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Moebius geometry of submanifolds in $\mathbb{S}^n$

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Abstract. In this paper we define a Moebius invariant metric and a Moebius invariant second fundamental form for submanifolds in $\mathbb{S}^n$ and show that in case of a hypersurface with $n \geq 4$ they determine the hypersurface up to Moebius transformations. Using these Moebius invariants we calculate the first variation of the moebius volume functional. We show that any minimal surface in $\mathbb{S}^n$ is also Moebius minimal and that the image in $\mathbb{S}^n$ of any minimal surface in $\mathbb{R}^n$ unter the inverse of a stereographic projection is also Moebius minimal. Finally we use the relations between Moebius invariants to classify all surfaces in $\mathbb{S}^3$ with vanishing Moebius form.

0. Introduction

In this paper we study submanifolds in $\mathbb{S}^n$ unter the Moebius group. We define a Moebius invariant metric g and a Moebius invariant 2-form $\mathbb{B}$ called the Moebius second fundamental form. We show that in case of hypersurface with $n \geq 4$ $\{g, \mathbb{B}\}$ form a complete invariant system which determines the hypersurface up to Moebius transformations (cf. Theorem 3.1). Using these Moebius invariants we calculate the Euler–Lagrange equations for the volume functional with respect to g (cf. Theorem 4.1). In case of surfaces in $\mathbb{S}^3$ a critical surface to this functional is exactly a Willmore surface, which is well-studied (cf. [2], [5], [8]). We show that any minimal surface in $\mathbb{S}^n$ is also Moebius minimal and that the image in $\mathbb{S}^n$ of any minimal surface in $\mathbb{R}^n$ under the inverse of a stereographic projection is also Moebius minimal, known for $n = 3$ (cf. [2], pp24). Finally we use the relations between Moebius invariants to classify all surfaces in $\mathbb{S}^3$ with vanishing Moebius form. We show that the image in $\mathbb{R}^3$ of such surface unter a stereographic projection is Moebius equivalent to a circular cylinder, or a torus of revolution, or a circular cone (cf. Theorem 5.1).

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1. The Moebius metric for submanifold in $\mathbb{S}^n$

Let $\mathbb{R}^{n+2}_1$ be the Lorentz space $\mathbb{R}^{n+2}$ with the inner product $\langle\ ,\ \rangle$ given by

$$\langle X, Y\rangle := -x_0y_0 + x_1y_1 + \cdots + x_{n+1}y_{n+1}, \tag{1.1}$$

where $X = (x_0, x_1, \cdots, x_{n+1})$, $Y = (y_0, y_1, \cdots, y_{n+1}) \in \mathbb{R}^{n+2}$. We denote by $\mathbf{C}^{n+1}_+$ the half cone in $\mathbb{R}^{n+2}_1$ and by $\mathbb{Q}^n$ the quadric in $\mathbb{R}P^{n+1}$:

$$\mathbf{C}^{n+1}_+ := \{X \in \mathbb{R}^{n+2}_1 \mid \langle X, X\rangle = 0, x_0 = p(X) > 0\}, \tag{1.2}$$

$$\mathbb{Q}^n := \{[X] \in \mathbb{R}P^{n+1} \mid \langle X, X\rangle = 0\}, \tag{1.3}$$

where $p : \mathbb{R}^{n+2}_1 \to \mathbb{R}$ is the projection of X to its first coordinate x_0. Let $O(n+1, 1)$ be the Lorentz group of $\mathbb{R}^{n+2}_1$ keeping the inner product $\langle, \rangle$ invariant. Then $O(n+1, 1)$ is a transformation group on $\mathbb{Q}^n$ defined by

$$\mathbb{T}([X]) := [X\mathbb{T}], \quad X \in \mathbf{C}^{n+1}_+, \mathbb{T} \in O(n+1, 1).$$

A classical theorem states that

Theorem 1.1. *Two submanifolds $x, \tilde{x} : \mathbf{M} \to \mathbb{S}^n$ are Moebius equivalent if and only if there exists $\mathbb{T} \in O(n+1, 1)$ such that $[1, \tilde{x}] = \mathbb{T}([1, x]) : \mathbf{M} \to \mathbb{Q}^n$.*

Let $x : \mathbf{M} \to \mathbb{S}^n$ be a m-dimensional submanifold in $\mathbb{S}^n$. We define $X = [1, x] : \mathbf{M} \to \mathbb{Q}^n$. For any local lift Z of the standard projection $\pi : \mathbf{C}^{n+1}_+ \to \mathbb{Q}^n$ we get a local lift $y := Z \circ X$ of $X : \mathbf{M} \to \mathbb{Q}^n$. Such lift y exists around each point of $\mathbf{M}$.

Let $y : U \to \mathbf{C}^{n+1}_+$ be a lift of $X = [1, x] : \mathbf{M} \to \mathbb{Q}^n$ defined in an open subset U of $\mathbf{M}$. Then we have $y = \lambda(1, x)$ for some smooth positive function $\lambda : U \to \mathbb{R}$. Thus $\langle dy, dy\rangle = \lambda^2 dx \cdot dx$ is a metric conformal to the induced metric $dx \cdot dx$ of $x : \mathbf{M} \to \mathbb{S}^n$. We denote by ∇, Δ and κ the gradient, the Laplacian operator and the normalized scalar curvature of the metric $\langle dy, dy\rangle$. Then we have

Theorem 1.2. *The 2-form*

$$g := (\langle \Delta y, \Delta y\rangle - m^2\kappa)\,\langle dy, dy\rangle \tag{1.4}$$

is a globally defined Moebius invariant of $x : \mathbf{M} \to \mathbb{S}^n$. Moreover, g is positive definite at any non-umbilic point of $\mathbf{M}$.

Proof. Let $\tilde{y}$ be another lift of $[1, x]$ defined on an open set V of $\mathbf{M}$ with $U \cap V \neq \phi$. We denote by $\tilde{\Delta}$ and $\tilde{\kappa}$ the Laplacian operator and the normalized scalar curvature of $\langle d\tilde{y}, d\tilde{y}\rangle$. Then we can find a smooth function $\tau : U \cap V \to \mathbb{R}$ such that $\tilde{y} = e^\tau y$ on $U \cap V$. Since $\langle y, y\rangle = 0$ and $\langle dy, y\rangle = 0$ we have

$$\langle d\tilde{y}, d\tilde{y}\rangle = e^{2\tau}\,\langle dy, dy\rangle\,. \tag{1.5}$$

Let $\{u^1, u^2, \cdots, u^n\}$ be a local coordinate system in $U \cap V$. For any function $F : \mathbf{M} \to \mathbb{R}$ we denote by F_i the derivative $\frac{\partial F}{\partial u^i}$. Then we have

$$\langle dy, dy\rangle = \sum_{i,j=1}^{m} \langle y_i, y_j\rangle du^i \otimes du^j := \sum_{i,j=1}^{m} g_{ij} du^i \otimes du^j. \tag{1.6}$$

We denote by (g^{ij}) the inverse matrix of (g_{ij}), by $\{\Gamma^k_{ij}\}$, $\{\tilde{\Gamma}^k_{ij}\}$ the Levi-Civita connection of $\langle dy, dy\rangle$ and $\langle d\tilde{y}, d\tilde{y}\rangle$ respectively. By a direct calculation we have

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \tau_i\delta^k_j + \tau_j\delta^k_i - \sum_{\lambda=1}^{m} \tau_\lambda g^{k\lambda} g_{ij}; \tag{1.7}$$

$$\tilde{\kappa} = e^{-2\tau}(\kappa - \frac{2}{m}\Delta\tau - \frac{m-2}{m}\|\nabla\tau\|^2); \tag{1.8}$$

$$\tilde{\Delta}\tilde{y} = e^{-\tau}\left\{(\Delta\tau + (m-1)\|\nabla\tau\|^2)y + \Delta y + m\sum_{i,k=1}^{m} \tau_i g^{ki} y_k\right\}. \tag{1.9}$$

Since $g_{ij} = \langle y_i, y_j\rangle$, we have

$$\begin{aligned}\langle \mathrm{Hess}(y)_{ij}, y_k\rangle :=& \left\langle y_{ij} - \sum_{\lambda=1}^{m} \Gamma^\lambda_{ij} y_\lambda, y_k\right\rangle \\ =& \langle y_{ij}, y_k\rangle - \frac{1}{2}((g_{kj})_i + (g_{ki})_j - (g_{ij})_k) = 0.\end{aligned}$$

In particular we get

$$\langle \Delta y, y_k\rangle = 0, \quad 1 \le k \le m. \tag{1.10}$$

Using (1.6) we get

$$\begin{aligned}\langle \Delta y, y\rangle =& \sum_{i,j=1}^{m} g^{ij}\left\langle y_{ij} - \sum_{\lambda=1}^{m} \Gamma^\lambda_{ij} y_\lambda, y\right\rangle = \sum_{i,j=1}^{m} g^{ij}\langle y_{ij}, y\rangle \\ =& -\sum_{i,j=1}^{m} g^{ij}\langle y_i, y_j\rangle = -m.\end{aligned} \tag{1.11}$$

It follows from (1.9) and (1.10) that

$$\left\langle \tilde{\Delta}\tilde{y}, \tilde{\Delta}\tilde{y}\right\rangle = e^{-2\tau}\left\{\langle \Delta y, \Delta y\rangle - 2m\Delta\tau - m(m-2)\|\nabla\tau\|^2\right\}. \tag{1.12}$$

Thus (1.5), (1.8) and (1.12) yield

$$\left(\left\langle \tilde{\Delta}\tilde{y}, \tilde{\Delta}\tilde{y}\right\rangle - m^2\tilde{\kappa}\right)\langle d\tilde{y}, d\tilde{y}\rangle = (\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle dy, dy\rangle. \tag{1.13}$$

From (1.13) and Theorem 1.1 we know that the 2-form g defined by (1.4) is a globally defined Moebius invariant of $x : \mathbf{M} \to \mathbb{S}^n$. To show that g is positive definite at any non-umbilical point we take $y := (1, x)$. Let $\{e_1, e_2, \cdots, e_m\}$ be an orthonormal basis for $T\mathbf{M}$ with dual basis $\{\theta_1, \theta_2, \cdots, \theta_m\}$ and let $\{e_{m+1}, \cdots, e_n\}$ be an orthonormal basis to the normal bundle of x in $\mathbb{S}^n$. We write the second fundamental form II of x as

$$II = \sum_{\alpha=m+1}^{n} \sum_{i,j=1}^{m} h_{ij}^{\alpha} \theta_i \otimes \theta_j e_\alpha.$$

Then from the structure equations for $x : \mathbf{M} \to \mathbb{S}^n$ we have

$$\Delta x = \sum_{\alpha=m+1}^{n} (\sum_{i=1}^{m} h_{ii}^{\alpha}) e_\alpha - mx. \tag{1.14}$$

From (1.13) and the Gauss equation we get

$$\begin{aligned}
&\langle \Delta y, \Delta y \rangle - m^2 \kappa \\
&= \sum_{\alpha=m+1}^{n} \left(\sum_{i=1}^{m} h_{ii}^{\alpha} \right)^2 + m^2 \\
&\quad - m^2 \left\{ 1 + \frac{1}{m(m-1)} (\sum_{\alpha=m+1}^{n} (\sum_{i=1}^{m} h_{ii}^{\alpha})^2 - \sum_{\alpha=m+1}^{n} \sum_{i,j=1}^{m} (h_{ij}^{\alpha})^2) \right\} \qquad (1.15) \\
&= \frac{m}{m-1} \left(\sum_{\alpha=m+1}^{n} \sum_{i,j=1}^{m} (h_{ij}^{\alpha})^2 - \frac{1}{m} \sum_{\alpha=m+1}^{n} \left(\sum_{i=1}^{m} h_{ii}^{\alpha} \right)^2 \right) \\
&= \frac{m}{m-1} \left\| II - \frac{1}{m} tr(II) I \right\|^2,
\end{aligned}$$

where $I := dx \cdot dx$ is the first fundamental form of x. This shows that g is positive definite at any non-umbilical point and that $g = 0$ at umbilical points. □

Definition 1.3. *The metric g defined by (1.4) is called the Moebius metric for $x : \mathbf{M} \to \mathbb{S}^n$.*

2. Moebius invariants for submanifolds in $\mathbb{S}^n$

In this section we assume that $x : \mathbf{M} \to \mathbb{S}^n$ is a connected submanifold without umbilical point. Since in this case the Moebius metric g is positive definite, there exist a unique lift $Y : \mathbf{M} \to \mathbf{C}_+^{n+1}$ of $[1, x] : \mathbf{M} \to \mathbb{Q}^n$ such that $g = \langle dY, dY \rangle$. We call Y the canonical lift of $[1, x]$. By taking $y := Y$ in (1.4) we get

$$\langle \Delta Y, \Delta Y \rangle = 1 + m^2 \kappa. \tag{2.1}$$

Let $\{E_1, E_2, \cdots, E_m\}$ be a local orthonormal basis for g with dual basis $\{\omega_1, \omega_2, \cdots, \omega_m\}$. For any function $F : \mathbf{M} \to \mathbb{R}$ we denote by F_i the partial derivative $E_i(F)$. Then from $g = \langle dY, dY \rangle$ we get

$$\left\langle Y_i, Y_j \right\rangle = \delta_{ij}, \quad 1 \leq i, j \leq m. \tag{2.2}$$

We define

$$N = -\frac{1}{m} \Delta Y - \frac{1}{2m^2} \langle \Delta Y, \Delta Y \rangle Y. \tag{2.3}$$

From the facts (cf. (1.10), (1.11)) that

$$\langle Y, Y \rangle = 0, \quad \langle Y, dY \rangle = 0, \tag{2.4}$$

$$\langle \Delta Y, Y \rangle = -m, \quad \langle \Delta Y, Y_k \rangle = 0, \quad 1 \leq k \leq m, \tag{2.5}$$

one can easily verify that

$$\langle N, Y \rangle = 1, \quad \langle N, N \rangle = \langle N, Y_k \rangle = 0, \quad 1 \leq k \leq m. \tag{2.6}$$

Thus $\text{span}\{N, Y\} \perp \text{span}\{Y_1, \cdots, Y_m\}$. We define

$$\mathbb{V} = \{\text{span}\{N, Y\} \oplus \text{span}\{Y_1, \cdots, Y_m\}\}^{\perp}. \tag{2.7}$$

Then $\mathbb{V}$ is a positive definite subspace of $\mathbb{R}^{n+2}_1$ such that

$$\mathbb{R}^{n+2}_1 = \text{span}\{N, Y\} \oplus \text{span}\{Y_1, \cdots, Y_m\} \oplus \mathbb{V}. \tag{2.8}$$

We call $\mathbb{V}$ the Moebius normal bundle for $x : \mathbf{M} \to \mathbb{S}^n$.

Let $\{E_{m+1}, \cdots, E_n\}$ be a local orthonormal basis for the bundle $\mathbb{V}$ over $\mathbf{M}$. Then $\{Y, N, Y_1, \cdots, Y_m, E_{m+1}, \cdots, E_n\}$ forms a moving frame in $\mathbb{R}^{n+2}_1$ along $\mathbf{M}$. By using the range of indices:

$$1 \leq i, j, k, \lambda \leq m; \quad m+1 \leq \alpha, \beta, \gamma \leq n;$$

and (2.4), (2.6) we can write the structure equations as follows:

$$dY = \sum_i \omega_i Y_i; \tag{2.9}$$

$$dN = \sum_i \psi_i Y_i + \sum_\alpha \phi_\alpha E_\alpha; \tag{2.10}$$

$$dY_i = -\psi_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_\alpha \omega_{i\alpha} E_\alpha; \tag{2.11}$$

$$dE_\alpha = -\phi_\alpha Y - \sum_i \omega_{i\alpha} Y_i + \sum_\beta \omega_{\alpha\beta} E_\beta; \tag{2.12}$$

where $\{\psi_i, \omega_{ij}, \omega_{i\alpha}, \phi_\alpha, \omega_{\alpha\beta}\}$ are 1-forms on **M** with $\omega_{ij} = -\omega_{ji}$ and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. By exterior differentiation of these equations we get

$$\sum_i \omega_i \wedge \psi_i = 0; \quad \sum_i \omega_{i\alpha} \wedge \omega_i = 0; \tag{2.13}$$

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j; \tag{2.14}$$

$$d\psi_i - \sum_j \omega_{ij} \wedge \psi_j = \sum_\alpha \omega_{i\alpha} \wedge \phi_\alpha; \tag{2.15}$$

$$d\phi_\alpha - \sum_\beta \omega_{\alpha\beta} \wedge \phi_\beta = -\sum_i \omega_{i\alpha} \wedge \psi_i; \tag{2.16}$$

$$d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\sum_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha} - \omega_i \wedge \psi_j - \psi_i \wedge \omega_j; \tag{2.17}$$

$$d\omega_{i\alpha} - \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha} - \sum_j \omega_{ij} \wedge \omega_{j\alpha} + \omega_i \wedge \phi_\alpha = 0; \tag{2.18}$$

$$d\omega_{\alpha\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} = -\sum_i \omega_{i\alpha} \wedge \omega_{i\beta}. \tag{2.19}$$

By (2.13) and the Cartan's Lemma we can write

$$\psi_i = \sum_j A_{ij}\omega_j, \;\; A_{ij} = A_{ji}; \quad \omega_{i\alpha} = \sum_j B^\alpha_{ij}\omega_j, \;\; B^\alpha_{ij} = B^\alpha_{ji}; \tag{2.20}$$

where $\{A_{ij}\}$ and $\{B^\alpha_{ij}\}$ are locally defined functions. It is clear that

$$\mathbb{A} := \sum_{i,j} A_{ij}\omega_i \otimes \omega_j; \quad \mathbb{B} := \sum_\alpha \sum_{ij} B^\alpha_{ij}\omega_i \otimes \omega_j E_\alpha; \tag{2.21}$$

$$\Phi := \sum_\alpha \phi_\alpha E_\alpha = \sum_{i\alpha} C^\alpha_i \omega_i E_\alpha \tag{2.22}$$

are Moebius invariants. We will call $\mathbb{B}$ the Moebius second fundamental from of x and Φ the Moebius form of x. We define

$$dC^\alpha_i + \sum_j C^\alpha_j \omega_{ji} + \sum_\beta C^\beta_i \omega_{\beta\alpha} = \sum_j C^\alpha_{i,j}\omega_j. \tag{2.23}$$

$$dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki} = \sum_k A_{ij,k}\omega_k. \tag{2.24}$$

$$dB^\alpha_{ij} + \sum_k B^\alpha_{ik}\omega_{kj} + \sum_k B^\alpha_{kj}\omega_{ki} + \sum_\beta B^\beta_{ij}\omega_{\beta\alpha} = \sum_k B^\alpha_{ij,k}\omega_k. \tag{2.25}$$

$$d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2}\sum_{k\lambda} R_{ijk\lambda}\omega_k \wedge \omega_\lambda, \;\; R_{ijk\lambda} = -R_{ij\lambda k}; \tag{2.26}$$

$$d\omega_{\alpha\beta} - \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} = -\frac{1}{2} \sum_{ij} R_{\alpha\beta ij} \omega_i \wedge \omega_j, \quad R_{\alpha\beta ij} = -R_{\alpha\beta ji}. \quad (2.27)$$

Then (2.15)–(2.19) are equivalent to the following equations:

$$A_{ij,k} - A_{ik,j} = \sum_{\alpha} (B^{\alpha}_{ik} C^{\alpha}_{j} - B^{\alpha}_{ij} C^{\alpha}_{k}); \quad (2.28)$$

$$C^{\alpha}_{i,j} - C^{\alpha}_{j,i} = \sum_{k} (B^{\alpha}_{ik} A_{kj} - B^{\alpha}_{kj} A_{ki}); \quad (2.29)$$

$$B^{\alpha}_{ij,k} - B^{\alpha}_{ik,j} = \delta_{ij} C^{\alpha}_{k} - \delta_{ik} C^{\alpha}_{j}; \quad (2.30)$$

$$R_{ijk\lambda} = \sum_{\alpha} (B^{\alpha}_{ik} B^{\alpha}_{j\lambda} - B^{\alpha}_{i\lambda} B^{\alpha}_{jk}) + (\delta_{ik} A_{j\lambda} + \delta_{j\lambda} A_{ik} - \delta_{i\lambda} A_{jk} - \delta_{jk} A_{i\lambda}); \quad (2.31)$$

$$R_{\alpha\beta ij} = \sum_{k} (B^{\alpha}_{ik} B^{\beta}_{kj} - B^{\beta}_{ik} B^{\alpha}_{kj}). \quad (2.32)$$

By (2.11) we have

$$\Delta Y = -tr(\mathbb{A}) Y - mN + \sum_{\alpha} \sum_{i} B^{\alpha}_{ii} E_{\alpha}.$$

It follows from (2.1) and (2.3) that

$$tr(\mathbb{A}) = \frac{1}{2m}(1 + m^2 \kappa); \quad \sum_{i} B^{\alpha}_{ii} = 0. \quad (2.33)$$

By taking trace in (2.30) and (2.31) we get

$$-\sum_{j} B^{\alpha}_{ij,j} = (m-1) C^{\alpha}_{i}; \quad (2.34)$$

$$R_{ij} = -\sum_{\alpha} \sum_{k} B^{\alpha}_{ik} B^{\alpha}_{kj} + tr(\mathbb{A}) \delta_{ij} + (m-2) A_{ij}. \quad (2.35)$$

Taking trace in (2.35) and using (2.33) we get

$$\sum_{\alpha} \sum_{ij} (B^{\alpha}_{ij})^2 = \frac{m-1}{m}. \quad (2.36)$$

From (2.34) and (2.35) we know that in case $m \geq 3$ all coefficients in the PDE system (2.9)–(2.12) are determined by the Moebius metric g, the Moebius second fundamental form $\mathbb{B}$ and the normal connection $\nabla^* := \{\omega_{\alpha\beta}\}$ in the Moebius normal bundle $\mathbb{V}$.

3. Fundamental theorem for hypersurfaces in $\mathbb{S}^n$

In this section we give the relations between the Moebius invariants introduced in §2 and $SO(n+1)$-invariants of $x : \mathbf{M} \to \mathbb{S}^n$. We give also a Moebius fundamental theorem for hypersurfaces in $\mathbb{S}^n$.

Let $x : \mathbf{M} \to \mathbb{S}^n$ be a submanifold without umbilical point in $\mathbb{S}^n$. Let $\{e_1, e_2, \cdots, e_m\}$ be a local orthonormal basis for $T\mathbf{M}$ with respect to $dx \cdot dx$. Let $\{\theta_1, \theta_2, \cdots, \theta_m\}$ be its dual basis. Let $\{e_{m+1}, \cdots, e_n\}$ be a local orthonormal basis for the normal bundle $N(\mathbf{M})$ of x in $\mathbb{S}^n$. We write the second fundamental form II and the mean curvature vector H of x by

$$II = \sum_\alpha \sum_{ij} h^\alpha_{ij} \theta_i \otimes \theta_j e_\alpha, \quad H = \frac{1}{m} \sum_\alpha \sum_i h^\alpha_{ii} e_\alpha := \sum_\alpha H^\alpha e_\alpha. \tag{3.1}$$

Then we have the structure equations for $x : \mathbf{M} \to \mathbb{S}^n$:

$$dx = \sum_i \theta_i e_i(x); \tag{3.2}$$

$$d(e_i(x)) = \sum_j \theta_{ij} e_j(x) + \sum_\alpha \sum_j h^\alpha_{ij} \theta_j e_\alpha - \theta_i x; \tag{3.3}$$

$$de_\alpha = -\sum_{ij} h^\alpha_{ij} \theta_j e_i + \sum_\beta \theta_{\alpha\beta} e_\beta; \tag{3.4}$$

where $\{\theta_{\alpha\beta}\}$ is the normal connection of x in $N(\mathbf{M})$. Since $Y = \rho(1, x)$ for

$$\rho := \sqrt{\frac{m}{m-1}} \| II - \frac{1}{m} tr(II) I \|,$$

we have

$$e_i(Y) = e_i(\log \rho) Y + \rho(0, e_i(x)). \tag{3.5}$$

Thus $\{E_i := \rho^{-1} e_i\}$ is a local orthonormal basis for the Moebius metric g. From (3.5) we get

$$\begin{aligned} dY_i &= (0, de_i) \pmod{Y, Y_k} \\ &= \sum_j \left(0, \sum_\alpha h^\alpha_{ij} e_\alpha - \delta_{ij} x\right) \rho^{-1} \omega_j \pmod{Y, Y_k}. \end{aligned} \tag{3.6}$$

Thus we have

$$\Delta Y = m\rho^{-1} \left(0, \sum_\alpha H^\alpha e_\alpha - x\right) \pmod{Y, Y_k}. \tag{3.7}$$

We define

$$E_\alpha = (H^\alpha, e_\alpha + H^\alpha x); \quad m+1 \le \alpha \le n. \tag{3.8}$$

Then we have

$$\langle E_\alpha, Y\rangle = \langle E_\alpha, Y_k\rangle = \langle E_\alpha, \Delta Y\rangle = 0, \quad \left\langle E_\alpha, E_\beta\right\rangle = \delta_{\alpha\beta}. \tag{3.9}$$

Thus $\{E_\alpha\}$ is a local orthonormal basis for the Moebius normal bundle $\mathbb{V}$. From (3.6), (3.8) and (2.11) we get

$$B^\alpha_{ij} = \rho^{-1}(h^\alpha_{ij} - H^\alpha\delta_{ij}), \tag{3.10}$$

where $\{B^\alpha_{ij}\}$ are the components of the tensor $\mathbb{B}$ with respect to $\{E_i\}$. Thus we have

$$\mathbb{B} = \sum_\alpha\sum_{ij}\rho(h^\alpha_{ij} - H^\alpha\delta_{ij})\theta_i\otimes\theta_j(H^\alpha, e_\alpha + H^\alpha x). \tag{3.11}$$

From (3.8) we have

$$dE_\alpha = (dH^\alpha, de_\alpha + dH^\alpha x + H^\alpha dx). \tag{3.12}$$

Thus we get

$$\omega_{\alpha\beta} = \left\langle dE_\alpha, E_\beta\right\rangle = \left\langle de_\alpha, e_\beta\right\rangle = \theta_{\alpha\beta}. \tag{3.13}$$

Therefore, the bundle map $f : N(M) \to \mathbb{V}$ defined by

$$f(e_\alpha) := (H^\alpha, e_\alpha + H^\alpha x)$$

preserves the inner product and the connection. In particular, the normal connection $\{\theta_{\alpha\beta}\}$ in $N(\mathbf{M})$ is a Moebius invariant.

By a direct calculation we get the following expression of the Moebius invariants $\mathbb{A}$ and $\Phi := \sum_{i\alpha} C^\alpha_i\omega_i E_\alpha$:

$$\begin{aligned} A_{ij} = &-\rho^{-2}\left(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum_\alpha H^\alpha h^\alpha_{ij}\right)\\ &-\frac{1}{2}\rho^{-2}\left(|\nabla\log\rho|^2 - 1 + \sum_\alpha(H^\alpha)^2\right)\delta_{ij}; \end{aligned} \tag{3.14}$$

$$C^\alpha_i = -\rho^{-2}\left(H^\alpha_{,i} + \sum_j(h^\alpha_{ij} - H^\alpha\delta_{ij})e_j(\log\rho)\right); \tag{3.15}$$

where $\{\mathrm{Hess}_{ij}\}$ and $\{H^\alpha_{,i}\}$ are Hessian-Matrix of $dx\cdot dx$ and the covariant derivative of the mean curvature vector field of x in the normal bundle $N(M)$ (with respect to the basis $\{e_i\}$ and $\{e_\alpha\}$).

Now we consider the case that $x : \mathbf{M} \to \mathbb{S}^n$ is a hypersurface, i.e. $m = n - 1$. In this case we get from (3.10) that

$$\mathbb{S} := \rho^{-1}(S - Hid) = \sum_{ij} B^n_{ij}\omega_i E_j, \tag{3.16}$$

where S is the Weingarten operator for $x : \mathbf{M} \to \mathbb{S}^n$. It follows from (3.12) that $\rho^{-1}(S - Hid)$ is a Moebius invariant which determines the tensor $\mathbb{B}$. We call $\rho^{-1}(S - Hid)$ the Moebius shape operator of x. Since for $m = n - 1 \geq 3$ all coefficients of the PDE system (2.9)–(2.12) are determined by $\{g, \rho^{-1}(S - Hid)\}$, we get the following theorem:

Theorem 3.1. *Two hypersurface $x, \tilde{x} : \mathbf{M} \to \mathbb{S}^n$ $(n \geq 4)$ are Moebius equivalent if and only if there exists a diffeomorphsim $\sigma : \mathbf{M} \to \mathbf{M}$ which preserves the Moebius metric g and the Moebius shape operator $\mathbb{S} = \rho^{-1}(S - Hid)$.*

Remark 3.2. In case n=3 a complete Moebius invariant system is given in [10].

4. The first variation of the Moebius volume functional

Let $x_0 : \mathbf{M} \to \mathbb{S}^n$ be a compact submanifold with boundary $\partial\mathbf{M}$. We denote by dM_0 the volume form of x_0 with respect to the metric $dx_0 \cdot dx_0$. Then we define the generalized Willmore functional $\mathbb{W}$ as the volume functional of the Moebius metric g:

$$\mathbb{W}(\mathbf{M}) := \left(\frac{m}{m-1}\right)^{\frac{m}{2}} \int_M \|II - HI\|^m dM_0 = Vol_g(\mathbf{M}), \tag{4.1}$$

where $I = dx_0 \cdot dx_0$, II and H is the first, the second fundamental form and the mean curvature vector of x_0 in $\mathbb{S}^n$ respectively. In this section we assume that x_0 has no umbilical point.

Let $x : \mathbf{M} \times \mathbb{R} \to \mathbb{S}^n$ be a smooth variation of x_0 such that $x(\cdot, t) = x_0$ and $dx_t(TM) = dx_0(TM)$ on $\partial\mathbf{M}$ for each (small) t. We call such variation an admissible variation of x_0. We note that the two boundary conditions for an admissible variation disappear if $\partial\mathbf{M} = \phi$. For each t we denote by $\{e_i\}$ a local orthonormal basis for $T\mathbf{M}$ with respect to $dx_t \cdot dx_t$ with dual basis $\{\theta_i\}$ and by $\{e_\alpha\}$ a local orthonormal basis for the normal bundle of x_t. Let $Y = \rho(1, x) : \mathbf{M} \times \mathbb{R} \to \mathbf{C}^{n+1}_+$ be the canonical lift of x_t and $g_t = \langle dY, dY\rangle$ be the Moebius metric of x_t. Let $\{E_i := \rho^{-1}e_i\}$ be a local orthonormal basis for g_t with dual basis $\{\omega_i = \rho\theta_i\}$. Using the Laplacian operator Δ of g_t and (2.3) we can define the map $N : \mathbf{M} \times \mathbb{R} \to \mathbb{R}^{n+2}_1$. Let $\{E_\alpha\}$ be the orthonormal basis for the Moebius normal bundle $\mathbb{V}_t$ of x_t defined by (2.7). We denote by d the differential operator on $\mathbf{M} \times \mathbb{R}$. Since $\{Y, N, Y_i, E_\alpha\}$ is a moving frame along $\mathbf{M} \times \mathbb{R}$, we can find 1-forms

$$\{V, V_\alpha, \Psi_i, \Phi_\alpha, \Omega_i, \Omega_{ij}, \Omega_{i\alpha}, \Omega_{\alpha\beta}\}$$

on $\mathbf{M}\times\mathbb{R}$ with $\Omega_{ij}=-\Omega_{ji}$ and $\Omega_{\alpha\beta}=-\Omega_{\beta\alpha}$ such that

$$dY = VY + \sum_i \Omega_i Y_i + \sum_\alpha V_\alpha E_\alpha; \tag{4.2}$$

$$dN = -VN + \sum_i \Psi_i Y_i + \sum_\alpha \Phi_\alpha E_\alpha; \tag{4.3}$$

$$dY_i = -\Psi_i Y - \Omega_i N + \sum_j \Omega_{ij} Y_j + \sum_\alpha \Omega_{i\alpha} E_\alpha; \tag{4.4}$$

$$dE_\alpha = -\Phi_\alpha Y - V_\alpha N - \sum_i \Omega_{i\alpha} Y_i + \sum_\beta \Omega_{\alpha\beta} E_\beta. \tag{4.5}$$

Taking the differential of (4.2)–(4.5) we get

$$dV = \sum_i \Psi_i \wedge \Omega_i + \sum_\alpha \Phi_\alpha \wedge V_\alpha; \tag{4.6}$$

$$d\Omega_i = \sum_j \Omega_{ij}\wedge\Omega_j + V\wedge\Omega_i - \sum_\alpha V_\alpha\wedge\Omega_{i\alpha}; \tag{4.7}$$

$$dV_\alpha = \sum_\beta \Omega_{\alpha\beta}\wedge V_\beta + \sum_i \Omega_i\wedge\Omega_{i\alpha} + V\wedge V_\alpha; \tag{4.8}$$

$$d\Psi_i = \sum_j \Omega_{ij}\wedge\Psi_j - \sum_\alpha \Phi_\alpha\wedge\Omega_{i\alpha} + \Psi_i\wedge V; \tag{4.9}$$

$$d\Phi_\alpha = \sum_\beta \Omega_{\alpha\beta}\wedge\Phi_\beta + \sum_i \Psi_i\wedge\Omega_{i\alpha} + \Phi_\alpha\wedge V; \tag{4.10}$$

$$d\Omega_{ij} = \sum_k \Omega_{ik}\wedge\Omega_{kj} - \sum_\alpha \Omega_{i\alpha}\wedge\Omega_{j\alpha} - \Psi_i\wedge\Omega_j - \Omega_i\wedge\Psi_j; \tag{4.11}$$

$$d\Omega_{i\alpha} = \sum_j \Omega_{ij}\wedge\Omega_{j\alpha} + \sum_\beta \Omega_{i\beta}\wedge\Omega_{\beta\alpha} - \Psi_i\wedge V_\alpha - \Omega_i\wedge\Phi_\alpha; \tag{4.12}$$

$$d\Omega_{\alpha\beta} = \sum_\gamma \Omega_{\alpha\gamma}\wedge\Omega_{\gamma\beta} - \sum_i \Omega_{i\alpha}\wedge\Omega_{i\beta} - \Phi_\alpha\wedge V_\beta - V_\alpha\wedge\Phi_\beta. \tag{4.13}$$

Since $Y=\rho(1,x)$, if we write the variation vector field of x in $T\mathbb{S}^n$ by

$$\frac{\partial x}{\partial t} = \rho^{-1}\left(\sum_i v_i e_i + \sum_\alpha v_\alpha e_\alpha\right), \tag{4.14}$$

then by (3.5) and (3.8) we can find a function $v:\mathbf{M}\times\mathbb{R}\to\mathbb{R}$ such that

$$\frac{\partial Y}{\partial t} = vY + \sum_i v_i Y_i + \sum_\alpha v_\alpha E_\alpha. \tag{4.15}$$

From (4.2) and the fact that $d = \sum_i \omega_i E_i + dt\frac{\partial}{\partial t}$ on $C^\infty(\mathbf{M}\times\mathbb{R})$ we get

$$V = v dt; \quad V_\alpha = v_\alpha dt; \quad \Omega_i = \omega_i + v_i dt. \tag{4.16}$$

Since $T^*(\mathbf{M}\times\mathbb{R}) = T^*\mathbf{M}\oplus T^*(\mathbb{R})$, we have the decomposition

$$\Psi_i = \psi_i + a_i dt; \quad \Phi_\alpha = \phi_\alpha + b_\alpha dt; \tag{4.17}$$

$$\Omega_{ij} = \omega_{ij} + P_{ij}dt; \;\; \Omega_{i\alpha} = \omega_{i\alpha} + L_{i\alpha}dt; \;\; \Omega_{\alpha\beta} = \omega_{\alpha\beta} + Q_{\alpha\beta}dt, \tag{4.18}$$

where $\{a_i, b_\alpha, P_{ij}, L_{i\alpha}, Q_{\alpha\beta}\}$ are local functions with $P_{ij} = -P_{ji}$ and $Q_{\alpha\beta} = -Q_{\beta\alpha}$. We denote by d_M the differential operator on $T^*\mathbf{M}$, then we have $d = d_M + dt\wedge\frac{\partial}{\partial t}$ on $T^*\mathbf{M}\oplus T^*(\mathbb{R})$. Using (4.7), (4.16), (4.18) and comparing the terms in $T^*\mathbf{M}\wedge dt$ we get

$$\frac{\partial\omega_i}{\partial t} = \sum_j\left(v_{i,j} + P_{ij} + v\delta_{ij} - \sum_\alpha v_\alpha B^\alpha_{ij}\right)\omega_j, \tag{4.19}$$

where $\{v_{i,j}\}$ is the covariant derivative of $\sum_i v_i E_i$ with respect to g_t. Here we have used the notations of Moebius invariants $\{B^\alpha_{ij}, A_{ij}, C^\alpha_i\}$ for x_t defined by §2. By the same way we get from (4.8) and (4.12) that

$$L_{i\alpha} = v_{\alpha,i} + \sum_j v_j B^\alpha_{ij}; \tag{4.20}$$

$$\frac{\partial\omega_{i\alpha}}{\partial t} = \sum_j\left(L_{i\alpha,j} + \sum_k P_{ik}B^\alpha_{kj} - \sum_\beta B^\beta_{ij}Q_{\beta\alpha} + A_{ij}v_\alpha + b_\alpha\delta_{ij} - v_i C^\alpha_j\right)\omega_j, \tag{4.21}$$

where $\{v_{\alpha,i}\}$ and $\{L_{i\alpha,j}\}$ are covariant derivatives of $\sum_\alpha v_\alpha E_\alpha$ and $\sum_{i\alpha} L_{i\alpha}\omega_i E_\alpha$ respectively. Using (4.21), (4.19), (4.20) and the second equation of (2.20) and (2.33) we get

$$\begin{aligned}\frac{\partial B^\alpha_{ij}}{\partial t} + vB^\alpha_{ij} =& v_{\alpha,ij} + \sum_k v_k B^\alpha_{ik,j} + \sum_k (P_{ik}B^\alpha_{kj} - P_{kj}B^\alpha_{ik}) \\ &- \sum_\beta B^\beta_{ij}Q_{\beta\alpha} + \sum_{k\beta} v_\beta B^\alpha_{ik}B^\beta_{kj} + A_{ij}v_\alpha + b_\alpha\delta_{ij} - v_i C^\alpha_j.\end{aligned} \tag{4.22}$$

It follows from (2.36) that

$$\begin{aligned}\frac{m-1}{m}v =& \sum_{ij\alpha} B^\alpha_{ij}v_{\alpha,ij} + \sum_{\alpha\beta}\sum_{ijk} B^\alpha_{ik}B^\alpha_{ij}B^\beta_{kj}v_\beta + \sum_{ij\alpha} A_{ij}B^\alpha_{ij}v_\alpha \\ &+ \sum_{ijk\alpha} v_k B^\alpha_{ik,j}B^\alpha_{ij} - \sum_{ij\alpha} v_i C^\alpha_j B^\alpha_{ij}.\end{aligned} \tag{4.23}$$

From (2.30) and (2.36) we get

$$\sum_{ijk\alpha} v_k B^\alpha_{ik,j} B^\alpha_{ij} = \sum_{ij\alpha} v_i C^\alpha_j B^\alpha_{ij}.$$

Thus we get from (4.23) that

$$\frac{m-1}{m} v = \sum_{ij\alpha} B^\alpha_{ij} v_{\alpha,ij} + \sum_{\alpha\beta}\sum_{ijk} B^\alpha_{ik} B^\alpha_{ij} B^\beta_{kj} v_\beta + \sum_{ij\alpha} A_{ij} B^\alpha_{ij} v_\alpha. \quad (4.24)$$

Now we calculate the variation of

$$V(t) = Vol(g_t) = \int_M \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_m = \int_M dM,$$

where dM is the volume form for g_t. From (4.19) and (4.24) we get

$$V'(t) = \int_M \left(\sum_i v_{i,i} + mv \right) dM = m \int_M v dM$$

$$= \frac{m^2}{m-1} \int_M \left\{ \sum_{ij\alpha} B^\alpha_{ij} v_{\alpha,ij} + \sum_{\alpha\beta}\sum_{ijk} B^\alpha_{ik} B^\alpha_{ij} B^\beta_{kj} v_\beta + \sum_{ij\alpha} A_{ij} B^\alpha_{ij} v_\alpha \right\} dM. \quad (4.25)$$

From the fact that $x(\cdot, t) = x_0$ and $dx_t(T\mathbf{M}) = dx_0(T\mathbf{M})$ on $\partial\mathbf{M}$ for all small t, we know that at each point on $\partial\mathbf{M}$ we have $v_i = v_\alpha = 0$ and

$$0 = \frac{\partial}{\partial t}(dx_t) = d\left(\frac{\partial x}{\partial t}\right) = \rho^{-1}\left(\sum_i dv_i e_i + \sum_\alpha dv_\alpha e_\alpha \right).$$

Thus we get also $v_{\alpha,i} = 0$ on $\partial\mathbf{M}$. It follows from (4.25) and Green's formula that

$$V'(t) = \frac{m^2}{m-1} \int_M \sum_\alpha \left\{ \sum_{ij} B^\alpha_{ij,ij} + \sum_\beta \sum_{ijk} B^\beta_{ik} B^\beta_{ij} B^\alpha_{kj} + \sum_{ij} A_{ij} B^\alpha_{ij} \right\} v_\alpha dM. \quad (4.26)$$

It follows from (4.14) and (4.26) that

Theorem 4.1. *The volume variation of the Moebius metric depends only on the normal component of the variation vector field. A submanifold $x : \mathbf{M} \to \mathbb{S}^n$ is a Moebius minimal submanifold (i.e. a critical submanifold to the Moebius volume functional) if and only if*

$$\sum_{ij} B^\alpha_{ij,ij} + \sum_\beta \sum_{ijk} B^\beta_{ik} B^\beta_{ij} B^\alpha_{kj} + \sum_{ij} A_{ij} B^\alpha_{ij} = 0, \quad m+1 \le \alpha \le n. \quad (4.27)$$

Using (2.34) and (2.35) we can write the Euler–Lagrange equations (4.27) as

$$\sum_i C^\alpha_{i,i} + \sum_{ij}\left(\frac{1}{m-1}R_{ij} - A_{ij}\right)B^\alpha_{ij} = 0, \quad m+1 \le \alpha \le n. \tag{4.28}$$

Theorem 4.2. *Any minimal surface in $\mathbb{S}^n$ is also Moebius minimal.*

Proof. Let $x : \mathbf{M} \to \mathbb{S}^n$ be a surface in $\mathbb{S}^n$. Let $\{e_1, e_2\}$ be a local orthonormal basis of $dx \cdot dx$ and $\{e_\alpha\}$ a local orthonormal basis for the normal bundle. We use the notations in §2 and §3. From (3.10) and (3.14) we get

$$\begin{aligned}\sum_{ij} A_{ij}B^\alpha_{ij} = &-\sum_{ij}\rho^{-2}(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho))B^\alpha_{ij} \\ &+ \rho^{-1}\sum_{ij\beta} H^\beta B^\beta_{ij}B^\alpha_{ij}.\end{aligned} \tag{4.29}$$

Since we have the following relations of connections

$$\omega_{ij} = \theta_{ij} + e_i(\log\rho)\theta_j - e_j(\log\rho)\theta_i; \quad \omega_{\alpha\beta} = \theta_{\alpha\beta},$$

a direct calculation implies

$$\sum_i C^\alpha_{i,i} = -\sum_{ij}\rho^{-2}(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho))B^\alpha_{ij} - \rho^{-3}\Delta H^\alpha, \tag{4.30}$$

where $\Delta H^\alpha := \sum_i H^\alpha_{,ii}$. Thus we have

$$\sum_i C^\alpha_{i,i} - \sum_{ij} A_{ij}B^\alpha_{ij} = -\rho^{-3}\left(\Delta H^\alpha + \rho^2\sum_{ij\beta} H^\beta B^\beta_{ij}B^\alpha_{ij}\right). \tag{4.31}$$

If x is a minimal surface, that is $H^\alpha \equiv 0$ and $R_{ij} = \kappa\delta_{ij}$, we know from (4.28) and (4.31) that x is Moebius minimal. □

Theorem 4.3. *Let $u : \mathbf{M} \to \mathbb{R}^n$ be a minimal surface in $\mathbb{R}^n$. Let $\sigma : \mathbb{R}^n \to \mathbb{S}^n$ be the inverse of a stereographic projection. Then $x := \sigma \circ u : \mathbf{M} \to \mathbb{S}^n$ is Moebius minimal.*

Proof. We may assume that σ is the inverse of the stereographic projection from the point $(0, \cdots, 0, -1)$ given by

$$\sigma(u) = \left(\frac{2u}{1+\|u\|^2}, \frac{1-\|u\|^2}{1+\|u\|^2}\right), \quad u \in \mathbb{R}^n. \tag{4.32}$$

Then we have

$$dx \cdot dx = \frac{4}{(1+\|u\|^2)^2} du \cdot du. \tag{4.33}$$

Let $\{\tilde{e}_1, \tilde{e}_2\}$ be a local orthonormal basis for $T\mathbf{M}$ with respect to $du \cdot du$ and $\{\tilde{e}_\alpha\}$ a local orthonormal basis for the normal bundle of u. We denote by $\{\tilde{h}^\alpha_{ij}\}$ the second fundamental form of u with respect to $\{\tilde{e}_i, \tilde{e}_\alpha\}$ and by $\sum_\alpha \tilde{H}^\alpha \tilde{e}_\alpha$ the mean curvature vector of u. Then the basis $\{e_i\}$ and $\{e_\alpha\}$ defined by

$$e_i = \frac{1 + \|u\|^2}{2}\tilde{e}_i, \quad e_\alpha = \frac{1 + \|u\|^2}{2} d\sigma(\tilde{e}_\alpha) \tag{4.34}$$

is a local orthonormal basis $T\mathbf{M}$ with respect to $dx \cdot dx$ and the normal bundle of x respectively. We use the notations in §2 and §3 for $x = \sigma \circ u$. Then we have the following relations between the second fundamental form of x and u:

$$h^\alpha_{ij} = \frac{1 + \|u\|^2}{2}\tilde{h}^\alpha_{ij} + \tilde{e}_\alpha \cdot u\delta_{ij}, \quad H^\alpha = \frac{1 + \|u\|^2}{2}\tilde{H}^\alpha + \tilde{e}_\alpha \cdot u. \tag{4.35}$$

Let $\{\theta_{ij}\}$ and $\{\theta_{\alpha\beta}\}$ (resp. $\{\tilde{\theta}_{ij}\}$ and $\{\tilde{\theta}_{\alpha\beta}\}$) be the Levi-Civita connection and the normal connection of $x : \mathbf{M} \to \mathbb{S}^n$ (resp. $u : \mathbf{M} \to \mathbb{R}^n$) with respect to $\{e_i, e_\alpha\}$ (resp. $\{\tilde{e}_i, \tilde{e}_\alpha\}$). Then we have

$$\theta_{ij} = \tilde{\theta}_{ij} + \frac{2u \cdot \tilde{e}_j(u)}{1 + \|u\|^2}\tilde{\theta}_i - \frac{2u \cdot \tilde{e}_i(u)}{1 + \|u\|^2}\tilde{\theta}_j, \quad \theta_{\alpha\beta} = \tilde{\theta}_{\alpha\beta}, \tag{4.36}$$

where $\{\tilde{\theta}_1, \tilde{\theta}_2\}$ is the dual basis for $\{\tilde{e}_i\}$. Now let Δ and $\tilde{\Delta}$ be the Laplacian operator with respect to x and u respectively. Then by a staightforward calculation we get

$$\begin{aligned} &\Delta H^\alpha + \rho^2 \sum_{ij\beta} H^\beta B^\beta_{ij} B^\alpha_{ij} \\ &= \left(\frac{1 + \|u\|^2}{2}\right)^3 \left\{\tilde{\Delta}\tilde{H}^\alpha + \sum_{ij\beta} \tilde{H}^\beta(\tilde{h}^\beta_{ij} - \tilde{H}^\beta\delta_{ij})\tilde{h}^\alpha_{ij}\right\}. \end{aligned} \tag{4.37}$$

Thus Theorem 4.3 follows from (4.31) and (4.37). □

Remark 4.4. Let $u : \mathbf{M}^m \to \mathbb{R}^n$ be a submanifold in $\mathbb{R}^n$. Then

$$\tilde{g} := \frac{m}{m-1}\left\{\sum_{ij\alpha}(\tilde{h}^\alpha_{ij})^2 - m\sum_\alpha(\tilde{H}^\alpha)^2\right\} du \cdot du$$

is a Moebius invariant metric in $\mathbb{R}^n$. In case m=2 the Euler-Lagrange equations to the variation of the volume functional of $\tilde{g}$ is given by

$$\tilde{\Delta}\tilde{H}^\alpha + \sum_{ij\beta} \tilde{H}^\beta(\tilde{h}^\beta_{ij} - \tilde{H}^\beta\delta_{ij})\tilde{h}^\alpha_{ij} = 0. \tag{4.38}$$

5. Special Moebius surfaces in $\mathbb{S}^3$

In this section we use the formulas given by §2 to classify all surfaces in $\mathbb{S}^3$ with vanishing Moebius form $\Phi := \phi_\alpha$ defined by (2.22).

Let $x : \mathbf{M} \to \mathbb{S}^3$ be a surface without umbilic point. We assume that for x the Moebius form $\Phi \equiv 0$. We use the notations in §2 and omit all α and β in the formulas because the codimension now is one. Since the Moebius shape operator $\mathbb{S}$ for a surface has eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$, we can find an orthonormal basis $\{E_1, E_2\}$ of the Moebous metric g such that

$$B_{11} = \frac{1}{2}, \quad B_{12} = B_{21} = 0, \quad B_{22} = -\frac{1}{2}. \tag{5.1}$$

From the assumption that $\phi = \sum_i C_i \omega_i \equiv 0$, we know from (2.30) that $\{B_{ij,k}\}$ are totally symmetric. Since

$$dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k, \tag{5.2}$$

we get $B_{11,2} = B_{22,1} = 0$. By taking $i = 1$ and $j = 2$ in (5.2) we get $\omega_{12} = 0$. Thus the Moebius metric g is flat. From (2.29) we know that the matrices (A_{ij}) and (B_{ij}) commute, which implies that

$$A_{11} = a, \quad A_{12} = A_{21} = 0, \quad A_{22} = b. \tag{5.3}$$

Let (u, v) be a coordinate system for $\mathbf{M}$ such that $E_1 = \frac{\partial}{\partial u}$ and $E_2 = \frac{\partial}{\partial v}$, then by (2.28) we know that a depends only on u and b depends only on v. Moreover, from (2.33) we have $a + b = \frac{1}{4}$, which implies that a and b are constant. Thus we can write the structure equations (2.9)–(2.12) as

$$N_u = aY_u, \quad N_v = bY_v; \tag{5.4}$$

$$Y_{uu} = -aY - N + \frac{1}{2}E, \quad Y_{uv} = 0, \quad Y_{vv} = -bY - N - \frac{1}{2}E; \tag{5.5}$$

$$E_u = -\frac{1}{2}Y_u, \quad E_v = \frac{1}{2}Y_v. \tag{5.6}$$

By (5.4) and (5.5) we know that $Y = f(u) + g(v)$ for some 1-variable functions f and g, which satisfy

$$f'''(u) + \left(2a + \frac{1}{4}\right) f'(u) = 0, \quad g'''(v) + \left(2b + \frac{1}{4}\right) g'(v) = 0. \tag{5.7}$$

We define $r = 2a + \frac{1}{4}$, then $2b + \frac{1}{4} = 1 - r$. By exchanging u and v if necessary we may assume that $r \le \frac{1}{2}$. Thus we have to consider the following three case: (i) $r = 0$; (ii) $0 < \gamma \le \frac{1}{2}$; (iii) $r < 0$.

In case (i) we can easily get a unique adapted solution (up to transformations in $O(4,1)$):

$$Y = \left(\frac{1}{2}u^2 + 1, \cos v, \sin v, u, -\frac{1}{2}u^2\right). \tag{5.8}$$

From the fact that $Y = \rho(1, x)$, we get the surface

$$x = \frac{1}{u^2+2}(2\cos v, 2\sin v, 2u, -u^2), \tag{5.9}$$

whose imgae unter the stereographic projection from the point $(0, 0, 0, -1)$ is the circular cylinder $\{(\cos v, \sin v, u), (u, v) \in \mathbb{R}^2\}$.

In case (ii) we get a unique adapted solution (up to transformations in $O(4,1)$):

$$\begin{aligned} Y =\Bigg(&\frac{1}{\sqrt{r(1-r)}}, \frac{1}{\sqrt{1-r}}\cos\left(\sqrt{1-r}v\right), \frac{1}{\sqrt{1-r}}\sin\left(\sqrt{1-r}v\right), \\ &\frac{1}{\sqrt{r}}\cos\left(\sqrt{r}u\right), \frac{1}{\sqrt{r}}\sin\left(\sqrt{r}u\right)\Bigg). \end{aligned} \tag{5.10}$$

The corresponding surface in $\mathbb{S}^3$ is the flat torus

$$\begin{aligned} x = \Bigg(&\sqrt{r}\cos\left(\sqrt{1-r}v\right), \sqrt{r}\sin\left(\sqrt{1-r}v\right), \\ &\sqrt{1-r}\cos\left(\sqrt{r}u\right), \sqrt{1-r}\sin\left(\sqrt{r}u\right)\Bigg). \end{aligned} \tag{5.11}$$

In case (iii) we get a unique adapted solution (up to transformations in $O(4,1)$):

$$\begin{aligned} Y =\Bigg(&\frac{1}{\sqrt{-r}}\cosh\left(\sqrt{-r}u\right), \frac{1}{\sqrt{1-r}}\cos\left(\sqrt{1-r}v\right) \\ &\frac{1}{\sqrt{1-r}}\sin\left(\sqrt{1-r}v\right), -\frac{1}{\sqrt{r(r-1)}}, \frac{1}{\sqrt{-r}}\sinh\left(\sqrt{-r}u\right)\Bigg). \end{aligned} \tag{5.12}$$

The corresponding surface in $\mathbb{S}^3$ is given by

$$\begin{aligned} x =&\frac{1}{\cosh\left(\sqrt{-r}u\right)}\Bigg(\sqrt{\frac{r}{r-1}}\cos\left(\sqrt{1-r}v\right), \\ &\sqrt{\frac{r}{r-1}}\sin\left(\sqrt{1-r}v\right), -\frac{1}{\sqrt{1-r}}, \sinh\left(\sqrt{-r}v\right)\Bigg), \end{aligned} \tag{5.13}$$

whose image unter the stereographic projection from the point $(0, 0, 0, -1)$ is the circular cone

$$\left\{ e^{-\sqrt{-r}u} \left(\sqrt{\frac{r}{r-1}} \cos\left(\sqrt{1-r}v\right), \sqrt{\frac{r}{r-1}} \sin\left(\sqrt{1-r}v\right), -\frac{1}{\sqrt{1-r}} \right), (u, v) \in \mathbb{R}^2 \right\}$$

in $\mathbb{R}^3$.

Thus we have the following classification theorem:

Theorem 5.1. *Any surface in $\mathbb{S}^3$ with vanishing Moebius invariant $\Phi := \phi_\alpha$ defined by (2.22) is Moebius equivalent to one of the surfaces given by (5.9), (5.11) and (5.13).*

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