} \le C\left(|I_\theta(u)|^{\frac{1}{p+1}} + 1\right),$$

so what remains is to estimate the first term $\langle u_{tt} - u_{xx}, z\rangle$ which after integration by parts is dominated by

$$|\langle u_{tt} - u_{xx}, z\rangle| \le \left|\int_0^{2\pi} x_1 u_x(\pi, t) - x_0 u_x(0, t)\, dt\right|.$$

A natural question is whether u verifies an adequate conservation law that will yield a good estimate for the latter term.

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References

[A-R] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)

[Ba-Be1] Bahri, A., Berestycki, H.: A perturbation method in critical point theory and applications. T.A.M.S. **267**, 1–32 (1981)

[Ba-Be2] Bahri, A., Berestycki, H.: Forced vibrations of superquadratic Hamiltonian systems. Acta Math. **152**, 143–197 (1984)

[Ba-L] Bahri, A., Lions, P.L.: Morse index of some min-max critical points I. Application to multiplicity results. Commun. Pure & App. Math. **41**, 1027–1037 (1988)

[Be] Berestycki, H.: Le nombre de solutions de certains problèmes semilineaires elliptiques. J. Funct. Anal. **40**, 1–29 (1981)

[B] Bolle, P.: On the Bolza problem. J. Diff. Equations **152** (2), 274–288 (1999)

[[Bo] Bolza, O.: Lectures on the Calculus of variations. reprinted by E.G. Stechert and Co, New York, 1931

[C-E] Clarke, F.H., Ekeland, I.: Nonlinear oscillations and boundary value problems for Hamiltonian systems. Arch. Rat. Mech. Ann. **78**, 315–333 (1982)

[E-G-T] Ekeland, I., Ghoussoub, N., Tehrani, H.: Multiple solutions for a classical problem in the calculus of variation. J. Differ. Eqs. **131**, no. 2, 229–243 (1996)

[R] Rabinowitz, P.H.: Minimax methods in critical point theory with applications to differential equations. C.B.M.S., A.M.S., **65**, (1986)

[S1] Struwe, M.: Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problem. Manuscripta Math. **32**, 335–364 (1980)

[S2] Struwe, M.: *Variational methods and their applications to non-linear partial differential equations and Hamiltonian systems*. Berlin–Heidelberg–New York: Springer-Verlag, 1990

[T1] Tanaka, K.: Morse indices at critical points related to the symmetric mountain pass theorem and applications. Comm. in part. diff. equations, **14** (1), 99–128 (1989)

[T2] Tanaka, K.: Infinitely many periodic solutions for the equation $u_{tt} - u_{xx} \pm |u|^{p-1}u = f(x,t)$. II. Trans. Amer. Math. Soc. **307**, 615–645 (1988)