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Asymptotic behaviour of families of real curves

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Abstract. We consider the family of fibres of a polynomial function f on a smooth noncompact algebraic real surface and we characterise the regular fibres of f which are atypical due to their asymptotic behaviour at infinity. We compare to the similar problem in the complex case.

1. Introduction

We focus on the following aspect of the study of families of algebraic varieties: Let $\{X_t\}_{t\in\mathbb{K}}$ be a one-parametre algebraic family of real (resp. complex) affine smooth curves, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $X := \cup_{t\in\mathbb{K}} X_t$ be the total space and suppose that it is a smooth algebraic manifold over $\mathbb{R}$, resp. $\mathbb{C}$. Denote by $\tau : X \to \mathbb{K}$ the projection on the parametre space and take a regular value $t_0 \in \mathbb{K}$ of τ. Locally, in some neighbourhood of t_0, we have that each member X_t of the family is a union of circles and lines ($\mathbb{K} = \mathbb{R}$), resp. a smooth non-compact Riemann surface ($\mathbb{K} = \mathbb{C}$). The following problem arises:

Give a criterion for such a family to be differentiably trivial.

The value $t_0 \in \mathbb{K}$ (resp. the curve X_{t_0}) is called *typical* if the map τ is a C^∞ fibration at t_0; otherwise, t_0 (resp. X_{t_0}) is called *atypical*. Let Λ_τ be the set of atypical values of τ. It is well-known that Λ_τ is finite, see for instance [Th], [Ve]. The problem posed above is equivalent to characterizing the atypical values of τ.

Our aim is to characterise those atypical values which are not critical values, in the real case $\mathbb{K} = \mathbb{R}$. The question wheather an improper submersion is a fibration was considered by L. Siebenmann in [KS, Essay II, §1] where he proves a sufficient condition in very large generality. In a recent paper [FP], the problem is posed in the particular case of a family $X_t = f^{-1}(t)$ defined as the fibres of a real polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$. As a matter of fact, the criterion given by the main result [FP, Theorem 4.2] cannot be true, as our Example 3.4 shows.

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With the above assumptions, we prove the following:

Theorem 1.1. *Let $\{X_t\}_{t\in\mathbb{R}}$ be an algebraic family of real curves and let t_0 be a regular value of $\tau : X \to \mathbb{R}$. Assume that the total space X is nonsingular. Then the curve X_{t_0} is typical if and only if the Euler characteristic $\chi(X_t)$ is constant when t varies within some neighbourhood of t_0 and there is no component of X_t which vanishes at infinity as t tends to t_0.*

The following simple example shows why we need X be nonsingular. We also show in Remark 2 how the statement can be extended for singular X.

Example 1.2. Take $X := \{x^2 + y^2 - z^2 = 0\} \cup \{z = 0\} \subset \mathbb{R}^3$, the union of a cone and a plane through the vertex of the cone. Take τ the projection on a line L through the origin. Then, for an adequate choice of L, the curve X_t, for $t \neq 0$, is the disjoint union of a line with an oval, but X_0 is just a line. When t tends to 0, the oval is "vanishing" in the origin, nevertheless the Euler characteristic is constant.

The criterion in our Theorem looks natural, since the constancy of the Euler characteristic and the non-vanishing are necessary conditions. Moreover, it has a striking similarity to certain criteria in the complex case, as we explain in the following.

For a family $X_t = f^{-1}(t)$ given by the fibres of a complex polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$, it has been proven by Hà, H.V. and Lê, D.T. [HL] that:

A reduced curve X_{t_0} is typical if and only if its Euler characteristic $\chi(X_{t_0})$ is equal to the Euler characteristic of a general fibre of f $(\mathbb{K} = \mathbb{C})$.

This gives the answer (within the considered class of families) to the problem stated above, i.e. a criterion for a fibre $f^{-1}(t)$ to be atypical, since it is known that a critical fibre is atypical (by a monodromy argument due to Lê, D.T.). An equivalent form of this criterion is the following, see [ST]:

A regular fibre X_{t_0} of a complex polynomial function is typical if and only if there are no vanishing cycles at infinity corresponding to this fibre.

A common idea of "non-vanishing" appears in both real and complex case. We shall explain in Section 4 the exact meaning of vanishing cycles and some further results in the complex case. In contrast to the complex, in the real case the two conditions (i.e. constancy of Euler characteristic, respectively non-vanishing condition) have to be considered together: neither of them implies that X_{t_0} is atypical. We show this by Example 3.1 (constancy of Euler characteristic holds but non-vanishing condition fails), respectively Example 3.2 (non-vanishing and "non-splitting" condition both hold but constancy of Euler characteristic fails).

2. Families of real curves

Let $X \subseteq \mathbb{R}^n$ be a smooth noncompact algebraic surface and let $f : X \to \mathbb{R}$ be the restriction of a polynomial function $F : \mathbb{R}^n \to \mathbb{R}$. We say that a regular value

$s \in f(X)$ is *typical* if f is a C^∞ trivial fibration at s; otherwise, s is called *atypical*. The set of atypical values of f is denoted by Λ_f.

Let $I :=]a, b[\subset f(X)$ be an open interval containing only typical values of f. The restriction $f : f^{-1}(I) \to I$ is a C^∞ trivial fibration and it restricts to a trivial fibration on any connected component $\mathcal{Y}$ of $f^{-1}(I)$.

Definition 2.1. *Let $\mathcal{Y}$ be a connected component of $f^{-1}(]a, b[)$, where $]a, b[\subset f(X) \setminus \Lambda_f$. Denote $Y_t := X_t \cap \mathcal{Y}$. We say that a point $p \in X$ is a limit point of the family $\{Y_t\}_{t\in]a,b[}$ when t tends to a if there exists a sequence of points $p_k \in \mathcal{Y}$, $k \in \mathbb{N}$, such that p_k tends to p and that $f(p_k)$ tends to a. We denote:*

$$\lim_{t\to a\ ,\ t>a} Y_t := \{p \in X \mid p \text{ is a limit point of } \{Y_t\}_t \text{ when } t \to a\ \}.$$

We define analogously $\lim_{t\to b\ ,\ t<b} Y_t$. It follows that $\lim_{s\to a,\ s>a} Y_s \subseteq f^{-1}(a)$ and $\lim_{s\to b,\ s<b} Y_s \subseteq f^{-1}(b)$.

Note that Y_t is connected, $\forall t \in]a, b[$.

Definition 2.2. *We say that the connected component Y_s of $f^{-1}(s)$* vanishes at infinity *when s tends to a, $s > a$, if $\lim_{s\to a,\ s>a} Y_s = \emptyset$.*

We have a similar notion when s tends to b, $s < b$.

Note that $\lim_{s\to a,\ s>a} Y_s = \emptyset$ if and only if $\lim_{s\to a,\ s>a} \inf\{\ \|z\| \mid z \in Y_s\ \} = \infty$.

The following lemma is an easy consequence of the definitions, hence we only give a hint.

Lemma 2.3. *Let $a \in f(X)$ be a regular value of f. In the above notations, we have:*

(i) *The limit $\lim_{s\to a,\ s>a} Y_s$ is either empty or equal to the union of some connected components of $f^{-1}(a)$.*

(ii) *Let $\{Y'_s\}_s$ be the family of curves corresponding to some connected component $\mathcal{Y}'$. If $\left(\lim_{s\to a,\ s>a} Y_s\right) \cap \left(\lim_{s\to a,\ s>a} Y'_s\right) \neq \emptyset$, then $\{Y_s\} = \{Y'_s\}$ (in particular the two limits coincide).*

Proof. Since a is a regular value, one can prove that the limit $\lim_{s\to a,\ s>a} Y_s$ is an open subset of $f^{-1}(a)$, by using local coordinates at points of $f^{-1}(a)$. But this limit is obviously a closed set. The rest is straightforward. □

Definition 2.4. *We say that the family $\{Y_s\}_s$* splits when *s* tends to *a, $s > a$, if the limit $\lim_{s\to a,s>a} Y_s$ contains at least two connected components of $f^{-1}(a)$.*

Let us suppose that $0 \in f(X)$ is a regular value of f. With the above definitions, we formulate the following **conditions**:

(B) The Betti numbers of the fibre X_t are constant for t within some neighbourhood of 0.

(E) The Euler characteristic $\chi(X_t)$ is constant for t within some neighbourhood of 0.
(nV) There is no connected component of X_t which vanishes at infinity when t tends to 0, $t < 0$ or $t > 0$.
(nS) There is no connected component of X_t which splits when t tends to 0, $t < 0$ or $t > 0$.

We state now the main result, which contains Theorem 1.1.

Theorem 2.5. *Let* $0 \in f(X)$ *be a regular value of* f. *Then the following are equivalent:*

(i) *The value* 0 *is a typical value of* f.
(ii) *Conditions* **(B)** *and* **(nV)** *are fulfilled.*
(iii) *Conditions* **(E)** *and* **(nV)** *are fulfilled.*
(iv) *Conditions* **(B)** *and* **(nS)** *are fulfilled.*

Remark. Example 3.3 shows that conditions **(E)** + **(nS)** do not imply condition (i).

Proof. It is clear that (ii)⇒(iii). It is also clear that (i)⇒**(nV)** and (i)⇒**(nS)**, hence (i)⇒(ii) and (i)⇒(iv).

(ii)⇒(i) Let $\mathcal{D}$ be a connected component of $f^{-1}([-\varepsilon, \varepsilon])$.

We first prove that, under the non-vanishing assumption **(nV)** and for small enough $\varepsilon > 0$, $\mathcal{D}$ contains at least one connected component of X_0. We may assume without dropping generality that $f(\mathcal{D}) \supset]0, \varepsilon]$. Then take a decreasing sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset]0, \varepsilon[$, $\varepsilon \to 0$. By the non-vanishing condition **(nV)**, one can choose a bounded sequence $\{p_k\}_{k\in\mathbb{N}}$ of points $p_k \in X_{\varepsilon_k} \cap \mathcal{D}$. Hence there exists a convergent sub-sequence. But the limit of this sub-sequence has to be on X_0. On the other hand, it is on $\mathcal{D}$ since $\mathcal{D}$ is closed. Applying now Lemma 2.3(i), we are done.

We next show that the restriction $f_| : \mathcal{D} \cap f^{-1}([-\varepsilon, \varepsilon]) \to [-\varepsilon, \varepsilon]$ is a C^∞ trivial fibration, for small enough ε.

In case that $\mathcal{D}$ contains a "circle" component $K \subset X_0$, $K \overset{\text{diffeo}}{\simeq} S^1$, we may take an open tubular neighbourhood T of K such that $T \cap X_0 = K$. Since K is compact, we get that $X_t \cap T$ is compact, for any small enough $|t| > 0$. Therefore the restriction $f_| : T \cap f^{-1}([-\varepsilon, \varepsilon]) \to [-\varepsilon, \varepsilon]$ is a proper submersion (for small enough $\varepsilon > 0$) and we may apply Ehresmann's Fibration Theorem to conclude that it is a C^∞ trivial fibration. It also follows that the total space $T \cap f^{-1}([-\varepsilon, \varepsilon])$ is connected, hence it coincides with $\mathcal{D}$, for small enough ε.

Consider finally the case when $\mathcal{D}$ contains a "line" component $L \subset X_0$, $L \overset{\text{diffeo}}{\simeq} \mathbb{R}$.

We have that $b_0(\mathcal{D} \cap X_t)$ is constant, for t in some neighbourhood of 0. Indeed, this number cannot decrease as $t \to 0$, by the non-vanishing condition **(nV)**. The constancy of Betti numbers **(B)** means that the sum $\sum_{\mathcal{D}} b_0(\mathcal{D} \cap X_t)$ over all connected components $\mathcal{D}$, is constant, which shows that $b_0(\mathcal{D} \cap X_t)$ cannot increase either. In particular the condition **(nS)** is fulfilled.

The Betti number $m := b_0(\mathcal{D} \cap X_t)$ has to be equal to 1, by the following reason. The sets $\mathcal{D} \cap f^{-1}(]0, \varepsilon])$ and $\mathcal{D} \cap f^{-1}([-\varepsilon, 0[)$ contain exactly m connected components, since the restriction of f on each of them is trivial. This gives m families $\{Y'_t\}_t$ from the positive side and another m families $\{Y''_t\}_t$ from the negative side, according to Definition 2.1. The limit of such a family must be a connected component of X_0, by Lemma 2.3(i) and the above discussion. Our fixed line component L of X_0 is the limit of a certain positive side family and also the limit of a certain negative side family. Now take the union $\mathcal{V}$ of these two families together with their limit L. Then $\mathcal{V}$ is a connected component of $f^{-1}([-\varepsilon, \varepsilon])$, since it is a closed set and disjoint from the other families and their limits (by Lemma 2.3(ii)). Therefore $\mathcal{V}$ coincides with $\mathcal{D}$ and the argument is now complete.

It remains to show that the restriction $f : \mathcal{D} \cap f^{-1}([-\varepsilon, \varepsilon]) \to [-\varepsilon, \varepsilon]$ is a C^∞ trivial fibration. This will follow from Proposition 2.7, by applying it locally on the base.

(iv)⇒(i) The proof follows the pattern of the case (ii)⇒(i) and we can safely leave it to the reader.

(iii)⇒(ii) First, we note that (iii)⇒**(nS)**. Indeed, condition **(nV)** implies that no "line" component of X_t vanishes at infinity when t tends to 0, and condition **(E)** means that the number of "line" components of X_t does not depend on t within some neighbourhood of 0. Thus, **(nS)** is satisfied, since otherwise we would have $\chi(X_0) > \chi(X_t)$ for some small t.

Conditions **(nS)** and **(nV)** show that the number of connected components of X_t is constant for t within a neighbourhood of 0. Together with **(E)**, this implies that **(B)** is satisfied. Note that **(nS)**+**(nV)** alone do not imply **(B)**, see Example 3.2. □

Note 2.6. The algebricity assumption insures that $X \subset \mathbb{R}^n$ is closed, that the Betti numbers of X_t are finite and that the set of atypical values of f is discrete. Now, if we drop the algebricity condition but suppose that these three conditions are fulfilled, then the proof of the equivalences (i)$\cdots$(iv) is still valid.

The next statement could be known, though we were not able to find a reference.

Proposition 2.7. *Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold of dimension $m+1$ and let $g : M \to \mathbb{R}^m$ be a smooth function. Assume that the function g has no critical values and that all the fibres $g^{-1}(t)$ are diffeomorphic to $\mathbb{R}$ and closed in $\mathbb{R}^n$. Then g is a C^∞ trivial fibration.*

In particular, $M \overset{\text{diffeo}}{\simeq} \mathbb{R}^{m+1}$.

Proof. We show that g is locally trivial over a point $p \in \mathbb{R}^m$. The result will follow since $\mathbb{R}^m$ is contractible. Fix a point $q \in g^{-1}(p)$. Since g is a submersion, we can find a submanifold $T \subset M$ such that it is transversal to the fibres of g, that $q \in T$ and that the restriction of g to T is a C^∞ diffeomorphism onto a small open ball $B \subset \mathbb{R}^m$ centered at p.

One can take a smooth vector field $w : g^{-1}(B) \to \mathbb{R}^n$ tangent to the fibres of g and without zeros. Moreover, one may take the unit tangent vector field (with respect to the Riemannian metric of $\mathbb{R}^n$). The fibres being closed and diffeomorphic to $\mathbb{R}$, this vector field defines a global flow $\psi : T \times \mathbb{R} \to g^{-1}(B)$, which is a diffeomorphism. Since T is diffeomorphic to B, it follows that $g^{-1}(B)$ is diffeomorphic to $B \times \mathbb{R}$. □

Remark. In the statement of Theorem 1.1 we may allow the total space X have singularities and use the following extension of the notion of “regular value": Let X be a real algebraic variety, possibly singular, and let $\tau : X \to \mathbb{R}$ denote an algebraic function. We say that $t_0 \in \mathbb{R}$ is a *regular value* for τ if there is $\varepsilon > 0$ such that the space $X(\varepsilon) := X \cap \tau^{-1}(]t_0 - \varepsilon, t_0 + \varepsilon[)$ is a manifold and t_0 is a usual regular value of the map τ on $X(\varepsilon)$.

The conditions in this definition imply that the singular locus of X is “far" from the fibre X_{t_0}. Example 1.2 shows why singularities of X have to be kept out.

3. Examples

We consider polynomials $f : \mathbb{R}^2 \to \mathbb{R}$ of the following type

$$f(x, y) := \alpha(y)x^2 + 2\beta(y)x + \gamma(y). \tag{1}$$

Let $A := \{y \in \mathbb{R} \mid \alpha(y) = 0\}$. We assume that $\varepsilon > 0$ is such that $I :=] - \varepsilon, \varepsilon[$ contains only regular values of f and that

$$\text{for any } y \in A \text{ we have: } \beta(y) = 0 \text{ and } |\gamma(y)| \geq \varepsilon. \tag{2}$$

Then for any $t \in I$, the equation $f = t$ in the variable x has two complex solutions $x_{1,2}(y, t)$. Let $\Delta(y, t) = \beta^2(y) - \alpha(y)\gamma(y) + t\alpha(y)$ and let us denote

$$\mathcal{D} := \{(y, t) \in \mathbb{R}^2 \mid \Delta(y, t) \geq 0\}, \quad \mathcal{K} := \{(y, t) \in \mathbb{R}^2 \mid \Delta(y, t) = 0\},$$

$$\mathcal{L}(s) := \{(y, t) \in \mathbb{R}^2 \mid t = s\} \text{ and } \mathcal{A} := \{(y, t) \in \mathbb{R}^2 \mid y \in A\}.$$

Then $x_{1,2}(y, t)$ are real numbers if and only if $(y, t) \in \mathcal{D}$. It is easy to see that if $(y, t) \in \mathcal{D}$ and y tends to a point in A, then $|x_{1,2}(y, t)|$ tends to infinity. Note also that $\mathcal{A} \subseteq \mathcal{K}$ and that $\mathcal{K} \setminus \mathcal{A} \subseteq \partial\mathcal{D}$ because $\dfrac{\partial\Delta}{\partial t} = \alpha(y) \neq 0$ for $y \notin A$.

For $t_0 \in I$, the topology of the fibre $f^{-1}(t_0)$ can be described using the projections

$$\{(x, y, t) \in \mathbb{R}^3 \mid f(x, y) = t\} \xrightarrow{\pi} \mathbb{R}^2 \to \mathbb{R}, \quad (x, y, t) \xmapsto{\pi} (y, t) \mapsto t.$$

More precisely, the connected components of the sets $\mathcal{F}(t_0) := \mathcal{D} \cap \mathcal{L}(t_0)$ and $\mathcal{F}(t_0) \setminus \mathcal{A}$ are segments and isolated points. By (2), if P is an isolated point of $\mathcal{F}(t_0)$ such that $P \in \mathcal{A}$, then $\pi^{-1}(P) = \emptyset$. Moreover, if Q is an isolated point of $\mathcal{F}(t_0)$ such that $Q \notin \mathcal{A}$, then $\pi^{-1}(Q)$ is an isolated point of $f^{-1}(t_0)$, hence a critical point of f; but t_0 is a regular value of f.

Now, consider the one-dimensional connected components of $\mathcal{F}(t_0) \setminus \mathcal{A}$. Let $\mathcal{J}$ be such a segment, let $\overline{\mathcal{J}}$ be its closure in $[-\infty, \infty] \times \mathbb{R}$ and let $n(\mathcal{J})$ be the number of endpoints of $\overline{\mathcal{J}}$ which are contained in $\mathcal{K} \setminus \mathcal{A}$.

If $(y, t_0) \in \mathcal{J} \setminus \partial\mathcal{J}$, then $\pi^{-1}(y, t_0)$ consists of two distinct points. Now assume that $(y, t_0) \in \mathcal{J} \setminus \partial\mathcal{J}$ tends to an endpoint Q of $\mathcal{J}$. There are three possibilities.

If $Q \in \mathcal{K} \setminus \mathcal{A}$, then the two points in $\pi^{-1}(y, t_0)$ tend to the (unique!) point $\pi^{-1}(Q)$.

If $Q \in \mathcal{A}$, then the two distinct points in $\pi^{-1}(y, t_0)$ tend to infinity, because their x-coordinate will be unbounded.

If $Q = (\pm\infty, t_0)$, then the two distinct points in $\pi^{-1}(y, t_0)$ tend to infinity, because their y-coordinate will be unbounded.

These imply the following.

(i) If $n(\mathcal{J}) = 2$, then $\pi^{-1}(\overline{\mathcal{J}})$ is diffeomorphic to a circle.
(ii) If $n(\mathcal{J}) = 1$, then $\pi^{-1}(\overline{\mathcal{J}})$ is diffeomorphic to a line.
(iii) If $n(\mathcal{J}) = 0$, then $\pi^{-1}(\overline{\mathcal{J}})$ is diffeomorphic to a disjoint union of two lines.

Thus, for $t \in I$, we can read the topology of the fibre $f^{-1}(t)$ from the pictures of $\mathcal{D}$, $\mathcal{K}$ and $\mathcal{A}$. Moreover, using our main result, we can decide if 0 is an atypical value of f or not.

Our first three examples use these considerations. We leave the details to the reader.

Example 3.1. The polynomial

$$\begin{aligned} f(x, y) := x^2 y^3 (y^2 - 25)^2 + 2xy(y^2 - 25)(y + 25) \\ - (y^4 + y^3 - 50y^2 - 51y + 575) \end{aligned}$$

has the property that 0 is an atypical value, but the Betti numbers of the fibres $f^{-1}(t)$ are constant, for $|t|$ small enough. Namely, all these fibers have 5 non–compact connected components. For this polynomial, 0 is a regular value and condition (2) is satisfied. Besides the lines in $\mathcal{A}$, the set $\mathcal{K}$ contains also the graph of the function

$$y \longmapsto \varphi(y) := \frac{-(y^2 - 25)^2 (y + 1)}{y} .$$

It is easily seen that this graph has two connected components, separated by the vertical asymptote $\{y = 0\}$. The set $\mathcal{D}$ consists of the lines in $\mathcal{A}$ and the region of the plane situated between the two connected components of the graph of φ. The only local extrema of the function φ are two local maximums, for $y = \pm 5$, and a local minimum, between -5 and -1. For $|t|$ sufficiently small, the equation $\varphi(y) = t$ has five (complex) solutions, say $a_j(t)$, $j = 1, \dots, 5$. One of these solutions, say $a_3(t)$, is a real one, for all t, while the other four are real if and only if $t \le 0$. Assume that

$$\lim_{t\to 0} a_1(t) = \lim_{t\to 0} a_2(t) = -5 \quad \text{and} \quad \lim_{t\to 0} a_4(t) = \lim_{t\to 0} a_5(t) = 5.$$

For $|t|$ sufficiently small and $t < 0$, the set $\mathcal{F}(t) \setminus \mathcal{A}$ has 5 connected components and each of them corresponds to a line component in $f^{-1}(t)$. Namely, we have:

$$\begin{aligned} \mathcal{F}(t) \setminus \mathcal{A} = ([a_1(t), -5) \times \{t\}) \cup ((-5, a_2(t)] \times \{t\}) \cup \\ \cup ([a_3(t), 0) \times \{t\}) \cup ((0, a_4(t)] \times \{t\}) \cup ([a_5(t), \infty) \times \{t\}) . \end{aligned}$$

We also have:

$$\mathcal{F}(0) \setminus \mathcal{A} = ([-1, 0) \times \{0\}) \cup ((0, 5) \times \{0\}) \cup ((5, \infty) \times \{0\}) .$$

Therefore, when $t < 0$ tends to 0, the line components in $f^{-1}(t)$ corresponding to the segments $([a_1(t), -5) \times \{t\}) \cup ((-5, a_2(t)] \times \{t\})$ will "vanish" at infinity since $\lim_{t \to 0} a_1(t) = \lim_{t \to 0} a_2(t) = -5 \in A$. Also, each of the line components in $f^{-1}(t)$ corresponding to the segments $((0, a_4(t)] \times \{t\}) \cup ([a_5(t), \infty) \times \{t\})$ will "split" in two line components for $t = 0$ since $\lim_{t \to 0} a_4(t) = \lim_{t \to 0} a_5(t) = 5 \in A$.

For $|t|$ sufficiently small and $t \geq 0$, the set $\mathcal{F}(t) \setminus \mathcal{A}$ has 3 connected components: one corresponds to a line component in $f^{-1}(t)$ and each of the other two corresponds to two line components in $f^{-1}(t)$. Namely, we have:

$$\mathcal{F}(t) \setminus \mathcal{A} = ([a_3(t), 0) \times \{t\}) \cup ((0, 5) \times \{t\}) \cup ((5, \infty) \times \{t\}) .$$

Thus, for $|t|$ sufficiently small, $f^{-1}(t)$ is a disjoint union of 5 line components. This means that the Betti numbers of $f^{-1}(t)$ do not depend on t, if $|t|$ is sufficiently small.

On the other hand, for $\varepsilon > 0$ sufficiently small, the restrictions $f : f^{-1}(-\varepsilon, 0) \to (-\varepsilon, 0)$ and $f : f^{-1}[0, \varepsilon) \to [0, \varepsilon)$ are easily seen to be C^∞ trivial fibrations, while $f : f^{-1}(-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ is not a topological fibration.

Example 3.2. The polynomial $f(x, y) := x^2y^2 + 2xy + (y^2 - 1)^2$ has the property that conditions **(nV)** + **(nS)** are satisfied, but 0 is an atypical value. Besides the line in $\mathcal{A}$, the set $\mathcal{K}$ contains also the graph of the function $\varphi(y) := y^4 - 2y^2$. This function has a local maximum, for $y = 0$, and two local minimums, for $y = \pm 1$. The set $\mathcal{D}$ consists of the line in $\mathcal{A}$ and the region of the plane situated above the graph of φ. For $t < 0$ with $|t|$ sufficiently small, the curve $f^{-1}(t)$ has two circle components. For $t \geq 0$ sufficiently small, the curve $f^{-1}(t)$ has two line components.

Example 3.3. The polynomial

$$\begin{aligned} f(x, y) := {} & x^2y^3(9 - y^2)^2 + 2xy(9 - y^2)(y^3 + y + 6) \\ & + 2(y^5 - 6y^3 + 6y^2 + 25y + 6) \end{aligned}$$

has the property that conditions **(E)** + **(nS)** are satisfied, but 0 is an atypical value. Besides the lines in $\mathcal{A}$, the set $\mathcal{K}$ contains also the graph of the function

$$y \longmapsto \varphi(y) := \frac{(y^2 - 1)(y^2 - 4)(y^2 - 9)}{y}.$$

This graph has two connected components, separated by the vertical asymptote $\{y = 0\}$. For $|t|$ sufficiently small, the equation $\varphi(y) = t$ has six real solutions. There exists $a \in]1, 2[$ and $b \in]2, 3[$ such that the local maxima of φ are $-b$ and a, and the local minima of φ are $-a$ and b. The set $\mathcal{D}$ consists of the lines in $\mathcal{A}$ and the region of the plane situated between the two connected components of the graph of φ. For $|t| \neq 0$ sufficiently small, the curve $f^{-1}(t)$ has a circle component and 4 line components. The curve $f^{-1}(0)$ has only 4 line components.

Example 3.4. This was suggested by Henry King. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := 2x^2y^3 - 9xy^2 + 12y$. Then f is a trivial C^∞ fibration because the map $F : \mathbb{R}^2 \to \mathbb{R}^2$,

$$F(x, y) := \left(f(x, y), \frac{x}{g(x, y)} \right) , \quad \text{where } \ g(x, y) := 2x^2y^2 - 9xy + 12 ,$$

is a diffeomorphism of order two ($F^{-1} = F$).

The criterion given in the paper [FP] for a regular fibre to be atypical is contradicted by the polynomial $h : \mathbb{R}^2 \to \mathbb{R}$, defined by $h(x, y) := f(x + y, y) = 2x^2y^3 + 4xy^4 - 9xy^2 + 2y^5 - 9y^3 + 12y$. For all $y \in \mathbb{R} \setminus \{0\}$, the discriminant (in variable x) of the equation $\frac{\partial h}{\partial y} = 0$ is $\Delta = 4y^2(4y^4 + 18y^2 + 9) > 0$. Therefore $\partial h/\partial y$ changes sign along the germ at infinity of the curve $C := \{\partial h/\partial y = 0\}$. Moreover, it is easy to see that along C, if y tends to 0, then $|x|$ tends to infinity and $h(x, y)$ tends to 0. According to [FP, Defintion 4.1], the value 0 is a "real critical value at infinity" for h and this would mean, by [FP, Theorem 4.2], that h is not locally trivial, in contrast to the explicit computation above.

4. Real versus complex

In the remainder we focus on families of fibres of polynomial functions $\mathbb{K}^2 \to \mathbb{K}$. We first introduce some notations in a larger context.

Given two $\mathbb{K}$-analytic functions $f, h : \mathbb{K}^n \to \mathbb{K}$, we say that the set:

$$\Gamma(f, h) := \text{closure}\{\text{Sing}\,(f, h) \setminus (\text{Sing}\, f \cup \text{Sing}\, h)\}$$

is the *polar locus* of (f, h). Let $\check{\mathbb{P}}^{n-1}$ be the set of all hyperplanes of the projective space $\mathbb{P}^{n-1}$ and let us identify a hyperplane $H \in \check{\mathbb{P}}^{n-1}$ by its defining linear form $l_H : \mathbb{K}^n \to \mathbb{K}$.

Polar curve lemma 4.1 ([Ti-1]). *There is an open dense $\Omega_f \subset \check{\mathbb{P}}^{n-1}$ (Zariski-open in the complex case) such that, for any $H \in \Omega_f$, $\Gamma_{\mathcal{S}}(l_H, f)$ is a curve or it is void.*

Identify $\mathbb{K}^n$ to its image by the embedding $\mathbb{K}^n \hookrightarrow \mathbb{K}^n \times \mathbb{K}$, $x \mapsto (x, f(x))$ and take the closure $\overline{\Gamma(l_H, f)}$ of $\Gamma(l_H, f) \subset \mathbb{K}^n \times \mathbb{K}$ within $\mathbb{P}^n \times \mathbb{K}$. Under the conditions of the above lemma, the intersection $\overline{\Gamma(l_H, f)} \cap H_\infty \times \mathbb{K} = \{(p_i, t_i)\}_i$ is a finite set, where H_∞ denotes the hyperplane at infinity $\mathbb{P}^n \setminus \mathbb{K}^n$ (coordinates of $\mathbb{K}^n$ are supposed fixed). Note that the values t_i might not be all distinct.

In the complex case $n = 2$ we have the following result, which does not hold over the reals.

Proposition 4.2. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a complex polynomial function. Let $H \in \Omega_f$ and let $\{(p_1, t_1), \ldots, (p_k, t_k)\} = \overline{\Gamma(l_H, f)} \cap H_\infty \times \mathbb{C}$. Denote $X_{D_i} := f^{-1}(D_i)$, where $D_i \subset \mathbb{C}$ is a closed disc centered at t_i, $i \in \{1, \ldots, k\}$, which does not contain any other atypical value of f besides t_i. Then:*

(i) *The union of* $\{t_1, \ldots, t_k\}$ *with the set of critical values of* f *is equal to the set of atypical values* Λ_f *of* f.

(ii) *Assume that* X_{t_i} *has only isolated singularities with the sum of their Milnor numbers denoted by* μ_i. *Let* $a_i \in \partial D_i$. *Then* X_{D_i} *is homotopy equivalent to* X_{a_i} *to which one attaches a number of* $\mu_i + \sum_{t_j=t_i} \lambda_j$ *cells of dimension 2, where the positive integers* λ_j *are defined bellow.*

Proof. (i) was shown in [HL] and [HN]. More generally, for polynomial functions $\mathbb{C}^n \to \mathbb{C}$ with isolated $\mathcal{W}$-singularities at infinity, it was shown in [ST, 3.4, 4.5].
(ii). By the theory of complex isolated singularities, taking a small ball B at an affine critical point of X_{t_i} one can show that $B \cap X_{D_i}$ is homotopy equivalent to $B \cap X_{a_i}$ to which one attaches a number of 2-cells equal to the Milnor number of the critical point (assuming that D_i is small enough).

In case of a singular point "at infinity", it was shown in [ST] that for each point (p_i, t_i), if one takes a small enough ball $B' \subset \mathbb{P}^2 \times \mathbb{C}$ at (p_i, t_i), than again $B' \cap X_{D_i}$ is homotopy equivalent to $B' \cap X_{a_i}$ to which one attaches λ_i cells of dimension 2 (for small enough D_i). The result then follows by patching the local contributions. The positive integer λ_i is defined as the local intersection multiplicity $\mathrm{int}_{(p_i,t_i)}(\overline{\Gamma(l_H, f)}, X_{t_i})$, see [ST, 3.4, 4.4] and [Ti-2]. □

We may suppose without loss of generality that f has connected general fibre. Then this fibre X_t is a bouquet of circles which are a basis of cycles in homology $H_1(X_t, \mathbb{Z})$. Starting with a general fibre X_t we may fill in the space $\mathbb{C}^2$ (homotopically) by adding a certain number of 2-cells for each singular point (in the affine space or at infinity). It follows that each such 2-cell kills a certain cycle, since the result is $\mathbb{C}^2$, hence contractible. One calls *vanishing cycles at infinity* those cycles that are killed in some neighbourhood of a point at infinity of type (p_i, t_i). Further details may be found in [ST].

Remarks. The problem of characterising Λ_f for a polynomial f of several complex variables has an answer only in special cases, see [NZ], [Pa], [ST], [Ti-1].

In case $n = 2$, $\mathbb{K} = \mathbb{C}$, the points (p_i, t_i) are called "singularities at infinity". There are several ways of characterising them, see [Du], [Ti-2].

In the real case, the polar curve criterion 4.2(i) does not work, as shown by Example 3.4.

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