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On Yau sequence over complete intersection surface singularities of Brieskorn type

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Abstract. In this paper, we study the Yau sequence concerning the minimal cycle over complete intersection surface singularities of Brieskorn type, and consider the relations between the minimal cycle A and the fundamental cycle Z. Further, we also give the coincidence between the canonical cycles and the fundamental cycles from the Yau sequence concerning the minimal cycle.

1. Introduction

After Artin's work [1], the complex normal surface singularity theories have been researching by many mathematicians (such as Wagreich, Brieskorn, Laufer, Saito, Wahl, Neumann, Yau, etc.). It is well known that the topological type of a complex normal surface singularity is determined by its resolution graph [14]. For a given resolution graph of a complex normal surface singularity, there are various types of complex structures which realize it. We are interested in finding the relations between analytic invariants and topological invariants [7,13,15,16].

Let (X, o) be the germ of a complex normal surface singularity and let $\pi : (\widetilde{X}, E) \to (X, o)$ be a good resolution, where $E = \pi^{-1}(o)$ denotes the exceptional divisor. Let $E = \bigcup_{i=1}^{r} E_i$ be the irreducible decomposition of E. Then $\sum_{i=1}^{r} E_i$ is a simple normal crossing divisor. A divisor on $\widetilde{X}$ supported in E is called a *cycle*. For any effective cycle $D = \sum_{i=1}^{r} d_i E_i$ ($d_i \in \mathbb{Z}, d_i \geq 0$ for any i) on E, $\chi(D)$ is defined by $\chi(D) = \dim_{\mathbb{C}} H^0(\widetilde{X}, \mathcal{O}_D) - \dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_D)$ where $\mathcal{O}_D = \mathcal{O}_{\widetilde{X}}/\mathcal{O}_{\widetilde{X}}(-D)$. From Riemann–Roch theorem, we have

$$\chi(D) = -\frac{1}{2}(D^2 + K_{\widetilde{X}} D), \tag{1.1}$$

where $K_{\widetilde{X}}$ is the canonical divisor on $\widetilde{X}$. For any irreducible component E_i, we have the adjunction formula

$$K_{\widetilde{X}} E_i = -E_i^2 + 2g(E_i) - 2 + 2\delta(E_i), \tag{1.2}$$

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where $g(E_i)$ is the geometric genus of E_i and $\delta(E_i)$ is the number of nodes and cusps on E_i [9]. The *arithmetic genus* $p_a(D)$ of D is defined by $p_a(D) = 1-\chi(D)$. It follows that if B, C are cycles, we have

$$p_a(B+C) = p_a(B) + p_a(C) - 1 + BC. \tag{1.3}$$

Among the non-zero effective cycles which have a non-positive intersection number with every irreducible component E_i of E, there is the smallest one, which is called the *fundamental cycle* Z_E on E. It is defined as follows (cf. [1]):

$$Z_E = \min \left\{ D = \sum_{i=1}^{r} a_i E_i \,\middle|\, a_i > 0 \text{ and } DE_i \leq 0 \text{ for any } i \right\}.$$

Obviously, $-Z_E^2$ is one of the most important numerically invariants of (X, o) and it is independent of the choice of resolutions. The arithmetic genus of Z_E is called the *fundamental genus* of (X, o), denoted by $p_f(X, o) := p_a(Z_E)$. This invariant is also independent of the choice of resolutions. Furthermore, there is the smallest non-zero effective cycle ($\leq Z_E$) whose arithmetic genus is equal to $p_f(X, o)$, which is called the *minimal cycle* A on E. It is defined as follows (cf. [9, Definition 3.1], [18, Definition 1.2]):

$$A = \min \left\{ D > 0 |\ p_a(D) = p_f(X, o),\ 0 < D \leq Z_E \right\}.$$

Yau gave the definition of Yau sequence concerning the minimal elliptic cycle for weakly elliptic singularities (cf. [19, Definition 3.3]) and showed which is important in his theories that if (X, o) is a numerically Gorenstein elliptic singularity, then

$$-K_{B_i} - (-K_{B_{i+1}}) = Z_{B_i}$$

for any i, where K_{B_i} is the canonical cycle on B_i and $\{Z_{B_i}\}$ is the Yau sequence (cf. Proof of Theorem 3.7 in [19]). As a generalization of the minimal elliptic cycle to the minimal cycle, Tomaru (cf. [18, Definition 5.1]) gave an analogue to Yau sequence concerning the minimal cycle for hypersurface singularities of Brieskorn type, and obtained a similar property for the case $p_f(X, o) \geq 2$ as Yau's theory, i.e.,

$$-K_{B_i} - (-K_{B_{i+1}}) = cZ_{B_i}, c \in \mathbb{Q}, \tag{1.4}$$

where $\{Z_{B_i}\}$ is the Yau sequence concerning the minimal cycle. It is well known that complete intersection surface singularity of Brieskorn type is a generalization of Brieskorn hypersurface singularity. It is a natural question to ask whether the equation (1.4) also holds for Brieskorn complete intersection surface singularity.

In this paper, we consider a germ $(W, o) \subset (\mathbb{C}^m, o)$ of a Brieskorn complete intersection surface singularity defined by

$$W = \{(x_1, x_2, \ldots, x_m) \in \mathbb{C}^m | q_{j1} x_1^{a_1} + \cdots + q_{jm} x_m^{a_m} = 0,\ j = 3, \ldots, m\},$$

where $a_i \geq 2$ are integers. We assume that (W, o) is an isolated singularity, this condition is equivalent to that every maximal minor of the matrix (q_{ji}) does not vanish (cf. [4, Section 7]). By Serre's criterion for normality, (W, o) is a normal surface singularity. Let $\pi : (\widetilde{W}, E) \to (W, o)$ be the good resolution of (W, o) with exceptional divisor E, Meng-Okuma (cf. [11]) constructed a good resolution of (W, o) by employing Konno-Nagashima's method (cf. [5]) and gave the concrete topological structure of (W, o), such as the weighted dual graph of the exceptional divisor E, the genus of the central curve E_0 in E, the fundamental genus $p_f(W, o)$ and the concrete description of the fundamental cycle Z_E in terms of $a_1, \ldots, a_m$. Following these results, we obtain a similar equality as (1.4) for (W, o), that is,

$$-K_{B_i} - (-K_{B_{i+1}}) = c \cdot Z_{B_i}, c \in \mathbb{Z}.$$

This paper is organized as follows. In Sect. 2, we introduce some notations and notions, and some fundamental results with respect to the minimal cycles. In Sect. 3 and 4, we give the relations between the minimal cycles and the fundamental cycles, and also consider a sequence given by Tomaru which is analogous to Yau sequence concerning the minimal cycle over Brieskorn complete intersection surface singularities, and give some new results on these singularities.

2. Preliminaries

In this section, we introduce some notations used throughout this paper, some fundamental results in terms of $a_1, a_2, \ldots, a_n$, and some fundamental facts on the minimal cycles over complex normal surface singularity.

2.1. *Some fundamental results*

Let $a_1, a_2, \ldots, a_m$ be positive integers. For $1 \leq i \leq m$, we define positive integers $d_m, d_{im}, \alpha_i, e_{im}$ as follows:

$$\begin{aligned} d_m &:= \operatorname{lcm}(a_1, \ldots, a_m), \\ d_{im} &:= \operatorname{lcm}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m), \\ \alpha_i &:= n_{im} := \frac{a_i}{\gcd(a_i, d_{im})}, \\ e_{im} &:= \frac{d_{im}}{\gcd(a_i, d_{im})}. \end{aligned}$$

In addition, we define integers $\beta_i := \mu_{im}$ by the following condition:

$$e_{im}\mu_{im} + 1 \equiv 0 \pmod{n_{im}}, \ 0 \leq \mu_{im} < n_{im}.$$

Let $\alpha = \prod_{i=1}^{m} \alpha_i$ and $\theta_0 = \min\{e_{mm}, \alpha\}$. We give the following Lemma 2.2 which implies the relation between the coefficient of the central curve E_0 in Z_E and α_i from the cyclic quotient singularity of type C_{α_i, β_i} for $i = 1, 2, \ldots, m-1$.

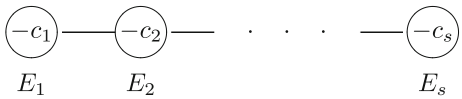

Fig. 1. The weighted dual graph of the minimal resolution of $C_{n,\mu}$

Remark 2.1. Let n and μ be positive integers that are relatively prime and $\mu < n$. Then the singularity of the quotient

$$\mathbb{C}^2 \Big/ \left\langle \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{\mu} \end{pmatrix} \right\rangle$$

is called the *cyclic quotient singularity of type* $C_{n,\mu}$, where ϵ_n denotes the primitive n-th root $\exp(2\pi\sqrt{-1}/n)$ of unity. If $n = 1$, $\mu = 0$, then the type $C_{1,0}$ means a non-singular point. For integers $c_i \geq 2$, $i = 1, 2, \ldots, s$, we put

$$[[c_1, \ldots, c_s]] := c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_s}}}.$$

Suppose $n/\mu = [[c_1, \ldots, c_s]]$, it is known (cf. [3]) that if $E = \bigcup_{i=1}^{s} E_i$ is the exceptional divisor of the minimal resolution of $C_{n,\mu}$, then $E_i \simeq \mathbb{P}^1$ and the weighted dual graph of E is chain-shaped as in Fig. 1.

It is well known that the complex structure of quotient surface singularity is determined by its resolution graph (cf. [2,10]).

Lemma 2.2. *Suppose that* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_m$. *Then* $\theta_0 \equiv 0 \pmod{\alpha_i}$ *for* $i \in \{1, 2, \ldots, m-1\}$.

Proof. If $\alpha \leq e_{mm}$, then the result is obvious following the assumption $2 \leq a_1 \leq a_2 \leq \cdots \leq a_m$. Suppose that $e_{mm} \leq \alpha$, then $\theta_0 = e_{mm}$. It suffices to prove that $\frac{a_i}{\gcd(a_i, d_{im})} \Big| \frac{d_{mm}}{\gcd(a_m, d_{mm})}$ for $i \in \{1, 2, \ldots, m-1\}$. We can easily see that

$$\frac{a_i \cdot d_{im}}{\gcd(a_i, d_{im})} = \operatorname{lcm}(a_i, d_{im}) = d_m,$$
$$\frac{a_m \cdot d_{mm}}{\gcd(a_m, d_{mm})} = \operatorname{lcm}(a_m, d_{mm}) = d_m.$$

Since $d_{im} \equiv 0 \pmod{a_m}$ for $i \in \{1, 2, \ldots, m-1\}$, and

$$a_m \cdot d_m = \frac{a_m \cdot a_i \cdot d_{im}}{\gcd(a_i, d_{im})}, \quad d_{im} \cdot d_m = \frac{d_{im} \cdot d_{mm} \cdot a_m}{\gcd(a_m, d_{mm})},$$

we have

$$\frac{a_m \cdot a_i \cdot d_{im}}{\gcd(a_i, d_{im})} \Big| \frac{d_{im} \cdot d_{mm} \cdot a_m}{\gcd(a_m, d_{mm})},$$

which implies that $e_{mm} \equiv 0 \pmod{\alpha_i}$ for $i \in \{1, 2, \ldots, m-1\}$. Thus, we obtain the assertion. $\square$

For any $x \in \mathbb{R}$, we put $\lceil x \rceil = \min\{n \in \mathbb{Z} | n \geq x\}$ and $\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x\}$. The following Lemma 2.3 is essentially following Hirzebruch resolutions for cyclic quotient singularities of type $C_{n,\mu}$ [3].

Lemma 2.3. *Let λ_0 be a positive integer and let n_i and μ_i be positive integers that are relatively prime with $\mu_i < n_i$. Suppose that $\epsilon_i := n_i/\mu_i = [[c_i, \ldots, c_s]]$ with $c_i \geq 2$ and $\lambda_i = \lceil \lambda_{i-1}/\epsilon_i \rceil$, $i = 1, 2, \ldots, s$, where*

$$[[c_i, \ldots, c_s]] = c_i - \cfrac{1}{c_{i+1} - \cfrac{1}{\ddots - \cfrac{1}{c_s}}}.$$

If $\lambda_0 \equiv 0 \pmod{n_1}$, then $\lambda_{s-1} \geq 2$ and $\lambda_s = \lambda_{s-1}/c_s$.

Proof. It is clear that $n_s/\mu_s = c_s$, that is $n_s = c_s \geq 2$, and

$$\frac{n_1}{\mu_1} = [[c_1, \ldots, c_s]] = c_1 - \frac{1}{n_2/\mu_2} = \frac{n_2 c_1 - \mu_2}{n_2}.$$

Since $\gcd(n_1, \mu_1) = 1$ and $\lambda_0 \equiv 0 \pmod{n_1}$, it follows that $\mu_1 = n_2$ and $\lambda_1 = \lceil \lambda_0/\epsilon_1 \rceil = \lambda_0 \mu_1/n_1$. Thus $\lambda_1 \equiv 0 \pmod{n_2}$. Further, since $\gcd(n_i, \mu_i) = 1$ for $i = 1, 2, \ldots, s$, we have

$$\begin{aligned}
&\mu_2 = n_3, \ldots, \mu_k = n_{k+1}, \ldots, \mu_{s-1} = n_s,\\
&\lambda_2 = \lambda_1 \mu_2/n_2, \ldots, \lambda_k = \lambda_{k-1}\mu_k/n_k, \ldots,\\
&\lambda_s = \lambda_{s-1}\mu_s/n_s = \lambda_{s-1}/c_s, k = 1, 2, \ldots, s.
\end{aligned}$$

It follows that $\lambda_{s-1} \equiv 0 \pmod{n_s}$. Thus, we obtain the assertion following the fact $n_s \geq 2$. □

From Lemma 1.2 in [5], we have the following remark.

Remark 2.4. If either $\lambda_0 \equiv 0 \pmod{n_1}$ or $\mu_1\lambda_0 + 1 \equiv 0 \pmod{n_1}$, then $\lambda_{i-1} + \lambda_{i+1} = \lambda_i c_i$ for $i = 1, 2, \ldots, s$. Furthermore, $\lambda_s c_s - \lambda_{s-1} = 0$ when $\lambda_0 \equiv 0 \pmod{n_1}$, and $\lambda_s c_s - \lambda_{s-1} = 1$ when $\mu_1\lambda_0 + 1 \equiv 0 \pmod{n_1}$.

In order to complete the proofs of our results, we also need the following results.

Lemma 2.5. ([18, Lemma 5.4]) *Let λ, ℓ and d be integers satisfying $\ell\lambda + 1 \equiv 0 \pmod{d}$ and $0 < \ell, \lambda < d$. For a non-negative integer t, let λ_t be an integer satisfying $\ell\lambda_t + 1 \equiv 0 \pmod{\ell t + d}$ and $0 < \lambda_t < \ell t + d$. If $\dfrac{d}{\lambda} = [[b_1, \ldots, b_r]]$, then $\dfrac{\ell t + d}{\lambda_t} = [[b_1, \ldots, b_r, \underbrace{2, \ldots, 2}_{t}]]$.*

Corollary 2.6. *Suppose that $n/\mu = [[b_1, \ldots, b_r, \underbrace{2, \ldots, 2}_{t}]]$, where t is a non-negative integer. Let e, p be integers defined by $e\mu + 1 \equiv 0 \pmod{n}$ with $0 < e < n$ and $ep + 1 \equiv 0 \pmod{n - et}$ with $0 \leq p < n - et$, respectively. Then $(n - et)/p = [[b_1, \ldots, b_r]]$.*

Proof. When $t = 0$, the assertion holds clearly. Assume that $t \geq 1$. Since $(n - e)\mu \equiv 1 \pmod n$, we have $n/(n-e) = [[\underbrace{2\ldots,2}_{t}, b_r, \ldots, b_1]]$ and

$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{\ddots\cfrac{1}{2-\frac{n}{n-e}}}}} = \frac{n-et}{n-e(t+1)} = [[b_r, \ldots, b_1]].$$

Furthermore, since $((n-et)-e)p \equiv 1 \pmod{n-et}$, we have $(n-et)/p = [[b_1, \ldots, b_r]]$. □

2.2. Minimal cycles over normal surface singularities

Let (X, o) be the germ of a complex normal surface singularity and $\pi : (\widetilde{X}, E) \to (X, o)$ a good resolution of (X, o), where $\pi^{-1}(o) = E = \bigcup_{i=1}^{r} E_i$ is the irreducible decomposition of E. Let Z_E be the fundamental cycle on E and D a non-zero effective cycle with $D < Z_E$. Then we can construct a computation sequence from D to Z_E as in [8]. For the relation between the arithmetic genus of D and the arithmetic genus of Z_E, we have the following Lemma.

Lemma 2.7. ([18, Lemma 1.1]) *Let D be a cycle on E such that $0 \leq D \leq Z_E$. Then $p_a(D) \leq p_a(Z_E) = p_f(X, o)$.*

Among the effective cycles ($\leq Z_E$), there is the smallest one whose arithmetic genus is equal to $p_a(Z_E)$, which is defined as follows.

Definition 2.8. ([9, 18]) Let A be an effective cycle on E satisfying $0 < A \leq Z_E$. Suppose $p_f(X, o) \geq 1$. Then A is said to be a *minimal cycle* on E if $p_a(A) = p_f(X, o)$ and $p_a(D) < p_f(X, o)$ for any cycle D with $0 \leq D < A$, that is,

$$A = \min\left\{D > 0 | p_a(D) = p_f(X, o), 0 < D \leq Z_E\right\}.$$

In 1977, Laufer showed that if (X, o) is an elliptic singularity (i.e., $p_f(X, o) = 1$), then A is the minimally elliptical cycle (cf. [9]). Further, the existence and the uniqueness of the minimal cycle A can be shown as in [9]. Also, Stevens (cf. [17]) defined the minimal cycle on the minimal resolution space and called it characteristic cycle for complex normal surface singularity (X, o). In fact, it is not easy to give the concrete descriptions of the minimal cycle A when $A \neq Z_E$ for the complex normal surface singularities. For the case $A = Z_E$, Tomaru (cf. [18]) proved that $A = Z_E$ if $\operatorname{lcm}(a_1, a_2) \leq a_3 < 2 \cdot \operatorname{lcm}(a_1, a_2)$ on the minimal resolution space for Brieskorn hypersurface singularity (V, o) with $p_f(V, o) \geq 1$, which is given as follows.

Theorem 2.9. ([18], Theorem 4.4) *Let $(\widetilde{V}, E) \to (V, o)$ be the minimal resolution with $p_f(V, o) \geq 1$, where (V, o) is the hypersurface singularity of Brieskorn type $\{(x_1, x_2, x_3) \in \mathbb{C}^3 | x_1^{a_1} + x_2^{a_2} = x_3^{a_3}\}$, if $\operatorname{lcm}(a_1, a_2) \leq a_3 < 2 \cdot \operatorname{lcm}(a_1, a_2)$, then $A = Z_E$ on E.*

Consequently, Meng et al. (cf. [12]) considered the Brieskorn complete intersection surface singularity (W, o) defined as in Sect. 1 and proved that $A = Z_E$ on the minimal resolution space if $\mathrm{lcm}(a_1, \ldots, a_{m-1}) \leq a_m < 2 \cdot \mathrm{lcm}(a_1, \ldots, a_{m-1})$, which is given as follows.

Theorem 2.10. ([12], Theorem 3.3) *Let $(\widetilde{W}, E) \to (W, o)$ be the minimal resolution, where (W, o) is the complete intersection surface singularity of Brieskorn type $\{(x_1, x_2, \ldots, x_m) \in \mathbb{C}^m | q_{j1} x_1^{a_1} + \cdots + q_{jm} x_m^{a_m} = 0,\ j = 3, \ldots, m\}$, if $\mathrm{lcm}(a_1, \ldots, a_{m-1}) \leq a_m < 2 \cdot \mathrm{lcm}(a_1, \ldots, a_{m-1})$, then $A = Z_E$ on E.*

Clearly, we always have $A \leq Z_E$. Since Z_E has been given the formula concretely, so in the case $A = Z_E$, they have the same status. However, for the case $A < Z_E$, it is useful to give the concrete descriptions of the minimal cycle A, which associate to (X, o) some new numerical invariants, such as the Yau cycle Y, $-Y^2$, $p_a(Y)$ and $\dim H^1(Y, \mathcal{O}_Y)$ [6].

3. Yau sequence concerning the minimal cycle over (W, o) when $Z_E = A$

Let $\pi : (\widetilde{X}, E) \to (X, o)$ be the minimal good resolution of a complex normal surface singularity (X, o), where $\pi^{-1}(o) = E = \bigcup_{i=1}^r E_i$ is the irreducible decomposition of the exceptional divisor E. Let Z_E and A be the fundamental cycle and minimal cycle on E, respectively. If $D = \sum_{i=1}^r d_i E_i$ is an effective cycle, we write $\mathrm{Supp} D = \bigcup E_i, d_i \neq 0$. Suppose $p_f(X, o) \geq 2$, Tomaru (cf. [18]) defined the following sequence concerning the minimal cycle which is an analogue to the Yau sequence concerning the minimal elliptic cycle (cf. [19, Definition 3.3]).

Definition 3.1. ([18, Definition 5.1]) If $Z_E A < 0$, we say that the Yau sequence concerning A is $\{Z_E\}$ and the length of the Yau sequence is 1.

Suppose $Z_E A = 0$. Let B_1 be the maximal connected subvariety of E such that $B_1 \supseteq \mathrm{Supp}\, A$ and $Z_E E_i = 0$ for any $E_i \subseteq B_1$. Since $Z_E^2 < 0$, B_1 is properly contained in E. Let Z_{B_1} be the fundamental cycle on B_1.

Suppose $Z_{B_1} A = 0$. Let B_2 be the maximal connected subvariety of B_1 such that $B_2 \supseteq \mathrm{Supp}\, A$ and $Z_{B_1} E_i = 0$ for any $E_i \subseteq B_2$. By the same argument as above, B_2 is properly contained in B_1.

We continue this process, if we obtain B_t with $Z_{B_t} A < 0$, we call $\{Z_{B_0} = Z_E,\ Z_{B_1}, \ldots, Z_{B_t}\}$ the *Yau sequence concerning* A of (X, o) and the *length* of the Yau sequence is $t + 1$. A connected component of $\bigcup_{E_i \not\subseteq \mathrm{Supp}\, A} E_i$ is called an *eliminative branch* of (X, o).

From Lemma 2.7 and the definition of minimal cycle, we know that for any non-zero effective cycle D with $A \leq D \leq Z_E$, we have $p_a(D) = p_f(X, o)$ for a complex normal surface singularity (X, o). Thus, if $\{Z_E = Z_{B_0}, Z_{B_1}, \ldots, Z_{B_t}\}$ is the Yau sequence of (X, o) and (X_{B_i}, o_i) is the complex normal surface singularity obtained by contracting B_i, $i = 1, \ldots, t$, we have $p_f(X_{B_1}, o_1) = \cdots = p_f(X_{B_t}, o_t) = p_f(X, o)$. Tomaru (cf. [18, 5]) showed that if (X, o) is a Brieskorn

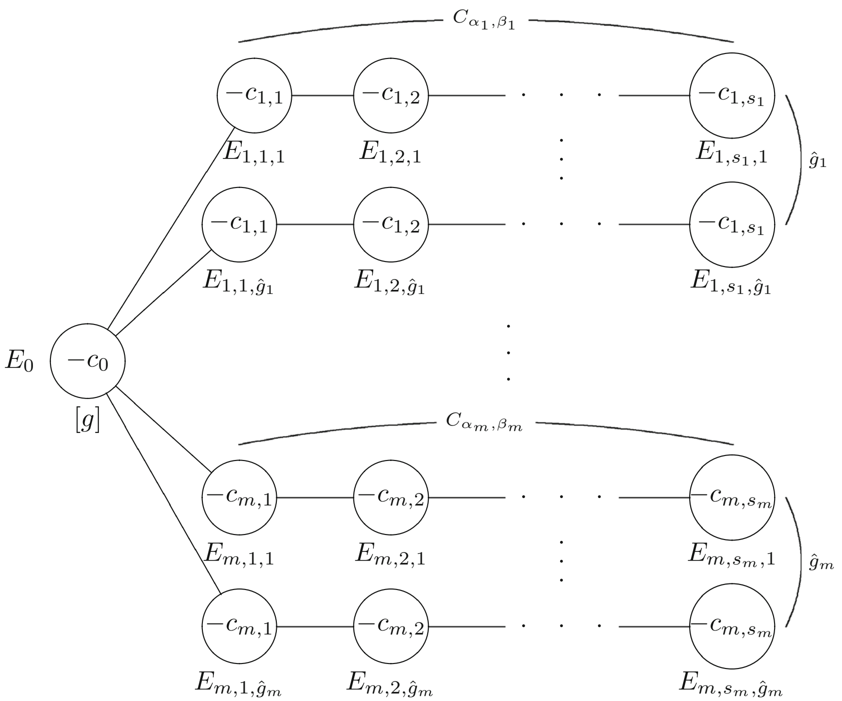

Fig. 2. The weighted dual graph of the exceptional divisor E

hypersurface singularity defined by $x_1^{a_1} + x_2^{a_2} = x_3^{a_3}$ $(2 \leq a_1 \leq a_2 \leq a_3)$ with $p_f(X, o) \geq 2$ in a restrictive situation, then

$$-K_{B_i} - (-K_{B_{i+1}}) = cZ_{B_i}, \ i = 0, 1, \ldots, t-1,$$

where $c \in \mathbb{Q}$ is a suitable positive rational number and K_{B_i} is the canonical cycle on B_i (A rational cycle K is called the *canonical cycle* if $KE_i = -K_{\widetilde{X}}E_i$ for all E_i, and the canonical cycle K exists such that $-K$ is a canonical divisor of $\widetilde{X}$ for Gorenstein surface singularity (cf. [19])). It is well known that the complete intersection surface singularity of Brieskorn type is the generalization of hypersurface singularity of Brieskorn type. In the following, we consider the Brieskorn complete intersection surface singularity (W, o) defined as in Section 1 with the assumption $2 \leq a_1 \leq a_2 \leq \cdots \leq a_m$, and give some new results.

Let $\pi : (\widetilde{W}, E) \to (W, o)$ be the minimal good resolution of (W, o) with exceptional divisor E. For $1 \leq i \leq m$, we define integers $\hat{g}$ and $\hat{g}_i$ as follows:

$$\hat{g} := \frac{a_1 \cdots a_m}{\operatorname{lcm}(a_1, \ldots, a_m)}, \ \hat{g}_i := \frac{a_1 \cdots a_{i-1} \cdot a_{i+1} \cdots a_m}{\operatorname{lcm}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)}.$$

Theorem 3.2. ([11, Theorem 4.4]) *Let g and $-c_0$ denote the genus and the self-intersection number of E_0, respectively. Then the weighted dual graph of the excep-*

tional set E is as in Fig. 2, where the invariants are as follows:

$$2g-2=(m-2)\hat{g}-\sum_{i=1}^{m}\hat{g}_i,$$

$$c_0=\sum_{w=1}^{m}\frac{\hat{g}_w\beta_w}{\alpha_w}+\frac{a_1\cdots a_m}{d_m^2},\ \ \beta_w/\alpha_w=\begin{cases}[[c_{w,1},\ldots,c_{w,s_w}]]^{-1} & if\ \alpha_w\geq 2,\\ 0 & if\ \alpha_w=1.\end{cases}$$

Theorem 3.3. ([11, Theorem 5.1]) *Let $\epsilon_{w,\nu}=[[c_{w,\nu},\ldots,c_{w,s_w}]]$ if $s_w>0$ and let*

$$Z_E=\theta_0E_0+\sum_{w=1}^{m}\sum_{\nu=1}^{s_w}\sum_{\xi=1}^{\hat{g}_w}\theta_{w,\nu,\xi}E_{w,\nu,\xi}.$$

Then θ_0 and the sequence $\{\theta_{w,\nu,\xi}\}$ are determined by the following:

$$\theta_{w,0,\xi}:=\theta_0:=\min\left(e_m,\prod_{w=1}^{m}\alpha_w\right),\theta_{w,\nu,\xi}:=\lceil\theta_{w,\nu-1,\xi}/\epsilon_{w,\nu}\rceil.$$

3.1. For the case $2\leq a_1\leq a_2\leq\cdots\leq a_m$

By Lemma 2.2, we know that $e_{mm}\equiv 0 \pmod{\alpha_i}$ for $i\in\{1,2,\ldots,m-1\}$, and following Definition 2.8, we obtain the following Theorem.

Theorem 3.4. *Suppose that $2\leq a_1\leq a_2\leq\cdots\leq a_{m-1}\leq a_m$ and $\alpha_w>1$ for any $w\in\{1,2,\ldots,m-1\}$, then*

$$p_a(Z_E-E_{w,s_w,\xi})<p_a(A),\xi=1,2,\ldots,\hat{g}_w,w=1,2,\ldots,m-1.$$

Proof. Following Definition 2.8, we have $p_a(A)=p_a(Z_E)$, that is,

$$Z_E^2+K_{\widetilde{W}}Z_E=A^2+K_{\widetilde{W}}A,$$

where $K_{\widetilde{W}}$ is the canonical divisor on $\widetilde{W}$. To prove $p_a(Z_E-E_{w,s_w,\xi})<p_a(A)$, $w\in\{1,2,\ldots,m-1\}$, by the adjunction formula (1.2), it suffices to prove that

$$-Z_EE_{w,s_w,\xi}+E_{w,s_w,\xi}^2+1<0.$$

From Theorem 3.3, we have

$$\begin{aligned}-Z_EE_{w,s_w,\xi}+E_{w,s_w,\xi}^2+1&=-\theta_{w,s_w-1,\xi}+\theta_{w,s_w,\xi}c_{w,s_w}-c_{w,s_w}+1\\&=-(\theta_{w,s_w-1,\xi}-1)+(\theta_{w,s_w,\xi}-1)c_{w,s_w}.\end{aligned}$$

If $e_{mm}\geq\prod_{w=1}^{m}\alpha_w$, then $\theta_0=\prod_{w=1}^{m}\alpha_w$. By Lemma 2.3, we have $\theta_{w,s_w-1,\xi}\geq 2$ and

$$\theta_{w,s_w,\xi}=\lceil\theta_{w,s_w-1,\xi}/c_{w,s_w}\rceil=\theta_{w,s_w-1,\xi}/c_{w,s_w}.$$

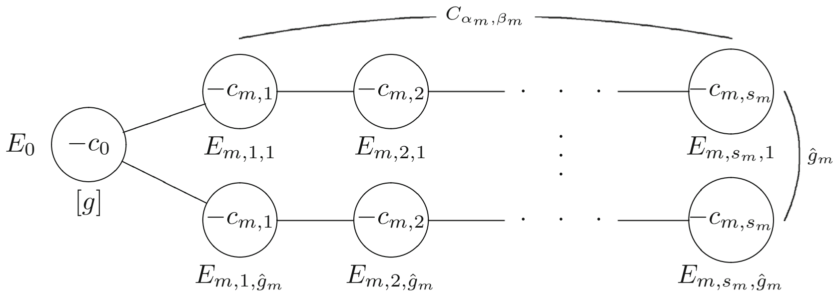

Fig. 3. The weighted dual graph of E for $a_{m-1} = \text{lcm}(a_1, \ldots, a_{m-2})$

Thus,

$$-Z_E E_{w,s_w,\xi} + E^2_{w,s_w,\xi} + 1 = -(\theta_{w,s_w-1,\xi} - 1) + (\theta_{w,s_w-1,\xi}/c_{w,s_w} - 1)c_{w,s_w}$$
$$= 1 - c_{w,s_w} < 0.$$

Similar for the case $e_{mm} \le \prod_{w=1}^{m} \alpha_w$ following Lemmas 2.2 and 2.3. Thus, we complete the proof. □

From Theorem 3.4, we note that the length of the Yau sequence concerning the minimal cycle A mainly depends on e_m, α_m and the structure of the cyclic quotient singularity C_{α_m,β_m} if we assume $2 \le a_1 \le a_2 \le \cdots \le a_{m-1} \le a_m$. For simplicity, we may first exclude the influences of the structures of the cyclic quotient singularities C_{α_i,β_i} for $i = 1, 2, \ldots, m-1$.

3.2. For the case $a_{m-1} \equiv 0 \pmod{\text{lcm}(a_1, \ldots, a_{m-2})}$

Assume that $a_{m-1} \equiv 0 \pmod{\text{lcm}(a_1, \ldots, a_{m-2})}$. Then we have $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 1$. However, there are many cases for the relations between e_m and α_m, and the structure of the C_{α_m,β_m}, where $\alpha_m/\beta_m = [[c_{m,1}, \ldots, c_{m,s_m}]]$, such as $e_{mm} \le \alpha_m$ or $\alpha_m \le e_{mm}$, and $[[c_{m,k}, \ldots, c_{m,s_m}]] = \frac{t+1}{t}$ or $[[c_{m,k}, \ldots, c_{m,s_m}]] \ne \frac{t+1}{t}$ for some positive integer t with $1 \le k \le s_m$. According to Definition 3.1, we should exclude some special cases satisfying $p_a(Z_E - E_{m,s_m,\xi}) \ne p_a(A)$ for $\xi \in \{1, 2, \ldots, \hat{g}_m\}$, and we obtain the following Theorem.

Theorem 3.5. *Suppose that $2 \le a_1 \le a_2 \le \cdots \le a_{m-1} \le a_m$ and $a_{m-1} \equiv 0 \pmod{\text{lcm}(a_1, \ldots, a_{m-2})}$. If $a_m/\beta_m = [[c_{m,1}, \ldots, c_{m,s_m}]]$ with $c_{m,s_m} > 2$, then $p_a(Z_E - E_{m,s_m,\xi}) < p_a(A)$ for $\xi = 1, 2, \ldots, \hat{g}_m$.*

Proof. Suppose $a_{m-1} \equiv 0 \pmod{\text{lcm}(a_1, \ldots, a_{m-2})}$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 1$. Thus the weighted dual graph of the exceptional divisor E is as in Fig. 3.

It is obvious that

$$\theta_0 = e_{mm} = \frac{a_{m-1}}{\text{gcd}(a_m, a_{m-1})} \le \prod_{w=1}^{m} \alpha_w = \alpha_m = \frac{a_m}{\text{gcd}(a_m, a_{m-1})}.$$

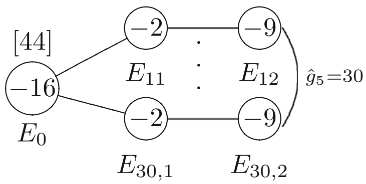

Fig. 4. The weighted dual graph of E for $a_1 = 2, a_2 = 3, a_3 = 5, a_4 = 30, a_5 = 34$

From Lemma 1.2 in [5], we have $Z_E E_{m,s_m,\xi} = -1$ or 0 for $\xi = 1, 2, \ldots, \hat{g}_m$. Since $c_{m,s_m} > 2$, following the formula (1.3), we have $p_a(A) = p_a(Z_E) > p_a(Z_E - E_{m,s_m,\xi})$ for $\xi = 1, 2, \ldots, \hat{g}_m$ if and only if $(Z_E - E_{m,s_m,\xi})E_{m,s_m,\xi} \geq 0$, i.e., $Z_E E_{m,s_m,\xi} \geq E^2_{m,s_m,\xi} = -c_{m,s_m}$. According to the assumption $c_{m,s_m} > 2$, we obtain the assertion. □

Remark 3.6. From Theorems 3.4 and 3.5, we note that the length of the Yau sequence concerning the minimal cycle A is 1 if $c_{m,s_m} > 2$. In other words, we have $Z_E = A$ if $c_{m,s_m} > 2$ and $2 \leq a_1 \leq a_2 \leq \cdots \leq a_{m-1} \leq a_m$.

In fact, by Theorems 3.4 and 3.5, we have the following corollary.

Corollary 3.7. *Suppose that* $2 \leq a_1 \leq a_2 \leq \cdots \leq a_{m-1} \leq a_m$. *If* $a_m/\beta_m = [[c_{m,1}, \ldots, c_{m,s_m}]]$ *with* $c_{m,s_m} > 2$, *then*

$$p_a(Z_E - E_{m,s_m,\xi}) < p_a(A), \xi = 1, 2, \ldots, \hat{g}_m.$$

Example 3.8. Let $a_1 = 2, a_2 = 3, a_3 = 5$ and $a_4 = \mathrm{lcm}(a_1, a_2, a_3) = 30, a_5 = 34$. Suppose that $(W, o) \subset (\mathbb{C}^5, o)$ is defined by

$$\{x_1^2 + x_2^3 = x_3^5, 2x_1^2 + 3x_2^3 = x_4^{30}, 5x_1^2 + 7x_2^3 = x_5^{34}\}.$$

Then the weighted dual graph of E on the minimal good resolution of (W, o) is as in Fig. 4. Furthermore, the fundamental cycle $Z_E = 15E_0 + 8\sum_{i=1}^{30} E_{i1} + \sum_{i=1}^{30} E_{i2}$, the fundamental genus $p_f(W, o) = 856$ and $-Z_E^2 = 30$. However, for any $E_{k2}, k = 1, 2, \ldots, 30$, we have $p_a(Z_E - E_{k2}) = 849 < p_a(A)$. In fact, we have $Z_E = A$.

4. Yau sequence concerning the minimal cycle over (W, o) when $Z_E \neq A$

According to Theorem 3.5 and Corollary 3.7, in order to study the length of the Yau sequence concerning the minimal cycle A, it is enough to consider the case $[[c_{m,k}, \ldots, c_{m,s_m}]] = [[2, 2, \ldots, 2]]$ for some $k \in \{1, 2, \ldots, s_m\}$. Obviously, if $a_{m-1} \equiv 0 \pmod{\mathrm{lcm}(a_1, a_2, \ldots, a_{m-2})}$ and $a_m \equiv 0 \pmod{a_{m-1}}$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = \alpha_m = 1$. This tells us that the length of the Yau sequence is always 1, that is $Z_E = A$. Without loss of generality, we may assume that $a_m \not\equiv 0 \pmod{a_{m-1}}$.

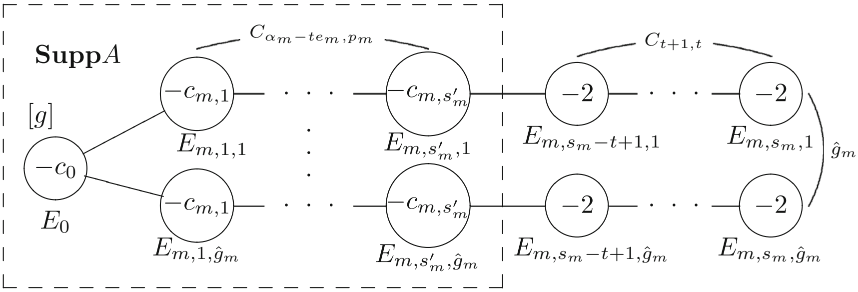

Fig. 5. The weighted dual graph of E

4.1. For the case $a_{m-1} \equiv 0 \pmod{\mathrm{lcm}(a_1, \ldots, a_{m-2})}$ *and* $a_m \not\equiv 0 \pmod{a_{m-1}}$

Tomaru ([18, Proposition 5.2]) condidered the Yau sequence concerning the minimal cycle A over Brieskorn hypersurface singularities under restrictive situation and we consider the Brieskorn complete intersection surface singularities (W, o) and obtained some new results. Suppose $p_f(W, o) \geq 2$, $2 \leq a_1 \leq a_2 \leq \cdots \leq a_{m-1} \leq a_m$ and $a_{m-1} \equiv 0 \pmod{\mathrm{lcm}(a_1, \ldots, a_{m-2})}$. Let t be a non-negative integer, and let p_m be a non-negative integer defined by

$$p_m e_m + 1 \equiv 0 \pmod{(\alpha_m - te_m)}$$

with $0 \leq p_m < \alpha_m - te_m$. By Theorem 3.2 and Corollary 2.6, we get the following theorem.

Theorem 4.1. *Assume that the length of the Yau sequence concerning the minmimal cycle* A *of* (W, o) *is* $t+1$ *with* $t \geq 1$, $Z_{B_t} = A$, *and* $E^2_{m,\nu,\xi} = -2$ *for each* $E_{m,\nu,\xi} \not\subseteq$ Supp A, *the coefficient of* $E_{m,\nu,\xi}$ *in* Z_E *is 1, where* $1 \leq \nu \leq s_m$, $1 \leq \xi \leq \hat{g}_m$. *Then the weighted dual graph of* E *is given as in Fig. 5, where* $s'_m = s_m - t$. *Furthermore,* $A = Z_E - \sum_{E_{m,\nu,\xi} \not\subseteq \mathrm{Supp}\, A} E_{m,\nu,\xi}$ *and* $Z_E^2 = -\hat{g}_m$.

Proof. Let $D = \sum_{E_{m,\nu,\xi} \not\subseteq \mathrm{Supp}\, A} E_{m,\nu,\xi}$. It is easy to see that $A + D \leq Z_E$ and the coefficient of any irreducible component of Supp$A$ in A which intersects an eliminative branch is always one. Since $Z_{B_t} = A$, $(A + D)E_i \leq 0$ for each irreducible component E_i of E, which implies $Z_E \leq A + D$. In fact, for each irreducible component E_i of Supp$D$, it is clear that $(A + D)E_i \leq 0$. On the other hand, for every irreducible component E_j of Supp$A$ = SuppZ_{B_t}, since $Z_{B_{t-1}} E_j = 0$, it is clear that $(A + D)E_j = AE_j + DE_j \leq 0$. Thus $Z_E = A + D$.

Since $t \geq 1$, $Z_E A = 0$, which implies that $-A^2 = AD$, i.e., the number of eliminative branches of (W, o). Since $a_{m-1} \equiv 0 \pmod{\mathrm{lcm}(a_1, \ldots, a_{m-2})}$, we have $\alpha_1 = \cdots = \alpha_{m-1} = 1$. Hence

$$Z_E^2 = Z_E(A + D) = (A + D)D = \hat{g}_m - 2\hat{g}_m = -\hat{g}_m$$

following $t \geq 1$ and Fig. 2. Furthermore, any eliminative branch is a chain whose component is a rational curve with self-intersection number -2. Following Corollary 2.6 and Theorem 3.2, we obtain that the weighted dual graph of E is as in Fig. 5. □

Theorem 4.2. ([11, Thoerem 5.4]) *If* $e_{mm} \geq \prod_{w=1}^{m} \alpha_w$, *then*

$$p_f(W,o) = \frac{1}{2}\prod_{w=1}^{m}\alpha_w \left\{(m-2)\hat{g} - \frac{\left(\prod_{w=1}^{m}\alpha_w - 1\right)\hat{g}}{d_m} - \sum_{w=1}^{m}\frac{\hat{g}_w}{\alpha_w}\right\} + 1.$$

If $e_{mm} \leq \prod_{w=1}^{m} \alpha_w$, *then*

$$p_f(W,o) = \frac{1}{2}e_{mm}\left\{(m-2)\hat{g} - \frac{(2\lceil e_{mm}/\alpha_m\rceil - 1)\hat{g}_m}{e_{mm}} - \sum_{w=1}^{m-1}\frac{\hat{g}_w}{\alpha_w}\right\} + 1.$$

Theorem 4.3. *In the situation of Theorem 4.1, assume that* $t \geq 1$ *and the Yau sequence of* (W,o) *is* $\{Z_{B_0} = Z_E, Z_{B_1}, \ldots, Z_{B_t}\}$. *Then*

$$-K_{B_i} - (-K_{B_{i+1}}) = \frac{2p_f(W,o) - 2 + \hat{g}_m}{\hat{g}_m} Z_{B_i},\ i = 0, 1, \ldots, t-1,$$

where K_{B_i} *is the canonical cycle on* B_i.

Proof. Since (W,o) is a Gorenstein singularity, the canonical cycle K on E exists. Thus we may write $-K$ as follows:

$$-K = \sum_{\nu=1}^{s_m-t}\sum_{\xi=1}^{\hat{g}_m} a_i E_{m,\nu,\xi} + \sum_{\nu=s_m-t+1}^{s_m}\sum_{\xi=1}^{\hat{g}_m} x_\nu E_{m,\nu,\xi},$$

where $\bigcup_{\nu=1}^{s_m-t}\bigcup_{\xi=1}^{\hat{g}_m} E_{m,\nu,\xi} = \operatorname{Supp} A$. Since $E_{m,\nu,\xi}^2 = -2$ for each $E_{m,\nu,\xi} \not\subseteq \operatorname{Supp} A$, it follows from (1.1) and (1.3) that

$$-KE_{m,s_m,\xi} = x_{s_m-1} - 2x_{s_m} = 0,\ -KE_{m,\nu,\xi} = x_{\nu-1} - 2x_\nu + x_{\nu+1} = 0$$

for $\nu = s_m-t+1, \ldots, s_m-1$, where x_{s_m-t} is the coefficient of $E_{m,s_m-t,\xi} \subset \operatorname{Supp} A$ in $-K$ which intersects $E_{m,s_m-t+1,\xi}$. Therefore,

$$-K = \sum_{\nu=1}^{s_m-t}\sum_{\xi=1}^{\hat{g}_m} a_i E_{m,\nu,\xi} + c\cdot \sum_{\nu=s_m-t+1}^{s_m}\sum_{\xi=1}^{\hat{g}_m}(s_m - \nu + 1)E_{m,\nu,\xi}, \tag{4.1}$$

where $c = x_{s_m}$. Similarly, following Definition 3.1, there is a constant c' such that

$$-K_{B_1} = \sum_{\nu=1}^{s_m-t}\sum_{\xi=1}^{\hat{g}_m} b_i E_{m,\nu,\xi} + c'\cdot \sum_{\nu=s_m-t+1}^{s_m-1}\sum_{\xi=1}^{\hat{g}_m}(s_m - \nu)E_{m,\nu,\xi}, \tag{4.2}$$

where K_{B_1} is the canonical cycle on B_1. Since $t \geq 1$ and from the assumption, it is easy to see that $Z_E A = 0$, $Z_E E_{m,s_m,\xi} = -1$ and $-K_{B_1}E_{m,s_m,\xi} = c'$. Then

$$(-K - (-K_{B_1}))E_j = c' Z_E E_j$$

for any irreducible component E_j of E, which implies that

$$-K - (-K_{B_1}) = c' Z_E, \tag{4.3}$$

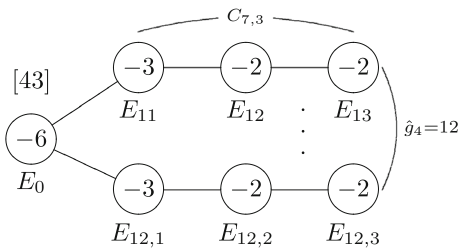

Fig. 6. The weighted dual graph of E for $a_1 = 3, a_2 = 4, a_3 = 12, a_4 = 42$

and then $c' \in \mathbb{Z}$ following the definition of canonical cycle. From (4.1), (4.2) and (4.3), we have $c = c' \in \mathbb{Z}$. Hence $-K - (-K_{B_1}) = cZ_E$. Since Z_{B_i} is the fundamental cycle on B_i, the coefficient of $E_{m,\nu,\xi}$ in Z_{B_i} is also 1 for every $E_{m,\nu,\xi} \not\subseteq \operatorname{Supp} A$. Continuing this process, we obtain that

$$-K_{B_i} - (-K_{B_{i+1}}) = cZ_{B_i}, \quad i = 0, 1, \dots, t-1,$$

where $-K_{B_0} = -K$ and $Z_{B_0} = Z_E$. Since $Z_E K_{B_1} = 0$, $-KZ_E = cZ_E^2$. From Theorem 4.1, we have

$$c = \frac{KZ_E}{-Z_E^2} = \frac{2p_f(W,o) - 2 - Z_E^2}{-Z_E^2} = \frac{2p_f(W,o) - 2 + \hat{g}_m}{\hat{g}_m}.$$

From Theorem 4.2, we can obtain the integer c. □

Remark 4.4. If $p_f(W,o) = 2$, then $\hat{g}_m \leq 2$ since $c \in \mathbb{Z}$. In fact, we have

$$\hat{g}_m | (2p_f(W,o) - 2).$$

Example 4.5. Let $a_1 = 3$, $a_2 = 4$ and $a_3 = \operatorname{lcm}(a_1, a_2) = 12$, $a_4 = 42$. Suppose that

$$(W,o) = \left(\{x_1^3 + x_2^4 + x_3^{12} = 0, 2x_1^3 + 3x_2^4 + x_4^{42} = 0\}, o\right) \subset (\mathbb{C}^4, o).$$

Then the weighted dual graph of the minimal good resolution of (W,o) is as in Fig. 6, the fundamental cycle $Z_E = 2E_0 + \sum_{i=1}^{12} \sum_{j=1}^{3} E_{ij}$, $Z_E^2 = -12$ and $p_f(W,o) = 91$. The minimal cycle $A = 2E_0 + \sum_{i=1}^{12} E_{i1}$. It is easy to see that $A = Z_E - \sum_{i=1}^{12} \sum_{j=2}^{3} E_{ij}$, $Z_E A = 0$, $B_1 = E_0 \cup (\cup_{i=1}^{12} \cup_{j=1}^{2} E_{ij})$ and $B_2 = E_0 \cup (\cup_{i=1}^{12} E_{i1})$. Then we have $Z_{B_1} = 2E_0 + \sum_{i=1}^{12} \sum_{j=1}^{2} E_{ij}$, $Z_{B_1} A = 0$ and $Z_{B_2} = A = 2E_0 + \sum_{i=1}^{12} E_{i1}$, $Z_{B_2} A < 0$. Hence the Yau sequence is $\{Z_E, Z_{B_1}, Z_{B_2}\}$ and the length of Yau sequence is 3. After computation, we have

$$-K = 111E_0 + 48\sum_{i=1}^{12} E_{i1} + 32\sum_{i=1}^{12} E_{i2} + 16\sum_{i=1}^{12} E_{i3},$$

$$-K_{B_1} = 79E_0 + 32\sum_{i=1}^{12} E_{i1} + 16\sum_{i=1}^{12} E_{i2},$$

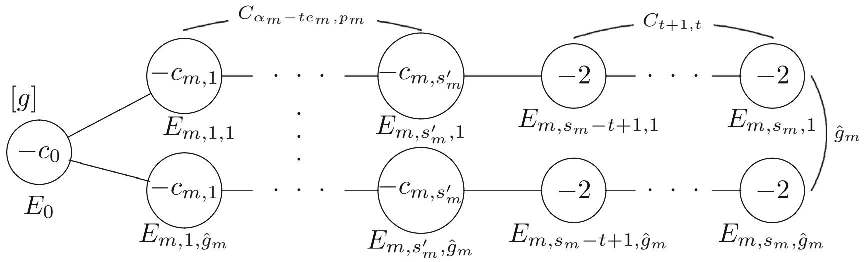

Fig. 7. The weighted dual graph of E

$$-K_{B_2} = 47E_0 + 16\sum_{i=1}^{12} E_{i1}.$$

It is clear that $c = 16$ and $-K - (-K_{B_1}) = 16Z_E$, $-K_{B_1} - (-K_{B_2}) = 16Z_{B_1}$ and $\hat{g}_4 | 2p_f(W, o) - 2$, i.e., $2p_f(W, o) - 2 = 15\hat{g}_4$.

Corollary 4.6. *Assume that the weighted dual graph of E is given as in Fig. 7, where $s'_m = s_m - t$ and $c_{m,s'_m} > 2$, and the coefficient of some $E_{m,\nu,\xi}$ in Z_E is 1 with $s_m - t + 1 \le \nu \le s_m$, $1 \le \xi \le \hat{g}_m$. Then the length of Yau sequence concerning the minimal cycle A is $t + 1$ and*

$$A = Z_E - \sum_{\nu=s_m-t+1}^{s_m} \sum_{\xi=1}^{\hat{g}_m} E_{m,\nu,\xi}.$$

Furthermore, we have $Z_E^2 = -\hat{g}_m$ and

$$-K_{B_i} - (-K_{B_{i+1}}) = \frac{2p_f(W, o) - 2 + \hat{g}_m}{\hat{g}_m} Z_{B_i}, \; i = 0, 1, \ldots, t-1,$$

where K_{B_i} is the canonical cycle on B_i, and $\{Z_{B_0} = Z_E, Z_{B_1}, \ldots, Z_{B_t}\}$ is the Yau sequence concerning the minimal cycle A.

4.2. *For the general case* $2 \le a_1 \le \cdots \le a_m$

According to Lemma 1.2 in [5], we know that if $e_{mm}\beta_m + 1 \equiv 0 \pmod{\alpha_m}$ and $a_{m-1} = \mathrm{lcm}(a_1, a_2, \ldots, a_{m-2})$, then for the fundamental cycle Z_E,

$$\theta_0 = e_{mm} = \frac{a_{m-1}}{\gcd(a_{m-1}, a_m)} \le \alpha_m = \frac{a_m}{\gcd(a_{m-1}, a_m)},$$

and $\theta_{m,s_m,\xi} = \lceil \theta_0/\alpha_m \rceil = 1, \xi = 1, 2, \ldots, \hat{g}_m$. Further, if $[[c_{m,k}, \ldots, c_{m,s_m}]] = [[2, 2, \ldots, 2]]$ for some $k \in \{1, 2, \ldots, s_m\}$, then $\theta_{m,\nu,\xi} = 1$ for $k \le \nu \le s_m$, $1 \le \xi \le \hat{g}_m$ following Lemma 1.2 in [5]. This means that we should consider the length of Yau sequence concerning minimal cycle A without the assumption $a_{m-1} = \mathrm{lcm}(a_1, a_2, \ldots, a_{m-2})$. That is, for a connected part containing the curve $E_{m,s_m,\xi}$ in the minimal resolution graph of C_{α_m,β_m} with all $E^2_{m,\nu,\xi} = -2$ and the coefficient of $E_{m,s_m,\xi}$ in Z_E is not 1 for $1 \le \xi \le \hat{g}_m$, then we obtain the following theorem.

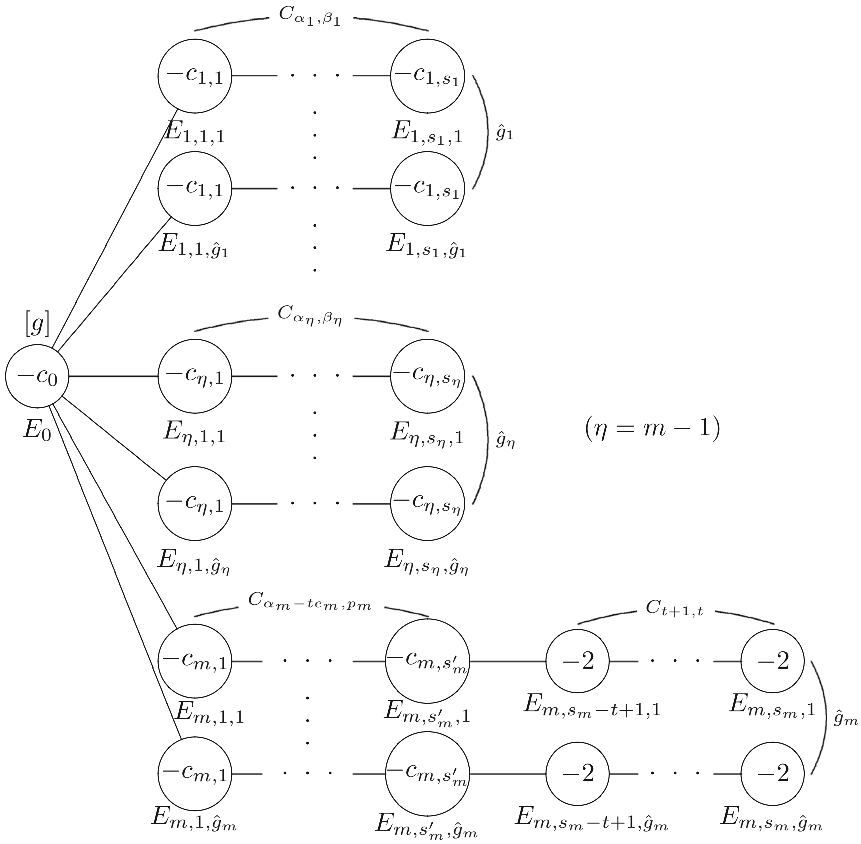

Fig. 8. The weighted dual graph of E

Theorem 4.7. *Assume that the weighted dual graph of E is given as in Fig. 8, where $s'_m = s_m - t$ and $c_{m,s'_m} > 2$, and the coefficient of $E_{m,s_m,\xi}$ in Z_E is not 1 with $1 \leq \xi \leq \hat{g}_m$. Then the length of Yau sequence concerning the minimal cycle A is $t+1$ and*

$$A = Z_E - \sum_{\nu=s_m-t+1}^{s_m} \sum_{\xi=1}^{\hat{g}_m} E_{m,\nu,\xi}.$$

Proof. If $t = 0$, then it is clear by Corollary 3.7. Assume that $t \geq 1$ and the coefficient of $E_{m,s_m,\xi}$ with $1 \leq \xi \leq \hat{g}_m$ in Z_E is $\theta_{s_m,\xi} := \theta_{m,s_m,\xi} \geq 2$. Since $c_{m,s'_m} > 2$, we have $Z_E E_{m,s_m,\xi} = -1$ following Lemma 1.2 in [5]. Thus, by (1.3), we have

$$p_a(Z_E - E_{m,s_m,\xi}) = p_a(Z_E) + p_a(-E_{m,s_m,\xi}) - 1 - Z_E E_{m,s_m,\xi} = p_a(Z_E).$$

Continuously, let $D = Z_E - \sum_{\nu=s_m-t+1}^{s_m} \sum_{\xi=1}^{\hat{g}_m} E_{m,\nu,\xi}$, following Lemma 1.2 in [5] and (1.3), we have

$$p_a\,(Z_E - D) = p_a(Z_E) + p_a(-D) - 1 - Z_E D$$

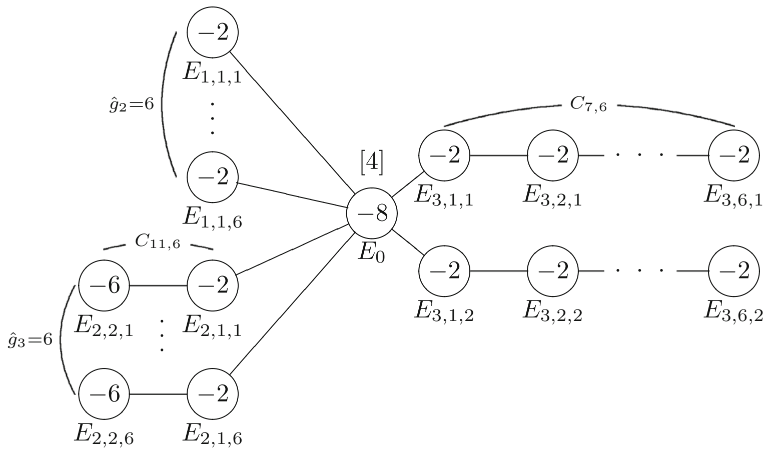

Fig. 9. The weighted dual graph of E for $a_1 = 3, a_2 = 4, a_3 = 22, a_4 = 42$

$$
\begin{aligned}
&= p_a(Z_E) + \frac{1}{2}D^2 - Z_E D \\
&= p_a(Z_E) - \hat{g}_m + \hat{g}_m \\
&= p_a(Z_E).
\end{aligned}
$$

Further, since $c_{m,s'_m} > 2$, according to Theorem 3.5, we have

$$p_a(D - E_{m,s'_m,\xi}) < p_a(Z_E) = p_a(A),\ 1 \le \xi \le \hat{g}_m.$$

By Definitions 2.8 and 3.1, we have

$$Z_{B_t} = Z_E - \sum_{\nu=s_m-t+1}^{s_m} \sum_{\xi=1}^{\hat{g}_m} E_{m,\nu,\xi} = A.$$

Hence we complete the proof. □

Let $a_1 = 3, a_2 = 4, a_3 = 22$ and $a_4 = 42$. Assume that

$$(W, o) = \left(\{x_1^3 + x_2^4 = x_3^{22}, 2x_1^3 + 3x_2^4 = x_4^{42}\}, o\right) \subset (\mathbb{C}^4, o).$$

Then the weighted dual graph of the minimal good resolution of (W, o) is as in Fig. 9 following Theorem 3.2. Further, by Theorems 3.3 and 4.2, we obtain that the fundamental cycle

$$
\begin{aligned}
Z_E =& 22E_0 + 11\sum_{\xi=1}^{6} E_{1,1,\xi} + 12\sum_{\xi=1}^{6} E_{2,1,\xi} + 2\sum_{\xi=1}^{6} E_{2,2,\xi} + 19\sum_{\xi=1}^{2} E_{3,1,\xi} \\
&+ 16\sum_{\xi=1}^{2} E_{3,2,\xi} + 13\sum_{\xi=1}^{2} E_{3,3,\xi} + 10\sum_{\xi=1}^{2} E_{3,4,\xi} + 7\sum_{\xi=1}^{2} E_{3,5,\xi} + 4\sum_{\xi=1}^{2} E_{3,6,\xi},
\end{aligned}
$$

and $p_a(Z_E) = 179$. Let

$$D = Z_E - \sum_{\nu=1}^{6}\sum_{\xi=1}^{2} E_{3,\nu,\xi} = 22E_0 + 11\sum_{\xi=1}^{6} E_{1,1,\xi} + 12\sum_{\xi=1}^{6} E_{2,1,\xi} + 2\sum_{\xi=1}^{6} E_{2,2,\xi}$$
$$+ 18\sum_{\xi=1}^{2} E_{3,1,\xi} + 15\sum_{\xi=1}^{2} E_{3,2,\xi} + 12\sum_{\xi=1}^{2} E_{3,3,\xi}$$
$$+ 9\sum_{\xi=1}^{2} E_{3,4,\xi} + 6\sum_{\xi=1}^{2} E_{3,5,\xi} + 3\sum_{\xi=1}^{2} E_{3,6,\xi},$$

then $p_a(D) = 179$. Furthermore, for any $E_{3,\nu,\xi}$, $\nu = 1, 2, \ldots, 6$, $\xi = 1, 2$, we have $p_a(D - E_{3,\nu,\xi}) < p_a(Z_E)$. Therefore, following Theorem 3.5 and Corollary 3.7, we obtain that the minimal cycle $A = D$.

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Declarations

Data Availability Statement The data used to support the findings of this study are included within the article.

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