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Rational fibered cubic fourfolds

Abstract. Some classes of cubic fourfolds are birational to fibrations over $\mathbb{P}^2$, where the fibers are rational surfaces. This is the case for cubics containing a plane (resp. an elliptic ruled surface), where the fibers are quadric surfaces (resp. del Pezzo sextic surfaces). It is known that the rationality of these cubic hypersurfaces is related to the rationality of these surfaces over the function field of $\mathbb{P}^2$ and to the existence of rational (multi)sections of the fibrations. We study, in the moduli space of cubic fourfolds, the intersection of the divisor $\mathcal{C}_8$ (resp. $\mathcal{C}_{18}$) with $\mathcal{C}_{14}, \mathcal{C}_{26}$ and $\mathcal{C}_{38}$, whose elements are known to be rational cubic fourfolds. We provide descriptions of the irreducible components of these intersections and give new explicit examples of rational cubics fibered in (quartic, quintic) del Pezzo surfaces or in quadric surfaces over $\mathbb{P}^2$. We also investigate the existence of rational sections for these fibrations. Under some mild assumptions on the singularities of the fibers, these properties can be translated in terms of Brauer classes on certain surfaces.

1. Introduction

A cubic fourfold is a smooth cubic hypersurface $X \subset \mathbb{P}^5$, that is the vanishing locus of a degree 3 homogeneous polynomial in 6 variables. Determining the rationality (or not) of X is a very challenging open problem in algebraic geometry.

The fourfold X is called *special* if it contains an algebraic surface S which is not homologous to a complete intersection. Hassett [12] defines the loci $\mathcal{C}_d$ of special cubic fourfolds of discriminant d and shows that these loci are non-empty irreducible divisors in the moduli space of cubic fourfolds $\mathcal{C}$, for $d > 6$ and $d \equiv 0, 2$ (mod 6). The values d hence make up an infinite sequence of integers. Moreover, for an infinite, proper subset of the set of the divisors $\mathcal{C}_d$, the Hodge structure of the nonspecial cohomology of the cubic fourfold is essentially the Hodge structure of the primitive cohomology of a K3 surface. This K3 surface is said to be *associated* to the special cubic fourfold. Hassett [12, Sect. 5] shows that special cubic fourfolds of discriminant d have an associated K3 surface if and only if $d \nmid 4, 9$, or any odd prime $p \equiv 2$ (mod 3). A natural suspicion is that any rational smooth cubic fourfold ought to have an associated K3 surface. For now, cubic fourfolds in $\mathcal{C}_{14}$, $\mathcal{C}_{26}, \mathcal{C}_{38}$ and $\mathcal{C}_{42}$ are proved to be rational (see [6,8,18] or very recently [19]).

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In this paper, we are interested in classes of cubic fourfolds that are birational to surface fibrations over $\mathbb{P}^2$. This is the case for cubics in $\mathcal{C}_8$ (resp. $\mathcal{C}_{18}$) containing a plane (resp. an elliptic ruled surface). The projection from the plane (resp. the linear system of quadrics through the elliptic ruled surface) gives the cubic a structure of quadric (resp. sextic del Pezzo) surface fibration over $\mathbb{P}^2$ ([1,11]). The study of these fibrations and in particular the rationality of the surfaces in question over the function field of $\mathbb{P}^2$ is strongly related to the birational geometry of the fibered cubic fourfolds (see [1,5,11]).

In fact, Hassett [11] identifies countably many divisors in $\mathcal{C}_8$ consisting of cubic fourfolds containing a plane whose Clifford invariant is trivial, which implies rationality. We call Clifford invariant the Brauer class $\alpha \in Br(S)$ of the $\mathbb{P}^1$-bundle over a smooth degree two K3 surface S, which is the relative Hilbert scheme of lines $\mathcal{H}(0, 1)$ in the fibers of the quadric fibration. This is well defined if the discriminant divisor of the quadric bundle is smooth [3,5]. The triviality of α is equivalent to the existence of a rational section of the associated quadric surface bundle over $\mathbb{P}^2$. Each of these loci is a codimension two subvariety in the moduli space of cubic fourfolds $\mathcal{C}$. On the other hand, in [1], the authors show other examples of rational cubic fourfolds in $\mathcal{C}_{18}$, fibered in del Pezzo sextics, parametrized by a countably infinite union of codimension two subvarieties. A generic cubic fourfold X in $\mathcal{C}_{18}$ contains an elliptic ruled surface T of degree 6. The ideal of quadrics through T define a rational map $X \dashrightarrow \mathbb{P}^2$ that displays the blow-up $\tilde{X}$ of X along T as a fibration in sextic del Pezzo surfaces over $\mathbb{P}^2$. Such a fibration is rational over the function field $\mathbb{C}(\mathbb{P}^2)$ when it has a rational section, and the codimension two loci described in [1] are exactly the loci were the associated del Pezzo fibration has a rational section. If the fibration has some mild *"good"* proprieties (see *Def 3.1 -* we will call these *good del Pezzo fibrations*), one can associate two Brauer classes to such a fibration: the class $\beta_2 \in Br(S)$ of a $\mathbb{P}^2$-bundle $\mathcal{H}(0, 3)$ over a smooth degree two K3 surface S and the class $\beta_3 \in Br(Z)$ of a $\mathbb{P}^1$-bundle $\mathcal{H}(0, 2)$ over a triple cover Z of $\mathbb{P}^2$. Here, $\mathcal{H}(0, 3)$ and $\mathcal{H}(0, 2)$ denote the relative Hilbert schemes over $\mathbb{P}^2$ parametrizing connected genus zero curves of respectively anticanonical degree three and two. The triviality of these Brauer classes β_2 or β_3 is equivalent to the existence of respectively a rational 2- or 3- multisection (see [14]) of the del Pezzo fibration. The existence of a rational section is equivalent to the triviality of both Brauer classes.

The goal of this paper is twofold. First, we study the geography of certain codimension two loci in the moduli space of cubic fourfold, obtained as intersections of $\mathcal{C}_8$ and $\mathcal{C}_{18}$ with other divisors that parametrize rational cubic fourfolds, $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$. We give a complete description of the irreducible components of these intersections. These results are obtained by studying the intersection lattices of cubic fourfolds contained in these loci.[1]

Theorem 1.1. (1) *The intersection $\mathcal{C}_8 \cap \mathcal{C}_{26}$ in the moduli space of cubic four-folds has eight irreducible components indexed by the discriminant $d \in \{29, 36, 48, 53, 61, 64, 68, 69\}$ of the intersection pairing of generic elements inside each component.*

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[1] In fact, for $\mathcal{C}_{14} \cap \mathcal{C}_8$ these are already known from [5]

(2) *The intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$ in the moduli space of cubic fourfolds has ten irreducible components indexed by the discriminant $d \in \{36, 45, 61, 68, 80, 85, 93, 96, 100, 101\}$ of the intersection pairing of generic elements inside each component.*

Theorem 1.2. (1) *The intersection $\mathcal{C}_{18} \cap \mathcal{C}_{14}$ has five irreducible components. Cubic fourfolds from these components have intersection lattice containing primitively embedded rank 3 sublattice of discriminant 84, 81, 72, 57 and 36.*
(2) *The intersection $\mathcal{C}_{18} \cap \mathcal{C}_{26}$ has eight irreducible components. Cubic fourfolds from these components have intersection lattice containing a primitively embedded rank 3 sublattice of discriminant 48, 81, 108, 129, 144, 153, 156 and 36.*
(3) *The intersection $\mathcal{C}_{18} \cap \mathcal{C}_{38}$ has eleven irreducible components. Cubic fourfolds from these components have intersection lattice containing a primitively embedded rank 3 sublattice of discriminant, 36, 81, 120, 153, 180, 201, 216, 225, 228, 45 and 57.*

On the other hand, applying the results of the first part of the paper, we study the existence of rational (multi-)sections of the surface fibrations in cubic fourfolds, and study the birational geometry related to the existence of such cycles.

First we consider $\mathcal{C}_8$. Using our preceding intersection theoretical results, we give new examples of rational cubic fourfolds in $\mathcal{C}_8$ fibered in quadric surfaces with no rational section, hence not contained in the divisors of $\mathcal{C}_8$ described by Hassett. We provide conditions on the intersection pairing of generic elements for the (non-)existence of the rational section (see Sect. 3). The first example of such a cubic had been given in [5].

Theorem 1.3. (1) *Four of the components in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{26}$ contain rational cubic fourfolds whose quadric fibration has no rational section.*
(2) *Five of the components in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$ in the moduli space of cubic fourfolds contain cubic fourfolds whose quadric fibration has no rational section.*

In order to translate this results in term of Brauer classes, we need the discriminant of the quadric fibration to be smooth. This is true for the generic cubic in $\mathcal{C}_8$ but since our examples lie in higher codimension, we need to construct at least a cubic with this property for each component.

Notably, with the help of Macaulay2, we produce an explicit rational example, with smooth discriminant divisor, with no rational section inside the intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$. Hence the Brauer class α on the degree two K3 surface is well defined and non-trivial.

For the case of $\mathcal{C}_{18}$, the scenario is far more complicated. Similarly to $\mathcal{C}_8$, we would like to consider examples of rational fibered cubic fourfolds such that the fibers are not rational over $\mathbb{P}^2$. Our starting idea was to produce an example of rational cubic fourfold fibrated in del Pezzo sextics, with no rational section over $\mathbb{P}^2$, using Macaulay2. But instead, we came across interesting examples of cubics fibered in quartic and quintic del Pezzo fibrations over $\mathbb{P}^2$. As far as we know, these are the first examples of this kind.

Theorem 1.4. (1) *The generic cubic fourfold in the locus of codimension 3 in*
  $\mathcal{C}$, *where the intersection lattice of cubics contains the following primitively*
  embedded lattice is birational to a quintic del Pezzo fibration over $\mathbb{P}^2$:

$$\begin{pmatrix} 3 & 5 & 1 & 6 \\ 5 & 13 & 0 & \tau \\ 1 & 0 & 3 & 0 \\ 6 & \tau & 0 & 18 \end{pmatrix},$$

  where $\tau \in \{9, 10, 11, 12, 13, 14\}$.
  (2) *The generic cubic fourfold in a locus of codimension 4, where the intersec-*
  tion lattices of cubics contain the following primitively embedded lattice, is
  birational to a del Pezzo quartic fibration over $\mathbb{P}^2$.

$$\begin{pmatrix} 3 & 5 & 1 & 1 & 6 \\ 5 & 13 & 0 & 0 & \tau \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 6 & \tau & 0 & 0 & 18 \end{pmatrix},$$

  with $\tau \in \{12, 13, 14\}$.

The existence of rational sections for these fibrations, and the related rationality
and unirationality conditions over the function field of $\mathbb{P}^2$ are discussed in Sect. 4.2.

Unfortunately we do not manage to find explicit examples of cubic fourfolds
fibrated in del Pezzo sextics, contained in the intersection of $\mathcal{C}_{18}$ with other divisors.
Nevertheless, if we assume that these do exist in each component, our results easily
allow to find the components where rational cubics with no rational sections should
live.

Theorem 1.5. *Let X be a cubic fourfold, and let* $\mathcal{D}_r$ *denote the codimension 2 locus*
in $\mathcal{C}_{18} \cap \mathcal{C}_m$ (m = 14, 26, 38), *where generically cubics have rank 3 intersection*
matrix of discriminant r. Suppose X lies in one of the following codimension 2 loci
of $\mathcal{C}$:

- *inside* $\mathcal{C}_{18} \cap \mathcal{C}_{14}$: $\mathcal{D}_{36}$ *and* $\mathcal{D}_{81}$;
- *inside* $\mathcal{C}_{18} \cap \mathcal{C}_{26}$: $\mathcal{D}_{108}$ *and* $\mathcal{D}_{153}$;
- *inside* $\mathcal{C}_{18} \cap \mathcal{C}_{38}$: $\mathcal{D}_{45}$, $\mathcal{D}_{81}$, $\mathcal{D}_{180}$, *and* $\mathcal{D}_{225}$.

Assume that the linear system $|2h - T|$ *induces a sextic del Pezzo fibration over*
$\mathbb{P}^2$. *Then the fibration has no rational section, and X is rational.*

If furthermore the fibration is *good*, then the Brauer class β_2 associated to the
cubics of Theorem 1.5 is non-trivial.

The structure of this paper is as follows. In Sect. 2, we give some notions and
introduce some Hodge Theory and lattices associated to cubic fourfolds. In Sect. 3,
we determine irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{26}$, $\mathcal{C}_8 \cap \mathcal{C}_{38}$ and study cubic fourfolds
in these intersections giving new examples of rational cubic fourfolds fibered in
quadric surfaces with a nontrivial Brauer class. In Sect. 4, we consider intersections

of $\mathcal{C}_{18}$ with $\mathcal{C}_{14}, \mathcal{C}_{26}$ and $\mathcal{C}_{38}$ determining their irreducible components. We use these results to produce new examples of rational cubic fourfolds that are birational to fibrations in del Pezzo surfaces, and discuss their birational geometry. Section 5 is the computational part of the paper, where the explicit examples that are necessary in the rest of the paper are constructed with Macaulay2. A code explaining the method to find other examples can be found attached.

2. Hodge theory for cubic fourfolds

Let X be a cubic fourfold over $\mathbb{C}$. Let us denote by $\mathcal{C}$ the moduli space of smooth cubic fourfolds in $\mathbb{P}^5$. It is a quasi-projective twenty-dimensional variety. The Hodge diamond of X is as follows

$$
\begin{array}{ccccccc}
& & & 1 & & & \\
& & 0 & & 0 & & \\
& 0 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 0 \\
0 & 1 & & 21 & & 1 & 0
\end{array}
$$

We focus on the middle cohomology of X, containing some nontrivial Hodge theoretic information.

Let L be the cohomology group $H^4(X, \mathbb{Z})$, known as the *cohomology lattice*, and $L_{prim} = H^4_{prim}(X, \mathbb{Z}) := \langle h^2 \rangle^\perp$ the *primitive cohomology lattice*, where $h \in H^2(X, \mathbb{Z})$ is the *hyperplane class* defined by the embedding $X \subset \mathbb{P}^5$. Note that L_{prim} is an even lattice (see [12, Sect. 2]).

We consider more precisely the lattice of integral middle Hodge classes of X:

$$
M(X) = H^{2,2}(X) \cap H^4(X, \mathbb{Z}) = H^2(X, \Omega_X^2) \cap H^4(X, \mathbb{Z})
$$

equipped with the intersection form $(-, -)$. $M(X)$ is a positive definite lattice by the Hodge-Riemann bilinear relations. For cubic fourfolds, the (integral) Hodge conjecture holds (see [22, Theorem 18] or [15, Corollary 0.3] for a recent proof), and rational, algebraic and homological equivalences coincide for cycles of codimension 2. This means that the cycle map $CH^2(X) \to H^4(X, \mathbb{Z})$ is injective, where $CH^2(X)$ is the Chow group of codimension 2 cycles on X up to rational equivalence (see [9, Sect. 5] or [7]). In particular by the Hodge conjecture the algebraic cycles are the (2, 2)-part of $H^4(X, \mathbb{Z})$. Denote by $d(M(X)) \in \mathbb{Z}$ the discriminant of the lattice $M(X)$, that is the determinant of the Gram matrix. The definition of *special* cubic fourfold introduced in Sect. 1 can be interpreted using $M(X)$: a smooth cubic fourfold X is *special* if and only if the rank of $M(X)$ is at least 2.

Definition 2.1. A *labelling* of a special cubic fourfold consists of a positive definite rank two saturated sublattice K_d, with $h^2 \in K_d \subseteq M(X)$. The discriminant d is the determinant of the intersection form on K_d.

Special cubic fourfolds form a countably infinite union of irreducible divisors $\mathcal{C}_d$ in $\mathcal{C}$ with a labelling of discriminant d, where d takes some integer values.

Only few $\mathcal{C}_d$'s can be defined explicitly in terms of particular surfaces contained in X. For example, $\mathcal{C}_8$ is defined as the locus of cubic fourfolds containing a plane, $\mathcal{C}_{14}$ is the closure of the one containing a quartic scroll or equivalently a quintic del del Pezzo surface, $\mathcal{C}_{18}$ is the closure of the locus of cubic fourfolds containing an elliptic ruled surface, and a few others.

We conclude this section with an important definition for the rest of the paper.

Definition 2.2. By a fibration in quadric (resp. sextic, quintic, quartic del Pezzo) surfaces over $\mathbb{P}^2$, we mean a morphism $\sigma : Z \to \mathbb{P}^2$ from a fourfold Z, whose generic fiber is a smooth quadric (resp. sextic, quintic, quartic del Pezzo) surface.

3. $\mathcal{C}_8$ and rational cubic fourfolds

3.1. Cubic fourfolds containing a plane

In this section, by lattice-theoretic calculations, we describe classes of rational cubic fourfolds containing a plane whose fibration in quadric surfaces has no rational section.

Let Y be a cubic fourfold in $\mathcal{C}_8$ containing a plane P. For h the hyperplane class of Y, the associated labelling K_8 has the following Gram matrix:

$$\begin{array}{cc} & \begin{array}{cc} h^2 & P \end{array} \\ \begin{array}{c} h^2 \\ P \end{array} & \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}. \end{array}$$

Let Q denote the class of the quadric surface residual to P so that $h^2 = P + Q$ and $\tilde{Y}$ the blow-up of Y along P. The projection from P resolves into a morphism

$$q \colon \tilde{Y} \to \mathbb{P}^2.$$

The fibers of this morphism are quadric surfaces in the class Q and cubic fourfolds containing a plane are birational to quadric surface bundles over $\mathbb{P}^2$ (see [11]). Hassett [11] identifies countably many divisors in $\mathcal{C}_8$ parametrizing rational cubic fourfolds. Each of these loci is a codimension two subvariety in the moduli space of cubic fourfolds $\mathcal{C}$.

Recall that the *discriminant divisor* E is defined as the locus over which q fails to be smooth. We say that the plane P is *good* if the fibers of q have at most isolated singularities. This is equivalent to having $E \subset \mathbb{P}^2$ smooth or also to having X not containing another plane intersecting P (see [21, Sect. 1 Lemma 2], [3, Sect. 1.5]). In this case, the double cover $S \to \mathbb{P}^2$ branched over E, a sextic curve, is a smooth K3 surface S of degree 2. If E is smooth, the relative Hilbert scheme of lines $\mathcal{H}(0, 1)$ of the morphism q is an étale $\mathbb{P}^1$-bundle over S (see [7, §5]). To such

an object we can associate a Brauer class $\alpha \in Br(S)$ which is trivial if and only if q has a rational section.

$$\begin{array}{ccc}
\mathcal{H}(0,1) & \xrightarrow{\;\mathbb{P}^1\;} & S \\
& \searrow & \Big\downarrow \scriptstyle{2:1} \\
\tilde{Y} & \xrightarrow{\;q\;} & \mathbb{P}^2 \supset E
\end{array}$$

Theorem 3.1. ([11]) *If there is a class $C \in M(Y)$ such that (C, Q) is odd then Y is rational over $\mathbb{C}$.*

In particular, q has a rational section if and only if there exists an algebraic cycle $C \in M(Y)$ such that $(C, Q) = 1$. In other words, the associated Brauer class α is trivial if and only if there exists an algebraic cycle $C \in M(Y)$ such that $(C, Q) = 1$. According to Hassett, the triviality of the Brauer class implies the rationality of Y over $\mathbb{C}$.

In the following, by lattice-theoretic calculations, we describe classes of rational cubic fourfolds containing a plane whose fibration in quadric surfaces has no rational section, respectively inside $\mathcal{C}_8 \cap \mathcal{C}_{26}$ and inside $\mathcal{C}_8 \cap \mathcal{C}_{38}$, hence not contained in the class of cubics described by Hassett.

3.2. $\mathcal{C}_8 \cap \mathcal{C}_{26}$:

A smooth cubic fourfold Y is in $\mathcal{C}_8$ or $\mathcal{C}_{26}$ if and only if $M(Y)$ has primitive sublattice $K_8 := \langle h^2, P \rangle$ or $K_{26} := \langle h^2, S_{26} \rangle$ with the following Gram matrix

$$\begin{array}{c@{}c}
 & \begin{array}{cc} h^2 & S_{26} \end{array} \\
\begin{array}{c} h^2 \\ S_{26} \end{array} & \begin{pmatrix} 3 & 7 \\ 7 & 25 \end{pmatrix},
\end{array}$$

such that P is a plane contained in a general element of $\mathcal{C}_8$ and S_{26} is a surface with one node obtained as the projection of a smooth del Pezzo surface $S \subset \mathbb{P}^7$ of degree seven from a line intersecting the secant variety of S transversally (see [18, Sect. 3]), contained in a general element of $\mathcal{C}_{26}$.

Thus a cubic fourfold $Y \in \mathcal{C}_8 \cap \mathcal{C}_{26}$ has a sublattice $\langle h^2, P, S_{26} \rangle \subset M(Y)$ with the following Gram matrix

$$\begin{array}{c@{}c}
 & \begin{array}{ccc} h^2 & P & S_{26} \end{array} \\
\begin{array}{c} h^2 \\ P \\ S_{26} \end{array} & \begin{pmatrix} 3 & 1 & 7 \\ 1 & 3 & \tau \\ 7 & \tau & 25 \end{pmatrix},
\end{array}$$

for some $\tau = (P, S_{26}) \in \mathbb{Z}$ depending on Y. The values of τ may be restricted following some properties and works of Voisin [21] or Yang and Yu [23].

τ	-1	0	1	2	3	4	5	6
$d(M_\tau)$	36	53	64	69	68	61	48	29

Fig. 1. Irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{26}$

Denote by M_τ the lattice of rank 3 whose bilinear form has the previous Gram matrix. Let $\mathcal{C}_{M_\tau} \subset \mathcal{C}$ be the locus of smooth cubic fourfolds such that there is a primitive embedding $M_\tau \subset M(Y)$ of lattices preserving h^2.

Proposition 3.2. *The irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{26}$ are the subvarieties of codimension two $\mathcal{C}_{M_\tau}$ for $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6\}$.*

Proof. We proceed as follows: first we find the set of possible values of τ for which $d(M_\tau) > 0$ and possibly nonempty. Then, for these values of τ, we prove that M_τ is saturated. Finally, we find the associated irreducible components $\mathcal{C}_{M_\tau}$ which are nonempty i.e. don't have roots, that is vectors of norm 2.

By construction, $\mathcal{C}_8 \cap \mathcal{C}_{26} = \cup_{\tau \in \mathbb{Z}} \mathcal{C}_{M_\tau}$. We determine which components of $\mathcal{C}_{M_\tau}$ are possibly nonempty. Since for $Y \in \mathcal{C}_8 \cap \mathcal{C}_{26}$, $M(Y)$ is a positive definite lattice, by Sylvester's criterion, the sublattice M_τ must have positive discriminant. As $d(M_\tau) = -3\tau^2 + 14\tau + 53$, the only values of τ making a positive discriminant are $\tau \in I = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$. Hence, $\mathcal{C}_8 \cap \mathcal{C}_{26} = \cup_{\tau \in I} \mathcal{C}_{M_\tau}$.

Then, we prove that $\mathcal{C}_{M_\tau}$ is empty for $\tau = -2, 7$ by finding primitive roots (that is, primitive vectors of norm 2) in $M_{\tau, prim} = \langle h^2 \rangle^{\perp}$. Indeed, the vectors $(1, -3, 0)$ and $(-3, 2, 1)$ form a basis for $M_{\tau, prim}$. For all $R_{a,b} \in M_{\tau, prim}$, $R_{a,b} = (a - 3b, -3a + 2b, b)$ for some $a, b \in \mathbb{Z}, \tau \in I$; for $\tau = -2$, we find primitive roots $\pm R_{0,1}$; for $\tau = 7$, we find primitive roots $\pm R_{1,1} = \pm(-2, -1, 1)$. Hence, by [21, Sect. 4 Proposition 1], $\mathcal{C}_{M_\tau}$ is empty for $\tau = -2, 7$. We are left with $\mathcal{C}_{M_\tau}$ possibly nonempty only for $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6\}$. The corresponding discriminants $d(M_\tau)$ are 36, 53, 64, 69, 68, 61, 48 and 29.

We prove now that the components $\mathcal{C}_{M_\tau}$ are irreducible. We first note that the rank of the lattice $M(X)$ is an upper-semicontinuous function on $\mathcal{C}$; the irreducible components of $\mathcal{C}_{M_\tau}$ correspond then to rank 3 overlattices B of M_τ (a finite index sublattice for some τ i.e. B/M_τ is a finite abelian group) which is primitively embedded into L. By standard lattice theory, we have that for an embedding $M_\tau \hookrightarrow B$ with finite index $[B : M_\tau] = |B/M_\tau|, |d(B)|.[B : M_\tau]^2 = |d(M_\tau)|$ (see [16] or [20]);

We will prove that no proper finite overlattices exist; for $\tau = 0, 2, 4, 6$ the discriminants of M_τ are squarefree, so there are no proper finite overlattices.

For the remaining cases, we can take h^2 and P as a part of a basis of the overlattice B. Let U be a vector that completes this to a basis such that $U = xh^2 + yP + zS_{26}$, with $x, y, z \in \mathbb{Q}$. Consider the natural embedding of M_τ in B that can be written as follows:

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -x/z \\ 0 & 1 & -y/z \\ 0 & 0 & 1/z \end{pmatrix} \in \mathcal{M}_{3,3}(\mathbb{Z}).$$

We can take $z = \frac{1}{n}$, for some $n \in \mathbb{Z}$ and $x' = nx$, $y' = yn \in \mathbb{Z}$; then $U = \frac{1}{n}(x'h^2 + y'P + S_{26})$. By adding multiples of h^2 and P, we may ensure that $0 \leq x', y' < n$.

Computing intersections, we have the following:

$$(U, h^2) = \frac{1}{n}(3x' + y' + 7) = a_0,$$

$$(U, P) = \frac{1}{n}(x' + 3y' + \tau) = b_0,$$

$$(U, U) = \frac{1}{n^2}(3x'^2 + 3y'^2 + 14x' + 2\tau y' + 2x'y' + 25) = c_0.$$

Hence, the Gram matrix of B is:

$$\begin{array}{c} \quad h^2 \quad P \quad U \\ \begin{array}{c} h^2 \\ P \\ U \end{array} \begin{pmatrix} 3 & 1 & a_0 \\ 1 & 3 & b_0 \\ a_0 & b_0 & c_0 \end{pmatrix}. \end{array}$$

Now we check each case separately for possible values of τ, n, x' and y':

(1) $\tau = -1$: We see that n can be 2, 3 or 6.
Remark that for all possible values of n, x' and y' other than ($n = 3$, $x' = 1$ and $y' = 2$), the Gram matrix of $B \notin \mathcal{M}_{3,3}(\mathbb{Z})$. For ($n = 3$, $x' = 1$ and $y' = 2$), B has the following Gram matrix:

$$\begin{pmatrix} 3 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 2 & 6 \end{pmatrix},$$

which has the root $(-2, 0, 1)$. Then no such overlattices exist. Thus $\mathcal{C}_{M_{-1}}$ is irreducible.

(2) $\tau = 1$: We see that n can be 2, 4 or 8.
Observe that for all possible values of n, x' and y', the Gram matrix of B is not in $\mathcal{M}_{3,3}(\mathbb{Z})$. Thus $\mathcal{C}_{M_1}$ is irreducible.

(3) $\tau = 3$: n can be 2.
We notice that for all possible values of x' and y', the Gram matrix of B is non-integral. Then no such overlattices exist. Thus $\mathcal{C}_{M_3}$ is irreducible.

(4) $\tau = 5$: We observe that n can be 2 or 4.
Remark that for $n = 4$, $|d(B)| = 3$ which is impossible by [24, Lemma 7.8]. Otherwise, for all possible values of x' and y', the Gram matrix of B is non-integral. Then no such overlattices exist. Thus $\mathcal{C}_{M_5}$ is irreducible.

To check the (non)emptiness of these $\mathcal{C}_{M_\tau}$, we proceed as follows; for all $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6\}$, M_τ is a positive definite saturated sublattice of rank 3:

$$h^2 \in M_\tau \subset M(Y) \subset L.$$

Furthermore, let $v = xh^2 + yP + zS_{26} \in M_\tau$ for $x, y, z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 3y^2 + 25z^2 + 2xy + 14xz + 2\tau yz, \tau \in \{-1, 0, 1, 2, 3, 4, 5, 6\}.$$

For these values of τ, there exists no $v \in M_\tau$ such that $(v, v) = 2$. Thus, by [23, Lemma 2.4], $\mathcal{C}_{M_\tau}$ is nonempty and has codimension two.

Note that the rank of the lattices is small compared to 21 so there is no issues with finding a primitive embedding in L.

$\square$

In the following, we address the question of the (non)triviality of the Brauer class.

Theorem 3.3. *Let Y be a general cubic fourfold in $\mathcal{C}_8 \cap \mathcal{C}_{26}$ (so that $M(Y)$ has rank 3) containing a good plane P. The Brauer class $\alpha \in Br(S)$ of Y is trivial if and only if τ is even.*

Proof. If τ is even, then a cycle $S_{26} + 3P \in M(Y)$ exists such that $(S_{26} + 3P, Q) = 1 - \tau \equiv 1 \pmod{2}$ (odd). Hence α is trivial by the application of the criterion of Hassett (see [11]).

If τ is odd, then M_τ has rank 3 and even discriminant, hence α is nontrivial (see [5, Proposition 2]). $\square$

Corollary 3.4. *The four irreducible components $\mathcal{C}_{M_\tau}$ of $\mathcal{C}_8 \cap \mathcal{C}_{26}$ corresponding to $\tau = -1$, 1, 3, 5 contain examples of rational cubic fourfolds whose associated quadric surface bundles do not have a rational section.*

3.3. $\mathcal{C}_8 \cap \mathcal{C}_{38}$

Using same methods as before, we compute the intersection between $\mathcal{C}_8$ and $\mathcal{C}_{38}$.

A cubic fourfold Y is in $\mathcal{C}_8$ or $\mathcal{C}_{38}$ if and only if $M(Y)$ has primitive sublattice $K_8 := \langle h^2, P \rangle$ or $K_{38} := \langle h^2, S_{38} \rangle$ with the following Gram matrix

$$\begin{array}{c} \\ h^2 \\ S_{38} \end{array} \begin{array}{c} h^2 \quad S_{38} \\ \begin{pmatrix} 3 & 10 \\ 10 & 46 \end{pmatrix}, \end{array}$$

such that P is a plane and S_{38} is the general degree 10 smooth surface of sectional genus 6 obtained as the image of $\mathbb{P}^2$ by the linear system of plane curves of degree 10 having 10 fixed triple points (see [18, Sect. 4]), contained in a general element of $\mathcal{C}_{38}$.

Hence, $Y \in \mathcal{C}_8 \cap \mathcal{C}_{38}$ has a sublattice $\langle h^2, P, S_{38} \rangle \subset M(Y)$ with Gram matrix:

$$\begin{array}{c} \\ h^2 \\ P \\ S_{38} \end{array} \begin{array}{c} h^2 \quad P \quad S_{38} \\ \begin{pmatrix} 3 & 1 & 10 \\ 1 & 3 & \tau \\ 10 & \tau & 46 \end{pmatrix}, \end{array}$$

for some $\tau \in \mathbb{Z}$ depending on Y.

Proposition 3.5. *The irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{38}$ are the subvarieties of codimension two $\mathcal{C}_{M_\tau}$ for $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$.*

Proof. As $d(M_\tau) = -3\tau^2 + 20\tau + 68$, the only values of τ inducing a positive discriminant are $\tau \in J = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Let $M_{\tau, prim} = \{(x, y, z) \in \mathbb{Z}^3 \mid 3x + y + 10z = 0\}$; indeed, the vectors $(1, -3, 0)$ and $(-3, -1, 1)$ form a basis for $M_{\tau, prim}$; for all $R_{a,b} \in M_{\tau, prim}$, $R_{a,b} = (a - 3b, -3a - b, b)$ for some $a, b \in \mathbb{Z}$, $\tau \in J$.

For $\tau = -2$, we find primitive roots $\pm R_{-1,1} = \pm(-4, 2, 1)$. Hence, by [21, Sect. 4 Proposition 1], $\mathcal{C}_{M_{-2}}$ is empty ; We are left with $\mathcal{C}_{M_\tau}$ possibly nonempty only for $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The corresponding discriminants $d(M_\tau)$ are 45, 68, 85, 96, 101, 100, 93, 80, 61, 36 and 5.

Similarly to the previous section, we prove that the components $\mathcal{C}_{M_\tau}$ are irreducible. For $\tau = 1, 3, 5, 7$ the discriminants are squarefree, so there are no proper finite overlattices. For the remaining cases, we can prove the any overlattice in $\mathcal{M}_{3,3}(\mathbb{Z})$ has a primitive root. Therefore no such proper overlattices exist.

Hence, for all $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, M_τ is a positive definite saturated sublattice of rank 3:

$$h^2 \in M_\tau \subset M(Y) \subset L.$$

Furthermore, let $v = xh^2 + yP + zS_{38} \in M_\tau$, for $x, y, z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 3y^2 + 46z^2 + 2xy + 20xz + 2\tau yz;$$

for $\tau = 9$, we have that $(-2h^2 - 2P + S_{38}, -2h^2 - 2P + S_{38}) = 2$. Then $\mathcal{C}_{M_9} \subset \mathcal{C}$ is empty (see [23, Lemma 2.4]). For the rest of the values of τ, there exists no $v \in M_\tau$ such that $(v, v) = 2$, $\mathcal{C}_{M_\tau} \subset \mathcal{C}$ are nonempty irreducible and have codimension 2 (see [23, Lemma 2.4]). $\qquad\square$

τ	-1	0	1	2	3	4	5	6	7	8
$d(M_\tau)$	45	68	85	96	101	100	93	80	61	36

Theorem 3.6. *Let Y be a general cubic fourfold in $\mathcal{C}_8 \cap \mathcal{C}_{38}$ (so that $M(Y)$ has rank 3) containing a good plane P. The Brauer class $\alpha \in Br(S)$ of Y is trivial if and only if τ is odd.*

Proof. If τ is odd, then a cycle $S_{38} + 5P \in M(Y)$ exists such that $(S_{38} + 5P, Q) = -\tau \equiv 1 \ (mod\ 2)$ (odd). Hence α is trivial by the application of the criterion (see [11]).

If τ is even, then M_τ has rank 3 and even discriminant, hence α is nontrivial (see [5, Proposition 2]). $\qquad\square$

Corollary 3.7. *$\mathcal{C}_8 \cap \mathcal{C}_{38}$ has five smooth irreducible components, corresponding to $\tau = 0, 2, 4, 6, 8$, containing examples of rational cubic fourfolds whose associated quadric surface bundles do not have a rational section.*

In Sect. 5.1, we construct an explicit example of rational smooth cubic fourfold containing a *good* plane in each irreducible component of the intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$. This means that it makes sense to consider the associated Brauer class $\alpha \in Br(S)$ on the degree 2 K3 surface. In particular, in our explicit example the associated Brauer class is nontrivial since the quadric bundle has no rational section.

4. $\mathcal{C}_{18}$ and rational cubic fourfolds

The goal of this section is twofold. First we describe the irreducible components of the intersection of $\mathcal{C}_{18}$ with the divisors $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, and $\mathcal{C}_{38}$. Then we use these intersection theoretical results to showcase loci of *rational* cubic fourfolds, which are fibered in del Pezzo surfaces. In each case, we consider gemetric conditions that imply the rationality of the fibers over the function field of $\mathbb{P}^2$ and hence of the associated cubic fourfolds.

4.1. Cubic fourfolds containing an elliptic ruled surface

A generic cubic fourfold in $\mathcal{C}_{18}$ contains an elliptic ruled surface of degree 6 (see [1]).

Let $X \in \mathcal{C}_{18}$ be a generic cubic fourfold containing an elliptic ruled surface T. The Gram matrix of the associated labelling K_{18} is as follows:

$$
\begin{array}{cc}
 & \begin{array}{cc} h^2 & T \end{array} \\
\begin{array}{c} h^2 \\ T \end{array} & \begin{pmatrix} 3 & 6 \\ 6 & 18 \end{pmatrix}.
\end{array}
$$

To construct T, we start by fixing two disjoint planes P' and P. Then, we choose 3 quadrics $\{Q_1, Q_2, Q_3\}$ containing the 2 disjoint planes. The intersection of the quadrics $Q_1 \cap Q_2 \cap Q_3$ has degree 8, and it is the union of T and of the two planes P and P'.

Let $\tilde{X}$ denote the blow-up of X along T and let

$$\pi : \tilde{X} \to \mathbb{P}^2$$

be the morphism induced by the linear system of quadrics containing T. For generic X, the generic fiber of π is a del Pezzo surface of degree 6.

Note that a singular del Pezzo surface is a surface **P** with *ADE* singularities and ample anticanonical class such that $K_S^2 = 6$. They are classified as follows (see [14, Sect. 2] or [1, Sect. 5] *for a complete description*):

- Type I: **P** has one A_1 singularity.
- Type II: **P** has one A_1 singularity obtained in a different way from Type I.
- Type III: **P** has two A_1 singularities.
- Type IV: **P** has one A_2 singularity.
- Type V: **P** has a A_1 and a A_2 singularity.

Type I and II occur in codimension one in the moduli stack of sextic del Pezzo surfaces. Type III and IV occur in codimension two; type V occurs in codimension three.

Definition 4.1. [1, Definition 11] Let $\mathbb{P}$ be a smooth complex projective surface. A good sextic del Pezzo fibration consists of a smooth fourfold Z and a flat projective morphism $\pi \colon Z \to \mathbb{P}^2$ with connected fibers such that the fibers are either smooth or singular sextic del Pezzo surfaces of Type I, II, III, IV or V. Let B_i denote the closure of the locus of Type i fibers in $\mathbb{P}^2$. B_i has the following properties:

- B_I is a non-singular curve;
- B_{II} is a curve, non-singular away from B_{IV};
- B_{III} is finite and coincides with the intersection of B_I and B_{II}, which is tranverse;
- B_{IV} is finite and B_{II} has cusps at B_{IV};
- B_V is empty.

The discriminant curve of a good sextic del Pezzo fibration π has two irreducible components, a smooth sextic curve B_I and a sextic curve B_{II} with 9 cusps (see [1,14] or [7] *for more details*). In this section, if a cubic $X \in \mathcal{C}_{18}$ has an associated good del Pezzo fibration, we will call it a *good* cubic. More generally, for a *good* del Pezzo fibration π we consider the following construction.

Recall that a smooth sextic del Pezzo surface can be described as the blow up of $\mathbb{P}^2$ in three general points. Let us now consider two different Hilbert schemes of curves in the fibers of π. Let us denote by $\mathcal{H}(0, 3)$ the relative Hilbert scheme of connected genus zero curves with anticanonical degree 3 contained in the fibers. There are two 2-dimensional families of such curves on a smooth del Pezzo sextic. One is given by the strict transforms of the lines in $\mathbb{P}^2$ that do not pass through any of the three blown up points p_1, p_2 and $p_3 \in \mathbb{P}^2$ of the corresponding del Pezzo surface. The second one is given by conics passing through the three base points. Hence the Stein factorization of $\mathcal{H}(0, 3) \to \mathbb{P}^2$ yields an étale $\mathbb{P}^2$-bundle $\mathcal{H}(0, 3)$ over a smooth degree two K3 surface S branched on B_I.

$$\begin{array}{ccc} \mathcal{H}(0, 3) & \xrightarrow{\ \mathbb{P}^2\ } & S \\[2mm] & \searrow & \downarrow {\scriptstyle 2:1} \\[2mm] \tilde{X} & \xrightarrow{\ \pi\ } & \mathbb{P}^2 \supset B_I \end{array}$$

We will denote by β_2 the Brauer class of this $\mathbb{P}^2$-bundle in $Br(S)$.

Let us consider now $\mathcal{H}(0, 2) \to \mathbb{P}^2$ the relative Hilbert scheme of connected genus zero curves with anticanonical degree 2 contained in the fibers. The Stein factorization yields an étale $\mathbb{P}^1$-bundle $\mathcal{H}(0, 2)$ over a non singular surface Z. In fact, associated to a *good* del Pezzo fibration π, there is a non-singular triple cover

$Z \to \mathbb{P}^2$ branched along a cuspidal sextic B_{II}. On every geometric fiber, the $\mathbb{P}^1$-bundle is given by the strict transform of the lines through each of the 3 base points of the corresponding del Pezzo sextic.

$$
\begin{array}{ccc}
\mathcal{H}(0,2) & \xrightarrow{\ \mathbb{P}^1\ } & Z \\
 & \searrow & \downarrow {\scriptstyle 3:1} \\
\tilde{X} & \xrightarrow{\ \pi\ } & \mathbb{P}^2 \supset B_{II}
\end{array}
$$

Similarly to the preceding case, we will denote by $\beta_3 \in Br(Z)$ the Brauer class of this $\mathbb{P}^1$-bundle.

For more details on the Brauer classes of a sextic del Pezzo surface see [2] or [13].

Furthermore, for a *good* sextic del Pezzo fibration $\pi : \tilde{X} \to \mathbb{P}^2$, we have the following (see [14, Proposition 5.20]).

Proposition 4.2. (1) *The Brauer class $\beta_2 \in Br(S)$ is trivial if and only if π has a rational 2-multisection.*

(2) *The Brauer class $\beta_3 \in Br(Z)$ is trivial if and only if π has a rational 3-multisection.*

It is known [1, Theorem 6] that the del Pezzo fibration associated to a good cubic fourfold $X \in \mathcal{C}_{18}$ always has a rational 3-section. It is worth to note that this rational 3-section does not come as the strict transform of a 2-dimensional algebraic cycle in X but rather as a surface obtained as the inverse image of a curve inside the elliptic ruled surface. This is surely of interest, but our analysis concentrates on 2-cycles on cubic fourfolds, since are the ones we can control better on special loci of the moduli space. This is why in the following we will concentrate on rational (multi)sections that are obtained from 2-cycles on the cubic fourfold. In this sense, it is worth recalling the following results ([17, Proposition 2.3] and [1, Proposition 8]).

Let $F \in M(X)$ denote the class of the fiber of the rational map given by quadrics through T, for a cubic fourfold $X \in \mathcal{C}_{18}$. That is,

$$F = 4h^2 - T.$$

Proposition 4.3. *The del Pezzo fibration $\pi : \tilde{X} \to \mathbb{P}^2$ has a rational section if and only if there is an algebraic cycle class $R \in H^{2,2}(X, \mathbb{Z})$ such that the intersection (R, F) is 1 or 2.*

Proposition 4.4. *Let $\mathcal{G} \to \mathbb{P}^2$ be a del Pezzo sextic fibration. The following are equivalent:*

(1) *$\mathcal{G}$ is rational over the function field $\mathbb{C}(\mathbb{P}^2)$;*

(2) *$\mathcal{G}$ admits an rational section $\mathbb{P}^2 \dashrightarrow \mathcal{G}$;*

(3) $\mathcal{G}$ admits a rational multi-section of degree prime to six;
(4) if $\mathcal{G}$ is good, the Brauer classes β_2 and β_3 are trivial.

In the next section we will describe the interplay between special surfaces contained in cubic fourfolds and irreducible components of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$.

4.2. $\mathcal{C}_{18} \cap \mathcal{C}_{14}$

Let X be a cubic fourfold in $\mathcal{C}_{18} \cap \mathcal{C}_{14}$; $M(X)$ has primitive sublattices $K_{18} := \langle h^2, T \rangle$ and $K_{14} := \langle h^2, D \rangle$ with its Gram matrix

$$\begin{array}{c} \begin{array}{cc} h^2 & D \end{array} \\ \begin{array}{c} h^2 \\ D \end{array} \begin{pmatrix} 3 & 5 \\ 5 & 13 \end{pmatrix}, \end{array}$$

such that D is (possibly a degeneration of) a quintic del Pezzo surface (see [8]), T is (possibly a degeneration of) an elliptic ruled surface.

Hence, $X \in \mathcal{C}_{18} \cap \mathcal{C}_{14}$ has a sublattice $\langle h^2, D, T \rangle \subset M(X)$ with the following Gram matrix

$$\begin{array}{c} \begin{array}{ccc} h^2 & D & T \end{array} \\ \begin{array}{c} h^2 \\ D \\ T \end{array} \begin{pmatrix} 3 & 5 & 6 \\ 5 & 13 & \tau \\ 6 & \tau & 18 \end{pmatrix}, \end{array}$$

for some $\tau \in \mathbb{Z}$ depending on X.

Denote by M_τ the lattice of rank 3 whose bilinear form has the previous Gram matrix (by abuse of notation, we use M_τ to denote the Gram matrix too) We will denote by $\mathcal{C}_{M_\tau} \subset \mathcal{C}$ the locus of cubic fourfolds such that there is a primitive embedding $M_\tau \subset M(X)$ of lattices preserving h^2.

Theorem 4.5. *The irreducible components of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$ are the subvarieties of codimension two $\mathcal{C}_{M_\tau}$ given by the following rank 3 g matrix*

$$M_\tau := \begin{array}{c} \begin{array}{ccc} h^2 & D & T \end{array} \\ \begin{array}{c} h^2 \\ D \\ T \end{array} \begin{pmatrix} 3 & 5 & 6 \\ 5 & 13 & \tau \\ 6 & \tau & 18 \end{pmatrix}, \end{array}$$

where $10 \le \tau \le 14$. Moreover, the associated discriminants $d(M_{10}), \dots, d(M_{14})$ are 84, 81, 72, 57, 36 respectively.

Proof. We proceed as follows: first we find the set of possible values of τ for which $d(M_\tau) > 0$. Then, for these values of τ, we prove that M_τ is saturated. Finally, we find the associated irreducible components $\mathcal{C}_{M_\tau}$ which are nonempty i.e. don't have roots, that is vectors of norm 2.

Note that M_τ, defined as a sublattice of a positive definite lattice, must have positive discriminant by Sylvester's criterion. As $d(M_\tau) = -3\tau^2 +$

$60\tau - 216$, the only values of τ making a positive discriminant are $\tau \in \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

For these values of τ, M_τ is a positive definite sublattice of rank 3 containing h^2. The corresponding discriminants $d(M_\tau)$ are 9, 36, 57, 72, 81, 84, 81, 72, 57, 36, 9. Note that there are isometries between the lattices M_5, M_6, M_7, M_8, M_9, and M_{15}, M_{14}, M_{13}, M_{12}, M_{11} respectively (see [24, Remark 7.7]). Thus, next, we may only consider M_τ for $\tau = 10, 11, 12, 13, 14, 15$.

We check whether $\mathcal{C}_{M_\tau}$ is irreducible. For $\tau = 5$, the discriminant $d(M_5)$ is squarefree, so there are no proper finite overlattices of M_5. For the remaining cases, we can take h^2 and D as a part of a basis of an overlattice B of M_τ and complete it with V such that $V = xh^2 + yD + zT$ with $x, y, z \in \mathbb{Q}$.

We can take $z = \frac{1}{n}$, for some $n \in \mathbb{Z}$ and $x' = nx$, $y' = yn \in \mathbb{Z}$; then $V = \frac{1}{n}(x'h^2 + y'D + T)$. By adding multiples of h^2 and D, we may ensure that $0 \leq x', y' < n$.

Note that $n = [B : M_\tau] = [B_{prim} : M_{\tau,prim}]$, with B_{prim} the finite proper overlattice of $M_{\tau,prim}$ (this follows from standard lattice theory).

Computing intersections:

$$(V, h^2) = \frac{1}{n}(3x' + 5y' + 6) = a_1,$$

$$(V, D) = \frac{1}{n}(5x' + 13y' + \tau) = b_1,$$

$$(V, V) = \frac{1}{n^2}(3(x')^2 + 13(y')^2 + 12x' + 2\tau y' + 10x'y' + 18) = c_1.$$

Then B has the following Gram matrix:

$$\begin{array}{c} \\ h^2 \\ D \\ V \end{array} \begin{array}{c} h^2 \quad D \quad V \\ \begin{pmatrix} 3 & 5 & a_1 \\ 5 & 13 & b_1 \\ a_1 & b_1 & c_1 \end{pmatrix}. \end{array}$$

Now we check each case separately for possible values of τ, n, x' and y'. For ($\tau = 10$, $n = 2$, $x' = 1$, $y' = 1$), B has the following Gram matrix

$$\begin{pmatrix} 3 & 5 & 7 \\ 5 & 13 & 14 \\ 7 & 14 & 19 \end{pmatrix},$$

of discriminant 21. The vectors $(-5, 3, 0)$ and $(-7, 0, 3)$ form a basis for B_{prim} which has the following Gram matrix:

$$\begin{pmatrix} 42 & 21 \\ 21 & 33 \end{pmatrix}.$$

Since B_{prim} is not even, no such overlattices exist and this component is irreducible. No overlattices exist in the following cases.

For ($\tau = 11$, $n = 3$, $x' = 2$, $y' = 0$), B has the following Gram matrix

$$\begin{pmatrix} 3 & 5 & 4 \\ 5 & 13 & 7 \\ 4 & 7 & 6 \end{pmatrix},$$

which has a root $(-3, 1, 1)$.

For ($\tau = 12$, $n = 2$, $x' = 1$, $y' = 1$), B has the following Gram matrix

$$\begin{pmatrix} 3 & 5 & 7 \\ 5 & 13 & 15 \\ 7 & 15 & 20 \end{pmatrix},$$

which has roots $\pm(-1, -1, 1)$.

For ($\tau = 12$, $n = 3$, $x' = 0$, $y' = 0$), B has the following Gram matrix

$$\begin{pmatrix} 3 & 5 & 2 \\ 5 & 13 & 4 \\ 2 & 4 & 2 \end{pmatrix},$$

which has roots $\pm(0, 0, 1)$.

For ($\tau = 12$, $n = 6$, $x' = 3$, $y' = 3$), B has the following Gram matrix

$$\begin{pmatrix} 3 & 5 & 5 \\ 5 & 13 & 11 \\ 5 & 11 & 10 \end{pmatrix},$$

which has a root $(-1, -1, 2)$.

For ($\tau = 14$, $n = 2$, $x' = 1$, $y' = 1$), the Gram matrix of B is:

$$\begin{pmatrix} 3 & 5 & 7 \\ 5 & 13 & 16 \\ 7 & 16 & 21 \end{pmatrix}.$$

B has roots such as $(0, -1, 1)$.

For ($\tau = 14$, $n = 3$, $x' = 2$, $y' = 0$), the Gram matrix of B is:

$$\begin{pmatrix} 3 & 5 & 4 \\ 5 & 13 & 8 \\ 4 & 8 & 6 \end{pmatrix}.$$

B has roots such as $(1, 1, -2)$.

For ($\tau = 14$, $n = 6$, $x' = 5$, $y' = 3$), the Gram matrix of B is:

$$\begin{pmatrix} 3 & 5 & 6 \\ 5 & 13 & 13 \\ 6 & 13 & 14 \end{pmatrix}.$$

B has roots such as $(-2, 0, 1)$.

For ($\tau = 15$, $n = 3$, $x' = 0$, $y' = 0$), B has the following form

$$\begin{pmatrix} 3 & 5 & 2 \\ 5 & 13 & 5 \\ 2 & 5 & 2 \end{pmatrix},$$

which has roots $\pm(0, 0, 1)$.

For all other possible values τ, n, x' and y', the Gram matrices of B are not in $\mathcal{M}_{3,3}(\mathbb{Z})$.

The lattice M_τ is saturated definite positive of rank 3 such that:

$$h^2 \in M_\tau \subset M(X) \subset L.$$

Let $v = xh^2 + yS_{14} + zT \in M_\tau$. For $x, y, z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 10y^2 + 18z^2 + 8xy + 12xz + 2\tau yz.$$

We prove that $\mathcal{C}_{M_{15}}$ is empty by finding roots. Indeed, $(-h^2 - D + T, -h^2 - D + T) = 2$; for the remaining values of τ, M_τ has no vectors v such that $(v, v) = 2$. Hence, $\mathcal{C}_{M_\tau} \subset \mathcal{C}$ is irreducible nonempty (see [23, Lemma 2.4]). □

In the rest of this section we will consider cubics living in (at least) codimension 1 loci inside $\mathcal{C}_{18}$, and hence the existence of the sextic del Pezzo fibration is not guaranteed by [1]. For some of these irreducible loci we will find explicit examples of cubics which are birational to fibrations in (quartic, quintic) del Pezzo surfaces over $\mathbb{P}^2$. This implies that the generic cubic has the same feature. For the loci where we do not show explicit examples, we tacitly assume that the generic cubic is birational to a sextic del Pezzo fibration.

Definition 4.6. Let $F \in M(X)$ be the class $4h^2 - T$. We will call a *1-sectional (resp. 2 or 3-sectional) cycle* a dimension 2 algebraic cycle $W \in M(X)$ such that we have $(W, F) = 1$ (resp. 2 or 3).

Let us assume that the class $F = 4h^2 - T \in M(X)$ gives a two-dimensional linear system of sextic del Pezzo surfaces. Under this assumption, we now check the existence of 1,2 and 3-sectional cycles in $M(X)$ for the corresponding del Pezzo fibration on each component of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$. Let $W_{a,b,c} \in M(X)$ be a cycle such that

$$W_{a,b,c} := ah^2 + bD + cT, \ \text{for } a, b, c \in \mathbb{Z},$$

with D a (degeneration of) quintic del Pezzo surface and T an elliptic ruled surface (or a degeneration of such surface).

We have that $(W_{a,b,c}, F) = 6a + (20 - \tau)b + 6c$. We will now check, for the possible values of τ, the existence of a 1, 2 or 3-sectional cycles.

In the following table, we collect our results about the existence of sectional cycles for the different components of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$. The symbol $\emptyset$ means that there are no sectional cycles of that given type.

τ	$(W_{a,b,c}, F) = 1$	$(W_{a,b,c}, F) = 2$	$(W_{a,b,c}, F) = 3$
10	$\emptyset$	$W_{0,2,-3}$	$\emptyset$
11	$\emptyset$	$\emptyset$	$W_{0,1,-1}$
12	$\emptyset$	$W_{0,1,-1}$	$\emptyset$
13	$W_{0,1,-1}$	$W_{0,2,-2}$	$W_{0,3,-3}$
14	$\emptyset$	$\emptyset$	$\emptyset$

Remark 4.7. (1) We observe that, by Proposition 4.3, the del Pezzo fibrations for $\tau = 11, 14$ have no rational section, hence - by Proposition 4.4 - $\tilde{X}$ in these cases is not rational on $\mathbb{C}(\mathbb{P}^2)$ but nevertheless rational since the cubic is contained in $\mathcal{C}_{14}$.

(2) Again by Proposition 4.4, this means that, if one of these cubics is good, then β_2 and β_3 are not both trivial for cubics in these components. Since by [1, Theorem 6] there is always a rational 3-section of the corresponding del Pezzo fibration, then by Proposition 4.2 the class β_3 is always trivial. This means that the class β_2 is nontrivial for good cubics in this component.

(3) Of course, the existence of a 1-sectional cycle is the strongest condition, and implies the existence of 2 and 3-sectional cycles.

(4) For the components with $\tau = 10$ and 12 it is worth observing that, by Proposition 4.3, the associated del Pezzo sextic fibrations have a rational section, though this does not come from a 1-sectional cycle in the cubic. It is nevertheless not hard to construct a rational section of the del Pezzo fibration, starting from the strict transform of a 2-sectional cycle, in the spirit of the proof of [17, Proposition 2.3].

The upshot is that for cubics in the component obtained with $\tau = 11, 14$, if they are birational to a fibration in del Pezzo sextics over $\mathbb{P}^2$, the rationality of the del Pezzo surfaces over the function field of $\mathbb{P}^2$ is not necessary for rationality, and these examples are not contained in the codimension 2 loci of rational cubics described in [1].

All the explicit examples of cubics X in the intersection $\mathcal{C}_{18} \cap \mathcal{C}_{14}$ that we were able to construct with Macaulay2 contain not only T, but also one of or both the disjoint planes P and P' that are residual to T in the complete intersection of the three quadrics (see Sect. 4.1). The consequence is that they are indeed birational to a del Pezzo fibration, but of lower degree. We will now give a theoretical description of these loci, where our explicit examples live.

In particular, let us consider the sublocus of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$ parametrizing cubics containing the elliptic ruled surface T, P, P' and a del Pezzo surface that intersects both P and P' along a smooth conic. In fact, a cubic X containing T and D automatically contains P and P', since $X \cap P$ (resp. $X \cap P'$) contains a plane cubic and a conic, hence $P \subset X$ (resp. $P' \subset X$). This locus has codimension 4 in the moduli space, and its generic element has $M(X)$ of rank 5. Such cubics can be found following instructions in the attached file using Macaulay2. Moreover, an explicit example is given in Sect. 5. Recall that the two planes intersect T along a cubic curve.

Proposition 4.8. *Let P and P' be two disjoint planes in $\mathbb{P}^5$, T the elliptic ruled surfaces obtained from P and P', and D a del Pezzo surface intersecting each of the planes in a smooth conic. Let $X \in \mathcal{C}_{18} \cap \mathcal{C}_{14}$ be a cubic containing T, P, P' and D. Then X has a sublattice $M_\tau := \langle h^2, D, P, P', T \rangle \subset M(X)$ of rank 5 whose bilinear form has the following Gram matrix:*

$$
\begin{array}{c@{\quad}c}
 & \begin{array}{ccccc} h^2 & D & P & P' & T \end{array} \\
\begin{array}{c} h^2 \\ D \\ P \\ P' \\ T \end{array} &
\left(\begin{array}{ccccc}
3 & 5 & 1 & 1 & 6 \\
5 & 13 & 0 & 0 & \tau \\
1 & 0 & 3 & 0 & 0 \\
1 & 0 & 0 & 3 & 0 \\
6 & \tau & 0 & 0 & 18
\end{array} \right),
\end{array}
$$

for some $\tau = (D, T) \in \mathbb{Z}$ depending on X.

Proof. Let S_1, S_2 be two smooth surfaces in a smooth cubic hypersurface such that the scheme-theoretic intersection $S_1 \cap S_2$ contains a smooth curve C of degree d and genus g. Then the multiplicity of intersection of S_1 and S_2 along C is given (see [8]) by:

$$mult_C(S_1, S_2) = 3d + K_{S_1}.C + K_{S_2}.C + 2 - 2g$$

where K_{S_i} denotes the canonical class of S_i.

In our case, we have that $K_P = K_{P'} = -3H$ and $K_D = -H$, with H the hyperplane class. Since the quintic del Pezzo surface D and the two disjoint planes intersect in a smooth conic curve C of genus 0, we have the following:

$$mult_C(D, P) = mult_C(D, P') = 3.2 - 3.2 - 2 + 2 = 0.$$

We do the same in order to compute the intersection $(T, P) = (T, P')$. Since T and the plane intersect in a curve C' of degree 3 and genus 1. The formula gives us:

$$mult_{C'}(T, P) = mult_{C'}(T, P') = 3.3 - 3.3 + 2 - 2 = 0.$$

$\square$

We will denote by $\mathcal{C}_{M_\tau} \subset \mathcal{C}$ the locus of cubic fourfolds such that there is a primitive embedding $M_\tau \subset M(X)$ of lattices preserving h^2.

Theorem 4.9. *The locus of smooth cubic fourfolds associated to the the following rank 5 g matrix*

$$
\begin{array}{c@{\quad}c}
 & \begin{array}{ccccc} h^2 & D & P & P' & T \end{array} \\
\begin{array}{c} h^2 \\ D \\ P \\ P' \\ T \end{array} &
\left(\begin{array}{ccccc}
3 & 5 & 1 & 1 & 6 \\
5 & 13 & 0 & 0 & \tau \\
1 & 0 & 3 & 0 & 0 \\
1 & 0 & 0 & 3 & 0 \\
6 & \tau & 0 & 0 & 18
\end{array} \right)
\end{array}
$$

with $\tau = (D, T)$, has exactly 3 irreducible components $\mathcal{C}_{M_{12}}$, $\mathcal{C}_{M_{13}}$ and $\mathcal{C}_{M_{14}}$. Moreover, the associated discriminants $d(M_{12})$, $d(M_{13})$ and $d(M_{14})$ are 108, 123 and 96 respectively.

Proof. Since M_τ is defined as a sublattice of a positive definite lattice, it must have positive discriminant by Sylvester's criterion. We find the set of possible values of τ for which $d(M_\tau) > 0$.

As $d(M_\tau) = -21\tau^2 + 540\tau - 3348$, the only values of τ making a positive discriminant are $\tau \in \{11, 12, 13, 14, 15\}$. For these values of τ, M_τ is a positive definite sublattice of rank 5 containing h^2 with discriminants $d(M_\tau)$ equal to 51, 108, 123, 96, 27 respectively.

We now prove that M_τ is saturated for the possible values of τ.

We will prove that no proper finite overlattices exist; for $\tau = 11$ the discriminant is squarefree, so there are no proper finite overlattices. For the remaining cases, we can take h^2, D, P and P' as a part of a basis of an overlattice B and complete it with V such that $V = xh^2 + yD + zP + tP' + uT$ with $x, y, z, t, u \in \mathbb{Q}$. Consider the natural embedding of M_τ in B that can be written as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & y \\ 0 & 0 & 1 & 0 & z \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & u \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & -x/u \\ 0 & 1 & 0 & 0 & -y/u \\ 0 & 0 & 1 & 0 & -z/u \\ 0 & 0 & 0 & 1 & -t/u \\ 0 & 0 & 0 & 0 & 1/u \end{pmatrix} \in \mathcal{M}_{5,5}(\mathbb{Z}).$$

We can take $u = \frac{1}{n}$, for some $n \in \mathbb{Z}$ and $x' = nx$, $y' = yn$, $z' = zn$, $t' = tn \in \mathbb{Z}$; then $V = \frac{1}{n}(x'h^2 + y'D + z'P + t'P' + T)$. By adding multiples of h^2, D, P and P', we may ensure that $0 \le x', y', z', t' < n$. Computing intersections, we have the following:

$$(V, h^2) = \frac{1}{n}(3x' + y' + z' + t' + 6) = a,$$

$$(V, D) = \frac{1}{n}(5x' + 13y' + \tau) = b,$$

$$(V, P) = \frac{1}{n}(x' + 3z') = c,$$

$$(V, P') = \frac{1}{n}(x' + 3t') = d,$$

$$(V, V) = \frac{1}{n^2}(3(x')^2 + 13(y')^2 + 3(z')^2 + 3(t')^2 + 10x'y' + 2y'z' + 2x't' + 12x'$$
$$+ 2\tau y' + 18) = e.$$

Then the Gram matrix of B is:

	h^2	D	P	P'	V
h^2	3	5	1	1	a
D	5	13	0	0	b
P	1	0	3	0	c
P'	1	0	0	3	d
V	a	b	c	d	e

Now we check each case separately for possible values of $\tau, n, x', y'\ z'$ and t'. The cases mentioned down below correspond to the ones such that $B \in \mathcal{M}_{5,5}(\mathbb{Z})$.

For all other possible values τ, n, x', y', z' and t', the Gram matrices of B are non-integers.

For ($\tau = 12$, $n = 3$, $x' = 0$, $y' = 0$, $z' = 0$, $t' = 0$), B has the following Gram matrix

$$\begin{pmatrix} 3 & 5 & 1 & 1 & 2 \\ 5 & 13 & 0 & 0 & 4 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 4 & 0 & 0 & 2 \end{pmatrix},$$

which has a primitive short root $\pm(2, 0, -1, -1, -2)$.

For ($\tau = 15$, $n = 3$, $x' = 0$, $y' = 0$, $z' = 0$, $t' = 0$), B has the following Gram matrix

$$\begin{pmatrix} 3 & 5 & 1 & 1 & 2 \\ 5 & 13 & 0 & 0 & 5 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 5 & 0 & 0 & 2 \end{pmatrix},$$

which has primitive roots $\pm(1, 0, 0, -1, -1)$.

Therefore, by [21, Sect. 4 Proposition 1], no overlattices exist in these two cases.

Thus, for $\tau = 11$, 12, 13, 14, 15, M_τ are saturated.

The saturated lattice M_τ is definite positive of rank 5 such that:

$$h^2 \in M_\tau \subset M(X) \subset L.$$

Let $v = xh^2 + yD + zP + tP' + uT \in M_\tau$. For $x, y, z, t, u \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 13y^2 + 3z^2 + 3t^2 + 18u^2 + 10xy + 2xz + 2xt + 12xt + 2\tau yu.$$

We prove that $\mathcal{C}_{M_{11}}$ and $\mathcal{C}_{M_{15}}$ are empty by finding roots. Indeed, $(5h^2 - D - 2P - 2P' - T, 5h^2 - D - 2P - 2P' - T) = 2$ is a short root for $\mathcal{C}_{M_{11}}$ and $(h^2 + D - T, h^2 + D - T) = 2$ is a short root for $\mathcal{C}_{M_{15}}$; for the remaining values of τ, M_τ has no vectors v such that $(v, v) = 2$. Hence, by [23, Lemma 2.4], $\mathcal{C}_{M_\tau} \subset \mathcal{C}$ is irreducible and nonempty for $\tau = 11$, 12, 13. $\qquad\square$

Remark 4.10. In Sect. 5 we give an example of cubic with $\tau = 12$. Examples with $\tau = 13, 14$ are easy to construct in the same way, and this is explained in the ancillary file.

Proposition 4.11. *Let X be a generic cubic fourfold in $\mathcal{C}_{M_{12}}$, $\mathcal{C}_{M_{13}}$ or $\mathcal{C}_{M_{14}}$. Then the full linear system $|2h - T - P - P'|$ defines a fibration in del Pezzo quartics over $\mathbb{P}^2$.*

Proof. The three quadrics that define $|2h - T|$ automatically contain also P and P', since T intersects the two planes along cubic curves. This means that $|2h - T - P - P'| = |2h - T|$. Clearly the fibers have degree 4 and it is easy to see that the anticanonical bundle of a fiber F is $\mathcal{O}_F(1)$. In order to show the smoothness of the generic fiber, we can construct explicit examples with at least one smooth fiber (and hence the generic fiber is smooth as well) inside each component of Theorem 4.9 (see Sect. 5, or the attached file for general method). This in turn implies that for the generic cubic in the three components, the generic quartic fiber is smooth. We can even compute that the fibers are (generically) smooth intersections of two quadrics in $\mathbb{P}^4$. □

Let $F \in M(X)$ be the class $4h^2 - P - P' - T$ of the fiber, a quartic del Pezzo surface. Let $W \in M(X)$ be a 2−cycle on X such that

$$W := ah^2 + bD + cP + dP' + eT, \text{ for } a, b, c, d, e \in \mathbb{Z}$$

We have that $(W, F) = -5a + b + (20 - \tau)c + d + 6e$. We observe that a rational 1−section exists always (i.e. $(W, F) = 1$) for cubics $\mathcal{C}_{M_{12}}$, $\mathcal{C}_{M_{13}}$ or $\mathcal{C}_{M_{14}}$. Recall that, if a quartic del Pezzo surface has k−rational points, then it is unirational over k. In our case, the quartic del Pezzo surface is unirational over the function field of $\mathbb{P}^2$ and hence unirational over $\mathbb{C}$. This is not surprising since the quartic del Pezzo fibration is birational to the cubic fourfold which is known to be unirational.

As far as the author knows, only one sufficient condition is known for the rationality of a del Pezzo quartic over an arbitrary field. If a quartic del Pezzo surface contains a line defined over the base field then it is rational (see [4, Proposition 5.4]).

We cannot exclude that a line defined over the base field inside the del Pezzo fibration exists, but we can give some evidence for its non existence.

Remark 4.12. The existence of a line over $\mathbb{P}^2$, contained in the quartic del Pezzo fibration birational to the cubic X is equivalent to saying that there exists an open subset $U \subset \mathbb{P}^2$, and a $\mathbb{P}^1$-bundle W over U contained in the family of quartic surfaces over the same open set U. Let us show that, under some hypotheses, this $\mathbb{P}^1$-bundle cannot be contained in the cubic fourfold. Suppose that W is contained in the cubic X (i.e. it does not come from the exceptional divisors of the base locus of the linear system $|2h - T|$) and consider the closure $\mathcal{W}$ of W in X. The Chow group $CH^1(X) = \mathbb{Z} = \langle h \rangle$ is infinite cyclic, generated by the hyperplane section. Hence the cycle $\mathcal{W}$ is a complete intersection of type $(3, d)$. We claim that, if a $(3, d)$ complete intersection has canonical singularities, then it cannot contain a 2-dimensional family of lines. The case $(3, 2)$ is classical and known since Fano. If $d \geq 3$, then the canonical bundle of the complete intersection is effective, and if the C.I. contained the family of lines, it would be uniruled, which is absurd. Of course if the complete intersection is highly singular, then the canonical class can be negative and this argument does not work.

Remark 4.13. (1) It is not hard to construct cubic fourfolds in $\mathcal{C}_{18} \cap \mathcal{C}_{14}$ containing only one plane. We can choose the same P, and a second disjoint plane P'' such that $P'' \cap D = \emptyset$ (see Sect. 5 for details). This means that in this case we

consider the locus of smooth cubic fourfolds whose lattice of 2-cycles have a sublattice as follows

$$
\begin{array}{c}
\begin{array}{cccc} h^2 & D & P & T \end{array} \\
\begin{array}{c} h^2 \\ D \\ P \\ T \end{array}
\left(\begin{array}{cccc}
3 & 5 & 1 & 6 \\
5 & 13 & 0 & \tau \\
1 & 0 & 3 & 0 \\
6 & \tau & 0 & 18
\end{array}\right),
\end{array}
$$

where $\tau = (D, T)$. A calculation similar to the previous ones shows that this locus has exactly 6 irreducible components $\mathcal{C}_{M_\tau}$ for $\tau \in \{9, 10, 11, 12, 13, 14\}$. An argument similar to Proposition 4.11, shows that these components give rise to fibrations in quintic surfaces over $\mathbb{P}^2$, given by $F' = 4h^2 - P - T$, with ample anticanonical bundle. We construct an explicit example in the component $\tau = 12$ for which the generic fiber is smooth, hence a quintic del Pezzo surface. Also in this case, one can check that the fibration has a rational section. This implies rationality. In fact, a classical theorem by Enriques states that a del Pezzo quintic defined over an infinite field k is always rational over k.

(2) Despite many attempts, we were unable to produce an example of a cubic fourfold in the intersection of $\mathcal{C}_{18}$ and $\mathcal{C}_{14}$ not containing any plane (i.e. with $rk(M(X)) = 3$). It is likely that non-generic cubics of this kind have a different geometric construction.

4.3. $\mathcal{C}_{18} \cap \mathcal{C}_{26}$

Let X be a cubic fourfold in $\mathcal{C}_{18} \cap \mathcal{C}_{26}$; $M(X)$ has primitive sublattices $K_{18} := \langle h^2, T \rangle$ and $K_{26} := \langle h^2, S_{26} \rangle$, such that h^2 is the square of the hyperplane class, T is (possibly a degeneration of) an elliptic ruled surface and S_{26} is a surface contained in a general element of $\mathcal{C}_{26}$ (as defined in Sect. 3.2). Hence $X \in \mathcal{C}_{18} \cap \mathcal{C}_{26}$ has a sublattice $M_\tau := \langle h^2, T, S_{26} \rangle \subset M(X)$ with Gram matrix:

$$
\begin{array}{c}
\begin{array}{ccc} h^2 & T & S_{26} \end{array} \\
\begin{array}{c} h^2 \\ T \\ S_{26} \end{array}
\left(\begin{array}{ccc}
3 & 6 & 7 \\
6 & 18 & \tau \\
7 & \tau & 25
\end{array}\right),
\end{array}
$$

for some $\tau \in \mathbb{Z}$ depending on X.

Theorem 4.14. *The intersection $\mathcal{C}_{18} \cap \mathcal{C}_{26}$ has exactly seven irreducible components $\mathcal{C}_{M_\tau}$ given by the following Gram matrix*

$$
\begin{array}{c}
\begin{array}{ccc} h^2 & T & S_{26} \end{array} \\
M_\tau := \begin{array}{c} h^2 \\ T \\ S_{26} \end{array}
\left(\begin{array}{ccc}
3 & 6 & 7 \\
1 & 18 & \tau \\
7 & \tau & 25
\end{array}\right)
or\ B_{12} := \left(\begin{array}{ccc}
3 & 6 & 8 \\
6 & 18 & 18 \\
8 & 18 & 24
\end{array}\right)
\end{array}
$$

where $\tau \in \{8, 9, 10, 11, 12, 13, 14\}$ and B_{12} is an overlattice of M_{12}. Moreover the corresponding discriminants of $M_8, .., M_{14}$ and B_{12} are 48, 81, 108, 129, 144, 153, 156 and 36 respectively.

Proof. The cubic fourfold $X \in \mathcal{C}_{18} \cap \mathcal{C}_{26}$ has a sublattice $M_\tau := \langle h^2, T, S_{26} \rangle \subset M(X)$ with the Gram matrix:

$$
\begin{array}{c} \\ h^2 \\ T \\ S_{26} \end{array}
\begin{array}{ccc}
h^2 & T & S_{26}
\end{array}
\left(
\begin{array}{ccc}
3 & 6 & 7 \\
6 & 18 & \tau \\
7 & \tau & 25
\end{array}
\right)
$$

For some $\tau \in \mathbb{Z}$ depending on X.

As $d(M_\tau) = -3(\tau^2 - 28\tau + 144)$, the only values making a positive discriminant are $\tau \in \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$.

Note that there are isometries between the lattices $M_7, M_8, M_9, M_{10}, M_{11}, M_{12}, M_{13}$ and $M_{21}, M_{20}, M_{19}, M_{18}, M_{17}, M_{16}, M_{15}$ respectively (see [24, Remark 7.7]). Thus, we may consider only M_τ for $\tau = 7, 8, 9, 10, 11, 12, 12, 13, 14$ of discriminant $d(M_\tau)$ respectively 9, 48, 81, 108, 129, 144, 153, 156.

For these values of τ, no overlattices exist for M_τ except for ($\tau = 12; n = 2; x' = 1; y' = 1$), an overlattice B_{12} exists given by

$$
B_{12} := \begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 18 \\ 8 & 18 & 24 \end{pmatrix}
$$

of discriminant 36.

For these values of τ, M_τ (or B_{12}) is a positive definite saturated sublattice of rank 3:

$$
h^2 \in M_\tau \subset M(X) \subset L
$$

Furthermore, let $v = xh^2 + yT + zS_{26} \in M_\tau$ $x, y, z \in \mathbb{Z}$, we get

$$
(v, v) = 3x^2 + 18y^2 + 25z^2 + 12xy + 14xz + 2\tau yz.
$$

M_7 has roots $\pm(5h^2 + T + S_{26})$. For the remaining values of τ, there exists no $v \in M_\tau$ such that $(v, v) = 2$. Then for these values of τ, $\mathcal{C}_{M_\tau} \subset \mathcal{C}$ is irreducible nonempty and has codimension 2 (see [23, Lemma 2.4]). $\square$

Let us assume that the class $4h^2 - T \in M(X)$ gives a two-dimensional linear system of sextic del Pezzo surfaces. Under this assumption, we now check the existence of 1,2 and 3-sectional cycles in $M(X)$ for the corresponding del Pezzo fibration.

For $W_{a,b,c}$ a cycle in $M(X)$ such that

$$
W_{a,b,c} = ah^2 + bT + cS_{26}, \text{ for } a, b, c \in \mathbb{Z},
$$

we have that $(W_{a,b,c}, F) = 6a + 6b + (28 - \tau)c$.

The following table resumes the situation, according to the different values of τ.

All the observations from Remark 4.7 hold true also in this case. The components with $\tau = 10, 13$ parametrize rational cubic fourfolds in $\mathcal{C}_{18}$ such that the associated del Pezzo fibration has no rational section. If the cubics are good, the Brauer class β_2 is nontrivial.

τ	$(W_{a,b,c}, F) = 1$	$(W_{a,b,c}, F) = 2$	$(W_{a,b,c}, F) = 3$
8	$\emptyset$	$W_{0,7,-2}$	$\emptyset$
9	$W_{0,16,-5}$	$W_{0,13,-4}$	$W_{0,10,-3}$
10	$\emptyset$	$\emptyset$	$\emptyset$
11	$W_{0,3,-1}$	$W_{0,6,-2}$	$W_{0,9,-3}$
12	$\emptyset$	$W_{0,3,-1}$	$\emptyset$
13	$\emptyset$	$\emptyset$	$W_{0,3,-1}$
14	$\emptyset$	$W_{0,5,-2}$	$\emptyset$

4.4. $\mathcal{C}_{18} \cap \mathcal{C}_{38}$

Let X be a cubic fourfold in $\mathcal{C}_{18} \cap \mathcal{C}_{38}$; $M(X)$ has primitive sublattices $K_{18} := \langle h^2, T \rangle$ and $K_{38} := \langle h^2, S_{38} \rangle$, such that T is (a degeneration of) an elliptic ruled surface and S_{38} is as defined in Sect. 3.3.

Hence $X \in \mathcal{C}_{18} \cap \mathcal{C}_{38}$ has a sublattice $M_\tau := \langle h^2, T, S_{38} \rangle \subset M(X)$ with Gram matrix:

$$
\begin{array}{c}
\\ h^2 \\ T \\ S_{38}
\end{array}
\begin{array}{c}
h^2 \quad T \quad S_{38} \\
\begin{pmatrix}
3 & 6 & 10 \\
6 & 18 & \tau \\
10 & \tau & 46
\end{pmatrix},
\end{array}
$$

for some $\tau \in \mathbb{Z}$ depending on X.

Theorem 4.15. *The irreducible components of $\mathcal{C}_{18} \cap \mathcal{C}_{38}$ are the subvarieties of codimension two $\mathcal{C}_{M_\tau}$ given by rank 3 lattices represented by*

$$
M_\tau := \begin{pmatrix} 3 & 6 & 10 \\ 6 & 18 & \tau \\ 10 & \tau & 46 \end{pmatrix} \text{ or } B_{16} := \begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 17 \\ 8 & 17 & 24 \end{pmatrix} \text{ or } B_{20} := \begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 19 \\ 8 & 19 & 26 \end{pmatrix}
$$

where $12 \leq \tau \leq 20$, B_{16} and B_{20} are overlattices of M_{16} and M_{20} respectively. Moreover the associated discriminants $d(M_{12}), .., d(M_{20}), d(B_{16})$ and $d(B_{20})$ are 36, 81, 120, 153, 180, 201, 216, 225, 228, 45, and 57 respectively.

Proof. As $d(M_\tau) = -3(\tau^2 - 40\tau + 324)$, the only values of τ making a positive discriminant are $\tau \in \{12, 13, \ldots, 28\}$. Note that there are isometries between the lattices $M_{12}, M_{13}, M_{14}, M_{15}, M_{16}, M_{17}, M_{18}, M_{19}$ and $M_{28}, M_{27}, M_{26}, M_{25}, M_{24}, M_{23}, M_{22}, M_{21}$ respectively (see [24, Remark 7.7]). Thus, we may consider only M_τ for $\tau = 12, 13, 14, 15, 16, 17, 18, 19, 20$. The corresponding discriminants $d(M_\tau)$ are 36, 81, 120, 153, 180, 201, 216, 225, 228.

Overlattices exist only for $\tau = 16$ and 20 such that :

$$
B_{16} := \begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 17 \\ 8 & 17 & 24 \end{pmatrix} \text{ and } B_{20} := \begin{pmatrix} 3 & 6 & 6 \\ 8 & 18 & 19 \\ 8 & 19 & 26 \end{pmatrix}.
$$

For each of these values of τ, M_τ (as well as B_{16} or B_{20}) is a positive definite saturated sublattice of rank 3 containing h^2.

Let $v = xh^2 + yT + zS_{38} \in M_\tau$, for x, y, $z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 18y^2 + 46z^2 + 12xy + 20xz + 2\tau yz.$$

There exists no $v \in M_\tau$ such that $(v, v) = 2$; thus $\mathcal{C}_{M_\tau} \subset \mathcal{C}$ is a nonempty irreducible component and has codimension 2 (see [24, Theorem 5.2]). □

We now look for 1, 2 and 3-sectional cycles inside cubics in the irreducible components of $\mathcal{C}_{18} \cap \mathcal{C}_{38}$, assuming that the linear system $|2h - T|$ defines a sextic del Pezzo fibration on $\mathbb{P}^2$.

For $W_{a,b,c}$ a cycle in $M(X)$ such that $W_{a,b,c} = ah^2 + bT + cS_{38}$, for a, b, $c \in \mathbb{Z}$ and $F = 4h^2 - T \in M(X)$, we have that

$$(W_{a,b,c}, F) = 6a + 6b + (40 - \tau)c.$$

τ	$(W_{a,b,c}, F) = 1$	$(W_{a,b,c}, F) = 2$	$(W_{a,b,c}, F) = 3$
12	∅	$W_{0,5,-1}$	∅
13	∅	∅	$W_{0,5,-1}$
14	∅	$W_{0,9,-2}$	∅
15	$W_{0,21,-5}$	$W_{0,17,-4}$	$W_{0,13,-3}$
16	∅	∅	∅
17	$W_{0,4,-1}$	$W_{0,8,-2}$	$W_{0,12,-3}$
18	∅	$W_{0,4,-1}$	∅
19	∅	∅	$W_{0,4,-1}$
20	∅	$W_{0,7,-2}$	∅

Cubic fourfolds from the components with $\tau = 13$, 16, 19 are examples of rational cubics in $\mathcal{C}_{18}$ s.t. the associated del Pezzo fibration has no rational section. If the cubics are good, then the Brauer class β_2 is non-trivial.

5. Explicit examples

In this section, we shall give explicit examples of rational cubic fourfolds which are fibered in quadric or del Pezzo surfaces. All our computations have been done using Macaulay2 [10]. We work over the finite field $\mathbb{F}_{3331}$ but our equations hold over fields of characteristic zero. Hence we will set $\mathbb{P}^5 := Proj(\mathbb{F}_{3331}[x_0, \ldots, x_5])$ and $\mathbb{P}^2 := Proj(\mathbb{F}_{3331}[t_0, \ldots, t_2])$.

5.1. Cubic fourfold in $\mathcal{C}_8$

We provide first an example of rational cubic fourfold in $\mathcal{C}_8 \cap \mathcal{C}_{38}$ containing a *good* plane with nontrivial *Brauer class* using Sect. 3.3. This is equivalent to the fact that the quadric fibration does not have a rational section. Let S_{38} be the smooth surface of degree 10 contained in a general element of $\mathcal{C}_{38}$ as defined in Sect. 3.3. It is given by the image of a plane via the linear system of curves of degree 10 with 10 general triple points (see [18]).

Let P be the plane whose ideal is generated by P_1, P_2 and P_3 as follows:

$$P_1 = x_2 + 884x_3 + 1526x_4 + 99x_5,$$
$$P_2 = x_1 + 363x_3 + 1053x_4 + 605x_5,$$
$$P_3 = x_0 + 229x_3 + 1382x_4 + 1193x_5.$$

One can compute that P and S_{38} intersect transversally in 2 points. They are contained in the cubic fourfold Y cut out by

$$
\begin{aligned}
C := {}& x_0^3 - 559x_0^2 x_1 - 647x_0 x_1^2 + 501x_1^3 + 1640x_0^2 x_2 - 878x_0 x_1 x_2 - 417x_1^2 x_2 \\
& + 1333x_0 x_2^2 - 289x_1 x_2^2 + 472x_2^3 + 103x_0^2 x_3 + 792x_0 x_1 x_3 + 183x_1^2 x_3 \\
& - 1078x_0 x_2 x_3 - 514x_1 x_2 x_3 + 1030x_2^2 x_3 + 886x_0 x_3^2 - 727x_1 x_3^2 - 1509x_2 x_3^2 \\
& + 609x_3^3 - 1146x_0^2 x_4 + 1639x_0 x_1 x_4 - 397x_1^2 x_4 + 744x_0 x_2 x_4 - 1035x_1 x_2 x_4 \\
& + 174x_2^2 x_4 + 3x_0 x_3 x_4 - 153x_1 x_3 x_4 - 239x_2 x_3 x_4 + 907x_3^2 x_4 - 771x_0 x_4^2 \\
& - 1025x_1 x_4^2 + 876x_2 x_4^2 + 633x_3 x_4^2 - 844x_4^3 - 505x_0^2 x_5 - 889x_0 x_1 x_5 - 30x_1^2 x_5 \\
& + 822x_0 x_2 x_5 - 30x_1 x_2 x_5 - 10x_2^2 x_5 + 159x_0 x_3 x_5 + 744x_1 x_3 x_5 - 851x_2 x_3 x_5 \\
& + 1187x_3^2 x_5 + 1473x_0 x_4 x_5 - 1372x_1 x_4 x_5 - 1106x_2 x_4 x_5 + 566x_3 x_4 x_5 + 957x_4^2 x_5 \\
& + 440x_0 x_5^2 - 714x_1 x_5^2 + 278x_2 x_5^2 - 957x_3 x_5^2
\end{aligned}
$$

Y is a smooth cubic hypersurface in $\mathbb{P}^5$ contained in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$. Let $\tilde{Y}$ be the blow-up of Y along P. The discriminant divisor $E \in \mathbb{P}^2$ of the map $q : \tilde{Y} \to \mathbb{P}^2$ is a smooth sextic curve defined as follows:

$$
\begin{aligned}
E : {}& t_0^6 + 1046t_0^5 t_1 - 804t_0^4 t_1^2 - 887t_0^3 t_1^3 + 1253t_0^2 t_1^4 + 58t_0 t_1^5 + 1280t_1^6 - 164t_0^5 t_2 \\
& + 786t_0^4 t_1 t_2 - 674t_0^3 t_1^2 t_2 + 960t_0^2 t_1^3 t_2 - 574t_0 t_1^4 t_2 - 524t_1^5 t_2 + 783t_0^4 t_2^2 \\
& + 1353t_0^3 t_1 t_2^2 - 1607t_0^2 t_1^2 t_2^2 - 686t_0 t_1^3 t_2^2 + 491t_1^4 t_2^2 - 379t_0^3 t_2^3 + 706t_0^2 t_1 t_2^3 \\
& - 602t_0 t_1^2 t_2^3 - 159t_1^3 t_2^3 + 784t_0^2 t_2^4 + 824t_0 t_1 t_2^4 - 854t_1^2 t_2^4 + 448t_0 t_2^5 + 1062t_1 t_2^5 \\
& + 760t_2^6 = 0
\end{aligned}
$$

Hence, Y is a smooth rational cubic fourfold containing a *good* plane P inside the irreducible component of $\mathcal{C}_8 \cap \mathcal{C}_{38}$ indexed by $\tau = (P, S_{38}) = 2$. By Corollary 3.7, the quadric surface bundle associated to cubic fourfolds in this component has no rational section.

5.2. Cubic fourfold in $\mathcal{C}_{18}$

We exhibit now examples of rational cubic fourfolds $X \in \mathcal{C}_{18} \cap \mathcal{C}_{14}$. The associated del Pezzo fibrations have rational sections in these cases.

5.2.1. A cubic fourfold with a quartic del Pezzo fibration
We will now construct an example of cubic fourfold containing an elliptic ruled surface T, the two planes Π_1 and Π_2 that we use to construct T, and a del Pezzo quintic D that intersects the two planes along smooth conics. From Sect. 4.2, we know the possible values of $\tau = (D, T)$ for cubic fourfolds of this kind. Our example will lie in the component with $\tau = 12$, but the same method allows to construct examples in each component (see the ancillary file).

Let D be a quintic del Pezzo surface in $\mathbb{P}^5$ as defined in [8, Sect. 4].

To construct T, we define three quadrics Q_1, Q_2 and Q_3 as follows:

$$\begin{aligned}
Q_1 = {} & x_1^2 - 884x_0x_2 + 1331x_1x_2 - 336x_2^2 + 538x_1x_3 - 895x_2x_3 - 538x_0x_4 \\
& -1580x_1x_4 + 531x_2x_4 - 644x_4^2 - 1405x_0x_5 - 1650x_1x_5 + 1251x_2x_5 \\
& -305x_3x_5 - 1097x_4x_5 - 131x_5^2, \\
Q_2 = {} & x_0x_1 - 871x_0x_2 - 512x_1x_2 - 1526x_2^2 - 1367x_1x_3 - 336x_2x_3 - 213x_0x_4 \\
& -304x_2x_4 - 644x_3x_4 + 813x_0x_5 - 169x_1x_5 - 564x_2x_5 + 1316x_3x_5 - 41x_4x_5 \\
& +99x_5^2, \\
Q_3 = {} & x_0^2 - 1020x_0x_2 - 102x_1x_2 - 948x_2^2 - 1580x_0x_3 - 1432x_1x_3 + 1471x_2x_3 \\
& -644x_3^2 + 1432x_0x_4 - 500x_2x_4 + 1256x_0x_5 - 1559x_1x_5 - 1110x_2x_5 \\
& +673x_3x_5 - 932x_4x_5 + 939x_5^2.
\end{aligned}$$

Each of these quadrics contains the two disjoint planes:

$$\Pi_1 = \{x_2 - 1188x_5 = x_1 - 1188x_4 = x_0 - 1188x_3 = 0\} \text{ and}$$
$$\Pi_2 = \{x_2 - 392x_5 = x_1 - 392x_4 = x_0 - 392x_3 = 0\}$$

T is obtained by saturating the ideal generated by Q_1, Q_2 and Q_3 with respect to the defining ideals of the planes Π_1 and Π_2.

The surfaces T and D are contained in the cubic fourfold X cut out by

$$\begin{aligned}
C' := {} & x_0^2x_1 + 1200x_0x_1^2 + 1052x_0x_1x_2 - 1200x_1^2x_2 - 1053x_1x_2^2 + 1516x_0x_1x_3 \\
& + 1133x_1^2x_3 + 1218x_0x_2x_3 + 696x_1x_2x_3 + 333x_2^2x_3 - 1459x_1x_3^2 \\
& - 1145x_2x_3^2 + 235x_0^2x_4 + 1537x_0x_1x_4 - 1420x_0x_2x_4 + 1093x_1x_2x_4 \\
& + 688x_2^2x_4 + 815x_0x_3x_4 + 1301x_1x_3x_4 - 161x_2x_3x_4 - 1309x_0x_4^2 - 398x_2x_4^2 \\
& - 1219x_0^2x_5 - 466x_0x_1x_5 + 768x_1^2x_5 - 1386x_0x_2x_5 - 1395x_1x_2x_5 \\
& - 606x_0x_3x_5 - 646x_1x_3x_5 + 1442x_2x_3x_5 + 644x_3^2x_5 - 1407x_0x_4x_5 \\
& - 255x_1x_4x_5 + 1394x_2x_4x_5 - 636x_3x_4x_5 - 8x_4^2x_5 + 129x_0x_5^2 - 1026x_1x_5^2 \\
& - 1392x_3x_5^2 + 1392x_4x_5^2.
\end{aligned}$$

X is a smooth irreducible subscheme of $\mathbb{P}^5$ of dimension 4 and degree 3. The surface T (resp. D) cuts a smooth cubic curve (resp. smooth conic) out of Π_1 and

Π_2. This implies directly that X contains the two planes, and that $(T, D) \geq 12$, since the intersection of the plane cubic and conic gives 6 points. A quick Macaulay2 calculation shows that T and D indeed intersect transversally in 12 points and this in turn means that our example belongs to the discriminant 108 component (see Theorem 4.9).

The linear system $|2h - T| = \mathbb{P}^2$ contains also Π_1 and Π_2 in its base locus, since $T \cap \Pi_i$ is a cubic curve for $i = 1, 2$, hence the fibers have degree 4. Let $\tilde{X}$ be the blow-up of X along T. The discriminant locus of the map $\pi \colon \tilde{X} \to \mathbb{P}^2$ given by $|2h - T|$ is a reducible curve of degree 12 with two irreducible components. The generic fiber is a del Pezzo surface of degree 4 (a smooth intersection of two quadrics).

5.2.2. Cubic fourfold with a quintic del Pezzo fibration   To construct T, we define three quadrics Q_1', Q_2' and Q_3' as follows:

$$
\begin{aligned}
Q_1' = {} & x_1^2 + 1065x_0x_2 + 175x_1x_2 - 70x_2^2 + 1044x_0x_3 - 1139x_1x_3 + 1091x_2x_3 \\
& -1140x_3^2 - 1024x_0x_4 + 1571x_1x_4 + 1184x_2x_4 + 1443x_3x_4 + 12x_4^2 \\
& -1027x_0x_5 + 802x_1x_5 - 1655x_2x_5 - 339x_3x_5 + 768x_4x_5 + 801x_5^2, \\
Q_2' = {} & x_0x_1 + 840x_0x_2 - 1358x_1x_2 + 77x_2^2 - 37x_0x_3 + 653x_1x_3 + 726x_2x_3 \\
& +653x_3^2 - 1360x_0x_4 - 618x_1x_4 - 759x_2x_4 + 1504x_3x_4 + 1364x_4^2 \\
& -161x_0x_5 + 1577x_1x_5 + 1014x_2x_5 - 1101x_3x_5 - 1415x_4x_5 + 1577x_5^2, \\
Q_3' = {} & x_0^2 + 1415x_0x_2 - 34x_1x_2 + 765x_2^2 - 615x_0x_3 - 1200x_1x_3 + 1008x_2x_3 \\
& -1200x_3^2 + 1194x_0x_4 - 1191x_1x_4 - 222x_2x_4 + 466x_3x_4 - 767x_4^2 \\
& -901x_0x_5 - 13x_1x_5 - 1058x_2x_5 - 1213x_3x_5 - 1314x_4x_5 - 13x_5^2.
\end{aligned}
$$

Each of these quadrics contains the two disjoint planes:

$$
\begin{aligned}
\Pi_1 &= \{x_2 - 1188x_5 = x_1 - 1188x_4 = x_0 - 1188x_3 = 0\} \text{ and} \\
O &= \{x_2 + x_4 = x_1 + x_3 + x_5 = x_0 - x_4 = 0\}
\end{aligned}
$$

T is obtained by saturating the ideal generated by Q_1, Q_2 and Q_3 with respect to the defining ideals of the planes Π_1 and O. While Π_1 is the same plane as in the preceding example (hence cutting out a cubic and a conic on T and D), O is a different plane, that still intersects T along a smooth cubic but has empty intersection with D. The surfaces T and D are contained in the cubic fourfold X of equation

$$
\begin{aligned}
C'' := {} & x_0^2x_1 - 634x_0x_1^2 - 983x_0x_1x_2 + 634x_1^2x_2 + 982x_1x_2^2 - 769x_0x_1x_3 - 1007x_1^2x_3 \\
& + 730x_0x_2x_3 - 208x_1x_2x_3 - 1249x_2^2x_3 - 528x_1x_3^2 - 911x_2x_3^2 - 572x_0^2x_4 \\
& + 371x_0x_1x_4 - 133x_0x_2x_4 + 59x_1x_2x_4 - 1255x_2^2x_4 - 913x_0x_3x_4 + 560x_1x_3x_4 \\
& + 1154x_2x_3x_4 + 1092x_0x_4^2 + 77x_2x_4^2 - 731x_0^2x_5 + 137x_0x_1x_5 - 57x_1^2x_5 \\
& - 1100x_0x_2x_5 - 878x_1x_2x_5 - 1079x_0x_3x_5 - 1578x_1x_3x_5 + 1172x_2x_3x_5 \\
& + 1441x_3^2x_5 + 127x_0x_4x_5 + 966x_1x_4x_5 + 1009x_2x_4x_5 + 238x_3x_4x_5 + 1652x_4^2x_5 \\
& + 1008x_0x_5^2 + 1175x_1x_5^2 + 1033x_3x_5^2 - 1033x_4x_5^2.
\end{aligned}
$$

X is a smooth irreducible subscheme of $\mathbb{P}^5$ of dimension 4 and degree 3 that contains Π_1 (by the same argument as the preceding example), T and D. A Macaulay2 computation shows that $(D, T) = 12$, and 6 of these points are contained in Π_1. The linear system $|2h - T| = \mathbb{P}^2$ contains also Π_1 in its base locus, since $T \cap \Pi_1$ is a cubic curve, hence the degree of the fibers of the induced map $X \to \mathbb{P}^2$ is five. The discriminant locus of the map $\tilde{X} \to \mathbb{P}^2$ is a reducible curve of degree 12 with two irreducible components: a singular curve of degree 6 and a smooth curve of degree 6. The generic fiber F is smooth, has degree 5, and it is not hard to show that $\mathcal{O}_F(1)$ is the anticanonical bundle. Hence the generic fiber is a del Pezzo quintic surface.

Declarations

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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