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Homogeneous nilmanifolds attached to representations of compact Lie groups

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Abstract. For each compact Lie algebra $\mathfrak{g}$ and each real representation V of $\mathfrak{g}$ we consider a two-step nilpotent Lie group $N(\mathfrak{g}, V)$, endowed with a natural left-invariant riemannian metric. The homogeneous nilmanifolds so obtained are precisely those which are naturally reductive. We study some geometric aspects of these manifolds, finding many parallels with H-type groups. We also obtain, within the class of manifolds $N(\mathfrak{g}, V)$, the first examples of non-weakly symmetric, naturally reductive spaces and new examples of non-commutative naturally reductive spaces.

1. Introduction

Two-step nilpotent Lie groups endowed with a left-invariant riemannian metric, often called *two-step homogeneous nilmanifolds*, have attracted considerable attention in the last twenty years, specially in riemannian geometry [31,54,17,18], harmonic analysis [33,12,4] and spectral geometry [24,15,16,44,48]. H-type groups (or generalized Heisenberg groups), introduced by A. Kaplan around 1980 [30], are a very special subclass of two-step homogeneous nilmanifolds. These spaces have provided examples and counterexamples to many questions and conjectures [32,47, 45,23,36,11,9,7,10].

Starting from a real representation (J, V) of a Clifford algebra $\mathsf{Cl}(\mathfrak{z})$, Kaplan constructs a two-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{z} \oplus V$ with center $\mathfrak{z}$ and Lie bracket defined on V by $\langle [v, w], z\rangle = \langle J_z v, w\rangle$ for all $v, w \in V$, $z \in \mathfrak{z}$, where $\langle , \rangle$ is a natural inner product on $\mathfrak{n}$. The corresponding H-type group is denoted by $(N, \langle , \rangle)$, where N is the simply connected Lie group with Lie algebra $\mathfrak{n}$, endowed with the left-invariant metric determined by $\langle , \rangle$.

We study in this work another subclass of two-step homogeneous nilmanifolds, with a construction analogous to that of H-type groups, but starting from a real representation (π, V) of a compact Lie algebra $\mathfrak{g}$. Indeed, let $\mathfrak{n} = \mathfrak{g} \oplus V$ be the two-step nilpotent Lie algebra with center $\mathfrak{g}$ and Lie bracket defined on V by $\langle [v, w], x\rangle = \langle \pi(x)v, w\rangle$ for all $v, w \in V$, $x \in \mathfrak{g}$, where $\langle , \rangle$ is a fixed $\mathfrak{g}$-invariant inner product on $\mathfrak{n}$ (see 3.1(iii)). We denote by $N(\mathfrak{g}, V)$ the simply connected Lie

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group with Lie algebra $\mathfrak{n} = \mathfrak{g} \oplus V$ and we endow it with the left-invariant metric determined by $\langle , \rangle$, obtaining a two-step homogeneous nilmanifold $(N(\mathfrak{g}, V), \langle , \rangle)$.

We shall prove, using a result due to C. Gordon [22], that the spaces $(N(\mathfrak{g}, V), \langle , \rangle)$ have a neat geometric characterization within the class of homogeneous nilmanifolds: they are precisely the naturally reductive ones (see Sect. 2). In particular, they also are riemannian g.o. spaces and D'Atri spaces [13,14,7].

We prove in Sect. 3 some partial results on the isometry classes of these two-step homogeneous nilmanifolds and on the isomorphism classes of the underlying two-step nilpotent Lie groups. Further, we compute the isotropy subgroup K of the isometry group of $(N(\mathfrak{g}, V), \langle , \rangle)$, which is given essentially by $K = G \times U$, where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and U is the group of orthogonal intertwining operators of V. The group U acts trivially on the center $\mathfrak{g}$ and each $g \in G$ acts on $\mathfrak{n} = \mathfrak{g} \oplus V$ by $(\mathrm{Ad}(g), \pi(g))$, where we also denote by π the corresponding representation of G on V. This is very similar to the H-type case, where essentially $K = \mathsf{Spin}(\mathfrak{z}) \times U$ (see [46]). We note that the isometry group of any simply connected homogeneous nilmanifold $(N, \langle , \rangle)$ is given by $\mathrm{I}(N, \langle , \rangle) = K \ltimes N$, where $K = \mathrm{Aut}(\mathfrak{n}) \cap \mathsf{O}(\mathfrak{n}, \langle , \rangle)$ is the isotropy subgroup of the identity element of N (see [54]).

The other goal of this paper is to study the notions of commutativity and weak symmetry within the class of naturally reductive manifolds $(N(\mathfrak{g}, V), \langle , \rangle)$. A *commutative space* is a connected riemannian homogeneous space M whose algebra of all $\mathrm{I}(M)^0$-invariant differential operators is commutative, where $\mathrm{I}(M)^0$ denotes the connected component of the full isometry group $\mathrm{I}(M)$. The notion of commutativity is strongly related to that of Gelfand pair, and it has been studied in several articles, see for instance [7,37–39,33,45,4,34,3,5,1].

Let T be any maximal torus of G and let $\tilde{V}$ denote a T-invariant complement in V of the zero weight space V_0, regarded naturally as a complex vector space. We prove in Sect. 4 the following characterization:

$N(\mathfrak{g}, V)$ is a commutative space if and only if the action of $T \times U$ on $\tilde{V}$ is multiplicity-free.

The action of a compact group on a complex vector space W is said to be *multiplicity free* if and only if all the isotypic components of the natural representation on the polynomial ring $\mathbb{C}[W]$ are irreducible.

Using this characterization, we shall prove that if $\mathfrak{g}$ is semisimple, V is irreducible of real type (i.e. the complexification $V_{\mathbb{C}}$ is also irreducible) and $\dim V > 3\,\mathrm{rank}(\mathfrak{g})$, then $N(\mathfrak{g}, V)$ is not a commutative space. This gives a large class of non-commutative, naturally reductive spaces; the first examples of this kind were given in [27,28]. We also obtain some new examples of commutative spaces.

Finally, we exhibit in Sect. 6 some applications to weakly symmetric spaces. A connected riemannian manifold M is said to be *weakly symmetric* if for any two points $p, q \in M$ there exists an isometry of M mapping p to q and q to p. These spaces, introduced by A. Selberg in [49], have been studied for instance in [6–8, 38,55,1]. It is proved in [49] that any weakly symmetric space is a commutative space (with respect to $\mathrm{I}(M)$-invariance, but this coincides with $\mathrm{I}(M)^0$-invariance for homogeneous nilmanifolds [5]). Thus, the non-commutative manifolds $N(\mathfrak{g}, V)$

described above are the first examples of non-weakly symmetric naturally reductive spaces.

We wish to note that we have recently become aware that certain one-dimensional solvable extensions of the two-step nilpotent Lie groups $N(\mathfrak{g}, V)$, with V irreducible, have been previously introduced by P. Eberlein and J. Heber in [19] for the purpose of constructing new Einstein solvmanifolds. Also, the curvature of these solvmanifolds has been studied in the thesis work of Sven Leukert (see [43]).

2. Description of naturally reductive homogeneous nilmanifolds via representations

Let M be a connected homogeneous riemannian manifold. Furthermore, let G be a Lie group acting transitively and effectively from the left by isometries on M and denote by K the isotropy subgroup of $p \in M$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Suppose $\mathfrak{m}$ is a vector space complement to $\mathfrak{k}$ in $\mathfrak{g}$ such that $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$ (i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a reductive decomposition). Thus we may identify $\mathfrak{m}$ with $\mathrm{T}_p M$ via the map $x \to \dot{\gamma}_x\ (0)$, where $\gamma_x(t) = \exp tx.p$. We denote by $\langle , \rangle$ the inner product on $\mathfrak{m}$ induced by the riemannian metric of M.

Definition 2.1. A manifold M is said to be *naturally reductive* if there exists a Lie group G and a subspace $\mathfrak{m}$ of $\mathfrak{g}$ with the properties described above such that

$$\langle [x, y]_{\mathfrak{m}}, z\rangle + \langle y, [x, z]_{\mathfrak{m}}\rangle = 0 \qquad \forall\, x, y, z \in \mathfrak{m}, \tag{1}$$

where $[x, y]_{\mathfrak{m}}$ denotes the projection of $[x, y]$ on $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

Condition (1) can be replaced by the following (see [35],p.192,196,201): any geodesic γ of M with $\gamma(0) = p$ is of the form $\gamma(t) = \exp tx.p$ for some $x \in \mathfrak{m}$. Clearly, any symmetric space is naturally reductive.

We consider a simply connected real nilpotent Lie group N endowed with a left-invariant riemannian metric, denoted by $(N, \langle , \rangle)$, where $\langle , \rangle$ is the inner product on the Lie algebra $\mathfrak{n}$ of N determined by the metric. The riemannian manifold $(N, \langle , \rangle)$ is said to be a (simply connected) *homogeneous nilmanifold*.

The full group of isometries of $(N, \langle , \rangle)$ is given by

$$\mathrm{I}(N, \langle , \rangle) = K \ltimes N \qquad \text{(semidirect product)}, \tag{2}$$

where $K = \mathrm{Aut}(\mathfrak{n}) \cap \mathrm{O}(\mathfrak{n}, \langle , \rangle)$ is the isotropy subgroup of the identity and N acts by left translations (see [54]). Thus, the structure of $\mathrm{I}(N, \langle , \rangle)$ is completely determined by K. Note that, since we always assume that N is simply connected, we make no distinction between automorphisms of N and $\mathfrak{n}$.

The following result follows from the proof of Theorem 3 in [54].

Theorem 2.2 ([54])**.** *Let N_1, N_2 be nilpotent Lie groups. Then $(N_1, \langle , \rangle_1)$ is isometric to $(N_2, \langle , \rangle_2)$ if and only if there exists an isomorphism of Lie algebras $A : \mathfrak{n}_1 \to \mathfrak{n}_2$ such that $\langle Ax, Ay\rangle_2 = \langle x, y\rangle_1$ for all $x, y \in \mathfrak{n}_1$.*

Let N be a two-step nilpotent Lie group and let $\langle , \rangle$ be an inner product on $\mathfrak{n}$. We call the corresponding $(N, \langle , \rangle)$ a (simply connected) *two-step homogeneous nilmanifold*. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and let $\mathfrak{n} = \mathfrak{z} \oplus V$ be the orthogonal decomposition. For each $x \in \mathfrak{z}$ we define $J_x : V \to V$ by

$$\langle J_x v, w \rangle = \langle x, [v, w] \rangle, \qquad v, w \in V. \tag{3}$$

Note that J_x is skew-symmetric for all $x \in \mathfrak{z}$ and $J : \mathfrak{z} \to \mathrm{End}(V)$ is a linear map. The maps $\{J_x\}_{x \in \mathfrak{z}}$ give the relationship between the Lie bracket of $\mathfrak{n}$ and the metric $\langle , \rangle$, thus they carry a lot of geometric information on the riemannian manifold $(N, \langle , \rangle)$ (see for example [31,17,18]). It is easy to prove that the isotropy subgroup K of $\mathrm{I}(N, \langle , \rangle)$ (see (2)) is given by

$$K = \{(\phi, T) \in \mathsf{O}(\mathfrak{z}, \langle , \rangle) \times \mathsf{O}(V, \langle , \rangle) : T J_x T^{-1} = J_{\phi x}, \quad x \in \mathfrak{z}\}. \tag{4}$$

Let $\mathfrak{k}$ be the Lie algebra of K. Thus $\mathfrak{k} = \mathrm{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n}, \langle , \rangle)$, i.e. the skew symmetric derivations of $(\mathfrak{n}. \langle , \rangle)$, and

$$\mathfrak{k} = \{(A, B) \in \mathfrak{so}(\mathfrak{z}, \langle , \rangle) \times \mathfrak{so}(V, \langle , \rangle) : B J_x - J_x B = J_{Ax}, \quad x \in \mathfrak{z}\}. \tag{5}$$

Remark 2.3. If $[\mathfrak{n}, \mathfrak{n}] \neq \mathfrak{z}$ then $N \simeq \mathbb{R}^k \times N_1$, where $N_1 = \exp([\mathfrak{n}, \mathfrak{n}] \oplus V)$ and $\mathbb{R}^k = \exp(\mathfrak{z} \cap [\mathfrak{n}, \mathfrak{n}]^{\perp})$ ($\exp : \mathfrak{n} \to N$ is the usual Lie exponential map). In this case, we will say that $(N, \langle , \rangle)$ has *euclidean factor*, since the direct product is also a product of riemannian manifolds. We have that $(N, \langle , \rangle)$ has euclidean factor if and only if there exists a nonzero $x \in \mathfrak{z}$ such that $J_x = 0$ (see [17, Proposition 2.7]).

Among the two-step homogeneous nilmanifolds the H-type groups are of particular significance. They were introduced by A. Kaplan in [30]. We say that $(N, \langle , \rangle)$ is an *H-type group* if $J_x^2 = -\langle x, x \rangle I$ for all $x \in \mathfrak{z}$. We next recall some general properties of H-type groups, following essentially [6] (see also [7]). Let $m = \dim \mathfrak{z}$ and let $\mathsf{Cl}(m)$ denote the Clifford algebra $\mathsf{Cl}(\mathfrak{z}, -| \; |^2)$. When $(N, \langle , \rangle)$ is H-type the action J of $\mathfrak{z}$ on V extends to a real representation of $\mathsf{Cl}(m)$. So V is a real $\mathsf{Cl}(m)$-module, and every real Clifford module arises in this way ([30]). The classification of H-type algebras up to isomorphism is given as follows:

(i) If $m \not\equiv 3 \pmod 4$, then $\mathsf{Cl}(m)$ has a unique irreducible module V_0. The general H-type algebra with a m-dimensional center is then obtained by taking $\mathfrak{n} = \mathfrak{z} \oplus (V_0)^p$ with $p \geq 1$.
(ii) If $m \equiv 3 \pmod 4$, then $\mathsf{Cl}(m)$ has two non-equivalent irreducible modules V_1 and V_2. The general H-type algebra with an m-dimensional center is obtained by taking $\mathfrak{n} = \mathfrak{z} \oplus (V_1)^p \oplus (V_2)^q$ with $p \geq q \geq 0$, $p + q \geq 1$, and only $V = (V_1)^p \oplus (V_2)^q$ and $V = (V_1)^q \oplus (V_2)^p$ lead to isomorphic H-type algebras.

In both cases $\mathfrak{n}$ can be endowed with a unique inner product (up to isometry) for which the H-type condition holds. If $x \in \mathfrak{z}$ is a unit vector, the map J_x defined in (3) extends to an element in K by setting it equal minus the reflection with respect to the hyperplane $x^{\perp}$ in $\mathfrak{z}$. The subgroup of K generated by the automorphisms $\{J_x\}_{x \in \mathfrak{z}}$ is isomorphic to the group $\mathsf{Pin}(m)$. If U denotes the group of orthogonal intertwining operators for the representation of $\mathsf{Cl}(m)$ on V, then $K^0 = \mathsf{Spin}(m) \times U$.

Natural reductivity on homogeneous nilmanifolds has been studied by C. Gordon in [22] (see also [32,52]). It is proved in [22] that if $(N, \langle,\rangle)$ is naturally reductive then N must be at most two-step nilpotent and the following characterization for naturally reductive two-step homogeneous nilmanifolds is given (see also [42] for an alternative proof of the following theorem using the theory of homogeneous structures developed in [51,52]).

Theorem 2.4. [22] *Let $(N, \langle,\rangle)$ be a two-step homogeneous nilmanifold without euclidean factor. $(N, \langle,\rangle)$ is naturally reductive if and only if*

(i) $J_{\mathfrak{z}} = \{J_x\}_{x\in\mathfrak{z}}$ *is a Lie subalgebra of* $\mathfrak{so}(V, \langle,\rangle)$.
(ii) $\tau_x \in \mathfrak{so}(\mathfrak{z}, \langle,\rangle)$ *for any* $x \in \mathfrak{z}$, *where* $\tau_x : \mathfrak{z} \to \mathfrak{z}$ *is given by* $J_xJ_y - J_yJ_x = J_{\tau_x y}$ *for all* $x, y \in \mathfrak{z}$.

Note that (ii) is equivalent to $(\tau_x, J_x) \in \mathfrak{k}$, the skew symmetric derivations of $\mathfrak{n}$ (see (5)).

Definition 2.5. *If* $\mathfrak{h}$ *is a Lie subalgebra (or just a subspace) of* $\mathrm{End}(V)$ *such that* $\mathfrak{h} \subset \mathfrak{so}(V, \langle,\rangle)$, *then we call* $\langle,\rangle$ *an* **$\mathfrak{h}$-invariant inner product**.

It follows from Theorem 2.4 that if $(N, \langle\,,\rangle)$ is naturally reductive, then the bilinear form τ given in (ii) defines a Lie algebra structure on $\mathfrak{z}$ and the map $J : \mathfrak{z} \to \mathrm{End}(V)$ becomes a real representation of the Lie algebra $(\mathfrak{z}, \tau)$ on V. Moreover, $\langle,\rangle|_{V\times V}$ is a $J_{\mathfrak{z}}$-invariant inner product and since $\tau_x \in \mathfrak{so}(\mathfrak{z}, \langle,\rangle)$ we have that $\langle,\rangle|_{\mathfrak{z}\times\mathfrak{z}}$ is $\mathrm{ad}\,\mathfrak{z}$-invariant, where ad denotes the adjoint representation of $(\mathfrak{z}, \tau)$.

Conversely, let $\mathfrak{g}$ be a real Lie algebra endowed with an $\mathrm{ad}\,\mathfrak{g}$-invariant inner product $\langle,\rangle_{\mathfrak{g}}$, and let (π, V) be a real faithful representation of $\mathfrak{g}$ endowed with a $\pi(\mathfrak{g})$-invariant inner product $\langle,\rangle_V$ and without trivial subrepresentations, that is, $\bigcap_{x\in\mathfrak{g}} \mathrm{Ker}\,\pi(x) = 0$. We define a two-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{g} \oplus V$ with Lie bracket given by

$$\begin{cases} [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{n}} = [\mathfrak{g}, V]_{\mathfrak{n}} = 0, \qquad [V, V]_{\mathfrak{n}} \subset \mathfrak{g}, \\ \langle [v, w]_{\mathfrak{n}}, x\rangle_{\mathfrak{g}} = \langle \pi(x)v, w\rangle_V \qquad \forall\, x \in \mathfrak{g},\ v, w \in V, \end{cases} \tag{6}$$

and we endow $\mathfrak{n}$ with the inner product $\langle,\rangle$ defined by

$$\langle,\rangle|_{\mathfrak{g}\times\mathfrak{g}} = \langle,\rangle_{\mathfrak{g}}, \qquad \langle,\rangle|_{V\times V} = \langle,\rangle_V, \qquad \langle\mathfrak{g}, V\rangle = 0. \tag{7}$$

Finally, we take N the simply connected Lie group having Lie algebra $\mathfrak{n}$ and we endow N with the left-invariant metric determined by $\langle,\rangle$, obtaining a two-step homogeneous nilmanifold $(N, \langle\,,\rangle)$.

Since (π, V) has no trivial subrepresentations, we have that $\mathfrak{g}$ is the center of $\mathfrak{n}$. Moreover, V is the orthogonal complement of $\mathfrak{g}$ and the transformations defined in (3) for $(N, \langle\,,\rangle)$ are precisely $\{\pi(x)\}_{x\in\mathfrak{g}}$. Thus $(N, \langle\,,\rangle)$ has no euclidean factor, since (π, V) is faithful (see Remark 2.3). It then follows from Theorem 2.4 that

$(N, \langle\, ,\rangle)$ is naturally reductive. In fact, $\pi(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{so}(V, \langle, \rangle_V)$ and since $\pi(x)\pi(y) - \pi(y)\pi(x) = \pi([x, y])$ for all $x, y \in \mathfrak{g}$, we have that $\tau_x = \mathrm{ad}\, x \in \mathfrak{so}(\mathfrak{g}, \langle, \rangle_{\mathfrak{g}})$ for all $x \in \mathfrak{g}$.

Remark 2.6. If a real Lie algebra $\mathfrak{g}$ admits an ad $\mathfrak{g}$-invariant inner product then $\mathfrak{g}$ is a *compact Lie algebra*, i.e. any of the following equivalent conditions hold (see [53]):

(i) $\mathfrak{g}$ is the Lie algebra of a compact Lie group.
(ii) The Killing form $B(x, y) = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$ is negatively semidefinite.
(iii) $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{c}$ with $\mathfrak{c}$ the center of $\mathfrak{g}$ and $\overline{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$ a compact semisimple Lie algebra (i.e. the Killing form of $\overline{\mathfrak{g}}$ is negative definite).

We have proved the following result.

Theorem 2.7. *Let $\mathfrak{g}$ be a compact Lie algebra endowed with an* ad $\mathfrak{g}$*-invariant inner product $\langle, \rangle_{\mathfrak{g}}$ and let (π, V) be a real faithful representation of $\mathfrak{g}$ without trivial subrepresentations and endowed with a $\pi(\mathfrak{g})$-invariant inner product $\langle, \rangle_V$. Then the two-step homogeneous nilmanifold $(N, \langle, \rangle)$ having Lie algebra $\mathfrak{n} = \mathfrak{g} \oplus V$ defined as in* (6)*, with $\langle, \rangle$ defined in* (7)*, is a naturally reductive space without euclidean factor. Moreover, any homogeneous nilmanifold $(N, \langle, \rangle)$ without euclidean factor which is naturally reductive can be constructed in this way.*

Clearly, this theorem states essentially the same as Theorem 2.4. However, we shall see in the next sections that the representation approach is very useful to study the naturally reductive two-step homogeneous nilmanifolds. We obtain a kind of classification of such spaces and we compute explicitly their isometry groups. Also, conditions for the commutativity of invariant integrable functions on N (or equivalently invariant differential operators) on these groups will be given in terms of representation theory, and this is the key to our study of commutative naturally reductive two-step homogeneous nilmanifolds in Sect. 4.

Remark 2.8. Suppose that the representation (π, V) of $\mathfrak{g}$ is not faithful or it has some nonzero trivial subrepresentation. We take the orthogonal decompositions

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathrm{Ker}\, \pi, \qquad V = V_1 \oplus \bigcap_{x \in \mathfrak{g}} \mathrm{Ker}\, \pi(x).$$

It is easy to see that $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ and V_1 is a $\mathfrak{g}$-invariant subspace of V, thus $(\pi_1 = \pi|_{\mathfrak{g}_1}, V_1)$ is a real faithful representation of $\mathfrak{g}_1$ without trivial subrepresentations. Moreover $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathbb{R}^k$, where $\mathfrak{n} = \mathfrak{g} \oplus V$, $\mathfrak{n}_1 = \mathfrak{g}_1 \oplus V_1$ and $\mathbb{R}^k$ is a central subspace of $\mathfrak{n}$ orthogonal to $[\mathfrak{n}, \mathfrak{n}]$. Henceforth, $(N, \langle, \rangle) = (N_1, \langle, \rangle|_{\mathfrak{n}_1 \times \mathfrak{n}_1}) \times \mathbb{R}^k$, where $\mathbb{R}^k$ becomes the euclidean factor of $(N, \langle, \rangle)$ (see Remark 2.3).

3. Two-step nilpotent Lie groups attached to representations of compact Lie algebras

In this section, we shall study in detail some properties of the two-step homogeneous nilmanifolds constructed as follows. In view of Theorem 2.7, these two-step homogeneous nilmanifolds are precisely the naturally reductive ones.

Definition 3.1. *We say that a triple* $(\mathfrak{g}, V, \langle , \rangle)$ *is a* **data set** *if,*

(i) $\mathfrak{g}$ *is a compact Lie algebra (see Remark 2.6),*
(ii) (π, V) *is a real faithful representation of* $\mathfrak{g}$ *without trivial subrepresentations, i.e.* $\bigcap_{x\in\mathfrak{g}} \operatorname{Ker}\pi(x) = 0$,
(iii) $\langle , \rangle$ *is a* $\mathfrak{g}$**-invariant** *inner product on* $\mathfrak{n} = \mathfrak{g} \oplus V$, *i.e.* $\langle , \rangle_{\mathfrak{g}} := \langle , \rangle|_{\mathfrak{g}\times\mathfrak{g}}$ *is* ad $\mathfrak{g}$*-invariant,* $\langle , \rangle_V := \langle , \rangle|_{V\times V}$ *is* $\pi(\mathfrak{g})$*-invariant and* $\langle \mathfrak{g}, V\rangle = 0$.

A data set $(\mathfrak{g}, V, \langle , \rangle)$ *determines a two-step nilpotent Lie group denoted by* $N(\mathfrak{g}, V)$ *having Lie algebra* $\mathfrak{n} = \mathfrak{g} \oplus V$, *with Lie bracket defined by (6). Finally, we endow* $N(\mathfrak{g}, V)$ *with the left-invariant metric determined by* $\langle , \rangle$, *obtaining a two-step homogeneous nilmanifold* $(N(\mathfrak{g}, V), \langle , \rangle)$.

Note that the construction of the group $N(\mathfrak{g}, V)$ could depend on the inner product $\langle , \rangle$, but as we shall prove in the following proposition, this does not happen.

Proposition 3.2. *Let* N *and* N' *denote the two-step nilpotent Lie groups corresponding to the data sets* $(\mathfrak{g}, V, \langle , \rangle)$ *and* $(\mathfrak{g}, V, \langle , \rangle')$ *respectively. Then* N *is isomorphic to* N'.

Proof. Since N and N' are simply connected by definition, it suffices to prove that their respective Lie algebras $\mathfrak{n}$ and $\mathfrak{n}'$ are isomorphic. The Lie brackets $[,]_{\mathfrak{n}}$ and $[,]_{\mathfrak{n}'}$ are defined by (6) using $\langle , \rangle = \langle , \rangle_{\mathfrak{g}} \oplus \langle , \rangle_V$ and $\langle , \rangle' = \langle , \rangle'_{\mathfrak{g}} \oplus \langle , \rangle'_V$ respectively. Suppose that

$$\langle x, y\rangle_{\mathfrak{g}} = \langle Px, y\rangle'_{\mathfrak{g}} \quad \forall\, x, y \in \mathfrak{g}, \qquad \langle v, w\rangle_V = \langle Qv, w\rangle'_V \quad \forall\, v, w \in V,$$

with P and Q positive definite symmetric transformations on $\mathfrak{g}$ and V with respect to $\langle , \rangle_{\mathfrak{g}}$, $\langle , \rangle'_{\mathfrak{g}}$ and $\langle , \rangle_V$, $\langle , \rangle'_V$ respectively.

If $x \in \mathfrak{g}$ and $v, w \in V$ then

$$\begin{aligned} \langle Q\pi(x)v, w\rangle'_V &= \langle \pi(x)v, w\rangle_V = -\langle v, \pi(x)w\rangle_V \\ &= -\langle Qv, \pi(x)w\rangle'_V = \langle \pi(x)Qv, w\rangle'_V, \end{aligned} \tag{8}$$

and this implies that $Q\pi(x) = \pi(x)Q$ for all $x \in \mathfrak{g}$, i.e. $Q \in \operatorname{End}_{\mathfrak{g}}(V)$, the intertwining operators of the representation (π, V) of $\mathfrak{g}$. Thus $Q^{\frac{1}{2}} \in \operatorname{End}_{\mathfrak{g}}(V)$, where $Q^{\frac{1}{2}}$ denotes the only symmetric square root of Q. We then have that $(P, Q^{\frac{1}{2}}) : \mathfrak{n} = \mathfrak{g}\oplus V \to \mathfrak{n}' = \mathfrak{g}\oplus V$ is an isomorphism of Lie algebras, i.e. $[Q^{\frac{1}{2}}v, Q^{\frac{1}{2}}w]_{\mathfrak{n}'} = P[v, w]_{\mathfrak{n}}$ for all $v, w \in V$. Indeed, if $x \in \mathfrak{g}$ and $v, w \in V$ then

$$\begin{aligned} \langle [Q^{\frac{1}{2}}v, Q^{\frac{1}{2}}w]_{\mathfrak{n}'}, x\rangle'_{\mathfrak{g}} &= \langle \pi(x)Q^{\frac{1}{2}}v, Q^{\frac{1}{2}}w\rangle'_V = \langle Q^{\frac{1}{2}}\pi(x)Q^{\frac{1}{2}}v, w\rangle'_V \\ &= \langle Q\pi(x)v, w\rangle'_V = \langle \pi(x)v, w\rangle_V \\ &= \langle [v, w]_{\mathfrak{n}}, x\rangle_{\mathfrak{g}} = \langle P[v, w]_{\mathfrak{n}}, x\rangle'_{\mathfrak{g}}, \end{aligned}$$

concluding the proof. □

Remark 3.3. (i) The construction of an H-type group is very similar to 3.1 (see Sect. 2). If (J, V) is a real representation of a Clifford algebra $\mathsf{Cl}(\mathfrak{z})$, then the corresponding H-type Lie algebra is given by $\mathfrak{n} = \mathfrak{z} \oplus V$, with Lie bracket defined as in (6) putting $\pi = J$. Moreover, any H-type algebra can be constructed in this way (see [30] or [31]).

(ii) It follows from the classification of naturally reductive H-type groups given in [32] (see also [52] for an alternative proof using homogeneous structures) that a group $N(\mathfrak{g}, V)$ is of H-type if and only if $\mathfrak{g} = \mathbb{R}$ and $V = \mathbb{R}^2 \oplus ... \oplus \mathbb{R}^2$ is any representation of $\mathbb{R}$ as in 3.1(ii), or $\mathfrak{g} = \mathfrak{s}u(2)$ and $V = \mathbb{C}^2 \oplus ... \oplus \mathbb{C}^2$, where $\mathbb{C}^2$ denotes the standard representation of $\mathfrak{s}u(2)$ regarded as a real representation. Note that these groups are respectively the Heisenberg groups and its quaternionic analogues, which are the Iwasawa N-groups associated to the simple Lie groups of real rank one $\mathsf{SU}(n, 1)$ and $\mathsf{Sp}(n, 1)$ respectively (see [9]).

Theorem 3.4. *Let* $(\mathfrak{g}, V, \langle, \rangle)$ *and* $(\mathfrak{g}', V', \langle, \rangle')$ *be two data sets as in 3.1. The corresponding two-step homogeneous nilmanifolds* $(N(\mathfrak{g}, V), \langle, \rangle)$ *and* $(N(\mathfrak{g}', V'), \langle, \rangle')$ *are isometric if and only if there exist an isometric isomorphism* $\phi : (\mathfrak{g}, \langle, \rangle) \to (\mathfrak{g}', \langle, \rangle')$ *and an isometry* $T : (V, \langle, \rangle) \to (V', \langle, \rangle')$ *such that*

$$T\pi(x)T^{-1} = \pi'(\phi x) \qquad \forall\, x \in \mathfrak{g}. \tag{9}$$

Proof. Suppose first that these groups are isometric. By Theorem 2.2 we have that there exists a Lie algebra isomorphism $A : \mathfrak{n} \to \mathfrak{n}'$ such that

$$\langle Ax, Ay\rangle' = \langle x, y\rangle \qquad \forall\, x, y \in \mathfrak{n}, \tag{10}$$

where $\mathfrak{n} = \mathfrak{g} \oplus V$ and $\mathfrak{n}' = \mathfrak{g}' \oplus V'$ are the Lie algebras of $N(\mathfrak{g}, V)$ and $N(\mathfrak{g}', V')$ respectively. Since $\mathfrak{g}$ and $\mathfrak{g}'$ are the centers of $\mathfrak{n}$ and $\mathfrak{n}'$, then $A\mathfrak{g} = \mathfrak{g}'$, and it follows from (10) that $AV = V'$. Thus A is of the form $A = (\phi, T)$ with $\phi : (\mathfrak{g}, \langle, \rangle) \to (\mathfrak{g}', \langle, \rangle')$ and $T : (V, \langle, \rangle) \to (V', \langle, \rangle')$ isometries. Since A is an isomorphism, we have that $[Tv, Tw] = \phi[v, w]$ for all $v, w \in V$, and thus it is easy to see that (9) holds. Furthermore, (9) implies that $\phi = (\pi')^{-1} \circ \mathrm{Ad}(T) \circ \pi$, and since $\pi : \mathfrak{g} \to \pi(\mathfrak{g}) \subset \mathrm{End}(V)$, $\mathrm{Ad}(T) : \mathrm{End}(V) \to \mathrm{End}(V')$ and $\pi' : \mathfrak{g} \to \pi'(\mathfrak{g}') \subset \mathrm{End}(V')$ are Lie algebra isomorphisms we obtain that $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra isomorphism.

Conversely, if there exist ϕ and T satisfying the properties stated in the theorem, it is easy to prove using (9) that

$$A := (\phi, T) : \mathfrak{n} = \mathfrak{g} \oplus V \to \mathfrak{n}' = \mathfrak{g}' \oplus V'$$

is a Lie algebra isomorphism satifying (10), since ϕ and T are isometries. Thus $(N(\mathfrak{g}, V), \langle, \rangle)$ and $(N(\mathfrak{g}', V'), \langle, \rangle')$ are isometric by Theorem 2.2. □

We deduce from Theorem 3.4 that if $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}'$, then a two-step homogeneous nilmanifold of the form $(N(\mathfrak{g}, V), \langle, \rangle)$ can never be isometric to another one of the form $(N(\mathfrak{g}', V'), \langle, \rangle')$. We then fix a compact Lie algebra $\mathfrak{g}$, and we study the isomorphism classes of nilpotent Lie groups $N(\mathfrak{g}, V)$ which can be constructed by using different representations V of $\mathfrak{g}$.

Proposition 3.5. *Let $\mathfrak{g}$ be a compact Lie algebra and let V and V' be representations of $\mathfrak{g}$ as in* 3.1(ii). *Let* $\mathrm{Inn}(\mathfrak{g})$ *denote the group of inner automorphisms of $\mathfrak{g}$.*

- (i) *If there exist $\phi \in \mathrm{Inn}(\mathfrak{g})$ and $T : V \to V'$ such that $T\pi(x)T^{-1} = \pi'(\phi x)$ for all $x \in \mathfrak{g}$, then $N(\mathfrak{g}, V) \simeq N(\mathfrak{g}, V')$. When $\mathfrak{g}$ is semisimple,* $\mathrm{Inn}(\mathfrak{g})$ *can be replaced by* $\mathrm{Aut}(\mathfrak{g})$.
- (ii) *In particular, if $V \simeq V'$ (equivalent to) then $N(\mathfrak{g}, V) \simeq N(\mathfrak{g}, V')$.*
- (iii) *Suppose that* $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. *If $\langle , \rangle$ and $\langle , \rangle'$ are $\mathfrak{g}$-invariant inner products on $\mathfrak{n} = \mathfrak{g} \oplus V$ and $\mathfrak{n}' = \mathfrak{g} \oplus V'$ respectively such that $(N(\mathfrak{g}, V), \langle , \rangle)$ is isometric to $(N(\mathfrak{g}, V'), \langle , \rangle')$, then $V \simeq V'$.*

Proof. (i) We fix on $\mathfrak{g}$ an ad $\mathfrak{g}$-invariant inner product $\langle , \rangle_{\mathfrak{g}}$. Also, we take a $\pi(\mathfrak{g})$-invariant inner product $\langle , \rangle_V$ on V and we consider the inner product $\langle , \rangle_{V'} = (T^{-1})^* \langle , \rangle_V$ on V'. The inner product $\langle , \rangle_{V'}$ is $\pi'(\mathfrak{g})$-invariant, since for all $x \in \mathfrak{g}$, $v', w' \in V'$ we have

$$\begin{aligned}\langle \pi'(x)v', w' \rangle' &= \langle T\pi(\phi^{-1}x)T^{-1}v', w' \rangle' = \langle \pi(\phi^{-1}x)T^{-1}v', T^{-1}w' \rangle \\ &= -\langle T^{-1}v', \pi(\phi^{-1}x)T^{-1}w' \rangle = -\langle T^{-1}v', T^{-1}\pi'(x)w' \rangle \\ &= -\langle v', \pi'(x)w' \rangle'.\end{aligned}$$

We construct the groups $N(\mathfrak{g}, V)$ and $N(\mathfrak{g}, V')$ using the inner products $\langle , \rangle = \langle , \rangle_{\mathfrak{g}} \oplus \langle , \rangle_V$ and $\langle , \rangle' = \langle , \rangle_{\mathfrak{g}} \oplus \langle , \rangle_{V'}$ respectively. By Proposition 3.2, these constructions do not depend on the invariant inner products chosen.

For all $x \in \mathfrak{g}$ and $v, w \in V$ we have that

$$\begin{aligned}\langle [Tv, Tw]_{\mathfrak{n}'}, x \rangle_{\mathfrak{g}} &= \langle \pi'(x)Tv, Tw \rangle_{V'} = \langle T\pi(\phi^{-1}x)v, Tw \rangle_{V'} \\ &= \langle \pi(\phi^{-1}x)v, w \rangle_V = \langle [v, w]_{\mathfrak{n}}, \phi^{-1}x \rangle_{\mathfrak{g}} = \langle \phi[v, w]_{\mathfrak{n}}, x \rangle_{\mathfrak{g}},\end{aligned}$$

thus $[Tv, Tw]_{\mathfrak{n}'} = \phi[v, w]_{\mathfrak{n}}$ for all $v, w \in V$ and hence $(\phi, T) : \mathfrak{n} = \mathfrak{g} \oplus V \to \mathfrak{n}' = \mathfrak{g} \oplus V'$ is a Lie algebra isomorphism. This implies that $N(\mathfrak{g}, V) \simeq N(\mathfrak{g}, V')$, since both groups are simply connected. Note that we have used $\mathrm{Inn}(\mathfrak{g}) \subset \mathrm{O}(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$, and if $\mathfrak{g}$ is semisimple then $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{O}(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$.

(ii) It follows from (i) putting $\phi = I$.

(iii) By Theorem 3.4 there exist $\phi \in \mathrm{Aut}(\mathfrak{g})$ and $T : V \to V'$ such that $T\pi(x)T^{-1} = \pi'(\phi x)$ for all $x \in \mathfrak{g}$. Since $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$, we have that there exist $x_1, \ldots, x_r \in \mathfrak{g}$ such that $\phi = e^{\mathrm{ad}\, x_1} \ldots e^{\mathrm{ad}\, x_r}$. If we put $T_i = e^{\pi'(x_i)}$, it is easy to see that $\pi' \circ e^{\mathrm{ad}\, x_i} = \mathrm{Ad}(T_i) \circ \pi'$, then

$$\pi'(\phi x) = T_r \ldots T_1 \pi'(x) T_1^{-1} \ldots T_r^{-1} \qquad \forall\, x \in \mathfrak{g}.$$

Henceforth

$$\pi(x) = T^{-1} T_r \ldots T_1 \pi'(x) T_1^{-1} \ldots T_r^{-1} T \qquad \forall\, x \in \mathfrak{g},$$

and this implies that $V \simeq V'$. $\square$

The following example shows that the converse of Proposition 3.5,(ii) is not valid, and that the condition $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ in (iii) can not be removed.

Example 3.6. We consider the real simple Lie algebra $\mathfrak{so}(8)$. Its complexification $\mathfrak{so}(8, \mathbb{C})$ is of type D_4, and the fundamental representations of $\mathfrak{so}(8)$ are

$$\mathbb{C}^8, \Lambda^2\mathbb{C}^8, \Delta^4_+, \Delta^4_-,$$

where $(\pi, \mathbb{C}^8)$ is the standard representation and (π_+, Δ^4_+), (π_-, Δ^4_-) denote the spin representations (see [2]). The spin representations are also 8-dimensional and of real type, i.e. they are complexifications of certain real representations $(\Delta^4_+)_{\mathbb{R}}, (\Delta^4_-)_{\mathbb{R}}$ of $\mathfrak{so}(8)$. It is well known that (π_+, Δ^4_+) and (π_-, Δ^4_-) can be obtained from $\mathbb{C}^8$ in the following way: there exists an outer automorphism ϕ of $\mathfrak{so}(8)$ such that

$$(\pi \circ \phi, \mathbb{C}^8) \simeq (\pi_+, \Delta^4_+).$$

This implies that the corresponding real representations $(\pi \circ \phi, \mathbb{R}^8)$ and $(\pi_+, (\Delta^4_+)_{\mathbb{R}})$ are also equivalent, and hence there exist $T : (\Delta^4_+)_{\mathbb{R}} \to \mathbb{R}^8$ satisfying

$$T\pi_+(x)T^{-1} = \pi(\phi x) \qquad \forall\, x \in \mathfrak{so}(8).$$

Using Proposition 3.5,(i) we obtain that $N(\mathfrak{so}(8), \mathbb{R}^8) \simeq N(\mathfrak{so}(8), (\Delta^4_+)_{\mathbb{R}})$, and analogously we have the same for $N(\mathfrak{so}(8), (\Delta^4_-)_{\mathbb{R}})$. However, the representations $\mathbb{R}^8, (\Delta^4_+)_{\mathbb{R}}, (\Delta^4_-)_{\mathbb{R}}$ are pairwise non-equivalent, since their respective complexifications are pairwise non-equivalent. We then obtain counterexamples to the converse of Proposition 3.5,(ii). Furthermore, if $\langle , \rangle$ is an $\mathfrak{so}(8)$-invariant inner product on $\mathfrak{n} = \mathfrak{so}(8) \oplus \mathbb{R}^8$, then it is easy to check that the inner product $\langle , \rangle' = (\phi, T)^*\langle , \rangle$ is also $\mathfrak{so}(8)$-invariant on $\mathfrak{n}' = \mathfrak{so}(8) \oplus (\Delta^4_+)_{\mathbb{R}}$. By Theorem 3.4 we obtain that $(N(\mathfrak{so}(8), \mathbb{R}^8), \langle , \rangle)$ is isometric to $(N(\mathfrak{so}(8), (\Delta^4_+)_{\mathbb{R}}), \langle , \rangle')$, and thus this provides a counterexample to Proposition 3.5,(iii), if we remove the condition $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$.

Remark 3.7. The situation in the example above is very similar to the H-type case. In fact, if $\dim \mathfrak{z} \equiv 3 \pmod 4$, then the algebra $\mathsf{Cl}(\mathfrak{z})$ has two non-equivalent irreducible modules V_1 and V_2. However, the corresponding H-type algebras $\mathfrak{n}_1 = \mathfrak{z} \oplus V_1$ and $\mathfrak{n}_2 = \mathfrak{z} \oplus V_2$ are isomorphic (see [30] and Sect. 2).

In the following theorem, we shall give some partial results about isometry classes of $\mathfrak{g}$-invariant metrics on a fixed group $N(\mathfrak{g}, V)$. Let B denote the Killing form of $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple then B is negative definite on $\mathfrak{g}$, since $\mathfrak{g}$ is compact. Thus $-B$ is an inner product on $\mathfrak{g}$ and any $\phi \in \mathrm{Aut}(\mathfrak{g})$ satisfies $\phi \in \mathsf{O}(\mathfrak{g}, -B)$. A left-invariant metric on $N(\mathfrak{g}, V)$ is said to be $\mathfrak{g}$-*invariant* if it is determined by a $\mathfrak{g}$-invariant inner product on $\mathfrak{n}$ (see 3.1(iii)).

Theorem 3.8. *Let $\langle , \rangle$ and $\langle , \rangle'$ be two $\mathfrak{g}$-invariant inner products on $\mathfrak{n} = \mathfrak{g} \oplus V$.*

- (i) *If $\langle x, y\rangle'_{\mathfrak{g}} = \langle \phi x, \phi y\rangle_{\mathfrak{g}}$ for all $x, y \in \mathfrak{g}$, then $(N(\mathfrak{g}, V), \langle , \rangle)$ is isometric to $(N(\mathfrak{g}, V), \langle , \rangle')$.*
- (ii) *If $\mathfrak{g}$ is simple, then $N(\mathfrak{g}, V)$ can be endowed with a unique $\mathfrak{g}$-invariant metric, up to isometry and scaling.*

(iii) *Suppose that* $\mathfrak{g}$ *is semisimple and*

$$\langle x, y\rangle = -B(Px, y), \qquad \langle x, y\rangle' = -B(P'x, y), \qquad \forall\, x, y \in \mathfrak{g}.$$

If $(N(\mathfrak{g}, V), \langle,\rangle)$ *is isometric to* $(N(\mathfrak{g}, V), \langle,\rangle')$ *then there exists* $\phi \in \mathrm{Aut}(\mathfrak{g})$ *such that* $\phi P \phi^{-1} = P'$ *and* $\langle x, y\rangle'_{\mathfrak{g}} = \langle \phi^{-1}x, \phi^{-1}y\rangle_{\mathfrak{g}}$ *for all* $x, y \in \mathfrak{g}$, .

(iv) *Suppose that* $\mathfrak{g}$ *is semisimple and* $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ *is the decomposition of* $\mathfrak{g}$ *into simple ideals and that* $\mathfrak{g}_i \not\simeq \mathfrak{g}_j$ *for all* $i \neq j$. *Then* $(N(\mathfrak{g}, V), \langle,\rangle)$ *is isometric to* $(N(\mathfrak{g}, V), \langle,\rangle')$ *if and only if* $\langle,\rangle_{\mathfrak{g}} = \langle,\rangle'_{\mathfrak{g}}$.

(v) *Under the hypothesis of* (iv)*, the* $\mathfrak{g}$*-invariant metrics on* $N(\mathfrak{g}, V)$*, up to isometry, are parametrized by*

$$\{(\lambda_1, ..., \lambda_k) : \lambda_i > 0\}.$$

Proof. (i) If $\langle v, w\rangle'_V = \langle Pv, w\rangle_V$ for all $v, w \in V$ then P is a positive definite symmetric transformation on V with respect to $\langle,\rangle_V$ and $\langle,\rangle_{V'}$. Since $\langle,\rangle_V$ and $\langle,\rangle_{V'}$ are $\mathfrak{g}$-invariant we have that $P \in \mathrm{End}_{\mathfrak{g}}(V)$ (see (8)), thus we also have $P^{\frac{1}{2}} \in \mathrm{End}_{\mathfrak{g}}(V)$. As in the proof of Proposition 3.5, (iii), if $\phi = e^{\mathrm{ad}\, x_1}...e^{\mathrm{ad}\, x_r}$ we take $T = e^{\pi(x_r)}...e^{\pi(x_1)} \in \mathrm{O}(V, \langle,\rangle_V)$ and thus we have that $T\pi(x)T^{-1} = \pi(\phi x)$ for all $x \in \mathfrak{g}$. This implies that $(\phi, P^{\frac{1}{2}}T) : \mathfrak{n} = \mathfrak{g} \oplus V \to \mathfrak{n} = \mathfrak{g} \oplus V$ determines an isometry between $(N(\mathfrak{g}, V), \langle,\rangle')$ and $(N(\mathfrak{g}, V), \langle,\rangle)$ (see Theorem 3.4).

(ii) Since $\mathfrak{g}$ is simple there is an unique $\mathrm{ad}\,\mathfrak{g}$-invariant inner product on $\mathfrak{g}$ up to scaling, thus the result follows from part (i), using $\phi = I$.

(iii) By Theorem 3.4 there exists an isometry $(\phi, T) : (\mathfrak{g} \oplus V, \langle,\rangle) \to (\mathfrak{g} \oplus V, \langle,\rangle')$ satisfying $T\pi(x)T^{-1} = \pi(\phi x)$ for all $x \in \mathfrak{g}$. Thus $\phi = \pi^{-1} \circ \mathrm{Ad}(T) \circ \pi \in \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{O}(\mathfrak{g}, -B)$ and for all $x, y \in \mathfrak{g}$ we have

$$-B(\phi P x, y) = -B(Px, \phi^{-1}y) = \langle x, \phi^{-1}y\rangle$$
$$= \langle \phi x, y\rangle' = -B(P'\phi x, y).$$

This implies that $\phi P = P'\phi$, concluding the proof of (iii).

(iv) It is easy to see that for all $i \neq j$, $\mathfrak{g}_i \perp \mathfrak{g}_j$ with respect to any $\mathfrak{g}$-invariant inner product. Thus P and P' must be of the form

$$P = \begin{bmatrix} \lambda_1 I_{\mathfrak{g}_1} & & \\ & \ddots & \\ & & \lambda_k I_{\mathfrak{g}_k} \end{bmatrix}, \; P' = \begin{bmatrix} \lambda'_1 I_{\mathfrak{g}_1} & & \\ & \ddots & \\ & & \lambda'_k I_{\mathfrak{g}_k} \end{bmatrix}, \quad \lambda_i, \lambda'_i > 0. \tag{11}$$

It follows from (iii) that there exists $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\phi P \phi^{-1} = P'$. The automorphism ϕ must preserve the ideals $\mathfrak{g}_i$, since they are pairwise non-isomorphic, thus $\lambda_i = \lambda'_i$ for all $i = 1, ..., k$. This implies that $P = P'$ and $\langle,\rangle_{\mathfrak{g}} = \langle,\rangle'_{\mathfrak{g}}$.

Conversely, if $\langle,\rangle_{\mathfrak{g}} = \langle,\rangle'_{\mathfrak{g}}$ then the corresponding groups are isometric by part (i).

(v) It follows clearly from (iv) and (11). □

Remark 3.9. The property in (v) is essentially different to the analogous in the H-type case. In fact, an H-type group can be endowed with a unique H-type metric, up to isometry. However, this is still satisfied by the groups $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ simple (see (ii)).

We shall now compute the isometry group of a two-step homogeneous nilmanifold $(N(\mathfrak{g}, V), \langle, \rangle)$, where $(\mathfrak{g}, V, \langle, \rangle)$ is a data set (see 3.1). Note that by (2), it suffices to compute the isotropy subgroup K of the isometry group.

We first consider the group $U := \{T \in K : T|_{\mathfrak{g}} = I\}$. It follows from (4) that $T \in U$ if and only if T is orthogonal and $T\pi(x)T^{-1} = \pi(x)$ for all $x \in \mathfrak{g}$, thus $U = \mathrm{End}_{\mathfrak{g}}(V) \cap \mathsf{O}(V, \langle, \rangle)$, where $\mathrm{End}_{\mathfrak{g}}(V)$ denotes the set of intertwining operators of the representation (π, V) of $\mathfrak{g}$. Suppose that

$$V = V_1^{r_1} \oplus ... \oplus V_k^{r_k}, \qquad V_i \text{ irreducible}, \qquad V_i \not\simeq V_j \quad \forall\, i \neq j,$$

i.e. the subspaces $V_l^{r_l} = V_l \oplus ... \oplus V_l$ (r_l copies) are the *isotypic components of* V. Since V_l is a real irreducible representation, we have that $\mathrm{End}_{\mathfrak{g}}(V_l)$ is a real division associative algebra, and thus $\mathrm{End}_{\mathfrak{g}}(V_l) = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the real and complex numbers and the quaternions respectively.

Definition 3.10. *An irreducible real representation V of $\mathfrak{g}$ is said to be of real type, complex type or quaternionic type if* $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$ *respectively (see [2] for further information).*

We then obtain

$$\mathrm{End}_{\mathfrak{g}}(V) = \mathfrak{g}l(r_1, \mathbb{F}_1) \oplus ... \oplus \mathfrak{g}l(r_k, \mathbb{F}_k),$$

where $\mathbb{F}_l = \mathbb{R}, \mathbb{C}, \mathbb{H}$ depending on the type of V_l, and $\mathfrak{g}l(r, \mathbb{F})$ denotes the Lie algebra of $(r \times r)$-matrixes with coefficients in the ring $\mathbb{F}$. Each $A = (a_{ij}) \in \mathfrak{g}l(r_l, \mathbb{F}_l)$ acts on $V_l^{r_l}$ by

$$A(v_1, ..., v_{r_l}) = (\sum_{i=1}^{r_l} a_{1i} v_i, ..., \sum_{i=1}^{r_l} a_{r_l i} v_i), \tag{12}$$

where $v_i \in V_l$ for $1 \leq i \leq r_l$. This implies that

$$U = U_1 \times ... \times U_k,$$

where $U_l = \mathsf{O}(r_l), \mathsf{U}(r_l), \mathsf{Sp}(r_l)$ depending on the type of V_l.

Before stating the main theorem, we need to describe the action of the center of $\mathfrak{g}$ on V.

Lemma 3.11. *Let $(\mathfrak{g}, V, \langle, \rangle)$ be a data set and let $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{c}$, with $\overline{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{c}$ the center of $\mathfrak{g}$. If $V = V_1 \oplus ... \oplus V_k$ is an orthogonal decomposition of V into $\mathfrak{g}$-irreducible subrepresentations, then for each $i = 1, ..., k$ there exists a skew-symmetric transformation $J_i : V_i \to V_i$ satisfying $J_i^2 = -I$ such that*

$$\pi(h) = \lambda_i(h) J_i \quad \text{for some } \lambda_i(h) \in \mathbb{R}, \qquad \forall\, h \in \mathfrak{c}.$$

Proof. We have that $\mathrm{End}_{\mathfrak{g}}(V_i) = \mathbb{R}, \mathbb{C}, \mathbb{H}$, thus the dimension of any abelian subspace of $\mathrm{End}_{\mathfrak{g}}(V_i)$ acting by skew-symmetric transformations must be at most one. Therefore, since $\pi(\mathfrak{c})|_{V_i} \subset \mathrm{End}_{\mathfrak{g}}(V_i)$ is abelian, there exists a skew-symmetric transformation $J_i \in \mathrm{End}_{\mathfrak{g}}(V_i)$ such that $\pi(\mathfrak{c})|_{V_i} \subset \mathbb{R}J_i$. Furthermore, the irreducibility of $\pi(\mathfrak{g})|_{V_i}$ implies that $J_i^2 = -\lambda^2 I$. We then may take a suitable multiple of J_i, concluding the proof. □

Theorem 3.12. *Let $(N(\mathfrak{g}, V), \langle , \rangle)$ be the two-step homogeneous nilmanifold corresponding to the data set $(\mathfrak{g}, V, \langle , \rangle)$ (see* 3.1*). We put $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{c}$ with $\overline{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{c}$ the center of $\mathfrak{g}$.*

(i) *The Lie algebra $\mathfrak{k} = \mathrm{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n}, \langle , \rangle)$ of the isotropy subgroup K of the isometry group of $(N(\mathfrak{g}, V), \langle , \rangle)$ is given by*

$$\mathfrak{k} = \overline{\mathfrak{g}} \oplus \mathfrak{u}, \qquad [\overline{\mathfrak{g}}, \mathfrak{u}] = 0,$$

where $\mathfrak{u} = \mathrm{End}_{\mathfrak{g}}(V) \cap \mathfrak{so}(V, \langle , \rangle)$ and $\overline{\mathfrak{g}}$ acts on $\mathfrak{n} = \mathfrak{g} \oplus V$ by $(\mathrm{ad}\, x, \pi(x))$ for all $x \in \overline{\mathfrak{g}}$.

(ii) *The connected component of the identity of K is*

$$K^0 = G \times U^0,$$

where $U = \mathrm{End}_{\mathfrak{g}}(V) \cap \mathsf{O}(V, \langle , \rangle)$, $G = \overline{G}/\mathrm{Ker}\,\pi$ and $\overline{G}$ is the simply connected Lie group with Lie algebra $\overline{\mathfrak{g}}$. The group U acts trivially on $\mathfrak{g}$ and if we also denote by π the corresponding representation of G on V, then each $g \in G$ acts on $\mathfrak{n} = \mathfrak{g} \oplus V$ by $(\mathrm{Ad}(g), \pi(g))$.

(iii) *If $V = V_1^{r_1} \oplus \ldots \oplus V_k^{r_k}$ with V_i irreducible and $V_i \not\simeq V_j$ for all $i \neq j$, then*

$$U = U_1 \times \ldots \times U_k,$$

where $U_i = \mathsf{O}(r_i), \mathsf{U}(r_i), \mathsf{Sp}(r_i)$ depending on the type of V_i, and U_i acts on $V_i^{r_i}$ as in (12).

(iv) *If $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$, then $K = G \times U$.*

Proof. (i) If D is an element of $\mathfrak{k}$ then D preserves the center $\mathfrak{g}$ of $\mathfrak{n}$, since D is a derivation of $\mathfrak{n}$, and it follows from the skew-symmetry of D that D also preserves the orthogonal complement V of $\mathfrak{g}$. We then suppose that $D = (A, B) \in \mathfrak{k}$ with $A : \mathfrak{g} \to \mathfrak{g}$ and $B : V \to V$. Using (5) we obtain

$$B\pi(x) - \pi(x)B = \pi(Ax) \qquad \forall\, x \in \mathfrak{g}.$$

We will denote by $[,]_{\mathfrak{n}}$ the Lie bracket of $\mathfrak{n}$ and by $[,]$ the Lie brackets of $\mathfrak{g}$ and $\mathrm{End}(V)$. If $x, y \in \mathfrak{g}$ then

$$\begin{aligned}\pi(A[x, y]) &= B\pi([x, y]) - \pi([x, y])B = B[\pi(x), \pi(y)] - [\pi(x), \pi(y)]B \\ &= [B, [\pi(x), \pi(y)]] = [[B, \pi(x)], \pi(y)] + [\pi(x), [B, \pi(y)]] \\ &= [\pi(Ax), \pi(y)] + [\pi(x), \pi(Ay)] = \pi([Ax, y] + [x, Ay]).\end{aligned}$$

Since π is faithful, then

$$A[x, y] = [Ax, y] + [x, Ay] \qquad \forall\, x, y \in \mathfrak{g},$$

obtaining that $A \in \mathrm{Der}(\mathfrak{g})$. Hence $\overline{\mathfrak{g}}$ and $\mathfrak{c}$ are A-invariant subspaces, and thus there exists $x_1 \in \overline{\mathfrak{g}}$ such that $A|_{\overline{\mathfrak{g}}} = \mathrm{ad}\, x_1$ (note that $\overline{\mathfrak{g}}$ is semisimple).

We also have that $(\mathrm{ad}\, x_1, \pi(x_1)) : \mathfrak{n} = \mathfrak{g} \oplus V \to \mathfrak{n} = \mathfrak{g} \oplus V$ is a skew-symmetric derivation of $\mathfrak{n}$. Indeed, $\langle , \rangle$ is a $\mathfrak{g}$-invariant inner product and

$$\pi(x_1)\pi(x) - \pi(x)\pi(x_1) = \pi([x_1, x]) \qquad \forall\, x \in \mathfrak{g},$$

(see (5)). We then consider the element of $\mathfrak{k}$ given by

$$(A', B') = (A - \mathrm{ad}\, x_1, B - \pi(x_1)),$$

which satisfies $A'|_{\overline{\mathfrak{g}}} \equiv 0$ and $A'\mathfrak{c} \subset \mathfrak{c}$.

We next prove that $A' = 0$. Let $h \in \mathfrak{c} - \{0\}$, thus $B'\pi(h) - \pi(h)B' = \pi(A'h)$. If $V = V_h \oplus \mathrm{Ker}\,\pi(h)$ is an orthogonal decomposition then $\pi(\mathfrak{g})$, and in particular $\pi(A'h)$ preserves the subspaces V_h and $\mathrm{Ker}\,\pi(h)$, since it commutes with $\pi(h)$.

For $v, w \in \mathrm{Ker}\,\pi(h)$ we have

$$\langle \pi(A'h)v, w\rangle_V = \langle B'\pi(h)v - \pi(h)B'v, w\rangle_V = \langle B'v, \pi(h)w\rangle_V = 0,$$

thus $\pi(A'h)|_{\mathrm{Ker}\,\pi(h)} \equiv 0$. Let $V_h = V_h^1 \oplus ... \oplus V_h^r$ be an orthogonal decomposition of V_h into $\mathfrak{g}$-irreducible subspaces. Fix an $i \in \{1, ..., r\}$. By Lemma 3.11 we have that $\pi(h)|_{V_h^i} = J_i$ (taking a suitable multiple of h) and $\pi(A'h)|_{V_h^i} = \lambda_i J_i$, for some $\lambda_i \in \mathbb{R}$. If we set $B_i' = p_i \circ B'|_{V_h^i} : V_h^i \to V_h^i$, where $p_i : V \to V_h^i$ is the orthogonal projection, then $B_i' J_i - J_i B_i' = \lambda_i J_i$. We then obtain $-J_i B_i' J_i - B_i' = \lambda_i I$, and thus $J_i^{-1} B_i' J_i = B_i' + \lambda_i I$. It follows from the fact that B_i' is skew-symmetric that $\lambda_i = 0$, and this happens for all $i = 1, ..., r$. Thus $\pi(A'h) = 0$ and, since π is faithful, we obtain that $A'h = 0$. This implies that $A' = 0$.

Henceforth, the element $D = (A, B) \in \mathfrak{k}$ is of the form

$$(A, B) = (\mathrm{ad}\, x_1, \pi(x_1)) + (0, B')$$

with $B' = B - \pi(x_1) \in \mathrm{End}_{\mathfrak{g}}(V) \cap \mathfrak{so}(V, \langle , \rangle_V) = \mathfrak{u}$. Since $\overline{\mathfrak{g}}$ and $\mathfrak{u}$ commute, then $\mathfrak{k} = \overline{\mathfrak{g}} \oplus \mathfrak{u}$ is a direct sum of Lie algebras. Note that we are identifying $\overline{\mathfrak{g}}$ with $\{(\mathrm{ad}\, x, \pi(x)) : x \in \overline{\mathfrak{g}}\} \subset \mathfrak{k}$.

(ii) We have that $\overline{G}$ is a compact semisimple Lie group. Each $g \in \overline{G}$ defines an element of K acting on $\mathfrak{n} = \mathfrak{g} \oplus V$ by $(\mathrm{Ad}(g), \pi(g))$, where Ad denotes the adjoint representation of $\overline{G}$. In fact, it is easy to see that $\pi(g)\pi(x)\pi(g)^{-1} = \pi(\mathrm{Ad}(g)x)$ for all $x \in \mathfrak{g}$, $g \in \overline{G}$ (see (4)). Since $\mathrm{Ker}\,\pi \subset \mathrm{center}(\overline{G})$, the kernel of the morphism $\overline{G} \to K$, $g \to (\mathrm{Ad}(g), \pi(g))$ is given precisely by $\mathrm{Ker}\,\pi$, which is a finite group. Thus, there is a connected subgroup of K isomorphic to $\overline{G}/\mathrm{Ker}\,\pi$, having Lie algebra $\overline{\mathfrak{g}}$. It follows from (i) that $K^0 = G \times U^0$.

(iii) The group U was obtained after Definition 3.10.

(iv) By (4) we have that

$$K = \{(\phi, T) \in \mathrm{O}(\mathfrak{g}, \langle , \rangle) \times \mathrm{O}(V, \langle , \rangle) : T\pi(x)T^{-1} = \pi(\phi x), \quad x \in \mathfrak{g}\}.$$

Hence, if $(\phi, T) \in K$ then $\phi = \pi^{-1} \circ \mathrm{Ad}(T) \circ \pi \in \mathrm{Aut}(\mathfrak{g})$, and since $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ there must exist $g \in G$ such that $\phi = \mathrm{Ad}(g)$. By (ii) we have that $(\mathrm{Ad}(g), \pi(g)) \in K$ and thus $\pi(g)^{-1}T \in U$. We then obtain that (ϕ, T) can be written as

$$(\phi, T) = (\mathrm{Ad}(g), \pi(g)).(I, \pi(g)^{-1}T),$$

proving that $K = G \times U$ (note that both subgroups commute). □

Remark 3.13. If $\mathfrak{n}$ is an H-type algebra then $\mathfrak{k} = \mathfrak{so}(\mathfrak{z}) \oplus \mathfrak{u}$, where each element of $\mathfrak{so}(\mathfrak{z})$ acts naturally on $\mathfrak{z}$ and it can be extended to $\mathfrak{n} = \mathfrak{z} \oplus V$ using the representation of $\mathrm{Cl}(\mathfrak{z})$ on V (see [47]). Moreover, we have that $K^0 = \mathrm{Spin}(\mathfrak{z}) \times U^0$, where the group U can be computed as described after Remark 3.9 (see [46] and Sect. 2).

4. Commutativity on manifolds $N(\mathfrak{g}, V)$

A *commutative space* is a connected riemannian homogeneous space M such that the algebra of all $\mathrm{I}(M)^0$-invariant differential operators is commutative, where $\mathrm{I}(M)^0$ denotes the connected component of the full isometry group $\mathrm{I}(M)$. It is well known that any symmetric space is commutative (see [21]; or else [25], p.293). Commutativity in the class of homogeneous nilmanifolds is strongly related to the notion of Gelfand pair. Let N be a nilpotent Lie group and let K be a compact group of automorphisms of N. We say that (K, N) is a *Gelfand pair* if the convolution algebra $\mathrm{L}^1_K(N)$ of K-invariant integrable functions on N is commutative. If $H = K \ltimes N$ then it is easy to prove that $\mathrm{L}^1_K(N)$ is isomorphic to $\mathrm{L}^1(H//K)$, the convolution algebra of K-bi-invariant integrable functions on H (see [40]). Thus (K, N) is a Gelfand pair precisely when (H, K) is a Gelfand pair in the usual sense (see [20], p. 36).

It is shown in [4] that if (K, N) is a Gelfand pair then N must be two-step nilpotent (or abelian). Note that this is analogous to C. Gordon's result on naturally reductive homogeneous nilmanifolds (see Sect. 2). We will thus assume that N is a two-step nilpotent Lie group.

In the following theorem we give the relationship between commutativity and Gelfand pairs. We shall first recall some preliminary facts and introduce some notation.

If $K \subset \mathrm{Aut}(N) \approx \mathrm{Aut}(\mathfrak{n})$ (we always assume that N is simply connected), we endow $\mathfrak{n}$ with a K-invariant inner product $\langle , \rangle$ and for each nonzero $x \in \mathfrak{z}$, we consider the Lie algebra $\mathfrak{n}_x = \mathbb{R}x \oplus V_x$, where $V_x = \{v \in V : [v, V] \perp x\}^{\perp} = (\mathrm{Ker}\, J_x)^{\perp}$, with defining Lie bracket $[v, w]_x = \langle [v, w], x\rangle x$ for all $v, w \in V_x$. It is clear that the group $N_x = \exp \mathfrak{n}_x$ is isomorphic to a Heisenberg group, unless $J_x = 0$ (i.e. $V_x = 0$), where $N_x \simeq \mathbb{R}$. We have that $K_x \subset \mathrm{Aut}(N_x)$, where $K_x = \{k \in K : kx = x\}$.

Definition 4.1. *Since $J_x : V_x \to V_x$ is invertible, there exists an orthogonal decomposition $V_x = V_1 \oplus ... \oplus V_r$ such that* $\dim V_i = 2$ *and*

$$J_x|_{V_i} = \begin{bmatrix} 0 & -c_i \\ c_i & 0 \end{bmatrix}, \qquad c_i \neq 0, \qquad \forall\, i = 1, ..., r.$$

If we take $J : V_x \to V_x$ *given by* $J|_{V_i} = \frac{1}{c_i} J_x|_{V_i}$, *then* $J^2 = -I$ *and thus* J *defines a complex structure on* V_x. *We denote by* $\tilde{V}_x$ *the corresponding complex vector space* (V_x, J).

It follows from (4) that the elements of K_x commute with J_x and hence they also commute with J, this implies that K_x acts by complex linear transformations on $\tilde{V}_x$.

A complex representation W of a compact Lie group K is said to be *multiplicity free* if the action of K (or equivalently of its complexification $K_{\mathbb{C}}$) on the polynomial ring $\mathbb{C}[W]$ given by $(k.p)(w) = p(k^{-1}w)$ is multiplicity free, i.e. its isotypic components are all irreducible (see [29,26] for further information).

Theorem 4.2. *If* N *is a two-step nilpotent Lie group,* K *is a compact subgroup of* $\mathrm{Aut}(N)$ *and* $H = K \ltimes N$, *then the following conditions are equivalent.*

- (i) *The algebra of* H^0*-invariant differential operators on* N *is commutative. In particular, if* K *is the isotropy subgroup of the isometry group of* $(N, \langle , \rangle)$, *this means that* $(N, \langle , \rangle)$ *is a commutative space.*
- (ii) (K^0, N) *is a Gelfand pair.*
- (iii) (K, N) *is a Gelfand pair.*
- (iv) (K_x, N_x) *is a Gelfand pair for any nonzero* $x \in \mathfrak{z}$.
- (v) *The action of* K_x *(or* K_x^0*) on the complex vector space* $\tilde{V}_x$ *defined in* (4.1) *is multiplicity free for any nonzero* $x \in \mathfrak{z}$.

It is well known that (i) is equivalent to the commutativity of the algebra $\mathrm{L}^1(H^0//K^0)$ (see [25],p.486), thus the equivalence of (i) and (ii) follows from the isomorphism $\mathrm{L}^1(H^0//K^0) \simeq \mathrm{L}^1_{K^0}(N)$. It is proved that (ii) and (iii) are equivalent in [3] and [5]. The equivalence of (iii) and (iv) is called *localization*, and it has been proved in [34] and [5]. Finally, conditions (iv) and (v) are equivalent by [4].

In this section, we shall study the commutativity within the class of the manifolds $(N(\mathfrak{g}, V), \langle , \rangle)$ introduced in 3.1, i.e. in the class of naturally reductive two-step homogeneous nilmanifolds (see Theorem 2.7). Equivalently, in view of Theorems 4.2, 3.12, we shall study conditions for $(G \times U^0, N(\mathfrak{g}, V))$ to be a Gelfand pair.

As a first step, we prove that the commutativity of $(N(\mathfrak{g}, V), \langle , \rangle)$ does not depend on the $\mathfrak{g}$-invariant metric $\langle , \rangle$.

Proposition 4.3. *If* $\langle , \rangle$ *and* $\langle , \rangle'$ *are two* $\mathfrak{g}$*-invariant inner products on* $\mathfrak{n} = \mathfrak{g} \oplus V$ *then* $(N(\mathfrak{g}, V), \langle , \rangle)$ *is a commutative space if and only if* $(N(\mathfrak{g}, V), \langle , \rangle')$ *is so.*

Proof. Let K and K' denote the corresponding isotropy subgroups. By Theorem 3.12 we have that $K^0 = G \times U^0$ and $(K')^0 = G \times (U')^0$, where $U = \mathrm{End}_{\mathfrak{g}}(V) \cap \mathrm{O}(V, \langle , \rangle)$ and $U' = \mathrm{End}_{\mathfrak{g}}(V) \cap \mathrm{O}(V, \langle , \rangle')$.

If $\langle v, w \rangle = \langle Qv, w \rangle'$ for all $v, w \in V$ then $Q \in \mathrm{End}_{\mathfrak{g}}(V)$ (see (8)) and hence the map $T \to Q^{\frac{1}{2}} T Q^{-\frac{1}{2}}$ is an isomorphism between U and U'. Moreover, since Q commutes with the action of G on V we have that this map is an isomorphism between $K^0|_V$ and $(K')^0|_V$. Henceforth, if $h \in \mathfrak{g}$ then the actions of K_h^0 and $(K')_h^0$

on $\tilde{V}_h$ are conjugate via $Q^{\frac{1}{2}}$. This implies that one of these actions is multiplicity free if and only if the other is so, hence the result follows from Theorem 4.2. □

Thus, we shall study the commutativity of a group $N(\mathfrak{g}, V)$, assuming that it is endowed with any $\mathfrak{g}$-invariant metric. We shall always use condition (v) of Theorem 4.2, thus we have to compute for $h \in \mathfrak{g}$ the stabilizer $K_h^0 = \{\varphi \in K^0 : \varphi h = h\}$, where K^0 is the connected component of the isotropy subgroup of $N(\mathfrak{g}, V)$.

Suppose that $\mathfrak{g}$ is semisimple. Let $h \in \mathfrak{g}$ be a regular element and let $\mathfrak{t}$ denote the only maximal torus of $\mathfrak{g}$ (maximal abelian subalgebra) containing h. We note that $\lambda \in \mathfrak{t}^*$ is called a *weight* of a real representation (π, V) if there exist $v, w \in V$ such that $\pi(h')v = \lambda(h')w$ and $\pi(h')w = -\lambda(h')v$ for all $h' \in \mathfrak{t}$ (see [2]). We can choose h such that $\lambda(h) \neq 0$ for all nonzero $\lambda \in P(V)$, where $P(V)$ denotes the set of weights of the representation V with respect to $\mathfrak{t}$. This implies that $\operatorname{Ker} \pi(h) = V_0$, the zero weight space of V, and thus $V = V_h \oplus V_0$. By Theorem 3.12 we have that $K^0 = G \times U^0$, where U acts trivially on $\mathfrak{g}$ and G acts by the adjoint representation on $\mathfrak{g}$. This implies that the Lie algebra of K_h^0 is $C_{\mathfrak{g}}(h) \oplus \mathfrak{u}$, where $C_{\mathfrak{g}}(h) = \{x \in \mathfrak{g} : [x, h] = 0\}$ is the centralizer of h in $\mathfrak{g}$. Since h is regular we have that $C_{\mathfrak{g}}(h) = \mathfrak{t}$ and thus, if T is the maximal torus of G with Lie algebra $\mathfrak{t}$ then

$$K_h^0 = T \times U^0, \tag{13}$$

where each $\exp h' \in T$ $(h' \in \mathfrak{t})$ acts on V by $e^{\pi(h')}$. We then obtain a necessary condition for $N(\mathfrak{g}, V)$ to be a commutative space: the action of $e^{\pi(\mathfrak{t})} \times U^0$ on $\tilde{V}_h$ must be multiplicity free (see Theorem 4.2,(v)), where $e^{\pi(\mathfrak{t})} = \{e^{\pi(h')} : h' \in \mathfrak{t}\}$. In the following theorem we prove that the condition above is also sufficient for the commutativity of $N(\mathfrak{g}, V)$ when $\mathfrak{g}$ is semisimple.

Theorem 4.4. *A group $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ semisimple is a commutative space if and only if the action of $e^{\pi(\mathfrak{t})} \times U^0$ on $\tilde{V}$ is multiplicity free, where $\mathfrak{t}$ is any maximal torus of $\mathfrak{g}$ and $\tilde{V}$ is the complex vector space $\tilde{V}_h$ defined in* 4.1 *for any $h \in \mathfrak{t}$ satisfying $\lambda(h) \neq 0$ for all nonzero weight λ of V. Note that $V = \tilde{V} \oplus V_0$, where V_0 denotes the zero weights space of the representation V with respect to $\mathfrak{t}$.*

Proof. If $N(\mathfrak{g}, V)$ is a commutative space we have proved above that this condition must be satisfied.

Conversely, suppose that the action of $e^{\pi(\mathfrak{t})} \times U^0$ on $\tilde{V}$ is multiplicity free. If $h_1 \in \mathfrak{g} - \{0\}$ we take $\mathfrak{t}_1$ any maximal torus of $\mathfrak{g}$ containing h_1. In view of Theorem 4.2 we have to prove that the action of K_{h_1} on $\tilde{V}_{h_1}$ is multiplicity free, where $V = V_{h_1} \oplus \operatorname{Ker} \pi(h_1)$.

There exists $A \in G$ such that $A\mathfrak{t}_1 = \mathfrak{t}$. We also denote by A the corresponding extension to $\mathfrak{n} = \mathfrak{g} \oplus V$ as an element of K (see Theorem 3.12). Since $A\pi(x)A^{-1} = \pi(Ax)$ for all $x \in \mathfrak{g}$ (see (4)) and A commutes with the action of U we have that

$$Ae^{\pi(h)}TA^{-1} = Ae^{\pi(h)}A^{-1}T = e^{A\pi(h)A^{-1}}T = e^{\pi(Ah)}T \qquad \forall\, h \in \mathfrak{t}_1, T \in U^0.$$

This implies that $A(e^{\pi(\mathfrak{t}_1)} \times U^0)A^{-1} = e^{\pi(\mathfrak{t})} \times U^0$. If $V = \tilde{V}^1 \oplus V_0^1$ denotes the decomposition as in the theorem for $\mathfrak{t}_1$, then $AV_0^1 = V_0$. Indeed, for all $v \in V_0^1$,

$$\pi(h)Av = A\pi(A^{-1}h)v = 0 \qquad \forall\, h \in \mathfrak{t},$$

thus $A\tilde{V}^1 = \tilde{V}$, since A is orthogonal. It is clear that the action of

$$A^{-1}(e^{\pi(\mathfrak{t})} \times U^0)A = e^{\pi(\mathfrak{t}_1)} \times U^0$$

on $A^{-1}\tilde{V} = \tilde{V}^1$ is also multiplicity free, and since $e^{\pi(\mathfrak{t}_1)} \times U^0 \subset K^0_{h_1}$ and $V_{h_1} \subset \tilde{V}^1$ (note that $h_1 \in \mathfrak{t}_1$ and $V^1_0 \subset \operatorname{Ker} \pi(h_1)$) we obtain that the action of $K^0_{h_1}$ on V_{h_1} is multiplicity free, as was to be shown. □

Using the characterization in the theorem above, we shall give now two families of examples of groups $N(\mathfrak{g}, V)$ which are commutative spaces. We first need a lemma about multiplicity free actions of a torus, which will be very useful.

Lemma 4.5. *Let $\mathbb{C}^*$ denote the multiplicative group $\mathbb{C} - \{0\}$. A complex representation W of an n-dimensional torus T^n is multiplicity free if and only if the set of weights $P(W) \subset \mathfrak{t}^*$ of W is $\mathbb{R}$-linearly independent. In particular, if W is multiplicity free then* $\dim_{\mathbb{C}} W \leq n$.

Proof. The complex irreducible representations of a torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ are all one-dimensional and of the form

$$\pi([x]) = e^{2\pi i\lambda(x)}, \qquad x \in \mathbb{R}^n,$$

for some $\lambda : \mathbb{R}^n \to \mathbb{R}$ given by

$$\lambda(x_1, ..., x_n) = \sum_{j=1}^{n} k_j x_j, \qquad (k_1, ..., k_n) \in \mathbb{Z}^n \tag{14}$$

(see [2],p.107). Thus, if (ρ, W) is an m-dimensional complex representation of T^n, then there exists a basis $\{w_1, ..., w_m\}$ of W such that

$$\rho([x])w_j = e^{2\pi i\lambda_j(x)}w_j, \qquad x \in \mathbb{R}^n,$$

where each λ_j is as in (14) for some element of $\mathbb{Z}^n$. The Lie algebra $\mathfrak{t}$ of T^n can be identified with $\mathbb{R}^n$ and its corresponding action on W is given by

$$x.w_j = 2\pi i\lambda_j(x)w_j, \qquad x \in \mathfrak{t} = \mathbb{R}^n.$$

We must study the action of $\mathfrak{t}$ on the polynomial ring $\mathbb{C}[W]$. We denote by z_j the element of $\mathbb{C}[W]$ given by $z_j(w) = a_j$, where $w = a_1w_1 + ... + a_mw_m$. Since the polynomial z_j is linear it is not hard to check that $(x.z_j)(w) = z_j(-x.w)$, thus

$$\begin{aligned} x.z_j(w) = z_j(-x.w) &= z_j(-2\pi i\lambda_1(x)a_1w_1 - ... - 2\pi i\lambda_m(x)a_mw_m) \\ &= -2\pi i\lambda_j(x)a_j = -2\pi i\lambda_j(x)z_j(w) \end{aligned}$$

and therefore

$$x.z_j = -2\pi i\lambda_j(x)z_j \qquad \forall\ j = 1, ..., m,\ x \in \mathfrak{t}. \tag{15}$$

Using that $\mathfrak{t}$ acts by derivations on $\mathbb{C}[W]$ (i.e. $x.(pq) = (x.p)q + p(x.q)$, where pq denotes ordinary multiplication in $\mathbb{C}[W]$) and (15) it is easy to see that any $x \in \mathfrak{t}$ acts on a monomial of $\mathbb{C}[W]$ by

$$x.(z_1^{k_1}...z_m^{k_m}) = -2\pi i(k_1\lambda_1(x) + ... + k_m\lambda_m(x))z_1^{k_1}...z_m^{k_m}.$$

We then obtain that $\mathbb{C}[W]$ will be multiplicity free if and only if $k_1\lambda_1+...+k_m\lambda_m \neq k_1'\lambda_1 + ... + k_m'\lambda_m$ for all $(k_1, ..., k_m) \neq (k_1', ..., k_m')$ $(\in (\mathbb{Z}_{\geq 0})^m)$. This condition is equivalent to the set $\{\lambda_1, ..., \lambda_m\} \subset \mathfrak{t}^*$ being $\mathbb{Z}$-linearly independent, and since the λ_i are integral (see (14)), we have that this is equivalent to $\{\lambda_1, ..., \lambda_m\}$ being $\mathbb{R}$-linearly independent, as it was to be shown. □

Example 4.6. Consider the group $N(\mathfrak{su}(n), \mathbb{C}^n)$, $n \geq 2$, where $\mathbb{C}^n$ is the standard representation of $\mathfrak{su}(n)$ regarded as a real representation. The subspace $\mathfrak{t}$ of $\mathfrak{su}(n)$ given by

$$\mathfrak{t} = \left\{ H = \begin{bmatrix} ih_1 & & \\ & \ddots & \\ & & ih_n \end{bmatrix} : \sum_j h_j = 0,\ h_j \in \mathbb{R} \right\}$$

is a maximal torus of $\mathfrak{su}(n)$. The representation $\mathbb{C}^n$ is of complex type, thus $e^{\pi(\mathfrak{t})} \times U^0 = e^{\pi(\mathfrak{t})} \times \mathrm{S}^1$ (see Definition 3.10). Furthermore, since $(\mathbb{C}^n)_0 = 0$, we have that $\tilde{V} = \mathbb{C}^n$. The Lie algebra of $e^{\pi(\mathfrak{t})} \times \mathrm{S}^1$ can be identified with $\mathfrak{t} \oplus \mathbb{R}$, and thus the weights of $\mathbb{C}^n$ are given by $P(\mathbb{C}^n) = \{\lambda_1 + \lambda, ..., \lambda_n + \lambda\}$, where $\lambda_j(H, r) = ih_j$ and $\lambda(H, r) = ir$ for all $H \in \mathfrak{t}$, $r \in \mathbb{R}$. Since $P(\mathbb{C}^n)$ is a linearly independent subset of $(\mathfrak{t} \oplus \mathbb{R})^*$, we obtain from Lemma 4.5 that the action of $e^{\pi(\mathfrak{t})} \times \mathrm{S}^1$ on $\mathbb{C}^n$ is multiplicity free and hence $N(\mathfrak{su}(n), \mathbb{C}^n)$ is a commutative space by Theorem 4.4.

Example 4.7. We consider the group $N(\mathfrak{so}(n), \mathbb{R}^n)$, $n \geq 2$, where $\mathbb{R}^n$ denotes the standard representation of $\mathfrak{so}(n)$. In this case $e^{\pi(\mathfrak{t})} \times U^0 = e^{\pi(\mathfrak{t})}$, since $\mathbb{R}^n$ is of real type. If $n = 2k + 1$ we choose the maximal torus of $\mathfrak{so}(n)$

$$\mathfrak{t} = \left\{ H = \begin{bmatrix} 0 & -h_1 & & & & \\ h_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -h_k & \\ & & & h_k & 0 & \\ & & & & & 0 \end{bmatrix} : h_j \in \mathbb{R} \right\},$$

and if $n = 2k$ we take the same $\mathfrak{t}$ but with the last row and column deleted. It is clear that in both cases we have to analyze the action of $e^{\pi(\mathfrak{t})}$ on $\tilde{V} = \mathbb{C}^k$ given by $e^{\pi(H)}.(c_1, ..., c_k) = (ih_1c_1, ..., ih_kc_k)$ (see 4.1). The Lie algebra of $e^{\pi(\mathfrak{t})}$ is $\mathfrak{t}$ and $P(\mathbb{C}^k) = \{\lambda_1, ..., \lambda_k\}$, where $\lambda_j(H) = ih_j$, thus $P(\mathbb{C}^k)$ is a linearly independent subset of $\mathfrak{t}^*$. By Lemma 4.5 we have that the action of $e^{\pi(\mathfrak{t})}$ on $\mathbb{C}^k$ is multiplicity free and thus $N(\mathfrak{so}(n), \mathbb{R}^n)$ is a commutative space (see Theorem 4.4).

Remark 4.8. It is easy to see that the group $N(\mathfrak{so}(n), \mathbb{R}^n)$ is precisely the so called *free two-step nilpotent Lie group on n generators*. These groups have been considered by many authors, see [4,55,50] for instance, and the commutativity has been proved in [4]. Moreover, it was also proved in this work that the only Gelfand pair of the form $(K, N(\mathfrak{so}(n), \mathbb{R}^n))$ is $(\mathsf{SO}(n), N(\mathfrak{so}(n), \mathbb{R}^n))$.

Lemma 4.9. *Let W be a complex representation of $\mathfrak{g}$ such that* $\dim_{\mathbb{C}} W_\lambda = 1$ *for all* $\lambda \in P(W) - \{0\}$. *Then* $\dim_{\mathbb{C}} W_0 \leq \operatorname{rank}(\mathfrak{g})$.

Proof. If $r = \operatorname{rank}(\mathfrak{g})$ we take $\Delta = \{\alpha_1, ..., \alpha_r\}$ the set of simple roots of $\mathfrak{g}$. Denote by $\lambda_1 \in P(W)$ the maximum weight of W and let $w_1 \in W_{\lambda_1} - \{0\}$. If $x_{-\alpha_{i_1}} ... x_{-\alpha_{i_t}} w_1 \in W_0$ with $x_{-\alpha_{i_j}} \in \mathfrak{g}_{-\alpha_{i_j}}$, then $x_{-\alpha_{i_2}} ... x_{-\alpha_{i_t}} w_1 \in W_{\alpha_{i_1}}$, and thus $x_{-\alpha_{i_1}} ... x_{-\alpha_{i_t}} w_1 \in x_{-\alpha_{i_1}} W_{\alpha_{i_1}}$. Since W_0 is $\mathbb{C}$-linearly generated by the elements of the form $x_{-\alpha_{i_1}} ... x_{-\alpha_{i_t}} w_1$ we have that

$$W_0 \subset < x_{-\alpha_1} W_{\alpha_1} \cup ... \cup x_{-\alpha_r} W_{\alpha_r} >_{\mathbb{C}} .$$

Now, using that $\dim W_{\alpha_i} \leq 1$ for all i we obtain that $\dim W_0 \leq r$. □

The following theorem gives a large family of non-commutative naturally reductive spaces. The first examples of this kind were given in [27,28].

Theorem 4.10. *If the group $N(\mathfrak{g}, V)$, with $\mathfrak{g}$ semisimple and V irreducible of real type (see Definition 3.10) is a commutative space, then* $\dim V \leq 3 \operatorname{rank}(\mathfrak{g})$.

Proof. We will use Theorem 4.4. In this case we have that $e^{\pi(\mathfrak{t})} \times U^0 = e^{\pi(\mathfrak{t})}$. If $V = \tilde{V} \oplus V_0$ as in the theorem we take a real basis of $\tilde{V}$

$$\tilde{V} = \{v_1, w_1, ..., v_n, w_n\}_{\mathbb{R}}$$

such that

$$\pi(h)|_{\{v_j, w_j\}_{\mathbb{R}}} = \begin{bmatrix} 0 & -\lambda_j(h) \\ \lambda_j(h) & 0 \end{bmatrix} \qquad \forall\, h \in \mathfrak{t}.$$

Thus, as a complex vector space, $\tilde{V} = \{v_1, ..., v_n\}_{\mathbb{C}}$ and the action is given by $\pi(h)v_j = i\lambda_j(h)v_j$ for all $h \in \mathfrak{t}$. Suppose that $N(\mathfrak{g}, V)$ is commutative. Since the action of $e^{\pi(\mathfrak{t})}$ on $\tilde{V}$ is multiplicity free, we obtain from Lemma 4.5 that $n \leq \dim \mathfrak{t}$ and $\{\lambda_1, ..., \lambda_n\}$ is a linearly independent subset of $\mathfrak{t}^*$. Since V is of real type, we have that its complexification $W = \mathbb{C} \otimes V$, which is naturally a complex representation of $\mathfrak{g}$, is also irreducible. Furthermore,

$$\dim_{\mathbb{C}} W - \dim_{\mathbb{C}} W_0 = \dim V - \dim V_0 = \dim_{\mathbb{R}} \tilde{V} = 2n \leq 2 \dim \mathfrak{t}.$$

We thus obtain that the complex representation W of $\mathfrak{g}$ satisfies:

$$\begin{array}{l} \dim_{\mathbb{C}} W \leq 2 \operatorname{rank}(\mathfrak{g}) + \dim_{\mathbb{C}} W_0. \\ \dim_{\mathbb{C}} W_\lambda = 1 \quad \forall\, \lambda \in P(W) - \{0\}, \end{array} \tag{16}$$

where W_λ denotes the λ-weight space of W. Now, using (16) and Lemma 4.9, we obtain that $\dim V = \dim_{\mathbb{C}} W \leq 3 \operatorname{rank} \mathfrak{g}$. □

As an example, we take $\mathfrak{g} = \mathfrak{su}(2)$. All the odd dimensional irreducible representations of $\mathfrak{su}(2)$ are of real type, but in view of Theorem 4.10, we have that only $N(\mathfrak{su}(2), V)$ with the 3-dimensional representation $V = \mathbb{R}^3$ is a commutative space. Note that $\mathfrak{su}(2) = \mathfrak{so}(3)$, and $V = \mathbb{R}^3$ is the standard representation of $\mathfrak{so}(3)$.

5. Applications to weakly symmetric spaces

A connected riemannian manifold M is said to be *weakly symmetric* if for any two points $p, q \in M$ there exists an isometry of M mapping p to q and q to p. This notion was introduced by A. Selberg in [49]. This is not the original definition given by Selberg, but it is equivalent to it (see [8]). It is easy to see that any symmetric space is weakly symmetric.

We note that the commutativity of a space (see Sect. 4) is defined sometimes with respect to the full isometry group $\mathrm{I}(M)$. The equivalence of these two notions is still an open problem. However, in the class of two-step homogeneous nilmanifolds both notions coincide (see Theorem 4.2,(ii),(iii) and [6]).

Theorem 5.1 **(**[49]**).** *Any weakly symmetric space M is a commutative space (with respect to* $\mathrm{I}(M)$*-invariance).*

The converse is known to be false, there are examples in [40,41] of modified H-type groups which are commutative spaces and not weakly symmetric. A motivation for the study of the commutativity and weak symmetry on manifolds $(N(\mathfrak{g}, V), \langle , \rangle)$ has been the fact that, up to now, there were no examples of non-weakly symmetric naturally reductive spaces. The following result provides a large family of such examples, and its proof follows from Theorems 4.10, 5.1.

Theorem 5.2. *Any $(N(\mathfrak{g}, V), \langle , \rangle)$ with $\mathfrak{g}$ semisimple, V irreducible of real type and* $\dim V > 3\,\mathrm{rank}(\mathfrak{g})$ *is a non-weakly symmetric naturally reductive space.*

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