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Instanton sheaves on Fano threefolds

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Abstract. Generalizing the definitions originally presented by Kuznetsov and Faenzi, we study (possibly non locally free) instanton sheaves of arbitrary rank on Fano threefolds. We classify rank 1 instanton sheaves and describe all curves whose structure sheaves are rank 0 instanton sheaves. In addition, we show that every rank 2 instanton sheaf is an elementary transformation of a locally free instanton sheaf along a rank 0 instanton sheaf. To complete the paper, we describe the moduli space of rank 2 instanton sheaves of charge 2 on a quadric threefold X and show that the full moduli space of rank 2 semistable sheaves on X with Chern classes $(c_1, c_2, c_3) = (-1, 2, 0)$ is connected and contains, besides the instanton component, just one other irreducible component which is also fully described.

Contents

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1. Introduction

In their seminal work [2], Atiyah, Drinfeld, Hitchin, and Manin presented the notion of *mathematical instantons*, rank 2 holomorphic vector bundles on $\mathbb{P}^3$ that correspond to anti-self-dual connections, a.k.a. instantons, on the sphere S^4. More precisely, a mathematical instanton of charge n is defined as a stable rank 2 vector bundle E with Chern classes $c_1(E) = 0,\ c_2(E) = n$ and such that $H^i(E(-2)) = 0$ for $i = 1, 2$.

In the following years, several authors presented different generalizations of mathematical instanton, first to odd-dimensional projective spaces [25], then to non-locally free sheaves of any rank on arbitrary projective spaces [18], and to other Fano threefolds besides $\mathbb{P}^3$ [7,11,23], and more recently to arbitrary polarized projective varieties [3,4].

In the present paper, we introduce a definition of instanton sheaves on a Fano threefold X of Picard rank one, compatible with all the aforementioned ones. Namely, a torsion-free sheaf E on X is said to be an *instanton sheaf* of charge n if it is μ-semistable and it satisfies $c_1(E) = -r_X \cdot \mathrm{rk}(E)/2, c_2(E) = n$ and $h^i(E(-q_X)) = 0$ for $i = 1, 2$, with $r_X,\ q_X$ integers such that $K_X \sim -(2q_X + r_X)H_X$ where K_X and H_X are, respectively the anti-canonical class and the ample generator of $\mathrm{Pic}(X)$. These requirements are nevertheless sufficient to guarantee that several expected properties for instanton sheaves still hold in this more general setting.

Once the notion of an instanton sheaf has been provided and their main features have been illustrated, we focus our attention on how they behave in families with a particular emphasis on the rank 2 case. Indeed, there exists a vast literature on moduli spaces of rank 2 instanton sheaves.

On $\mathbb{P}^3$, the moduli space $\mathcal{I}(n)$ of rank 2 instanton bundles on $\mathbb{P}^3$ has been proved to be an irreducible [32,33] smooth [19] affine [8] variety of dimension $8n - 3$. A more comprehensive picture of $\overline{\mathcal{I}(n)}$, the closure of $\mathcal{I}(n)$ within the Gieseker–Maruyama moduli scheme $\mathcal{M}_{\mathbb{P}^3}(0, n, 0)$ of rank 2 semistable sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = 0,\ c_2 = n,\ c_3 = 0$, can then be obtained taking into account also the non locally free instanton sheaves, as shown in [21,22]. Moreover, the moduli space $\mathcal{L}(n)$ of all rank 2 instanton sheaves of charge n was shown in [20] to be connected for $n \leq 4$.

Moduli spaces of rank 2 instanton bundles on other Fano threefolds X have been widely inspected as well: among them, we mention for example [9,26,31]. In these works, a frequently used technique to construct instantons and study their moduli is the so-called *Serre correspondence*. This latter establishes a correspondence between global sections of rank 2 reflexive sheaves on X and locally Cohen Macaulay (l.c.m.) curves on X; it is then possible to deduce geometrical properties of moduli spaces of sheaves from the geometry of the Hilbert scheme of curves on X.

In order to carry on these pursuits, we present in this paper a more general form of the Serre correspondence that applies to torsion-free sheaves. This will allow us to describe in greater detail the geometry of the curves corresponding to the non locally free rank 2 instanton sheaves, and then apply these results to study moduli spaces of rank 2 instanton sheaves on the quadric threefold.

Here is the plan for the paper. In Sect. 2 we set up the notation for the rest of the paper by recalling the classification of Fano threefolds of Picard rank one, and some features of (semi)stable sheaves. Section 3 is dedicated to formulating the Serre correspondence for torsion-free sheaves, generalizing the classical correspondence presented in [1,13,14].

Instanton sheaves on Fano threefolds, the main characters of our tale, are then introduced in Sect. 4. After going over some basic properties and examples of instanton sheaves, we also introduce the notion of *rank 0 instanton sheaves*, that is 1-dimensional sheaves T on X satisfying the cohomological vanishing $H^i(T(-q_X)) = 0,\ i = 0, 1$; for $X = \mathbb{P}^3$, rank 0 instantons were originally introduced in [15] and further studied in [12], and play a key role in the study of non-reflexive instanton sheaves via a procedure known as *elementary transformation*. We present a classification of the rank 0 instantons T of the form $T \simeq \mathcal{O}_C$ where $C \subset X$ an l.c.m. curve (that will be therefore referred to as an *instanton curve*) and as a direct consequence of this result we obtain a classification of rank 1 instanton sheaves, see Proposition 21.

The first main result of the paper is a classification of rank 2 instanton sheaves, see Theorem 24. To be precise, we prove that in this case, an instanton sheaf E is either locally free or its singular locus has pure dimension one. This implies in particular that every non locally free rank 2 instanton sheaf E is obtained via an elementary transformation of a locally free instanton sheaf F along a rank 0 instanton sheaf T; if this occurs we moreover have $E^{\vee\vee} \simeq F$ and $\mathrm{Sing}(E) = \mathrm{Supp}(T)$.

We complete Sect. 4 with a detailed description of the Serre correspondence for non locally free rank 2 instanton sheaves. In particular, we describe how curves corresponding to locally free instanton sheaves behave under elementary transformation; in this way, we can also relate the geometry of the curves corresponding to non locally free instanton sheaves to the singularities of these sheaves.

This formulation of the Serre correspondence is the main tool used in Sect. 5 to study the moduli space of instantons of charge 2 on the quadric threefold $X \subset \mathbb{P}^4$. The Serre correspondence was used in [31] to describe the moduli space $\mathcal{I}(2)$ of instanton bundles of charge 2 on X and to prove that the latter is an irreducible smooth variety of dimension 6. Our study of the families of curves corresponding to non locally free instanton sheaves allows us to prove that these always deform to locally free instanton sheaves and that they are parameterized by an irreducible divisor in $\overline{\mathcal{I}(2)}$.

Still relying on the Serre correspondence, we give a complete description of the moduli space $\mathcal{M}_X(2; -1, 2, 0)$ of rank 2 semistable sheaves with Chern classes $c_1 = -1,\ c_2 = 2,\ c_3 = 0$ in Sect. 6. We prove that this moduli space consists of two irreducible components, namely $\overline{\mathcal{I}(2)}$, and a 10-dimensional irreducible component $\mathcal{C}$ whose general point is the kernel of an epimorphism $F \twoheadrightarrow \mathcal{O}_p$, with

F a stable reflexive sheaf with $c_1(F) = -1,\ c_2(F) = 2,\ c_3(F) = 2$ and $p \in X$ a point. We will finally show that $\overline{\mathcal{I}(2)} \cap \mathcal{C} \neq \emptyset$ proving the connectedness of $\mathcal{M}_X(2; -1, 2, 0)$.

2. Background and notation

2.1. Classification of Fano threefolds

Let X be a smooth 3-dimensional projective variety whose Picard group has rank one. Letting H_X denote the ample generator of $\mathrm{Pic}(X)$, the canonical class K_X can be written in the form

$$K_X = -i_X H_X \qquad i_X \in \mathbb{Z};$$

X is said to be *Fano* whenever $i_X > 0$. For each Fano threefold X, we define the following numerical invariants:

- The **index**, defined as the integer i_X;
- The **degree** $d_X := H_X^3$;

In addition, we let q_X and r_X denote the quotient and the remainder of the division of i_X by 2, so that we can write $i_X = 2q_X + r_X$, with $q_X \geq 0$ and $r_X \in \{0, 1\}$.

The cohomology groups of a Fano threefold X satisfy the following properties. All the groups $H^{i,i}(X)$ have dimension one, and for this reason throughout the entire article, we will write the Chern classes of any sheaf $F \in Coh(X)$ as integers. By Kodaira vanishing we then compute:

$$h^i(\mathcal{O}_X(k)) = 0,\ i = 1, 2,\ k \in \mathbb{Z}$$
$$h^{0,p}(X) = h^{p,0}(X) = 0,\ p > 0.$$

Fano threefolds with Picard rank one were classified by Iskovskikh and Mukai [17,28]. Recall that $i_X \leq 4$, and

- If $i_X = 4$, then $X \simeq \mathbb{P}^3$;
- If $i_X = 3$, then X is a smooth quadric hypersurface in $\mathbb{P}^4$.
- There are five families of Fano threefolds with $i_X = 2$, up to deformation; these families are classified according to its degree $d_X \in \{1, 2, 3, 4, 5\}$.
- There are ten deformation families of Fano threefolds with $i_X = 1$, which are also classified according to their degrees d_X taking all even values between 2 and 22, except 18.

Note that even if in some cases we have different isomorphism classes of Fano threefolds, these belong to the same deformation family.

2.2. Stability of sheaves

Let E be a coherent sheaf on a non-singular projective variety X with $\mathrm{Pic}(X) = \mathbb{Z}$; let $E(t) := E \otimes H_X^{\otimes t}$, where H_X denotes the ample generator of $\mathrm{Pic}(X)$.

Recall that E is *(semi)stable* if it is pure and, for every proper subsheaf $F \subset E$ we have

$$p_F(t) < \ (\leq)\ p_E(t),$$

where $p_E(t)$ denotes the reduced Hilbert polynomial of the sheaf E. Furthermore, when E is a torsion-free sheaf, E is *μ-(semi)stable* if for every proper subsheaf $F \subset E$ such that E/F is also torsion-free we have

$$\frac{c_1(F)\cdot H_X^{d-1}}{\mathrm{rk}(F)} < \ (\leq)\ \frac{c_1(E)\cdot H_X^{d-1}}{\mathrm{rk}(E)}.$$

Remark that for torsion-free sheaves:

$$\mu - \text{stability} \implies \text{stability} \implies \text{semistability} \implies \mu - \text{semistability},$$

see [16, Lemma 1.2.13]. In addition, E is μ-(semi)stable if and only if $E^\vee$ is μ-(semi)stable.

Here is a simple characterization of μ-(semi)stable rank 2 sheaves, which generalizes well-known results for reflexive sheaves, cf. [29, Lemma II.1.2.5]. Recall that a torsion-free sheaf E is said to be *normalized* if $c_1(E) \in \{0, -1, \ldots, -\mathrm{rk}(E)+1\}$; every torsion-free sheaf can be normalized after a twist by $\mathcal{O}_X(k)$ for some suitable integer k. Recall also that a normalized torsion-free sheaf with $c_1(E) < 0$ is μ-stable if and only if it is μ-semistable.

Lemma 1. *Let X be a non-singular projective variety X of dimension 3 with* $\mathrm{Pic}(X) = \mathbb{Z}$*, and let E be a normalized torsion-free sheaf with* $\mathrm{rk}(E) \geq 2$.

(1) *Assuming $c_1(E) = 0$, we have that*
  (1.1) *if E is μ-stable then $h^3(E \otimes \omega_X) = 0$;*
  (1.2) *if E is μ-semistable then $h^3(E \otimes \omega_X(1)) = 0$;*
  (1.3) *the converse of (1.1) and (1.2) hold when* $\mathrm{rk}(E) = 2$.
(2) *Assuming $c_1(E) < 0$, we have that*
  (2.1) *if E is μ-semistable then $h^3(E \otimes \omega_X) = 0$;*
  (2.2) *the converse of (2.1) holds when* $\mathrm{rk}(E) = 2$.

Proof. By Serre duality, we know that $H^3(E \otimes \omega_X) \simeq \mathrm{Hom}(E, \mathcal{O}_X)^*$. If $h^3(E \otimes \omega_X) > 0$, then there is a non-trivial morphism $\varphi : E \to \mathcal{O}_X$; let $F := \ker(\varphi)$. Since $c_1(F) = -c_1(\mathrm{im}(\varphi)) \geq 0$, we conclude that E cannot be μ-stable.

Conversely, assume that $\mathrm{rk}(E) = 2$; if E is not μ-stable, let F be a destabilizing subsheaf; since E has rank 2, we must have that both F and $G := E/F$ are rank 1 torsion-free sheaves. It follows that $G = \mathcal{I}_\Gamma(k)$ for some subscheme $\Gamma \subset X$ and $k \leq 0$, thus there exists a monomorphism $G \hookrightarrow \mathcal{O}_X$; composing the epimorphism $E \twoheadrightarrow G$ with the latter, we obtain a non-trivial morphism $E \to \mathcal{O}_X$, showing that $h^3(E \otimes \omega_X) > 0$.

The proofs for items (1.2) and (2) are completely analogous. □

We will need the following result regarding 1-dimensional sheaves.

Lemma 2. *Let T be a pure 1-dimensional sheaf with $\chi(T(t)) = d \cdot (t + e)$. If $h^0(T(-e)) = 0$, then T is semistable.*

Proof. If $S \hookrightarrow T$ is a subsheaf, then $h^0(S(-e)) = 0$; if we set $\chi(S(t)) = s \cdot t + x$, then $\chi(S(-e)) = -se + x = -h^1(S(-e)) \leq 0$, thus $x \leq se$; note that $s > 0$ because T has pure dimension 1. It follows that

$$\frac{\chi(T(t))}{d} - \frac{\chi(S(t))}{s} = e - \frac{x}{s} \geq 0,$$

proving that T is semistable. □

3. Serre correspondence for torsion-free sheaves

The so-called *Serre correspondence* is one of the most efficient tools to construct and study rank 2 sheaves on a threefold X.

Recall from [14, Theorem 4.1] (which generalizes [13, Theorem 1.1]) that this is a correspondence relating pairs (C, ξ) consisting of a curve C in X and a global section ξ of a twist of the dualizing sheaf $\omega_C := \mathcal{E}xt^2(\mathcal{O}_C, \omega_X)$, with pairs (E, s) consisting of a rank 2 reflexive sheaf E and a global section $s \in H^0(E)$ whose cokernel is torsion-free. Another version of the Serre correspondence was given by Arrondo in [1, Theorem 1.1], including locally free sheaves of higher rank.

The main goal of this section is to consider a generalization of the Serre correspondence for torsion-free sheaves on projective varieties generalizing all of the results mentioned above.

Theorem 3. *Let X be a non-singular, projective variety and let L be a line bundle on X such that $H^1(L^\vee) = H^2(L^\vee) = 0$. There is a correspondence between*

- *Sets $(E, s_1, \dots, s_{r-1})$ consisting of a rank r torsion-free sheaf E with $\det(E) = L$, and global sections $s_1, \dots, s_{r-1} \in H^0(E)$ whose dependency locus has codimension at least 2;*
- *Sets $(C, \xi_1, \dots, \xi_{r-1})$ consisting of a codimension 2 subscheme $C \subset X$ and sections $\xi_1, \dots, \xi_{r-1} \in H^0(\omega_C \otimes \omega_X^{-1} \otimes L^{-1})$.*

Proof. Starting with a set $(E, s_1, \dots, s_{r-1})$ as described in the first item, we form a morphism

$$\sigma := (s_1, \dots, s_{r-1}) : \mathcal{O}_X^{\oplus(r-1)} \longrightarrow E;$$

the hypothesis on $(s_1, \dots, s_{r-1})$, which means that the common zeros of s_i have codimension at least 2, imply that σ is injective and $\mathrm{coker}(\sigma)$ is a torsion-free sheaf. It follows that $\mathrm{coker}(s) \simeq \mathcal{I}_C \otimes L$, where C is a (possibly empty) subscheme of codimension at least 2, which is precisely the dependency locus of $(s_1, \dots, s_{r-1})$, and L is a line bundle. Therefore, we obtain a short exact sequence of the form

$$0 \longrightarrow \mathcal{O}_X^{\oplus(r-1)} \xrightarrow{\sigma} E \longrightarrow \mathcal{I}_C \otimes L \longrightarrow 0; \tag{1}$$

in addition, this exact sequence defines an extension class

$$\xi \in \mathrm{Ext}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X^{\oplus(r-1)}).$$

Using the spectral sequence for local-to-global Ext

$$H^p(\mathcal{E}xt^q(\mathcal{I}_C, \mathcal{O}_X)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{I}_C, \mathcal{O}_X) \tag{2}$$

one checks that the hypothesis $H^1(L^\vee) = H^2(L^\vee) = 0$ yields the first of the following isomorphisms

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X) &\simeq H^0(\mathcal{E}xt^1(\mathcal{I}_C \otimes L, \mathcal{O}_X)) \\ &\simeq H^0(\mathcal{E}xt^2(\mathcal{O}_C, \omega_X) \otimes \omega_X^{-1} \otimes L^\vee). \end{aligned} \tag{3}$$

This means that the extension class ξ can be regarded as $r-1$ sections $\xi_1, \dots, \xi_{r-1} \in H^0(\omega_C \otimes \omega_X^{-1} \otimes L^{-1})$. We have thus obtained a set $(C, \xi_1, \dots, \xi_{r-1})$ as described in the second item.

Conversely, given a set $(C, \xi_1, \dots, \xi_{r-1})$ we can use the isomorphisms in display (3) to re-interpret $\xi_1, \dots, \xi_{r-1}$ as an extension class in $\mathrm{Ext}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X^{\oplus(r-1)})$ leading to an exact sequence as in display (1), which yields a set $(E, s_1, \dots, s_{r-1})$. □

In general, the abelian groups $\mathrm{Ext}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X)$ and $H^0(\mathcal{E}xt^1(\mathcal{I}_C \otimes L, \mathcal{O}_X))$ are related via the following exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(L^\vee) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X) \longrightarrow H^0(\mathcal{E}xt^1(\mathcal{I}_C \otimes L, \mathcal{O}_X)) \\ \longrightarrow H^2(L^\vee); \end{aligned}$$

here, we used the isomorphism $\mathcal{H}om(\mathcal{I}_C \otimes L, \mathcal{O}_X) \simeq L^\vee$. Therefore, if one only assumed that $H^2(L^\vee) = 0$, then every set $(C, \xi_1, \dots, \xi_{r-1})$ defines an extension class in $\mathrm{Ext}^1(\mathcal{I}_C \otimes L, \mathcal{O}_X^{\oplus(r-1)})$ and thus a torsion-free sheaf of rank r.

In this paper, we will only be concerned with threefolds, so we fix $\dim(X) = 3$ once and for all. Moreover, we will mostly consider only rank 2 sheaves.

Remark 4. Fix $r = 2$.

(1) E is reflexive if and only if the scheme C is locally Cohen–Macaulay (l.c.m.) and $\xi : \mathcal{O}_X \to \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_X) \otimes L^{-1}$ only vanishes on a 0-dimensional subscheme $Z \subset C$. In addition, Z coincides with the singular locus of E.
(2) E is locally free if and only if the scheme C is locally complete intersection (l.c.i.) and $\xi : \mathcal{O}_X \to \omega_C \otimes \omega_X^{-1} \otimes L^{-1}$ is non vanishing.

Detailed explanations for these claims can be found in the classical references [14, Theorem 4.1] and [13, Theorem 1.1], respectively.

When E is not reflexive, set $T_E := E^{\vee\vee}/E$ and consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \iota\circ s} & & & & \\
0 & \longrightarrow & E & \xrightarrow{\iota} & E^{\vee\vee} & \xrightarrow{q} & T_E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}_C \otimes L & \longrightarrow & G & \longrightarrow & T_E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{4}$$

where ι is the canonical embedding of E in its double-dual $E^{\vee\vee}$ (E is indeed assumed to be torsion free) $G := \mathrm{coker}(\iota(s))$ and $L = \det(E)$; we argue that G is torsion-free, so that $G \simeq \mathcal{I}_{C'} \otimes L$ for some l.c.m. curve $C' \subset C$.

Indeed, assume that G is not torsion-free and assume that $P \hookrightarrow G$ is the maximal torsion subsheaf, so that G/P is torsion-free; the exact sequence in the middle column implies that $\mathcal{E}xt^p(G, \mathcal{O}_X) = 0$ for $p > 1$, thus $\mathcal{E}xt^p(P, \mathcal{O}_X) = 0$ for $p > 1$ as well (since $\mathcal{E}xt^3(G/P, \mathcal{O}_X) = 0$); it follows that P and $\mathcal{E}xt^1(P, \mathcal{O}_X)$ must be both sheaves of pure dimension 2. On the other hand, since $\dim T_E \leq 1$, one can dualize the exact sequence in the bottom line and conclude that $G^\vee \simeq L^\vee$ and

$$0 \longrightarrow \mathcal{E}xt^1(G, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1(\mathcal{I}_C \otimes L, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^2(T_E, \mathcal{O}_X) \longrightarrow 0, \tag{5}$$

since $\mathcal{E}xt^p(G, \mathcal{O}_X) = 0$ for $p > 1$. This means that $\dim \mathcal{E}xt^1(G, \mathcal{O}_X) \leq 1$, which is impossible. If ever this was the case, we would indeed have an inclusion $\mathcal{E}xt^1(G/P, \mathcal{O}_X) \hookrightarrow \mathcal{E}xt^1(G, \mathcal{O}_X)$ whose cokernel is a subsheaf of $\mathcal{E}xt^1(P, \mathcal{O}_X)$ of dimension at most one; this is a contradiction since we proved that $\mathcal{E}xt^1(P, \mathcal{O}_X)$ has pure dimension 2. Therefore, G does not admit a torsion subsheaf.

In general, the quotient sheaf T_E is not pure dimensional. Our next result characterizes those torsion-free sheaves E for which T_E has pure dimension 1.

Lemma 5. *Assume that the pairs (E, s) and (C, ξ) correspond via the Serre correspondence outlined in Theorem 3. The scheme C is l.c.m. if and only if T_E is a pure 1-dimensional sheaf.*

Proof. Dualizing the bottom line of the diagram in display (4) yields the first of the following isomorphisms

$$\mathcal{E}xt^3(T_E, \mathcal{O}_X) \simeq \mathcal{E}xt^2(\mathcal{I}_C \otimes L, \mathcal{O}_X) \simeq \mathcal{E}xt^3(\mathcal{O}_C, \mathcal{O}_X);$$

the leftmost sheaf is 0-dimensional, so one can disregard the twist by the line bundle L. Therefore, T_E contains a 0-dimensional subsheaf if and only if $\mathcal{O}_C$ also does, meaning that C is not l.c.m. $\square$

Let $U \hookrightarrow \mathcal{O}_C$ be the maximal 0-dimensional subsheaf, so that $\mathcal{E}xt^3(\mathcal{O}_C, \mathcal{O}_X) \simeq \mathcal{E}xt^3(U, \mathcal{O}_X)$. As a by-product of the previous proof, we also conclude that $\mathcal{E}xt^3(T_E, \mathcal{O}_X) \simeq \mathcal{E}xt^3(U, \mathcal{O}_X)$. In other words, the 0-dimensional components of the support of T_E are always contained in the 0-dimensional components of C, regardless of the choice of section s.

4. Instanton sheaves on Fano threefolds

Let X be a Fano threefold of Picard rank one and of index i_X, following all the notation and definitions posed in Sect. 2.

The key point of the present paper is the introduction of the following definition, which generalizes Faenzi's ([11, Definition 1]) and Kuznetsov's ([23, Definition 1.1]) definitions of instanton bundles on a Fano threefold.

Definition 6. An *instanton sheaf* E on X is a μ-semistable sheaf with first Chern class $c_1(E) = -r_X \cdot \mathrm{rk}(E)/2$ and such that:

$$H^1(E(-q_X)) = H^2(E(-q_X)) = 0. \tag{6}$$

The *charge* of E is defined to be $c_2(E)$.

The μ-semistability condition rules out $\mathcal{O}_X(-r_X)^{\oplus r} \oplus \mathcal{O}_X^{\oplus r}$ as instanton sheaf when $r_X = 1$ (i.e. when i_X is odd); however, $\mathcal{O}_X^{\oplus r}$ is considered an instanton sheaf when $r_X = 0$ (i.e. when i_X is even).

Remark 7. When $X = \mathbb{P}^3$ this definition is, in general, a bit more restrictive than the definition of instanton sheaves adopted in [12,18,20,22]; in these references, an instanton sheaf on $\mathbb{P}^3$ was defined as a torsion-free sheaf E with $c_1(E) = 0$ and

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

Using this definition, one can find examples of instanton sheaves of rank 4 and larger that are not μ-semistable, see [18, Example 3]. However, both definitions are equivalent when $\mathrm{rk}(E) = 2$, since every rank 2 sheaf on $\mathbb{P}^3$ satisfying the conditions above is automatically μ-semistable.

The following technical results will be useful later on.

Lemma 8. *Let E be an instanton sheaf of rank r.*

(1) $H^0(E(-n)) = 0,\ \forall\, n \geq 1 - r_X$ *and* $H^3(E(n)) = 0,\ \forall\, n \geq -i_X + 1$.
(2) $H^i(E(-q_X)) = 0,\ \mathrm{Ext}^i(E, \mathcal{O}_X(-q_X - r_X)) = 0,\ \forall\, i$.

In particular, we conclude that $\chi(E(-q_X)) = 0$.

Proof. Since E is a rank r μ-semistable sheaf, $\mathrm{Hom}(\mathcal{O}_X(n), E) = H^0(E(-n)) = 0$ whenever $n > \frac{-r_X}{2}$ and since $r_X = 0$ or 1, this happens if and only if $n \geq 1 - r_X$. An equivalent argument leads to $\mathrm{Hom}(E, \mathcal{O}_X(n)) = 0$ whenever $n < \frac{-r_X}{2}$ so that, by duality, $\mathrm{Ext}^3(\mathcal{O}_X(n), E) \simeq H^3(E(-n)) = 0$ for $n < i_X - \frac{r_X}{2}$ that is to say, whenever $n \leq i_X - 1$. Since $1 - r_X \leq q_X \leq i_X - 1$ we get $H^i(E(-q_X)) = 0$, for $i = 0, 3$ and this together with (6) leads to $H^i(E(-q_X)) = 0, \ \forall\, i$. By Serre duality, we have isomorphisms

$$H^i(E(-q_X)) \simeq \mathrm{Ext}^{3-i}(E(-q_X), \omega_X))^* \simeq \mathrm{Ext}^{3-i}(E, \mathcal{O}_X(-q_X - r_X))^*,$$

therefore $\mathrm{Ext}^i(E, \mathcal{O}_X(-q_X - r_X)) = 0, \ \forall\, i$. □

From these computations, we determine the value of the third Chern class of instanton sheaves.

Corollary 9. *Let E be an instanton sheaf of rank r.*

(1) *If* $i_X > 1, \ \chi(E(-q_X)) = \frac{c_3(E)}{2} = 0$, *hence* $c_3(E) = 0$.
(2) *If* $i_X = 1, \ \chi(E(-q_X)) = \chi(E) = (r-2) + \frac{c_3(E)}{2} = 0$ *hence* $c_3(E) = 2(2-r)$.

Proof. $\chi(E(-q_X)) = 0$ by Lemma 8; by Grothendieck–Riemann–Roch theorem we compute $\chi(E(-q_X)) = \frac{c_3(E)}{2}$ whenever $i_X > 1$ so that $c_3(E) = 0$ and $\chi(E) = (r-2) + \frac{c_3(E)}{2}$ for $i_X = 1$ so that $c_3(E) = 2(2-r)$. □

The main motivation behind Definition 6 is that non locally free instanton sheaves naturally arise as degenerations of locally free ones. When $X = \mathbb{P}^3$, this phenomenon has been studied in [21,22]. To see it in greater generality, let us consider some explicit examples of rank 2 instanton sheaves.

Let $C := L_1 \sqcup \cdots \sqcup L_n$ be the disjoint union of lines in X, and set $G := \mathcal{I}_C(q_X - 1)$; note that for $p = 1, 2$, we have

$$H^p(G(-q_X)) = H^p(\mathcal{I}_C(-1)) \simeq \bigoplus_{k=1}^{n} H^{p-1}(\mathcal{O}_{L_k}(-1)) = 0.$$

We can then consider extensions of the form

$$0 \longrightarrow \mathcal{O}_X(-q_X - r_X + 1) \longrightarrow E \longrightarrow \mathcal{I}_C(q_X - 1) \longrightarrow 0; \tag{7}$$

clearly, $c_1(E) = -r_X$ and one easily checks that $H^p(E(-q_X)) = 0$ for $p = 1, 2$. Applying Lemma 1, we verify that E is μ-semistable when $i_X \geq 2$; however, such sheaves are always properly μ-semistable when $i_X = 2$ and are not μ-semistable when $i_X = 1$. Therefore, E is a rank 2 instanton sheaf provided $i_X \geq 2$. Inspired by the traditional nomenclature for $X = \mathbb{P}^3$, instanton sheaves given by an extension as in display (7) are called *'t Hooft instantons*; the charge of a 't Hooft instanton sheaf corresponding to n lines is $n - 1$

Example 10. Here is an example of a family of rank 2 locally free instanton sheaves degenerating into a non locally free one. Assume that $i_X \geq 2$, and let C be a disjoint union of $n \geq 2$ lines in X, as above.

Since

$$\mathrm{Ext}^1(\mathcal{I}_C(q_X - 1), \mathcal{O}_X(-q_X - r_X + 1)) = \mathrm{Ext}^1(\mathcal{I}_C, \omega_X(2)) \simeq H^0(\omega_C(2))$$

$$= \bigoplus_{k=1}^{n} H^0(\mathcal{O}_{L_k}),$$

we can consider extension classes $\xi_t = (1, \ldots, 1, t)$ with $t \in \mathbb{C}$, inducing a family of instanton sheaves E_t, parametrized by $\mathbb{C}$.

Note that E_t is locally free when $t \neq 0$ since ξ_t is nonvanishing in this case. On the other hand, ξ_0 vanishes along L_n, so the corresponding 't Hooft instanton sheaf E_0 is not locally free. Note that E_0 satisfies the following short exact sequence

$$0 \longrightarrow E_0 \longrightarrow F \longrightarrow \mathcal{O}_L(q_X - 1) \longrightarrow 0,$$

where F is a locally free 't Hooft instanton sheaf of charge $n - 1$.

When i_X is even, instanton sheaves of rank larger than 2 can easily be produced using rank 2 instantons and ideal sheaves, via the following claim.

Lemma 11. *Assume that $i_X = 2, 4$, so that $r_X = 0$. If E_1 and E_2 are instanton sheaves, then any extension of E_1 by E_2 is also an instanton sheaf.*

Proof. If E is an extension of E_1 by E_2, then it is easy to check that E satisfies the cohomological conditions in Definition 6. Since E_1 and E_2 are μ-semistable sheaves with vanishing slopes, then so is E. □

Next, we consider the generalization of a definition first introduced in [15, Definition 6.1] for $X = \mathbb{P}^3$, and further studied in [12,20].

Definition 12. A *rank 0 instanton sheaf* on a Fano threefold X is a pure 1-dimensional sheaf T satisfying $h^0(T(-q_X)) = h^1(T(-q_X)) = 0$.

If T is a rank 0 instanton sheaf on X, then $\chi(T(t)) = d \cdot (t + q_X)$, and the coefficient d is called the *degree* of T. Moreover, Lemma 2 implies that T is always semistable.

Proposition 13. *If T is a rank 0 instanton sheaf, then so is $T^{\mathrm{D}} \otimes \omega_X^{-1}(-r_X)$, where $T^{\mathrm{D}} := \mathcal{E}xt^2(T, \omega_X)$.*

Proof. Note first that T^{D} is a pure 1-dimensional sheaf and, by definition,

$$H^p(T^{\mathrm{D}} \otimes \omega_X^{-1}(-r_X - q_X)) = H^p(\mathcal{E}xt^2(T, \mathcal{O}_X(-q_X - r_X))), \quad p = 0, 1.$$

Using the spectral sequence (2) for local to global Exts, we check that

$$H^0(\mathcal{E}xt^2(T, \mathcal{O}_X(-q_X - r_X))) \simeq \mathrm{Ext}^2(T, \mathcal{O}_X(-q_X - r_X)) \text{ and}$$

$$H^1(\mathcal{E}xt^2(T, \mathcal{O}_X(-q_X - r_X))) \oplus H^0(\mathcal{E}xt^3(T, \mathcal{O}_X(-q_X - r_X))) \\ \simeq \mathrm{Ext}^3(T, \mathcal{O}_X(-q_X - r_X)).$$

Serre duality yields the isomorphisms (for $p = 0, 1$)

$$\mathrm{Ext}^{p+2}(T, \mathcal{O}_X(-q_X - r_X)) \simeq H^{1-p}(T(-q_X))^*$$

and the latter vanishes by the instantonic condition on T. □

Lemma 14. *If T is a rank 0 instanton sheaf, then $h^0(T(-q_X - n)) = 0$ and $h^1(T(-q_X + n)) = 0$ for every $n \geq 0$.*

Proof. Given a rank 0 instanton sheaf T, let $S \subset X$ be a hyperplane section transversal to the support of T (i.e. $\dim(\mathrm{Supp}(T) \cap S) = 0$), so that $\mathcal{T}or^1(T, \mathcal{O}_S) = 0$. This implies that we can twist the exact sequence $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$ by $T(k)$ to obtain the short exact sequence

$$0 \longrightarrow T(k-1) \longrightarrow T(k) \longrightarrow T \otimes \mathcal{O}_S(k) \longrightarrow 0.$$

Taking cohomology, we conclude that $h^0(T(k-1)) = 0$ whenever $h^0(T(k)) = 0$, while $h^1(T(k)) = 0$ whenever $h^1(T(k-1)) = 0$, since $\dim(T \otimes \mathcal{O}_S) = 0$. The desired claims follow by induction. □

We are now interested in detecting when a locally Cohen-Macaulay curve C is such that the structure sheaf $\mathcal{O}_C$ is a rank 0 instanton. We refer to a curve C of such a kind as an *instanton curve*.

Lemma 15. *Let X be a Fano threefold of Picard rank one.*

(1) *There are no instanton curves when $i_X = 1, 4$.*
(2) *When $i_X = 2, 3$, every instanton curve C of degree d fits in a short exact sequence of the form*

$$0 \longrightarrow \mathcal{O}_l \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C'} \longrightarrow 0 \tag{8}$$

where $l \subset X$ is a line and C' is an instanton curve of degree $d - 1$.

Proof. The fact that there are no instanton curves on a Fano threefold X of index $i_X = 1$ is simply because we can not have a projective algebraic curve $C \subset X$ such that $H^0(\mathcal{O}_C) = 0$ (the restriction map $H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_C)$ is necessarily an injection).

Let us now treat the cases $q_X > 0$. Consider an instanton curve $C \subset X$ of degree d; if C is reduced, then

$$h^0(\mathcal{O}_C) = \chi(\mathcal{O}_C) + h^1(\mathcal{O}_C) \geq d \cdot q_X,$$

thus C has at least $d \cdot q_X$ connected components.

If $q_X = 2$ (i.e. $X = \mathbb{P}^3$), then this is impossible since C can have at most d connected components.

If $q_X = 1$, then C is a reduced curve of degree d with at least d connected components; the only possibility is for C to be a disjoint union of lines which ensures that $\mathcal{O}_C$ fits in a short exact sequence of the form (8).

If C is not reduced, let C_{red} be its reduction. This latter is a l.c.m. curve satisfying $h^1(\mathcal{O}_{C_{\mathrm{red}}}(-q_X)) = 0$ and $h^0(\mathcal{O}_{C_{\mathrm{red}}}(-q_X)) = 0$ (since $q_X > 0$ and C_{red} is reduced and l.c.m.): in other words C_{red} is an instanton curve.

In particular, there are no instanton curves C for $i_X = 1, 4$ and for $i_X = 2, 3$, C_{red} is a disjoint union of $d' < d$ lines. It follows that

$$\mathcal{O}_C \simeq \bigoplus_{j=1}^{d'} \mathcal{O}_{C_j}$$

where $\ell_j := (C_j)_{\mathrm{red}}$ are disjoint lines, and $H^i(\mathcal{O}_{C_j}(-1)),\ i = 0, 1$. To prove our claim it is therefore enough to show that each $\mathcal{O}_{C_j}$ fits in a short exact sequence of the form (8); this is obtained applying Lemma 16 below. □

Lemma 16. *Let C be a multiple structure of degree d on a line $\ell \subset X$. Suppose that $H^i(\mathcal{O}_C(-1)) = 0$ for $i = 0, 1$. Then C fits in a short exact sequence of the form*

$$0 \to \mathcal{O}_\ell \to \mathcal{O}_C \to \mathcal{O}_{C'} \to 0$$

where C' is a multiple structure of degree $d - 1$ on ℓ such that $H^i(\mathcal{O}_{C'}(-1)) = 0$ for $i = 0, 1$.

Proof. According to [5], a curve C satisfying the hypotheses above admits a filtration:

$$l = C_1 \subset C_2 \subset \ldots \subset C_m = C \tag{9}$$

where each C_j is a multiple structure on ℓ whose sheaf of ideals $\mathcal{I}_{C_j}$ is the kernel of a surjection $\mathcal{I}_{C_{j-1}} \twoheadrightarrow L_{j-1}$ with L_{j-1} a vector bundle on ℓ. Furthermore there exist induced generically surjective morphisms $L_i \otimes L_j \to L_{i+j}$ for each $i, j \in \{1, \ldots m\}$ and, in particular, generically surjective maps $L_1^{\otimes j} \to L_j$. Note that since in our case $C_1 = C_{\mathrm{red}}$ is the line ℓ, each vector bundle L_j splits as $L_j = \bigoplus_{i=1}^{k_j} \mathcal{O}_\ell(a_j^i),\ k_j \in \{1, 2\}$ (as l has codimension 2). Therefore, $C = C_m$ fits into a sort exact sequence of the form

$$0 \to L_{m-1} \to \mathcal{O}_C \to \mathcal{O}_{C_{m-1}} \to 0;$$

in order to prove the lemma it is therefore sufficient to show that each summand of L_{m-1} has degree 0.

Consider the second step C_2 of the filtration (9). This must satisfy $H^1(\mathcal{O}_{C_2}(-1)) = 0$ so that $a_1^i \geq 0$ for $1 \leq i \leq k_1$. Since we have a generically surjective morphism $L_1^{\otimes m-1} \to L_{m-1}$ we deduce that $a_{m-1}^i \geq 0,\ 1 \leq i \leq k_{m-1}$; but as $H^0(\mathcal{O}_C(-1)) = 0,\ H^0(L_{m-1}(-1)) = 0$ we have $a_{m-1}^i \leq 0,\ 1 \leq i \leq k_{m-1}$. The only possibility is thus $a_{m-1}^i = 0$ for $1 \leq i \leq k_{m-1}$. □

From now on the l.c.m. curve of degree d constructed "inductively" via the short exact sequences of the form (8), will be referred to as *degree d line arrangements.*

Remark 17. From the proof of Lemma 15 we learn that the degree d lines arrangements are the only degree d l.c.m. curves C such that $\chi(\mathcal{O}_C) = d$. In particular we notice that for a degree d l.c.m. curve C, since $\chi(\mathcal{O}_C(-1)) = -h^1(\mathcal{O}_C(-1)) = -d + \chi(\mathcal{O}_C) \leq 0$, we always have $\chi(\mathcal{O}_C) \leq d$ and equality holds if and only if C is a degree d line arrangement.

One of the main reasons that justify our interest in rank 0 instanton sheaves is that these sheaves play a primary role in the study of non-reflexive instanton sheaves. It is indeed possible to construct instanton sheaves out of non-reflexive ones performing an *elementary transformation* along a rank 0 instanton sheaf.

We recall that the elementary transformation consists of the following procedure: let F be a reflexive instanton sheaf, let T be a rank 0 instanton sheaf, and consider an epimorphism $q : F \twoheadrightarrow T$. It is easy to check that $E := \ker q$ is also an instanton sheaf. Indeed, consider the exact sequence

$$0 \longrightarrow E \longrightarrow F \xrightarrow{q} T \longrightarrow 0; \tag{10}$$

E is μ-semistable because F is μ-semistable and $\mu(F) = \mu(E)$; the exact sequence in cohomology (here, $p = 1, 2$)

$$H^{p-1}(T(-q_X)) \longrightarrow H^p(E(-q_X)) \longrightarrow H^p(F(-q_X))$$

implies that $h^p(E(-q_X)) = 0$ ($p = 1, 2$) since T and F are instanton sheaves.

It is almost immediate to prove that whenever E is the elementary transformation of F along T, then the following holds

Lemma 18. *Let E be an instanton sheaf obtained by elementary transformation of a reflexive instanton F along a rank 0 instanton T. Then $F \simeq E^{\vee\vee}$.*

Proof. Applying the functor $\mathcal{H}om(\,\cdot\,, \mathcal{O}_X)$ to (10), we get $E^\vee \simeq F^\vee$ (since T is one-dimensional) hence $E^{\vee\vee} \simeq F^{\vee\vee} \simeq F$. □

As it turns out not all the non-reflexive instanton sheaves are necessarily obtained in this way. We can indeed prove that for a non reflexive instanton sheaf E, $E^{\vee\vee}/E$ is always purely one-dimensional but not necessarily a rank 0 instanton sheaf; accordingly $E^{\vee\vee}$ is not necessarily an instanton sheaf either.

Proposition 19. *Let E be a non reflexive instanton sheaf of rank $r > 0$. Then the following hold:*

(1) *$T_E := E^{\vee\vee}/E$ has pure dimension one;*
(2) *E has homological dimension 1;*
(3) *$H^p(E^{\vee\vee}(-q_X)) = 0$, $p = 0, 2, 3$*
(4) *$E^{\vee\vee}$ is an instanton if and only if T is a rank 0 instanton sheaf. This condition is equivalent to $c_3(E^{\vee\vee}) = 0$ for $i_X > 1$ and $c_3(E^{\vee\vee}) = 2(2-r)$ for $i_X = 1$.*

Proof. Since E is torsion-free, it injects in its double dual, leading to a short exact sequence:

$$0 \to E \to E^{\vee\vee} \to T_E \to 0 \tag{11}$$

where $T_E := E^{\vee\vee}/E$ is a torsion sheaf supported on the locus of points where E fails to be reflexive. Applying the functor $\mathcal{H}om(\,\cdot\,, \mathcal{O}_X)$ to (11) we obtain an exact sequence $0 \to E^\vee \to E^{\vee\vee} \to \mathcal{E}xt^1(T_E, \mathcal{O}_X) \to \mathcal{E}xt^1(E^{\vee\vee}, \mathcal{O}_X)$ from which we deduce that $\dim(T_E) \le 1$. Indeed, since $E^\vee$ is reflexive, $E^\vee \simeq E^{\vee\vee}$ which implies that $\mathcal{E}xt^1(T_E, \mathcal{O}_X)$ injects in $\mathcal{E}xt^1(E^{\vee\vee}, \mathcal{O}_X)$; but this can clearly not happen if ever T_E had dimension > 1. If ever this was the case, $\mathcal{E}xt^1(T_E, \mathcal{O}_X)$ would have dimension bigger than one as well, leading to a contradiction since $\mathcal{E}xt^1(E^{\vee\vee}, \mathcal{O}_X)$

is zero-dimensional due to the reflexivity of $E^{\vee\vee}$. Twisting now (11) by $\mathcal{O}_X(-q_X)$ and taking cohomology we get an exact sequence:

$$H^0(E^{\vee\vee}(-q_X)) \to H^0(T_E(-q_X)) \to H^1(E(-q_X)).$$

The left side term vanishes since $E^{\vee\vee}$ is μ-semistable, the right side term vanishes since E is an instanton; accordingly $H^0(T_E(-q_X)) = 0$ thus $H^0(T_E(-n)) = 0$, $\forall n \geq q_X$ (apply 14) which allows us to conclude that T_E has pure dimension one ending the proof of (1). As a consequence of (1) we get that $\mathcal{E}xt^2(E, \mathcal{O}_X) \simeq \mathcal{E}xt^3(T_E, \mathcal{O}_X) = 0$ and since moreover $0 = \mathcal{E}xt^3(E^{\vee\vee}, \mathcal{O}_X)$ surjects onto $\mathcal{E}xt^3(E, \mathcal{O}_X)$, we can conclude that E has homological dimension 1, proving (2). The long exact sequence in cohomology from (11) twisted by $\mathcal{O}_X(-q_X)$ now leads to

$$H^i(E(-q_X)) \simeq H^i(E^{\vee\vee}(-q_X)) = 0,\ i = 2, 3, \quad H^1(E^{\vee\vee}(-q_X)) \\ = H^1(T_E(-q_X)).$$

These equalities lead to (3) (as we have already pointed out that $H^0(E^{\vee\vee}(-q_X))$ vanishes by μ-semistability) and ensure that $E^{\vee\vee}$ is an instanton if and only if T_E is a rank 0 instanton. Finally, we compute that $\chi(T_E(-q_X)) = \frac{c_3(E^{\vee\vee})}{2}$ for $i_X > 1$ (resp. $\chi(T_E) = \frac{c_3(E^{\vee\vee})}{2} + r - 2$ for $i_X = 1$) which holds if and only if $c_3(E^{\vee\vee}) = 0$ (resp. if and only if $c_3(E^{\vee\vee}) = 2(2 - r)$). □

4.1. Classification of rank 1 instanton sheaves

According to our definition, rank 1 instanton sheaves can only occur when $r_X = 0$, so i_X is even. Since $\mathrm{Pic}(X) = \mathbb{Z}$, a locally free (or equivalently reflexive) instanton sheaf of rank one is uniquely determined by its first Chern class; therefore, the only instanton line bundle is $\mathcal{O}_X$. Let us now consider the non locally free case.

Lemma 20. *Let L be a non locally free instanton sheaf of rank one. Then $L^{\vee\vee} \simeq \mathcal{O}_X$ and $L^{\vee\vee}/L$ is a rank 0 instanton sheaf.*

Proof. Applying Proposition 19 (recall that in rank one reflexivity is equivalent to local freeness) we have that L always fits in a short exact sequence of the form:

$$0 \to L \to L^{\vee\vee} \to T \to 0 \tag{12}$$

with T being a torsion sheaf of pure dimension one.Accordingly, $L^{\vee\vee}$ is a rank one reflexive sheaf with $c_1(L^{\vee\vee}) = c_1(L) = 0$, that is to say $L^{\vee\vee} \simeq \mathcal{O}_X$. Since $\mathcal{O}_X$ is an instanton, one can easily check that T is a rank 0 instanton sheaf. □

We, therefore, understand that the classification of rank 1 instanton sheaves reduces to the classification of rank 0 instanton sheaves T admitting an epimorphism $\mathcal{O}_X \twoheadrightarrow T$.

Proposition 21. *Let L be a rank one instanton sheaf of charge d on a Fano threefold X with Picard rank one. The following hold:*

(1) *if* $i_X = 4$, *then* $d = 0$ *and* $L \simeq \mathcal{O}_X$;
(2) *if* $i_X = 2$, *we have* $L \simeq \mathcal{O}_X$ *whenever* $d = 0$ *whilst for* $d > 0$ L *always fits in a short exact sequence of the form:*

$$0 \to L \to L' \to \mathcal{O}_\ell \to 0$$

for a line $\ell \subset X$ *and* L' *a rank one instanton sheaf of charge* $d - 1$.

Proof. By Lemma 20, the classification of rank one instanton sheaves L of charge d, reduces to the classification of degree d rank 0 instanton sheaves T admitting a surjection $\mathcal{O}_X \twoheadrightarrow T$, that is to say, to the classification of degree d l.c.m. curves $C \subset X$ such that $\mathcal{O}_C$ is a rank 0 instanton. But this means that $H^i(\mathcal{O}_C(-2)) = 0$, $i = 0, 1$ for $i_X = 4$ and $H^i(\mathcal{O}_C(-1))$, $i = 0, 1$ for $i_X = 2$. The arguments used to prove Lemma 15 (i) show then that there are no rank 1 non locally free instantons of rank one on Fano threefolds of index 4. Similarly, from Lemma 15 (2) and Remark 17, we know that the only degree d l.c.m. curves such that $H^i(\mathcal{O}_C(-1)) = 0$ are the degree d lines arrangements. This proves the point (2). □

Remark 22. Proposition 21 can be rephrased by saying that each rank one instanton L of charge $d > 0$, on a Fano threefold X of index $i_X = 2$, is always isomorphic to $\mathcal{I}_C$ for C a degree d line arrangement. Recall that a curve C of such a kind is supported by $d' \leq d$ disjoint lines and can be constructed "inductively" from an extension:

$$0 \to \mathcal{O}_\ell \to \mathcal{O}_C \to \mathcal{O}'_C \to 0$$

with C' a degree $d - 1$ line arrangement. As a consequence, for $i_X = 2$, rank one instantons of strictly positive charge coincides with ideal sheaves of instanton curves.

4.2. Classification of rank 2 instanton sheaves

In general, the double dual $E^{\vee\vee}$ of a torsion-free sheaf E is a reflexive (possibly non locally free) sheaf; if E is an instanton sheaf, we have that $c_1(E^{\vee\vee}) = -r_X$ and that $E^{\vee\vee}$ is μ-semistable, but $E^{\vee\vee}$ may not satisfy the instantonic vanishing conditions.

The main result of this section guarantees that $E^{\vee\vee}$ is a locally free instanton sheaf when $\mathrm{rk}(E) = 2$. Recall that, since X has Picard rank one, we have in this case an isomorphism:

$$E^{\vee\vee} \simeq E^\vee(-r_X). \tag{13}$$

In particular, this implies that Serre's duality establishes isomorphisms:

$$H^i(E^{\vee\vee}(n)) \simeq H^{3-i}(E^\vee(-n - i_X))^* \simeq H^{3-i}(E^{\vee\vee}(-2q_X - n))^*, \; i = 0, 3. \tag{14}$$

When X is a Fano threefold, Lemma 1 says that a normalized rank 2 torsion-free sheaf E on X is μ-semistable if and only if $h^3(E(-i_X + 1)) = 0$. In addition, note that μ-semistability implies that $h^0(E(r_X-1))=0$, when we assume that $c_1(E)=-r_X$.

Let us now focus on rank 2 instantons. To begin with, we show that, in the rank 2 case, the reflexivity of an instanton implies its local freeness.

Lemma 23. *Let E be a rank 2 reflexive instanton. Then E is locally free.*

Proof. Applying Corollary 9, $\chi(E(-q_X))=0$ leads to $c_3(E)=0$. The proof of [14, Proposition 2.6] applies verbatim to arbitrary Fano threefolds of Picard rank one which allows us to conclude that $c_3(E)=0$ if and only if E is locally free. □

We are finally ready to prove our classification of non locally free rank 2 instanton sheaves.

Theorem 24. *Let E be a rank 2 instanton sheaf. Then $E^{\vee\vee}$ is an instanton bundle and $T_E:=E^{\vee\vee}/E$ is a rank 0 instanton sheaf whenever $T_E\neq 0$.*

Proof. By Propositions 19 and 23 it is enough to prove that for a non locally free instanton E, $h^1(E^{\vee\vee}(-q_X))=0$. Consider the local-to-global spectral sequence

$$E_2^{p,q}=H^p(\mathcal{E}xt^q(E^{\vee\vee},\mathcal{O}_X(-q_X-r_X)))\Rightarrow \operatorname{Ext}^{p+q}(E^{\vee\vee},\mathcal{O}_X(-q_X-r_X)).$$

$\mathcal{H}om(E^{\vee\vee},\mathcal{O}_X(-q_X-r_X))\simeq E^{\vee\vee}(-q_X)$ hence $E^{p,0}=0$ for $p\neq 1$ which implies that the spectral sequence already degenerates at the $r=2$ sheet. Therefore

$$\begin{aligned}&\operatorname{Ext}^1(E^{\vee\vee},\mathcal{O}_X(-q_X-r_X))\\ &\quad\simeq H^0(\mathcal{E}xt^1(E^{\vee\vee},\mathcal{O}_X(-q_X-r_X)))\oplus H^1(E^{\vee\vee}(-q_X));\end{aligned}$$

since $\operatorname{Ext}^1(E^{\vee\vee},\mathcal{O}_X(-r_X-q_X))\simeq H^2(E^{\vee\vee}(-q_X))^*=0$, we then deduce that $H^0(\mathcal{E}xt^1(E^{\vee\vee},\mathcal{O}_X(-q_X-r_X)))=0$ and $H^1((E^{\vee\vee}(-q_X))=0$. This ensures that $E^{\vee\vee}$ is an instanton bundle. □

Remark 25. Theorem 24 generalizes [9, Proposition 3.1, Theorem 3.5] and [34, Theorem 1.2], [35, Theorem 1.3].

Remark 26. It is worth pointing out that if in the definition of instanton, we replace μ-semistability with a weaker cohomological condition, Theorem 24 no longer holds. We can prove this with the following counterexample. Let us consider a Fano threefold X of index $i_X=1$ and a sheaf $E\in Coh(X)$ defined by:

$$0\to E\to \mathcal{O}_X\oplus\mathcal{O}_X(-1)\to\mathcal{O}_p\to 0$$

with $p\in X$ a point. E is a μ-unstable rank 2 torsion free sheaf (it is destabilized by $\mathcal{I}_p$) such that $H^i(E)=0$ for $0\le i\le 2$, but $E^{\vee\vee}/E\simeq\mathcal{O}_p$.

Corollary 27. *Let E be a rank 2 non-locally free instanton. Then the sheaf $S_E:=\mathcal{E}xt^1(E,\mathcal{O}_X(-r_X))$ is a rank 0 instanton.*

Proof. Whenever E is not locally free, it fits in an exact sequence of the form (11) and since $E^{\vee\vee}$ is locally free, we get an isomorphism:

$$\mathcal{E}xt^1(E, \mathcal{O}_X(-r_X)) \simeq \mathcal{E}xt^2(T_E, \mathcal{O}_X(-r_X)).$$

Since T_E is a rank 0 instanton, by Lemma 13, $T_E^{\mathrm{D}} \otimes \omega_X^{-1}(-r_X)$ is a rank 0 instanton as well. The statement of the corollary then follows from the fact that we have an isomorphism $\mathcal{E}xt^2(T_E, \mathcal{O}_X(-r_X)) \simeq T_E^{\mathrm{D}} \otimes \omega_X^{-1}(-r_X)$. □

Remark 28. If E is a rank 2 non locally free instanton, we then have an equality $\mathrm{Supp}(T_E) = \mathrm{Sing}(E)$ for $T_E := E^{\vee\vee}/E$. Note that this might not hold for arbitrary rank since a priori $\mathrm{Supp}(T_E) \subset \mathrm{Sing}(E)$ and equality holds if and only if $E^{\vee\vee}$ is locally free.

Summing up these last results, we can affirm that in rank 2 an instanton sheaf E is either locally free or $\mathrm{Sing}(E)$ has pure dimension one, and E is obtained by elementary transformation of an instanton bundle along a rank 0 instanton supported on $\mathrm{Sing}(E)$. Elementary transformation of rank 2 instanton bundles has been widely used in [22] to construct and study families of non-locally free instantons on $X = \mathbb{P}^3$. Most importantly for the present paper, Faenzi made a very interesting use of this construction in [11]: in loc. cit. elementary transformation is indeed used to prove the existence of rank 2 instanton bundle of charge k, $\forall\, k \geq 2$ on Fano threefolds of index 2. Mimicking this approach, we can state the following theorem:

Theorem 29. *Let X be a Fano threefold of index i_X. Assume that F is a rank 2 locally free instanton sheaf, and $\ell \subset X$ is a line such that the following hypotheses hold:*

- *F is unobstructed, i.e.* $\mathrm{Ext}^2(F, F) = 0$*;*
- $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_\ell(q_X - 1) \oplus \mathcal{O}_\ell(i_X - q_X - 1)$ *and* $F|_\ell \simeq \mathcal{O}_\ell \oplus \mathcal{O}_\ell(-r_X)$.

Then X admits rank 2 locally free instanton sheaves of charge k for every $k \geq c_2(F)$.

Proof. The induction argument presented in [11, Theorem D] applies to any Fano threefold X carrying an instanton bundle F and a line ℓ satisfying the hypotheses of the theorem. We summarize here the main steps of the proof. For any pair (F, ℓ) as above, we get the existence of an epimorphism $\phi : F \twoheadrightarrow \mathcal{O}_\ell(q_X - 1)$ whose kernel E is a non locally free instanton (as $\mathcal{O}_\ell(q_X - 1)$ is a rank 0 instanton) of charge $c_2(F) + 1$. One then proves that the assumptions made on (F, ℓ) ensure then that $\mathrm{Ext}^2(E, E) = 0$ and that a general deformation of E is an instanton bundle. This is done by showing at first that a general non-locally free deformation of E is still an instanton singular along a line and obtained from a deformation of (ϕ, F, ℓ). We then compute that the family of instantons whose singular locus is a line has dimensions strictly less than $\mathrm{ext}^1(E, E)$. As a consequence, E deforms to a locally free sheaf, and by semicontinuity, a general deformation is a non-obstructed instanton bundle. By induction, we, therefore, get the existence of an unobstructed instanton bundle of rank 2 for each charge $\geq c_2(F)$. □

The theorem always applies on Fano threefold of index $i_X \geq 2$.

- On $X \simeq \mathbb{P}^3$ a t'Hooft instanton of charge 1 and a general line lead to the existence of rank 2 instantons of charge k for each $k \geq 1$. The same result is proved, with different techniques in [22].
- For $i_X = 3$ the spinor bundle and any line $\ell \subset X$ ensure the existence of instanton bundles of charge $k \geq 1$.
- The case $i_X = 2$ was treated in [11] where the existence of instanton bundles of charge $k \geq 2$ is proved.

Fano threefolds of index one for which the theorem applies are treated in [6, Theorem 3.7].

We end the section characterizing the dimensions of the intermediate cohomology groups $H^i(E(n))$, $i = 1, 2$ of rank 2 instantons (the groups $H^i(E(n))$ for $i = 0, 3$ where studied in Lemma 8).

Lemma 30. *Let E be a rank 2 instanton and let m_0 denote the smallest integer such that $m_0 H$ is very ample. Then the following hold:*

(1) $H^1(E(-q_X - n)) = 0$ *for* $n = m_0$ *and* $\forall n \geq 2m_0$
(2) $H^2(E(-q_X + n)) = 0$ *for* $n = m_0$ *and* $\forall n \geq 2m_0$

Proof. Let D be a general element in the linear system $|m_0 H|$. By generality assumption $E|_D$ is μ-semistable (see [27]), therefore $H^0(E|_D(-q_X)) = 0$, and $H^2(E|_D(-q_X + m_0)) \simeq Hom(E|_D, \mathcal{O}_D(-q_X - r_X))^* = 0$. Taking cohomology in the short exact sequence:

$$0 \to E(-q_X - m_0) \to E(-q_X) \to E|_D(-q_X) \to 0 \tag{15}$$

we therefore get that $H^1(E(-q_X - m_0)) = 0$; twisting then (15) by $\mathcal{O}_X(m_0)$ and taking cohomology, we obtain $H^2(E(-q_X + m_0))$. These arguments apply to the letter to any general divisor $D \in |nH|$, $n \geq 2m_0$ since under these assumptions D is very ample and $E|_D$ is μ-semistable. □

4.3. Stability of rank 2 instanton sheaves

Clearly, when i_X is odd, every rank 2 instanton sheaf E on X is μ-stable, simply because μ-stability coincides with μ-semistability when $c_1(E) = -1$.

In [20], the authors showed that every nontrivial rank 2 instanton sheaf on $X = \mathbb{P}^3$ is stable, so that $\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$ is the only properly semistable (meaning semistable but not stable) rank 2 instanton sheaf. In addition, a rank 2 instanton sheaf is properly μ-semistable only when $E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$.

The situation is quite different for Fano threefolds of index 2. Indeed, let E be a properly μ-semistable rank 2 instanton sheaf on a Fano threefold X with $i_X = 2$. When E is locally free, this is equivalent to say that $h^0(E) > 0$; choosing a non-trivial section $s \in H^0(E)$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{I}_C \longrightarrow 0, \tag{16}$$

and it is easy to check that $\mathcal{I}_C$ is a rank one instanton, meaning that C is an instanton curve; recall that the latter have been classified in Lemma 15. Summing up, we proved the following claim.

Lemma 31. *Let E be a rank 2 locally free instanton sheaf on a Fano threefold X of index $i_X = 2$. If E is properly μ-semistable, then E fits into an exact sequence as in display (16) where C is an instanton curve. In particular, such sheaves are not semistable.*

When E is not locally free, we have that $E^{\vee\vee}$ is a properly μ-semistable locally free instanton sheaf of rank 2; taking the unique (up to scalar multiple) nontrivial section $s \in H^0(E^{\vee\vee})$, we get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_{C_1} & \longrightarrow & \mathcal{O}_X & \overset{\phi}{\searrow} & & & \\
 & & \downarrow & & \downarrow s & & & & \\
0 & \longrightarrow & E & \longrightarrow & E^{\vee\vee} & \xrightarrow{q} & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow Id & & \\
0 & \longrightarrow & \mathcal{I}_{C_2} & \longrightarrow & \mathcal{I}_C & \longrightarrow & \mathrm{coker}(\phi) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{17}$$

Here $\phi = q \circ s$; clearly, the kernel of ϕ is the ideal sheaf of a pure 1-dimensional scheme, which we denote by C_1; remark that if $q = 0$, then C_1 is empty. Since the cokernel of the inclusion $\mathcal{I}_{C_1} \hookrightarrow E$ must also be torsion-free, we complete the leftmost column. Note that

$$\begin{aligned} H^1(\mathcal{I}_{C_1}(-1)) = H^2(\mathcal{I}_{C_2}(-1)) = 0 \iff H^0(\mathcal{O}_{C_1}(-1)) \\ = H^1(\mathcal{O}_{C_2}(-1)) = 0. \end{aligned} \tag{18}$$

Lemma 32. *Let E be a rank 2 instanton sheaf on a Fano threefold of index 2. If E is properly semistable, then E is S-equivalent to $\mathcal{I}_{C_1} \oplus \mathcal{I}_{C_2}$ where C_1 and C_2 are instanton curves of the same degree.*

Proof. As we have seen above, the hypothesis imply that E must be an extension of ideal sheaves $\mathcal{I}_{C_1}$ and $\mathcal{I}_{C_2}$ satisfying

$$\chi(\mathcal{I}_{C_j}(t)) = \frac{1}{2}\chi(E(t)) = \chi(\mathcal{O}_X(t)) - \frac{c_2(E)}{2}(t+1);$$

(recall indeed that instantons on Fano threefolds of index 2 have $c_3 = 0$). In particular, $\chi(\mathcal{I}_{C_j}(-1)) = 0$; the vanishings in display (18) imply that C_1 and C_2 must be instanton curves. □

From Lemma 32 we deduce the following corollaries.

Corollary 33. *On a Fano threefold X of index 2, there are no properly semistable rank 2 instanton sheaves of odd charge.*

Proof. Each semistable instanton E is S-equivalent to a sheaf of the form $\mathcal{I}_{C_1} \oplus \mathcal{I}_{C_i}$ where C_1 and C_2 are instanton curves of the same degree d; thus $c_2(E) = 2d$. □

Corollary 34. *On a Fano threefold X of index 2, there exist no properly semistable locally free instanton sheaves of charge > 0.*

Proof. If E is properly semistable and $c_2 > 0$ then E fits in

$$0 \to \mathcal{I}_{C_1} \to E \to \mathcal{I}_{C_2} \to 0$$

with C_i being instanton curves of degree $\deg(C_1) = \deg(C_2) = \frac{c_2(E)}{2}$. From this short exact sequence, we compute that E has depth 2 along all points $x \in C_1$ hence E can not be locally free. □

At the beginning of the section, we observed that there are no properly μ-semistable non locally free instanton sheaves on Fano threefolds of odd index (since on these varieties μ-semistable rank 2 bundles are μ-stable) and that each properly μ-semistable non locally instanton E on $\mathbb{P}^3$ satisfies $E^{\vee\vee} \simeq \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$. The Fano threefolds X of index $i_X = 2$ are the only ones carrying families of properly μ-semistable non locally free instanton sheaves E such that $c_2(E^{\vee\vee}) > 0$. Moreover, even if every properly μ-semistable instanton bundle F with $c_2(F) > 0$ is Gieseker unstable, the instanton sheaves E obtained as an elementary transformation of F along rank 0 instantons might be semistable or even stable.

Lemma 35. *Let (F, T, q) be, respectively, a properly μ-semistable rank 2 instanton bundle F of charge $n > 0$ a rank 0 instanton T of degree d and an epimorphism $q : F \twoheadrightarrow T$. Let E be the sheaf defined as $E := \ker(q)$. Then E is stable, resp. properly semi-stable, if and only if $\forall s \in H^0(F),\ s \neq 0, \mathrm{ch}_2(\mathrm{im}(q \circ s)) > \frac{n+d}{2}$, resp. if and only if T is the structure sheaf of an instanton curve and $\forall s \in H^0(F),\ s \neq 0$ $\mathrm{im}(q \circ s)$ is an instanton curve of degree $\frac{n+d}{2}$.*

Proof. As usual, we start considering the short exact sequence

$$0 \to E \to F \xrightarrow{q} T \to 0;$$

note that the charge of E is $n + d$.

Considering a diagram analogous to the one in display (17), we see that every subsheaf of E with trivial determinant is the ideal sheaf $\mathcal{I}_B$ of a scheme B such that $\mathcal{O}_B = \mathrm{im}(q \circ s)$ for a (hence for each) non zero section $s \in H^0(F) \simeq \mathbb{C}$. Denoting by d' and x the degree and the Euler characteristic of the curve B, respectively, we have:

$$\begin{aligned}\frac{1}{2}P_E(t) - P_{\mathcal{I}_B}(t) &= -\frac{n+d}{2}(t+1) + d't + x \\ &= \left(d' - \frac{(n+d)}{2}\right)t + x' - \frac{(n+d)}{2}\end{aligned}$$

It is therefore clear that if ever $d' > \frac{n+d}{2}$, then E is stable whilst $d' < \frac{n+d}{2}$ leads to the unstability of E. In the case $d' = \frac{(n+d)}{2}$, we can never have stability since $x' \leq \frac{(n+d)}{2}$ and equality occurs if and only if B is an instanton curve (or, equivalently, a degree d' line arrangement). Indeed for a l.c.m. curve B of degree

d', since $h^0(\mathcal{O}_B(-1)) = 0$, $\chi(\mathcal{O}_B(-1)) = -d' + x' \leq 0$ and equality holds if and only if B is an instanton curve.

This shows that E is stable, resp. properly semistable, if and only if for each non-zero global section $s \in H^0(F)$, $\mathrm{ch}_2(\mathrm{im}(q \circ s)) > \frac{n+d}{2}$ (resp. $\mathrm{ch}_2(\mathrm{im}(q \circ s)) = \frac{n+d}{2}$ and $\mathrm{im}(q \circ s)$ is an instanton curve). To conclude the proof of the proposition we still need to show that if E is properly semistable then T itself is the structure sheaf of an instanton curve. Once again we consider the diagram (17) induced by $s \in H^0(F)$ (recall that $F \simeq E^{\vee\vee}$); since $\mathrm{im}(q \circ s)$ is the structure sheaf of an instanton curve, then $\mathrm{coker}(q \circ s)$ must be a rank 0 instanton sheaf as well; moreover, by the fact that $\mathcal{I}_C$ surjects onto $\mathrm{coker}(q \circ s)$, we deduce that $\mathrm{coker}(q \circ s)$ must as well be isomorphic to $\mathcal{O}_{B'}$ with B' being an instanton curve of degree $\frac{(d-n)}{2}$. This last assertion is because a rank 0 instanton sheaf T' on a l.c.m. curve C has always degree (as a $\mathcal{O}_C$-module) $\deg(C) - \chi(\mathcal{O}_C) \geq 0$ with equality holding if and only if C is an instanton curve (indeed, whenever $P_C(t) = \deg(C)t + \deg(C)$, $h^0(\mathcal{O}_C(-1)) = 0$ implies $h^1(\mathcal{O}_C(-1)) = 0$) and $T' \simeq \mathcal{O}_C$. Therefore T is the structure sheaf of an instanton curve given by an extension of $\mathcal{O}_{B'}$ by $\mathcal{O}_B$. □

4.4. *Instantons via Serre correspondence*

Let X be a Fano threefold of Picard rank 1, index i_X, and take a rank 2 instanton sheaf E of charge c_2. Following the Serre correspondence outlined in Sect. 3, we choose a section $s \in H^0(E(n))$ with torsion-free cokernel and obtain a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-n) \longrightarrow E \longrightarrow \mathcal{I}_C(n - r_X) \longrightarrow 0$$

which yields a l.c.m. curve $C \subset X$; its arithmetic genus $p_a(C)$ and the degree d are given by

$$p_a(C) = 1 - [(d_X n^2 - d_X n r_X + c_2)(\frac{i_X}{2} + r_X - n) + \frac{1}{2} r_X (n r_X - n^2 - c_2)] \tag{19}$$

$$d = d_X n^2 - d_X n r_X + c_2; \tag{20}$$

moreover, its sheaf of ideals will satisfy the cohomological conditions:

$$H^0(\mathcal{I}_C(n - r_X - 1)) = 0,\ H^i(\mathcal{I}_C(n - r_X - q_X)) = 0,\ i = 1, 2. \tag{21}$$

Our aim now is to characterize in detail those curves that "Serre correspond" to non locally free instanton sheaves of rank 2.

We consider therefore a non locally free instanton sheaf E and we let n be a non-negative integer such that $h^0(E(n) > 0$; take $s \in H^0(E(n))$ such that $\mathrm{coker}(s)$ is a torsion-free sheaf, and let $s^{\vee\vee} \in H^0(E^{\vee\vee}(n))$ be the image of s via the injective map $H^0(E(n)) \hookrightarrow H^0(E^{\vee\vee}(n))$. According to the argument just below the diagram in display (4), we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_X(-n) & = & \mathcal{O}_X(-n) & & & & \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s^{\vee\vee}} & & & & \\
0 & \to & E & \to & E^{\vee\vee} & \to & T_E & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{I}_C(n-r_X) & \to & \mathcal{I}_{C'}(n-r_X) & \to & T_E & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{22}$$

where C and C' are the curves corresponding to the pairs (E, s) and $(E^{\vee\vee}, s^{\vee\vee})$, respectively; Lemma 5 guarantees that C' is l.c.m. To figure out the associated extension classes, we note that the short exact sequence in display (5) can be rewritten as follows

$$0 \to \omega_{C'}(r_X + i_X - 2n) \to \omega_C(r_X + i_X - 2n) \to \mathcal{E}xt^2(T_E(n), \mathcal{O}_X) \to 0. \tag{23}$$

Lemma 36. *If (C, ξ) corresponds to a pair $(E(n), s)$ where E is a non locally free instanton sheaf of rank 2, and n is a non-negative integer, then the pair (C', ξ') corresponding to $(E^{\vee\vee}(n), s^{\vee\vee})$ satisfies the following conditions:*

(1) $0 \to \mathcal{I}_C \to \mathcal{I}_{C'} \to T_E(r_X - n) \to 0$;
(2) ξ *is the image of* $\xi' \in H^0(\omega_{C'}(i_X + r_X - 2n))$ *under the inclusion:*

$$0 \to H^0(\omega_{C'}(i_X + r_X - 2n)) \xrightarrow{\iota} H^0(\omega_C(i_X + r_X - 2n))$$

Proof. The first item is just the bottom line of the diagram in display (22). As for the second item, applying $\mathcal{H}om(\,\cdot\,\mathcal{O}_X(r_X - 2n))$ to the short exact sequence in condition (1) of the statement and taking global sections, we obtain:

$$\begin{aligned} 0 \to H^0(\omega_{C'}(i_X + r_X - 2n)) &\xrightarrow{\iota} H^0(\omega_C(i_X + r_X - 2n)) \\ &\to H^0(\mathcal{E}xt^2(T_E(n), \mathcal{O}_X)), \end{aligned} \tag{24}$$

where the rightmost and middle terms are isomorphic to $\mathrm{Ext}^1(\mathcal{I}_{C'}(2n - r_X), \mathcal{O}_X)$ and $\mathrm{Ext}^1(\mathcal{I}_C(2n - r_X), \mathcal{O}_X)$, respectively. The middle column of the diagram in display (22) corresponds to a section $\xi' \in H^0(\omega_{C'}(i_X + r_X - 2n))$; its image $\iota(\xi') \in H^0(\omega_C(i_X + r_X - 2n))$ will precisely correspond, when regarded as an extension, to the first column of the same diagram. □

Remark 37. Note that whenever we are given a pair of l.c.m curve C, C' whose sheaves of ideals fit $0 \to \mathcal{I}_C \to \mathcal{I}_{C'} \to T_E(r_X - n) \to 0$, we obtain a short exact sequence like the one in display (23) and, taking global sections, a short exact sequence like the one in display (24). In particular, if $n \geq r_X + q_X$, then $H^0(\mathcal{E}xt^2(T_E(n), \mathcal{O}_X)) = 0$ (recall that $\mathcal{E}xt^2(T, \mathcal{O}_X(-r_X))$ is a rank 0 instanton) which means that there are no instanton bundles corresponding to the curve C and every $\xi \in H^0(\omega_C(r_X + i_X - 2n))$ corresponds to a non locally free instanton sheaf that is singular along $\mathrm{Supp}(T_E)$. The only cases in which a curve can correspond both to locally free and to non locally free instanton sheaves occur therefore for $i_X = 3, 4$ and $n = 1$. Since a curve C of such a kind satisfies (cf. 21) $H^i(\mathcal{I}_C(-1)) = H^{i-1}(\mathcal{O}_C(-1)) = 0$, $i = 1, 2$, from remark 17, we see that the only curves corresponding both locally free and non-locally free rank 2 instantons of charge n are the lines arrangement of degree $n+1-r_X$ (degree of C is computed applying 19). An example of a family of instanton bundles corresponding to a degree n line arrangement C and degenerating to a non locally free instanton, still corresponding to C was exhibited in example 10.

Next, we consider the reverse construction: let (C', ξ') be a pair consisting of a l.c.i. curve C' satisfying

$$H^i(\mathcal{I}_{C'}(n - r_X - q_X)) = 0, \; i = 1, 2,$$

and a nowhere vanishing section $\xi' \in H^0(\omega_{C'}(i_X + r_X - 2n))$. Considering the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-n) \xrightarrow{r} F \longrightarrow \mathcal{I}_{C'}(n - r_X) \longrightarrow 0 \tag{25}$$

given by regarding ξ' as a class in $\mathrm{Ext}^1(\mathcal{I}_{C'}(n - r_X), \mathcal{O}_X(-n))$ and the second part of Remark 4, it follows that the rank 2 sheaf F in the corresponding pair $(F(n), r)$ is a locally free instanton sheaf.

Lemma 38. *Any pair (C, ξ) consisting of*

(1) *A l.c.m. curve C containing C' such that $T := \mathcal{I}_{C'}(n - r_X)/\mathcal{I}_C(n - r_X)$ is a rank 0 instanton sheaf;*
(2) *A section $\xi \in H^0(\omega_C(i_X + r_X - 2n))$ lying in the image of the induced map*

$$H^0(\omega_{C'}(i_X + r_X - 2n)) \hookrightarrow H^0(\omega_C(i_X + r_X - 2n))$$

corresponds to a pair $(E(n), s)$ where E is a non locally free rank 2 instanton sheaf which is singular along $\mathrm{Supp}(T)$.

Proof. The sequence in display 25 and the way the curve C is chosen provides us with the diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{O}_X(-n) & & & & \\ & & & & \downarrow r & & & & \\ & & & & F & \overset{q}{\searrow} & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{I}_C(n-r_X) & \longrightarrow & \mathcal{I}_{C'}(n-r_X) & \longrightarrow & T & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{26}$$

The sheaf $E := \ker(q)$ satisfies two short exact sequences; the first one

$$0 \longrightarrow E \longrightarrow F \longrightarrow T \longrightarrow 0$$

implies that E is an instanton sheaf, while the second

$$0 \longrightarrow \mathcal{O}_X(-n) \longrightarrow E \longrightarrow \mathcal{I}_C(n-r_X) \longrightarrow 0$$

induces the section $\xi \in H^0(\omega_C(i_X + r_X - 2n))$ which vanishes on $\mathrm{Supp}(T)$, and therefore lies in the image of the map given in the statement of the lemma. □

The reason why we chose to portray in detail how the Serre correspondence works for rank 2 instantons is simply because we are mainly concerned with moduli spaces of rank 2 sheaves. Of course, these arguments can be generalized to instantons of arbitrary rank. Doing so we can in particular provide examples of non-locally free reflexive instantons of rank > 2 (we recall indeed that in rank 2 the reflexivity of instantons is equivalent to their local freeness).

Example 39. We can construct a non locally free reflexive instanton on X being either $\mathbb{P}^3$ or a quadric threefold as follows. Let $C \subset X$ be a smooth rational curve of degree $d \geq 4$ and we consider two linearly independent sections $\xi_i \in H^0(\mathcal{O}_C((d-2)\cdot p))$, $i = 1, 2$, for p a point on C, whose zero loci intersect along a 0-dimensional scheme $Z \subset C$ of length $d' < d-2$. Since $\mathcal{O}_C((d-2)\cdot p) \simeq \omega_C(1)$, these two sections correspond to an extension class in $\mathrm{Ext}^1(\mathcal{I}_C(2-r_X), \mathcal{O}_X(-1)^{\oplus 2})$, thus giving rise to a short exact sequence

$$0 \to \mathcal{O}_X(-1)^{\oplus 2} \to E \to \mathcal{I}_C(2-r_X) \to 0; \tag{27}$$

we argue that the middle term E is the sheaf we are looking for.

Indeed, the exact sequence in display (27) yield

$$H^i(E(-q_X)) \simeq H^i(\mathcal{I}_C) = 0, \ \forall\, i.$$

Dualizing the same exact sequence and recalling that $\mathcal{E}xt^1(\mathcal{I}_C(2-r_X), \mathcal{O}_X) \simeq \omega_C(2)$, we obtain:

$$0 \to \mathcal{O}_X(r_X-2) \to E^\vee \to \mathcal{O}_X(1)^{\oplus 2} \xrightarrow{\xi} \omega_C(2) \to \mathcal{E}xt^1(E, \mathcal{O}_X) \to 0$$

where the morphism ξ is defined by the two sections $(\xi_1, \xi_2) \in H^0(\omega_C(1)^{\oplus 2})$ we started with. By construction, ξ fails to be surjective along Z, so that $\mathcal{E}xt^1(E, \mathcal{O}_X)$ is supported on Z. This, together with the vanishing of $\mathcal{E}xt^i(E, \mathcal{O}_X)$ for $i > 1$ implies that E is reflexive with $\mathrm{Sing}(E) = Z$, thus non locally free.

Finally, to see that E is μ-semistable it is enough to check that $H^0(E(r_X-1)) = H^0(E^\vee(r_X-1)) = 0$, cf. [29, Remark 1.2.6]. $H^0(E(r_X-1))$ vanishes since it is isomorphic to $H^0(\mathcal{I}_C(1-2r_X)) = 0$ (this is clearly zero for $r_X = 1$ whilst for $r_X = 0$ it is ensured by the fact that C is not a planar curve). Setting $F := \ker(\xi)$, note that $H^0(E^\vee(r_X-1)) = H^0(F(r_X-1))$, and that the latter coincides with the kernel of the induced map

$$H^0(\mathcal{O}_X(r_X)^{\oplus 2}) \xrightarrow{(\xi_1,\xi_2)} H^0(\omega_C(r_X+1)),$$

given by multiplication by the sections $\xi_i,\ i = 1, 2$.

When $r_X = 0$ (i.e. $X = \mathbb{P}^3$) the fact that ξ_i are linearly independent is enough to guarantee that this map is injective, thus $H^0(F(r_X-1)) = 0$, as desired.

If $r_X = 1$, one must argue that (ξ_1, ξ_2) does have not a syzygy $(\sigma_1, \sigma_2) \in H^0(\mathcal{O}_C(d \cdot p))$ of degree d that lies in the image of the restriction map

$$H^0(\mathcal{O}_X(1)^{\oplus 2}) \to H^0(\mathcal{O}_C(1)^{\oplus 2}) \simeq H^0(\mathcal{O}_C(d \cdot p)).$$

This seems to be a generic condition when $d - d'$ is sufficiently large, but we have not been able to prove it.

5. Instanton sheaves on quadric threefolds

Let V be a 5-dimensional vector space and consider a smooth quadric hypersurface $X \subset \mathbb{P}(V) \simeq \mathbb{P}^4$. X is the only Fano 3-fold of Picard rank one and index 3, therefore an instanton sheaf E on X is defined as a torsion-free μ-semistable sheaf with $c_1(E) = -1$ and such that:

$$H^i(E(-1)) = 0,\ i = 1, 2. \tag{28}$$

Recall that since $c_1(E)$ is odd, every instanton sheaf on X is actually μ-stable; this ensures the vanishing of $H^i(E(-1))$ for $i = 0, 3$ as well (cf. Lemma 8). From now on we will only be concerned with instanton sheaves of rank 2 (therefore when referring to an instanton sheaf we will always imply that its rank is 2).

The Chern character of a rank 2 instanton E of charge n is:

$$\mathrm{ch}(E) = \left(2, -[H],\ (1-n)[l],\ \frac{-1}{3} + \frac{n}{2}\right) \tag{29}$$

(by Corollary 9, $c_3(E) = 0$); applying Riemann–Roch, we compute the Hilbert polynomial of E:

$$P_E(t) = \frac{2}{3}t^3 + 2t^2 + \left(\frac{7}{3} - n\right)t + (1 - n) \tag{30}$$

In this section, we present some results on instanton sheaves on X. We will focus our attention on instanton sheaves E of charge 2, emphasizing the relation that these sheaves have with the curves corresponding to global sections of $E(1)$ via Serre correspondence. The Serre correspondence allows us not only to describe the instanton moduli space but to obtain also a complete picture of the entire Gieseker–Maruyama moduli space $\mathcal{M} := \mathcal{M}_X(2, -1, 2, 0)$ of semistable rank 2 sheaves with Chern classes $(c_1, c_2, c_3) = (-1, 2, 0)$, together with its relation with the Hilbert scheme $\mathrm{Hilb}_{2t+2}(X)$.

5.1. Instanton sheaves of charge 1

Since every instanton sheaf E is a μ-semistable sheaf with $c_1(E) = -1$, the Bogomolov inequality implies that $c_2(E) \geq 1$. In the case $c_2 = 1$, a well-known example of an instanton bundle is provided by the so-called *spinor bundle* which will be henceforth denoted by $\mathcal{S}$.

Recall that it can be defined as follows, cf. [30, Definition 1.3] for which we refer to for all details in this paragraph. There is an embedding $s : X \to \mathbb{G}(1,3)$, the grassmannian of lines in $\mathbb{P}^3$ and $\mathcal{S} := s^*U$, where U is the universal bundle on $\mathbb{G}(1,3)$. This is a μ-stable rank 2 bundle on X with $c_1(\mathcal{S}) = -1$ and $c_2(\mathcal{S}) = 1$. In addition, $\mathcal{S}$ is rigid [30, Theorem 2.1], and every μ-stable rank 2 bundle E on X with $c_1(E) = -1$ and $c_2(E) = 1$ is isomorphic to the spinor bundle $\mathcal{S}$.

Since $h^0(\mathcal{S}(1)) = 4$, Serre construction provides the following short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{S} \longrightarrow \mathcal{I}_\ell \longrightarrow 0 \tag{31}$$

where ℓ is a line in X. It is then easy to see that $\mathcal{S}$ is a rank 2 instanton bundle of charge 1. This observation allows us to give the following characterization of the family $F(X)$ of lines on X. Since $h^0(\mathcal{S}) = 0$ (by stability) we deduce then that $\forall s \in H^0(\mathcal{S}(1)),\ s \neq 0$, $\mathrm{coker}(s)$ is torsion-free and isomorphic to $\mathcal{I}_l(1)$ for a line $l \subset X$ (this last assertion follows from a simple Chern character computation). Conversely, $\forall\, [l] \in F(X)$, every sheaf fitting in a non-split short exact sequence of the form (31) is a μ-stable vector bundle with $c_1 = -1,\ c_2 = 1$ and is, therefore, isomorphic to $\mathcal{S}$. Accordingly $F(X) \simeq \mathbb{P}^3 \simeq \mathbb{P}(H^0(\mathcal{S}(1)))$.

Proposition 40. *Every rank 2 instanton sheaf of charge 1 on X is isomorphic to the spinor bundle.*

Proof. Let E be a rank 2 instanton sheaf of charge 1. If E is reflexive, then it must be locally free and therefore it is isomorphic to the spinor bundle.

If E is not reflexive, then Theorem 24 implies that $E^{\vee\vee}$ is a locally free instanton sheaf of charge $c_2(E^{\vee\vee}) \geq 1$. However, $c_2(E^{\vee\vee}) + \deg(T_E) = c_2(E) = 1$, thus in fact $c_2(E^{\vee\vee}) = 1$ and $\deg(T_E) = 0$. It follows that $T_E = 0$, contradicting the hypothesis that E was not reflexive. □

The following result will also be useful later on.

Lemma 41. *Every μ-stable rank 2 reflexive sheaf with Chern classes $c_1 = -1$ and $c_2 = 1$ is isomorphic to the spinor bundle.*

Proof. Let F be a μ-stable rank 2 reflexive sheaf with $c_1(F) = -1$ and $c_2(F) = 1$, so that

$$\chi(F) = \frac{c_3(F)}{2} = h^2(F) - h^1(F)$$

since $h^0(F) = h^3(F) = 0$ by μ-stability. We claim that $h^1(F(n)) = 0,\ \forall n \leq 0$.

Indeed, take a general hyperplane section $Q \in |\mathcal{O}_X(1)|$ and consider the restriction sequence

$$0 \longrightarrow F(-1) \longrightarrow F \longrightarrow F|_Q \longrightarrow 0, \tag{32}$$

with $F|_Q$ being a μ-semistable locally free sheaf on Q (the reflexivity of F implies indeed that $\mathrm{Sing}(F)$ is zero dimensional hence, for Q general, $F|_Q$ is locally free) with $c_1(F|_Q) = (-1, -1)$. The μ-semistability of $F|_Q$ leads to the vanishing of $H^0(F|_Q(n)),\ \forall n \leq 0$ and of $H^2(F|_Q(n)),\ \forall n \geq -1$. In addition, since $\chi(F|_Q) = 1 - c_2(F) = 0$, we conclude that $h^1(F|_Q) = 0$; Serre duality then implies that $h^1(F|_Q(-1)) = 0$. From the fact that $h^1(F|_Q) = h^2(F|_Q(-1)) = 0$, we deduce that $F|_Q$ is 1-regular which implies that $h^1(F|_Q(n)) = 0,\ \forall n \geq 0$. Since by Serre duality $h^1(F|_Q(n)) = h^1(F|_Q(-n-1))$, we can finally conclude that $h^1(F|_Q(n)) = 0,\ \forall n \in \mathbb{Z}$. Twisting the sequence in display (32) and taking cohomology, we thus get $h^1(F(n)) = h^1(F(n+1)),\ \forall n \leq -1$; but from the reflexivity of F, $H^1(F(n)) = 0$ for $n \ll 0$ hence $h^1(F(n)) = 0,\ \forall n \leq 0$, as desired.

It follows that $h^2(F) = \dfrac{c_3(F)}{2}$; since $h^i(F|_Q(n)) = 0$ for $i = 1, 2$ and $\forall n \geq -1$ we also get, inductively, that $h^2(F(n)) = h^2(F(n+1))\ \forall n \geq -2$ so that $h^2(F(n)) = \dfrac{c_3(F)}{2},\ \forall n \geq -2$. By the Serre vanishing theorem, we must have that $h^2(F(n)) = 0$ when $n \gg 0$, thus in fact $h^2(F(n)) = 0$ for every $n \geq -2$, and hence $c_3(F) = 0$, implying that F must be locally free. But every rank 2 locally free sheaf with $c_1 = -1$ and $c_2 = 1$ on X is a spinor bundle. □

We end this preliminary section summoning some properties of $F(X)$, the family of lines on X. We have already recalled that $F(X) \simeq \mathbb{P}^3 \simeq \mathbb{P}(H^0(\mathcal{S}(1))$. One "geometric" way to realize $F(X)$ as $\mathbb{P}^3$ is the following. We start by constructing X as a hyperplane section of the Grassmannian $\mathbb{G}(1,3) \subset \mathbb{P}^5$ of lines in $\mathbb{P}^3$. Recall now that we have 2 families of planes contained in $\mathbb{G}(1,3)$: we have planes corresponding to families of lines passing through a point (we call them planes of type I), and planes parameterizing families of lines contained in a plane $\mathbb{P}^2 \subset \mathbb{P}^3$ (these will be referred to as planes of type II). For each line $l \subset X$ there exists a unique pair of planes $(\Delta_I, \ \Delta_{II})$ with Δ_I of type I, Δ_{II} of type II, containing l; these planes are both parameterized by a 3-dimensional linear space $\mathbb{P}^3$.

Several of our next results will rely on the geometry of linear spaces of lines; for this reason, we recall here briefly some of their fundamental property. We have two families of pencils of lines in $F(X)$. Consider indeed a pencil $\mathbb{P}^1 \subset F(X)$ and denote by $l_0, \ l_1$ a pair of generators. If ever $l_0 \cap l_1 = \emptyset$, then the entire $\mathbb{P}^1$ is a ruling in the quadric surface $Q := \langle l_0, l_1 \rangle \cap X$. In particular, we must have that *any* pair of lines in $\mathbb{P}^1$ are disjoint hence Q must be smooth since we have no disjoint lines in a singular hyperplane section of X. For the same reason, we deduce that if ever $l_0 \cap l_1 \neq \emptyset$, then any pair of lines in $\mathbb{P}^1$ must intersect so that this family must coincide with the family of lines on a singular hyperplane section of X.

This implies in particular that these lines all have the form $\overline{qp}$ with p fixed and q varying along a conic. From these observations, we deduce that there exists a morphism

$$\mathbb{G}(1, F(X)) \xrightarrow{\gamma} \mathbb{P}^{4*}$$

and that moreover, denoting by $\mathbb{P}^{4*}_{sm} := \mathbb{P}^{4*} \backslash X^*$ the open subscheme of smooth hyperplane sections and by $\mathcal{U} := \gamma^{-1}(\mathbb{P}^{4*}_{sm})$ (this latter is, by construction, open in $\mathbb{G}(1, F(X))$ hence irreducible), $\gamma|_{\mathcal{U}}$ is a degree 2 covering over $\mathbb{P}^{4*}_{sm}$.

Finally we recall that $\forall \ l \subset X$ we have a hyperplane $\mathbb{P}^2 \subset F(X)$ of lines meeting l (isomorphic to the family of planes in $\mathbb{P}^4$ containing l) and that all the hyperplanes in $F(X)$ are of this form.

5.2. Instantons of charge 2

Our study of the moduli space $\mathcal{M}$ starts with the study of $\mathcal{L}(2)$, the open subscheme parameterizing rank 2 instanton sheaves of charge 2. We will prove the following:

Theorem 42. $\mathcal{L}(2)$ *is a smooth, irreducible, 6-dimensional open subscheme of* $\mathcal{M}$ *whose general element is a locally free instanton sheaf. Its closure* $\overline{\mathcal{L}(2)}$ *is an irreducible component of* $\mathcal{M}$. *The moduli space* $\mathcal{M}$ *is smooth along* $\mathcal{L}(2)$.

The moduli space $\mathcal{I}(2)$ of locally free instanton sheaves of charge 2 was studied in [31]. In loc. cit the authors proved the following.

Theorem 43. [31, Theorem 4.1] *The moduli space* $\mathcal{I}(2)$ *is locally a trivial algebraic fibration over* $(\mathbb{P}^4)^*_{sm}$ *with fibre being two disjoint copies of* $\mathbb{P}^2 \setminus C_2$, *for a smooth conic* C_2. *In particular, it is a Stein manifold of dimension 6, rational irreducible, and smooth.*

The key ingredient of this result is the description of the families of curves arising as zero loci of global sections of $E(1)$ for $[E] \in \mathcal{I}(2)$.

Proposition 44. [31, Proposition 4.4] *The zero set* $V(s)$ *of a global section* s *of* $E(1)$ *is a divisor of type* $(2, 0)$ *on a smooth hyperplane section* $Q \subset X$ *(and hence it is either the union of two disjoint lines or a double line of arithmetic genus -1). The zero sets* $V(s), \ V(t)$ *of two sections* s, t *of* $E(1)$ *lie on the same smooth quadric* Q *and cut a system* g^1_2 *without base point.*

This characterisation of the linear spaces $\mathbb{P}(H^0(E(1)))$ implies indeed the existence of a morphism:

$$\mathcal{I}(2) \xrightarrow{\phi} \mathbb{P}^{4*}_{sm},$$

mapping a point $[E] \in \mathcal{I}(2)$ to the quadric surface containing all the curves $V(s),\ s \in H^0(E(1))$. The fiber of ϕ over Q consists of the base point free pencils of divisors of type (2, 0) or (0, 2) on Q, namely of two copies of $\mathbb{P}^2 \backslash C_2$ where C_2 is a smooth conic. The pencils of divisors of type (2, 0) are indeed parameterized by the projective space $\mathbb{G}(1, |\mathcal{O}_Q(2,0)|) \simeq |\mathcal{O}_Q(2,0)|^* \simeq \mathbb{P}^2$; inside this projective space, the locus of pencils with a base point identifies with C_2, the smooth conic of lines tangent to $\Gamma_2 \subset |\mathcal{O}_Q(2,0)|$, the conic parameterizing double lines.

Remark 45. Note that, by construction, the morphism ϕ factors through a morphism $\phi_{\mathcal{U}} : \mathcal{I}(2) \to \mathcal{U}$ where we recall that $\mathcal{U} \subset \mathbb{G}(1, F(X))$ is defined as the open subset parameterizing rulings of smooth hyperplane sections of X.

We now pass to the study of non locally free instantons $[E] \in \mathcal{M}$. By Theorem 24, if E is a non locally free instanton, $E^{\vee\vee}$ is an instanton bundle of charge $c_2(E^{\vee\vee}) < c_2(E)$ and $T_E := E^{\vee\vee}/E$ is a rank 0 instanton of degree $c_2(E) - c_2(E^{\vee\vee})$. Since $c_2(E) = 2$ and the minimal charge of an instanton sheaf on X is 1, the only possibility is that $E^{\vee\vee} \simeq \mathcal{S}$ so that T_E is a rank 0 instanton of degree 1. It is not difficult to prove that $T_E \simeq \mathcal{O}_l$ for a line $l \subset X$. Since $P_{T_E}(t) = t + 1$ and as $h^1(T_E(-1))$ implies $h^1(T_E) = 0$, we have $h^0(T_E) = 1$. For $s \in H^0(T_E)$, the image $\mathrm{im}(s)$ of the corresponding morphism $\mathcal{O}_X \xrightarrow{s} T_E$ must therefore be of the form $\mathcal{O}_C$ for C a degree one l.c.m curve. But this means $C \simeq l$ for a line $l \subset X$ and since $P_{\mathcal{O}_l} = P_{T_E}$ we conclude that $T_E \simeq \mathcal{O}_l$. Summing up, each non locally free instanton E of charge 2 is defined by a short exact sequence of the form:

$$0 \to E \to \mathcal{S} \xrightarrow{q} \mathcal{O}_l \to 0. \tag{33}$$

Our next aim is to formulate results similar to Proposition 44 and Theorem 43 for non locally free instanton sheaf. We start describing the families of curves corresponding to global sections of $E(1)$.

Proposition 46. *Let E be a non locally free instanton of charge 2 singular along a line l. Then $H^0(E(1)) \simeq \mathbb{C}^2$ and $\forall s \in H^0(E(1)),\ s \neq 0$, $\mathrm{coker}(s) \simeq \mathcal{I}_{l' \cup l}$ with l' varying in a ruling of a smooth hyperplane section of X containing l.*

Proof. Let us start with the computation of $H^0(E(1))$. Twisting (33) and taking global sections, we obtain a linear map $H^0(\mathcal{S}(1)) \to H^0(\mathcal{O}_l(1))$ that can not be injective, (as $h^0(\mathcal{S}(1)) = 4$) hence $H^0(E(1)) \neq 0$. Denote now by ι the inclusion $\iota : H^0(E(1)) \hookrightarrow H^0(\mathcal{S}(1))$. Since every non-zero element in $H^0(\mathcal{S}(1))$ has torsion-free cokernel, the same holds for any non-zero $s \in H^0(E(1))$; this implies that $\mathrm{coker}(s) \simeq \mathcal{I}_Y(1)$ for a l.c.m subscheme $Y \subset X$.

For any $s \in H^0(E(1)),\ s \neq 0$, we have $\mathrm{coker}(\iota(s)) \simeq \mathcal{I}_{l'}(1)$ for a line $l' \subset X$ and we get a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \iota(s)} & & \downarrow & & \\ 0 & \longrightarrow & E(1) & \longrightarrow & \mathcal{S}(1) & \xrightarrow{q} & \mathcal{O}_l(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathcal{I}_Y(1) & \longrightarrow & \mathcal{I}_{l'}(1) & \longrightarrow & \mathcal{O}_l(1) & \longrightarrow & 0 \end{array} \tag{34}$$

From it we compute that $P_Y(n) = 2n+2$ and we deduce that $\mathrm{Supp}(Y) = l' \cup l$. Since Y must be l.c.m. and as $\mathcal{I}_{l'}$ surjects onto $\mathcal{O}_l$, the only possibilities are either $l' = l$, in which case $\mathcal{I}_l|_l \simeq \mathcal{O}_l \oplus \mathcal{O}_l(-1)$ and Y is a double structure on l with arithmetic genus -1 and, or $l' \cap l = \emptyset$ in which case $\mathcal{I}_{l'}|_l \simeq \mathcal{O}_l$ and Y is simply the union of l and l'. In each of these cases, the scheme Y is contained in the unique hyperplane $\langle Y \rangle \simeq \mathbb{P}^3$ thus, from the first column of (34) we compute that $h^0(E(1)) = 2$. To complete the proof of the proposition we still need to describe the pencil $\mathbb{P}(H^0(E(1)))$. By construction the space of section $r_l \in H^0(\mathcal{S}(1))$ vanishing on l locates a point in $\mathbb{P}(\iota(H^0(E(1))) \simeq \mathbb{P}^1$ (since $r_l \otimes \mathcal{O}_l$=0); the arguments previously presented show that a generic element in $\iota(H^0(E(1)))$ corresponds to a line disjoint from l. From the discussion held at the end of section 5.1, the pencil $\mathbb{P}(\iota(H^0(E(1))))$ must therefore coincide with the ruling of a smooth hyperplane section Q of X containing l and the curves corresponding to non zero sections of $E(1)$ are thus all of the form $l \cup l'$ with l' varying in $\mathbb{P}(\iota(H^0(E(1))))$. □

Remark 47. Proposition 46 allows us to deduce that, given a l.c.m. curve Y with Hilbert polynomial $2t+2$ and $\mathrm{Supp}(Y) = l \cup l'$, a pair $(s, E(1))$ with E a non locally free instanton singular along l, corresponds to a pair (Y, ξ) with $\xi \in H^0(\omega_Y(2)) \simeq H^0(\mathcal{O}_l) \oplus H^0(\mathcal{O}'_l)$ of the form $(0, e),\ e \neq 0$.

We now want to understand how non locally free instantons behave in families. We consider therefore the set $\mathcal{D}(1,1) := \mathcal{L}(2) \setminus \mathcal{I}(2)$ that parameterizes non locally free instantons.

Proposition 48. *$\mathcal{D}(1,1)$ is a locally closed subscheme of $\mathcal{M}$; it is smooth, irreducible and of dimension 5.*

Proof. By semicontinuity, the instanton locus $\mathcal{L}(2)$ is open in $\mathcal{M}$; since being non locally free is a closed condition, we have that $\mathcal{D}(1,1)$ is locally closed in $\mathcal{M}$. To prove the rest of the proposition we mimic the proof of Theorem 43. From Proposition 46 we know that for $[E] \in \mathcal{D}(1,1)$, the curves in the pencil $\mathbb{P}(H^0(E(1))) \subset \mathrm{Hilb}_{2t+2}(X)$ are of the form $l \cup l'$ with $l = \mathrm{Sing}(E)$ and with the l's varying in a ruling of a smooth hyperplane section of X.

As it was the case for $\mathcal{I}(2)$, $\mathcal{D}(1,1)$ is endowed as well with a surjective map $\mathcal{D}(1,1) \xrightarrow{\phi^{(1,1)}} \mathbb{P}^{4*}_{sm}$ fitting in a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(1,1) & \xrightarrow{\phi^{(1,1)}} & \mathbb{P}^{4*}_{sm} \\ {\scriptstyle \phi^{(1,1)}_{\mathcal{U}}}\downarrow & \nearrow & \\ \mathcal{U} & & \end{array}$$

but this time the fibre over a ruling $|\mathcal{O}_Q(1,0)| \simeq \mathbb{P}^1 \in \mathcal{U},\ [Q] \in (\mathbb{P}^4)^*_{sm}$ consists of pencils of divisors of type $(2,0)$ with a base point. Each fiber of $\phi^{(1,1)}_{\mathcal{U}}$ is, therefore, isomorphic to $C_2 \subset \mathbb{G}(1, |\mathcal{O}_Q(2,0)|)$; the smooth conic parameterizing the tangents to the locus of singular divisors in $|\mathcal{O}_Q(2,0)|$. This proves that $\mathcal{D}(1,1)$ is smooth irreducible and of dimension equal to five. □

Remark 49. Applying arguments equivalent to Lemma 9.3 of [13], we can make the following considerations. Let $[E]$ be a point corresponding to an instanton and let us consider the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} E(1) \to \mathcal{I}_Y(1) \to 0$$

induced by $s \in H^0(E(1))$. From this short exact sequence, we deduce that the image of $t \in H^0(E(1))$ in $H^0(\mathcal{I}_Y(1))$ gives an equation for the hyperplane $\langle Y \rangle$ and that moreover for any $t \in H^0(E(1))$ independent from s, $E(1)/(s,t) \simeq \mathcal{I}_{Y,Q}(1)$, for $Q := X \cap \langle Y \rangle$. We have therefore a well defined linear map $H^0(E(1)) \to H^0(\mathcal{O}_Q(2,0))$ that maps each $s \in H^0(E(1))$ to the form defining $V(s)$ on Q.

5.3. A description of $\mathcal{L}(2)$ via Serre correspondence

For the moment we just know that $\mathcal{I}(2)$ and $\mathcal{D}(1,1)$ are locally trivial fibrations over $\mathcal{U}$. Using Serre correspondence we are now going to show that actually, the entire $\mathcal{L}(2)$ identifies with a $\mathbb{P}^2$-bundle over $\mathcal{U}$ and that $\mathcal{D}(1,1)$ and $\mathcal{I}(2)$ are, respectively, a divisor and an open subset of $\mathcal{L}(2)$. The key ingredient to prove this is the Serre correspondence which enables us to collect information about the geometry of $\mathcal{L}(2)$ by studying the geometric properties of the families of the corresponding curves. Our starting point is therefore the inspection of the open $\mathcal{H} \subset \mathrm{Hilb}_{2t+2}(X)$ that parameterises locally Cohen Macaulay curves. Note that any locally Cohen Macaulay curve with Hilbert's polynomial $2t+2$ is indeed either the union of two disjoint lines or a double structure on a line of arithmetic genus -1.

Lemma 50. *$\forall\, [Y] \in \mathcal{H}$, $h^0(\mathcal{N}_{Y/X}) = 6$ and $h^1(\mathcal{N}_{Y/X}) = 0$.*

Proof. We first show that for $[Y] \in \mathcal{H}$, Y lies in a unique smooth hyperplane section $Q \subset X$.

If $Y = l_1 \cup l_2$, $l_1 \cap l_2 = \emptyset$ then the only hyperplane containing Y is $\langle Y\rangle = \langle l_1, l_2\rangle$. If otherwise Y is a double line supported on l, we have that $\mathcal{I}_Y$ fits in the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{I}_l \to \mathcal{O}_l \to 0$$

from which we compute that $h^0(\mathcal{I}_Y(1)) \neq 0$ (since $H^0(\mathcal{I}_l(1))$ can not inject in $H^0(\mathcal{O}_l(1))$) and $h^0(\mathcal{I}_Y(1)) < 2$ (since no planar l.c.m. curve has negative arithmetic genus). Thus $h^0(\mathcal{I}_Y(1)) = 1$ and Y is contained in a unique hyperplane section Q of X. Q must be smooth since a degree 2l.c.m. curves on a singular hyperplane section of X is planar. The only degree 2l.c.m. curve on a cone of vertex p over a smooth conic C indeed are C itself or cones over degree 2 divisors $D \subset C$ of C. A curve of this former kind is always contained in a plane of the form $\langle p, L_D\rangle$, where L_D is the unique line spanned by D if D is reduced, whilst $L_D = \mathbb{T}_q C$ if D is supported on a unique point q instead.

Let us now compute $h^i(\mathcal{N}_{Y/X})$, $i = 0, 1$. Consider the smooth quadric surface $Q := \langle Y\rangle \cap X$ and denote by L_A, L_B the two generators of $\mathrm{Pic}(Q)$. The only degree 2 effective divisors in Q having arithmetic genus -1 belong either to the class $2L_A$ or to $2L_B$.

Without loss of generality, we suppose then $Y \sim 2L_A$ and we consider

$$0 \to \mathcal{O}_Q \to \mathcal{O}_Q(2L_A) \to \mathcal{O}_Y(2L_A) \to 0.$$

From this short exact sequence, since $H^i(\mathcal{O}_Q) = H^i(\mathcal{O}_Q(L_A)) = 0,\ \forall\, i \geq 1$, we compute $h^0(\mathcal{O}_Y(2L_A)) = 2$ and $h^1(\mathcal{O}_Y(2L_A)) = h^1(\mathcal{N}_{Y/Q}) = 0$.

We finally consider:

$$0 \to \mathcal{N}_{Y/Q} \to \mathcal{N}_{Y/X} \to \mathcal{N}_{Q/X}|_Y \to 0.$$

$\mathcal{N}_{Q/X}|_Y$ is isomorphic to $\mathcal{O}_Y(1)$ and from

$$0 \to \mathcal{O}_l(1) \to \mathcal{O}_Y(1) \to \mathcal{O}_l(1) \to 0$$

we obtain $h^0(\mathcal{O}_Y(1)) = 4$ and $h^1(\mathcal{O}_Y(1)) = 0$.

From these arguments we deduce the vanishing of $H^1(\mathcal{N}_{Y/X})$, which implies the smoothness of $\mathrm{Hilb}_{2t+2}(X)$ at $[Y]$, and we compute that $\mathrm{Hilb}_{2t+2}(X)$ has dimension $6 = h^0(\mathcal{N}_{Y/X}) = h^0(\mathcal{O}_Y(1)) + h^0(O_Y(2L_A))$ at $[Y]$. □

Lemma 51. *$\mathcal{H}$ is a $\mathbb{P}^2$-bundle over $\mathcal{U} \subset \mathbb{G}(1, F(X))$.*

Proof. Denote by $\mathcal{T}_{\mathcal{U}}$ the restriction of the tautological bundle over $\mathbb{G}(1, F(X))$ to $\mathcal{U}$. Take then the rank 3 vector bundle $\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^\vee)$ and the projective bundle

$$\mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^\vee)) \to \mathcal{U}.$$

For a point $h \in \mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^\vee))$ we denote by $l_{1,h}$, $l_{2,h}$ the corresponding (possibly coincident) lines. The incidence correspondence $\Sigma \subset \mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^\vee)) \times X$

$$\Sigma := \{(h, p) \in \mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^\vee)) \times X \mid p \in l_{1,h} \cup l_{2,h}\}$$

induces a bijective morphism $\mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^\vee)) \to \mathcal{H}$ that, since $\mathcal{H}$ and $\mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^\vee))$ are smooth, is, therefore, an isomorphism (this is due to Zariski main theorem). □

From now on we denote by $\pi_{\mathcal{H}} : \mathcal{H} \to \mathcal{U}$ the standard projection.

Let us now pass to the study of $\mathcal{L}(2)$. To begin with we show how we can deduce the smoothness of $\mathcal{M}$ along $\mathcal{L}(2)$ from the smoothness of $\mathcal{H}$.

Lemma 52. *For any* $[E] \in \mathcal{I}(2)$, $\operatorname{ext}^1(E, E) = 6$ *and* $\operatorname{ext}^2(E, E) = 0$.

Proof. Consider the short exact sequence:

$$0 \to \mathcal{O}_X(-1) \to E \to \mathcal{I}_Y \to 0. \tag{35}$$

Applying $\operatorname{Hom}(E, \cdot)$ we obtain an exact sequence of vector spaces

$$\operatorname{Ext}^2(E, \mathcal{O}_X(-1)) \to \operatorname{Ext}^2(E, E) \to \operatorname{Ext}^2(E, \mathcal{I}_Y).$$

The left side term is zero since it is dual to $H^1(E(-2)) \simeq H^1(\mathcal{I}_Y(-2)) = 0$. Let us now prove that the right side term vanishes as well. By stability and by Lemma 30, we have $h^i(E(1)) = h^{3-i}(E(-3)) = 0$ for $i = 2, 3$; as moreover $P_E(1) = 2$ and $h^0(E(1)) = 2$, we conclude that $h^1(E(1)) = 0$ as well. This implies the vanishing of $\operatorname{Ext}^i(E, \mathcal{O}_X) \simeq \operatorname{Ext}^i(\mathcal{O}_X, E(1))$ for $i = 1, 2$ which leads to an isomorphism $\operatorname{Ext}^2(E, \mathcal{I}_Y) \simeq \operatorname{Ext}^1(E, \mathcal{O}_Y)$. But E is locally free, therefore:

$$\operatorname{Ext}^1(E, \mathcal{O}_Y) \simeq H^1(\mathcal{H}om(E, \mathcal{O}_Y)) \simeq H^1(E|_Y^\vee) \simeq H^1(\mathcal{N}_{Y/X}) = 0$$

(the isomorphism $E|_Y^\vee \simeq \mathcal{N}_{Y/X}$ is obtained tensoring (35) for $\mathcal{O}_Y$ and the vanishing of $H^1(\mathcal{N}_{Y/X})$ is due to Lemma (50)). Therefore $\operatorname{Ext}^2(E, \mathcal{I}_Y) = 0$ which implies $\operatorname{Ext}^2(E, E) = 0$. Now, the stability of E leads to $\operatorname{Hom}(E, E) \simeq \mathbb{C}$ and $\operatorname{Ext}^3(E, E) \simeq \operatorname{Hom}(E, E(-3))^* = 0$. Since E has homological dimension one, we can apply an argument equivalent to [14, Proposition 3.4], obtaining:

$$\chi(E, E) = \frac{3}{2} c_1(E)^2 - 6c_2(E) + 4 = -5. \tag{36}$$

which allows to conclude that $\operatorname{ext}^1(E, E) = 6$. This ensures that the moduli space $\mathcal{M}$ is smooth along $\mathcal{I}(2)$ and that $\overline{\mathcal{I}(2)}$ is the only component passing through any point in $\mathcal{I}(2)$. □

We pass now to the case of non locally free instantons.

Proposition 53. $\mathcal{M}$ *is smooth of dimension 6 at any point* $[E] \in \mathcal{D}(1, 1)$.

Proof. We know that E fits in a short exact sequence:

$$0 \to E \to \mathcal{S} \to \mathcal{O}_l \to 0 \tag{37}$$

where $E^{\vee\vee} \simeq \mathcal{S}$ and $l = Sing(E)$. Applying $\operatorname{Hom}(\cdot, E)$ we end up with a sequence of vector spaces:

$$\operatorname{Ext}^2(\mathcal{S}, E) \to \operatorname{Ext}^2(E, E) \to \operatorname{Ext}^3(\mathcal{O}_l, E);$$

$\operatorname{Ext}^3(\mathcal{O}_l, E) \simeq \operatorname{Hom}(E, \mathcal{O}_l(-3))^* \simeq \operatorname{Hom}(E|_l, \mathcal{O}_l(-3))^*$. Tensoring (37) for $\otimes\, \mathcal{O}_l$ we obtain:

$$0 \to \mathcal{T}or_1(\mathcal{O}_l, \mathcal{O}_l) \xrightarrow{\alpha} E|_l \xrightarrow{\beta} \mathcal{S}|_l \xrightarrow{\gamma} \mathcal{O}_l \to 0$$

and consequently:

$$0 \to \mathcal{T}or_1(\mathcal{O}_l, \mathcal{O}_l) \xrightarrow{\alpha} E|_l \xrightarrow{\beta} Im(\beta) \to 0, \tag{38}$$
$$0 \to Im(\beta) \to \mathcal{S}|_l \xrightarrow{\gamma} \mathcal{O}_l \to 0. \tag{39}$$

$\mathcal{T}or_1(\mathcal{O}_l, \mathcal{O}_l) \simeq \mathcal{N}_{l/X}^\vee \simeq \mathcal{O}_l \oplus \mathcal{O}_l(-1)$, thus $\mathrm{Hom}(\mathcal{T}or_1(\mathcal{O}_l, \mathcal{O}_l), \mathcal{O}_l(-3)) = 0$ so that $\mathrm{Hom}(E|_l, \mathcal{O}_l(-3)) \simeq \mathrm{Hom}(Im(\beta), \mathcal{O}_l(-3))$. From (39), $Im(\beta)$ is a rank one torsion-free sheaf of degree -1; but l is a line, therefore $Im(\beta)$ is a line bundle of degree -1, so that $Im(\beta) \simeq \mathcal{O}_l(-1)$. From this we deduce $\mathrm{Hom}(Im(\beta), \mathcal{O}_l(-3)) = 0$ and consequently that $\mathrm{Hom}(E|_l, \mathcal{O}_l(-3)) \simeq \mathrm{Ext}^3(\mathcal{O}_l, E) = 0$.

To prove the vanishing of $\mathrm{Ext}^2(\mathcal{S}, E)$, we apply $\mathrm{Hom}(\mathcal{S}, \cdot)$ to (37), getting:

$$\mathrm{Ext}^1(\mathcal{S}, \mathcal{O}_l) \to \mathrm{Ext}^2(\mathcal{S}, E) \to \mathrm{Ext}^2(\mathcal{S}, \mathcal{S}).$$

The space $\mathrm{Ext}^2(\mathcal{S}, \mathcal{S})$ is zero. Formula (36) (that holds for every rank 2 sheaf having homological dimension at most 1) leads indeed to $\chi(\mathcal{S}, \mathcal{S}) = 1$; since, by stability, $\mathrm{Hom}(\mathcal{S}, \mathcal{S}) \simeq \mathbb{C}$, $\mathrm{Ext}^3(\mathcal{S}, \mathcal{S}) = 0$ and as $\mathrm{Ext}^1(\mathcal{S}, \mathcal{S}) \simeq H^1(\mathcal{S}^* \otimes \mathcal{S}) = 0$ (this is proven e.g. in [30, Theorem 2.10]) we conclude that $\mathrm{ext}^2(\mathcal{S}, \mathcal{S}) = 0$. Now, since $\mathcal{S}$ is locally free, $\mathrm{Ext}^1(\mathcal{S}, \mathcal{O}_l) \simeq H^1(\mathcal{H}om(\mathcal{S}, \mathcal{O}_l))$ and this latter vanishes again due to $\mathcal{S}|_l \simeq \mathcal{O}_l \oplus \mathcal{O}_l(-1)$. These computations yield $\mathrm{Ext}^2(E, E) = 0$ implying the smoothness of $\mathcal{M}$ at E. Also this time the stability of E ensures that $\mathrm{hom}(E, E) = 1$ and $\mathrm{ext}^3(E, E) = 0$, and once again, applying formula (36), we compute $\chi(E, E) = \frac{3}{2}c_1(E)^2 - 6c_2(E) + 4 = -5$. This implies that $\mathrm{ext}^1(E, E) = 6$, ending our proof. □

We consider now the scheme $\mathcal{B}$ parameterizing the pencils of curves $\mathbb{P}(H^0(E(1))$ for $[E] \in \mathcal{L}(2)$. $\mathcal{B}$ identifies with the Grassmann bundle:

$$\mathcal{B} := G_2(\mathrm{Sym}^2(\mathcal{T}_\mathcal{U}^\vee)) \simeq \mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_\mathcal{U}^\vee)^\vee) \xrightarrow{\pi_\mathcal{B}} \mathcal{U}. \tag{40}$$

To see that we indeed have the identification above notice that the dual of the tautological quotient bundles on $\mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_\mathcal{U}^\vee)^\vee)$ and $G_2(\mathrm{Sym}^2(\mathcal{T}_\mathcal{U}^\vee))$ induce, respectively, morphisms $\psi : \mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_\mathcal{U}^\vee)^\vee) \to G_2(\mathrm{Sym}^2(\mathcal{T}_\mathcal{U}^\vee)))$ and $\phi : G_2(\mathrm{Sym}^2(\mathcal{T}_\mathcal{U}^\vee)) \to \mathbb{P}(\mathrm{Sym}^2(\mathcal{T}_\mathcal{U}^\vee)^\vee)$ and that are inverses of each other. By construction, $\mathcal{B}$ is a smooth and irreducible 6-dimensional variety. Our next goal is to show that $\mathcal{B}$ is isomorphic to $\mathcal{L}(2)$. To prove this we will construct a projective bundle $\mathbb{P}(\mathcal{E}) \to \mathcal{B}$ that carries a family of instantons and such that the induced morphism $\mathbb{P}(\mathcal{E}) \to \mathcal{L}(2)$ factors trough an isomorphism $\mathcal{B} \to \mathcal{L}(2)$.

We start by considering the universal curve $\mathbf{Y} \subset \mathcal{H} \times X$ and the relative ext sheaf $\mathcal{E} := \mathcal{E}xt^1_{p_1}(\mathcal{I}_\mathbf{Y}(1), \mathcal{O}_{\mathcal{H}\times X}) \in Coh(\mathcal{H})$, where p_1 is the projection onto the first factor (here for $\mathcal{F} \in Coh(\mathcal{H} \times X)$ we define $\mathcal{F}(n) := \mathcal{F} \otimes {p_2}^*\mathcal{O}_X(n)$.)

Proposition 54. *$\mathcal{E}$ is a rank 2 vector bundle on $\mathcal{H}$ and the projective bundle $\mathbb{P}(\mathcal{E})$ admits the structure of a $\mathbb{P}^1$ bundle over $\mathcal{B}$.*

Proof. Recall that $\mathcal{E} := R^1(p_{1*}\mathcal{H}om(\mathcal{I}_{\mathbf{Y}}(1), \cdot))(\mathcal{O}_{\mathcal{H}\times X})$ hence, from the spectral sequence $R^p p_{1*}(\mathcal{E}xt^q(\mathcal{I}_{\mathbf{Y}}(1), \mathcal{O}_{\mathcal{H}\times X})) \Rightarrow \mathcal{E}xt^{p+q}_{p_1}(\mathcal{I}_{\mathbf{Y}}(1), \mathcal{O}_{\mathcal{H}\times X})$ we obtain an exact sequence:

$$0 \to R^1 p_{1*}(\mathcal{H}om(\mathcal{I}_{\mathbf{Y}}(1), \mathcal{O}_{\mathcal{H}\times X})) \to \mathcal{E} \to$$
$$\to p_{1*}(\mathcal{E}xt^1(\mathcal{I}_{\mathbf{Y}}(1), \mathcal{O}_{\mathcal{H}\times X})) \to R^2 p_{1*}(\mathcal{H}om(\mathcal{I}_{\mathbf{Y}}(1), \mathcal{O}_{\mathcal{H}\times X})) \to 0.$$

As $R^i p_{1*}(\mathcal{H}om(\mathcal{I}_{\mathbf{Y}}(1), \mathcal{O}_{\mathcal{H}\times X})) = 0,\ i = 1, 2$, we get

$$\mathcal{E} \simeq p_{1*}(\mathcal{E}xt^1(\mathcal{I}_{\mathbf{Y}}(1), \mathcal{O}_{\mathcal{H}\times X})) \simeq p_{1*}(\tilde{\omega}_{\mathbf{Y}}(2)),$$

where $\tilde{\omega}_{\mathbf{Y}}$ is the relative dualizing sheaf. $\forall\, [Y] \in \mathcal{H},\ h^0(\omega_Y(2)) = 2$ and since $\mathcal{H}$ is integral, we can conclude that $\mathcal{E}$ is a rank 2 vector bundle.

The isomorphism $\mathcal{E} \simeq p_{1*}(\tilde{\omega}_{\mathbf{Y}}(2))$ also implies that $\mathcal{E}$ commutes with base change, and since $p_{1*}\mathcal{H}om(\mathcal{I}_{\mathbf{Y}}(1), \mathcal{O}_{\mathcal{H}\times X}) = 0$, from [24, Corollary 4.5] we get the existence of a universal extension on $\mathbb{P}(\mathcal{E}) \times X$:

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})\times X} \otimes {p_1}^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to \hat{\mathbf{E}} \to \mathcal{I}_{\hat{\mathbf{Y}}} \to 0 \tag{41}$$

where $\hat{\mathbf{Y}} \subset \mathbb{P}(\mathcal{E}) \times X$ is the pullback of the universal curve $\mathbf{Y}$. Twisting and applying the functor p_{1*} we obtain a short exact sequence of vector bundles on $\mathbb{P}(\mathcal{E})$:

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to p_{1*}(\hat{\mathbf{E}}(1)) \to \pi^*_{\mathcal{E}}(p_{1*}(\mathcal{I}_{\mathbf{Y}}(1))) \to 0 \tag{42}$$

($\pi_{\mathcal{E}} : \mathbb{P}(\mathcal{E}) \to \mathcal{H}$ is the standard projection). We claim that the rank 2 vector bundle $p_{1*}(\hat{\mathbf{E}}(1))$ on $\mathbb{P}(\mathcal{E})$ induces a morphism $\mathbb{P}(\mathcal{E}) \to \mathcal{B}$. This is obtained via a relative version of the argument presented in Remark 49.

Take an affine cover $V_i = Spec(A_i)$ of $\mathbb{P}(\mathcal{E})$ that is trivialising for $p_{1*}(\hat{\mathbf{E}}(1))$; on each V_i we have

$$p_{1*}(\hat{\mathbf{E}}(1)))|_{V_i} \simeq H^0(\hat{\mathbf{E}}(1)|_{V_i\times X})^{\sim} \simeq (A_i^2)^{\sim}.$$

For any non zero $s_i \in H^0(\hat{\mathbf{E}}(1)|_{V_i\times X})$, its image in $H^0(\mathcal{I}_{\hat{\mathbf{Y}}}(1)|_{V_i\times X})$ determines a family $\hat{\mathbf{Q}}_i$ of hyperplane sections whose fibre over $v \in V_i$ is $\langle \hat{\mathbf{Y}}_v \rangle \cap X$. Moreover for any pair s_i, t_i of generators of $H^0(\hat{\mathbf{E}}(1)|_{V_i\times X})$, we have isomorphisms $(\hat{\mathbf{E}}(1)|_{V_i\times X})/(s_i, t_i) \simeq \mathcal{I}_{\hat{\mathbf{Y}}_i, \hat{\mathbf{Q}}_i}(1)$ ($\hat{\mathbf{Y}}_i$ being the restriction of $\hat{\mathbf{Y}}$ to $V_i \times X$). We get therefore injective A_i-linear maps $p_{1*}(\hat{\mathbf{E}}(1))|_{V_i} \hookrightarrow (\pi_{\mathcal{H}} \circ \pi_{\mathcal{E}})^*(\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^{\vee})|_{V_i})$ that glue defining an injective morphism $p_{1*}(\hat{\mathbf{E}}(1)) \hookrightarrow (\pi_{\mathcal{H}} \circ \pi_{\mathcal{E}})^*\,\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^{\vee})$. By the universal property of $\mathcal{B}$, $\pi_{\mathcal{H}} \circ \pi_{\mathcal{E}}$ factors therefore trough a morphism $\rho : \mathbb{P}(\mathcal{E}) \to \mathcal{B}$.

We finally show that $\mathbb{P}(\mathcal{E}) \xrightarrow{\rho} \mathcal{B}$ is a projective bundle. Denote by $\mathcal{T}_{\mathcal{B}} \subset \pi^*_{\mathcal{B}}(\mathrm{Sym}^2(\mathcal{T}_{\mathcal{U}}^{\vee}))$ the tautological rank 2 sub-bundle. By construction $\rho^*(\mathcal{T}_{\mathcal{B}}) \simeq p_{1*}(\hat{\mathbf{E}}(1))$ and since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \hookrightarrow p_{1*}(\hat{\mathbf{E}}(1))$, we get that ρ factors through a morphism $\mathbb{P}(\mathcal{E}) \xrightarrow{\rho'} \mathbb{P}(\mathcal{T}_{\mathcal{B}})$ such that $\rho'^*(\mathcal{O}_{\mathbb{P}(\mathcal{T}_{\mathcal{B}})}(-1)) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. ρ' is the morphism mapping a point (Y, e), $[Y] \in \mathcal{H}$, $e \in \mathrm{Ext}^1(\mathcal{I}_Y(1), \mathcal{O}_X)$ to $([\mathbb{P}(H^0(E(1)))], Y) \in \mathbb{P}(\mathcal{T}_{\mathcal{B}})$, $E = \hat{\mathbf{E}}_{(Y,e)}$. ρ' is a bijective morphism between smooth varieties, therefore it is an isomorphism. □

Remark 55. The variety $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{T}_{\mathcal{B}})$ identifies with the following incidence variety

$$\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{T}_{\mathcal{B}}) \simeq \{(Y, \mathbb{P}^1) \in \mathcal{H} \times \mathcal{B} \mid [Y] \in \mathbb{P}^1\}$$

From the proof of Proposition 54, we learn in particular that $\mathbb{P}(\mathcal{E})$ carries a family of instantons $\hat{\mathbf{E}}$. Accordingly, we have the following:

Corollary 56. *There exists a morphism $\psi : \mathbb{P}(\mathcal{E}) \to \mathcal{L}(2)$ that locally, in the étale topology, has the structure of a $\mathbb{P}^1$-bundle.*

Proof. The family $\hat{\mathbf{E}}$ induces a morphism $\psi : \mathbb{P}(\mathcal{E}) \to \mathcal{M}$ that, by Propositions 44 and 46, surjects onto $\mathcal{L}(2)$. To prove the rest of the current proposition, we argue as in [26, Lemma 5.3]. We start considering an étale cover $\mathbf{W}_i \to \mathcal{L}(2)$ of $\mathcal{L}(2)$ such that each $\mathbf{W}_i \times X$ carries a universal sheaf $\mathbf{E}_i$. Define $\mathbf{G}_i := p_{1*}(\mathbf{E}_i(1))$. This is a rank 2 vector bundle on $\mathbf{W}_i$. Denote by $\mathcal{E}_{\mathbf{W}_i}$ the pullback of $\mathcal{E}$ to $\mathbb{P}(\mathcal{E}) \times_{\mathcal{L}(2)} \mathbf{W}_i$, so that $\mathbb{P}(\mathcal{E}) \times_{\mathcal{L}(2)} \mathbf{W}_i \simeq \mathbb{P}(\mathcal{E}_{\mathbf{W}_i})$. Define ψ_i as the induced morphism $\mathbb{P}(\mathcal{E}_{\mathbf{W}_i}) \to \mathbf{W}_i$, and let $\hat{\mathbf{E}}_{\mathbf{W}_i}$ denote the pullback of $\hat{\mathbf{E}}$ to $\mathbb{P}(\mathcal{E}_{\mathbf{W}_i})$; by the universal property of $\mathbf{E}_i$, $\psi_i^*(\mathbf{E}_i) \simeq \hat{\mathbf{E}}_{\mathbf{W}_i} \times \mathbf{L}$, for some line bundle $\mathbf{L}$ on $\mathbb{P}(\mathcal{E}_{\mathbf{W}_i})$. Observe now that pulling back (42) to $\mathbb{P}(\mathcal{E}_{\mathbf{W}_i})$, we obtain an injection $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathbf{W}_i})}(1) \otimes \mathbf{L}^* \hookrightarrow \psi_i^*(\mathbf{G}_i)$; this induces a morphism $\mathbb{P}(\mathcal{E}_{\mathbf{W}_i}) \to \mathbb{P}(\mathbf{G}_i)$ and once again since this is a bijective morphism between smooth varieties, we conclude that it is an isomorphism. □

From the irreducibility of $\mathbb{P}(\mathcal{E})$ we deduce the irreducibility of $\mathcal{L}(2)$; this observation together with Lemma 52 and Proposition 53 lead to the following claim.

Corollary 57. *$\mathcal{L}(2)$ is a smooth and irreducible scheme of dimension 6.*

Next, we argue that $\mathcal{L}(2)$ is not just locally a fibration over the open subset $\mathcal{U} \subset \mathbb{G}(1, F(X))$ (see Remark 45), but that actually, it is a projective bundle isomorphic to the scheme $\mathcal{B}$ defined in display (40).

Proposition 58. *There exists an isomorphism $\mathcal{L}(2) \xrightarrow{\zeta} \mathcal{B}$ such that $\rho = \zeta \circ \psi$.*

Proof. Set theoretically, ζ is the map sending $[E] \in \mathcal{L}(2)$ to the pencil of curves defined by $\mathbb{P}(H^0(E(1)))$. Let us check that it is actually a morphism of schemes. For each open $V \subset \mathcal{B}$, $\zeta^{-1}(V) = \psi(\rho^{-1}(V))$ is open in $\mathcal{L}(2)$ since ψ is open (this is a consequence of Proposition 56). We have then a morphism $\mathcal{O}_{\mathcal{B}}(V) \to \mathcal{O}_{\mathcal{L}(2)}(\zeta^{-1}(V))$ induced by $\mathcal{O}_{\mathcal{B}}(V) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\rho^{-1}(V))$: indeed from Corollary 56, $\psi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} \simeq \mathcal{O}_{\mathcal{L}(2)}$, thus $\mathcal{O}_{\mathcal{L}(2)}(\zeta^{-1}(V)) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\psi^{-1}(\zeta^{-1}(V))) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\rho^{-1}(V))$. ζ is bijective by construction thus, by the smoothness of $\mathcal{L}(2)$ and $\mathcal{B}$, is an isomorphism. □

We now denote by $\overline{\mathcal{I}(2)}$ the closure of $\mathcal{I}(2)$ in $\mathcal{M}$ and by $\overline{\mathcal{I}(2)}^{inst}$ its open subscheme parameterizing instantons.

Corollary 59. $\mathcal{L}(2) \simeq \overline{\mathcal{I}(2)}^{inst}$.

Proof. $\mathcal{L}(2)$ is a smooth irreducible 6-dimensional variety that contains the moduli $\mathcal{I}(2) = \mathcal{L}(2) \setminus \mathcal{D}(1,1)$ as an open dense subset. Therefore we have equalities $\mathcal{L}(2) = \overline{\mathcal{L}(2)} \cap \mathcal{L}(2) = \overline{\mathcal{I}(2)} \cap \mathcal{L}(2) := \overline{\mathcal{I}(2)}^{inst}$. Note that $\mathcal{I}(2)$ identifies with the following open subset of $\mathcal{L}(2)$: for $\mathring{\mathcal{B}}$, the open subset of base point free pencils, we have $\mathcal{I}(2) = \zeta^{-1}(\mathring{\mathcal{B}})$. □

Corollary 60. $\mathcal{D}(1,1)$ *is contained in* $\overline{\mathcal{I}(2)}$*; in particular a general deformation of a non locally free instanton* $[E]$ *in* $\mathcal{L}(2)$ *is an instanton bundle.*

Proof. $\mathcal{D}(1,1) \subset \overline{\mathcal{I}(2)}$ is an immediate consequence of Corollary 59.

The possibility to deform $[E] \in \mathcal{D}(1,1)$ to an instanton bundle is due to the smoothness of $\mathcal{L}(2)$. Note in particular that we have the following. The locus of pencils of curves with a base point is a smooth and irreducible divisor $\mathcal{Z} \subset \mathcal{B} = \mathcal{B}\backslash\mathring{\mathcal{B}}$ and it is the image of $\mathcal{D}(1,1)$ trough ζ. A deformation of E to an instanton bundle, for $[E] \in \mathcal{D}(1,1)$ corresponds therefore to a deformation of $[\mathbb{P}^1(H^0(E(1)))] \in \mathcal{Z}$ along a direction normal to $\mathcal{Z}$ (the smoothness of $\mathcal{B}$ and $\mathcal{Z}$ implies that such a deformation is always possible). □

6. The moduli space $\mathcal{M}_X(2; -1, 2, 0)$

In this section, we provide a full description of $\mathcal{M} := \mathcal{M}_X(2; -1, 2, 0)$, the moduli space of semistable rank 2 sheaves with Chern classes $(-1, 2, 0)$, on the quadric threefold X. In the previous section we proved that the closure $\overline{\mathcal{L}(2)}$ of the instanton moduli space is an irreducible component of $\mathcal{M}$; to complete our description of $\mathcal{M}$ we pass then to the study of the closed subscheme $\mathcal{C} := \mathcal{M} \setminus \mathcal{L}(2) \subset \mathcal{M}$ consisting of the non instanton sheaves in $\mathcal{M}$. Such sheaves can be characterized as follows.

Proposition 61. *Each sheaf* E *corresponding to a point* $[E] \in \mathcal{C}$ *is obtained by elementary transformation of a* μ*-stable sheaf* F *with Chern classes* $(-1, 2, 2)$ *along a point. Conversely, for each sheaf* F *such that* $[F] \in \mathcal{M}_X(2; -1, 2, 2)$ *the kernel of a surjection* $F \twoheadrightarrow \mathcal{O}_p$ *locates a point in* $\mathcal{C}$

Proof. Let us take a non instanton sheaf E and consider $E^{\vee\vee}$. This latter must be a μ-stable reflexive sheaf having $c_1(E^{\vee\vee}) = -1$ and by Lemma 41, either $c_2(E^{\vee\vee}) = 1$ and $E^{\vee\vee} \simeq \mathcal{S}$ or $c_2(E^{\vee\vee}) = 2$. In the first case, $E^{\vee\vee}/E$ is a one-dimensional sheaf T with Hilbert polynomial $n+1$; we denote by T_0 the maximal zero-dimensional subsheaf of T and by T_1 the quotient $T_1 := T/T_0$. T_1 is thus a line bundle on a line $l \subset X$ and since $\mathcal{S}$ surjects onto T_1, $\mathcal{S}|_l \simeq \mathcal{O}_l(-1) \oplus \mathcal{O}_l$ and $[E] \notin \mathcal{L}(2)$, we conclude that $T_1 \simeq \mathcal{O}_l(-1)$ and that $T_0 \simeq \mathcal{O}_p,\ p = \mathrm{Supp}(T_0)$. Denote by F the kernel of the surection $\mathcal{S} \twoheadrightarrow \mathcal{O}_l(-1)$; this is a μ-stable sheaf with Chern classes $(-1, 2, 2)$ and from the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & \mathcal{S} & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{O}_l(-1) & \longrightarrow & 0
\end{array}$$

we see that $E \simeq \ker(F \twoheadrightarrow \mathcal{O}_p)$. Suppose now that $c_2(E^{\vee\vee}) = 2$. In this case $T := E^{\vee\vee}/E$ is a zero-dimensional and has Chern character $\mathrm{ch}(T) = (0, 0, 0, \frac{c_3(E^{\vee\vee})}{2})$. Applying [10, Theorem 2.2] the spectrum of $E^{\vee\vee}$ can only consists of the integer $k = -2 = -1 - \frac{c_3}{2}$; see Remark 62 below.

Indeed, recall that for a stable rank 2 reflexive sheaf F with $c_1(F) = -1$, the spectrum consists of $c_2(F) - 1$ integers $k_1, \ldots, k_{c_2(F)-1}$ satisfying $\sum_{i=1}^{c_2(F)-1} k_i = -\frac{c_3(F)}{2} - c_2(F) + 1$ and such that, for $\mathcal{H} := \sum_i \mathcal{O}_{\mathbb{P}^1}(k_i)$, we have:

$$h^1(F(j)) = h^0(\mathcal{H}(j+1)), \text{ for } j \leq 0, \quad h^2(F(j)) = h^1(\mathcal{H}(j+1)), \text{ for } j \geq -2 \tag{43}$$

Since $c_3(E^{\vee\vee}) \geq 0$ and $c_2(E^{\vee\vee}) = 2$, the spectrum of $E^{\vee\vee}$ must consist of a unique negative integer k. Such a k must be < -1 otherwise we would have $E \simeq E^{\vee\vee}$ and E would be an instanton by (43), a contradiction. If $k < -2$ the integers $-2, \ldots, k$ would belong to the spectrum as well, which again is impossible. Thus the only possibility is that $k = -2$. This implies that $c_3(E) = 2$ hence that $T \simeq \mathcal{O}_p, \ p = \mathrm{Supp}(T)$.

For the converse implication, we just need to check that the elementary transformation E of a sheaf F, $[F] \in \mathcal{M}_X(2; -1, 2, 2)$ along a point p is indeed semistable. Arguing as above, for $[F] \in \mathcal{M}_X(2; -1, 2, 2)$ we have that $F^{\vee\vee}$ is reflexive with $c_1(F^{\vee\vee}) = -1$ and $c_2(F^{\vee\vee}) = 1$ or 2. In the first case $F^{\vee\vee} = \mathcal{S}$; in the second, applying again [10, Theorem 2.2], we get $c_3(F^{\vee\vee}) = 2$ hence $F \simeq F^{\vee\vee}$. In both cases $F^{\vee\vee}$ is μ-stable therefore, if $E = \ker(F \twoheadrightarrow \mathcal{O}_p)$, E is μ-stable as well since $E^{\vee\vee} \simeq F^{\vee\vee}$. □

Remark 62. Theorem 2.2 of [10], as stated in loc. cit., presents an inaccuracy. Given a stable rank 2 reflexive sheaf E on the quadric X, we have indeed that item 2(b) of the cited theorem holds whenever $c_1(E) = 0$; however, when $c_1(E) = -1$, having -2 in the spectrum of E no longer implies that -1 belongs to the spectrum as well.

To see this, following the proof of Theorem 2.2 of [10], adopting the same notation, let us consider the module

$$R := \ker \left\{ \oplus_l H^2(E(l)) \xrightarrow{x} \oplus_l H^2(E(l+1)) \right\}$$

where $x \in H^0(\mathcal{O}_X(1))$ is a general linear form defining a hyperplane section $Q_2 \subset X$; in addition, let us consider the graded module

$$N := \mathrm{im} \left\{ \oplus_l H^1(E(l)) \xrightarrow{\rho} \oplus_l H^1(E|_{Q_2}(l)), \right.$$

where ρ denotes the cohomology map induced by the restriction morphism $E \to E|_{Q_2}$.

Denote by R_l (respectively N_l) the graded component of R (respectively N) of degree l and define $r_l := \dim(R_l)$ and $n_l := \dim(N_l)$. The following identities hold:

$$r_{l+1} = h^2(E(l)) - h^2(E(l+1)) = \#\{k \in \text{spectrum of } E \mid k \leq -l-3\},$$
$$n_l = h^1(E(l)) - h^1(E(l-1)) = \#\{k \in \text{spectrum of } E \mid k \geq -l-1\}.$$

Let us look at $r_{-1} - r_0$, which coincides with the number of times -1 occurs in the spectrum of E. By [10, Lemma 2.1 item (2)], if $c_1(E) = -1, r_0 \neq 0$ (which occurs if ever we have an integer ≤ -2 in the spectrum of E) does not imply $r_{-1} > r_0$ whenever r_0 is not strictly less than $h^1(E|_{Q_2}(-1))$; this is the case, for example, of the sheaves in $\mathcal{R}(2; -1, 2, 2)$. In particular, since

$$n_0 + r_0 = n_{-1} + r_{-1} = h^1(E|_{Q_2}(-1)) = c_2 - 1,$$

cf. proof of [10, Theorem 2.2], if $c_2(E) = 2$, then it is clear that none of the summands can be strictly larger than 1 so that, if ever $r_0 \neq 0$, we must have $r_{-1} = r_0 = 1$.

Now, if $[E] \in \mathcal{C}$ and $c_2(E^{\vee\vee}) = 2$, the spectrum of $E^{\vee\vee}$ must consist of a unique negative integer $k = -\frac{c_3(E)^{\vee\vee}}{2} - 1$; if ever $k = -1$ we would have $E \simeq E^{\vee\vee}$ (as $c_3(E^{\vee\vee}) = 0$) and $h^1(E(-1)) = h^2(E(-1)) = 0$, which is a contradiction. If $k < -2$, by [10, Lemma 2.1 item 1(b)], $-2, \ldots, k$ would all belong to the spectrum which again is impossible. Therefore the only possibility is that $k = -2$ hence $c_3(E^{\vee\vee}) = 2$.

Once again we will use the Serre correspondence to deduce the geometric properties of $\mathcal{C}$ from the geometry of the family of the corresponding curves; these curves will still belong to the Hilbert scheme $\mathrm{Hilb}_{2t+2}(X)$ but this time they won't be l.c.m.

The study of $\mathcal{C}$ will lead us to prove the main result of this section:

Theorem 63. *The moduli space $\mathcal{M}$ is connected and consists of two irreducible components:*

(1) *A 6-dimensional component $\overline{\mathcal{L}(2)}$ given by the closure of the open subset of instanton sheaves;*
(2) *A 10-dimensional irreducible component $\mathcal{C}$ consisting of non instanton sheaves.*

In addition, $\mathcal{M}$ is generically smooth along both components.

6.1. The moduli space $\mathcal{M}_X(2; -1, 2, 2)$

In order to better understand the geometry of $\mathcal{C}$, we first need to study $\mathcal{M}_X(2; -1, 2, 2)$. From the proof of Proposition 61 we have already learned that we have two families of sheaves in $\mathcal{M}_X(2; -1, 2, 2)$:

Lemma 64. *Let F be a rank 2 semistable sheaf F with Chern classes* $(-1, 2, 2)$. *Then F is μ-stable and either F is reflexive and* $\mathrm{Sing}(F)$ *is zero dimensional of length 2, or $F^{\vee\vee} \simeq \mathcal{S}$ and $\mathcal{S}/F \simeq \mathcal{O}_l(-1)$ for a line $l \subset X$.*

We are going to prove the following:

Theorem 65. $\mathcal{M}_X(2; -1, 2, 2)$ *is a smooth 6-dimensional irreducible variety isomorphic to* $\mathbb{G}(1, \mathbb{P}(V))$.

Also on this occasion, our main tool to study sheaves in $\mathcal{M}_X(2;-1,2,2)$ is the Serre correspondence. Towards the rest of the section, we denote by $\mathcal{R}(2;-1,2,2)$ the open subset of $\mathcal{M}_X(2;-1,2,2)$ parameterizing reflexive sheaves.

Lemma 66. *Let* $[F]$ *be a point in* $\mathcal{M}_X(2;-1,2,2)$. *Then* $h^0(F(1)) = 3$ *and for each* $s \in H^0(E(1))$, $s \neq 0$, $\operatorname{coker}(s) \simeq \mathcal{I}_C(1)$ *for a conic* $C \subset X$.

Proof. Let us start by considering the case $[F] \in \mathcal{R}(2;-1,2,2)$. We show that $F(1)$ always admits global sections. By Riemann–Roch $\chi(F(1)) = 3$ and by stability $h^3(F(1)) = 0$. As already claimed in the proof of Proposition 61, the spectrum of F consists just of the integer -2 (due to [10, Theorem 2.2]) which implies the following:

$$h^1(F(j)) = h^0(\mathcal{O}_{\mathbb{P}^1}(j-1)),\ \forall\, j \leq 0,\ h^2(F(j)) = h^1(\mathcal{O}_{\mathbb{P}^1}(j-1)),\ \forall\, j \geq -2. \tag{44}$$

This means that $h^2(F(1)) = 0$ hence we necessarily have $h^0(F(1)) > 0$. Now, since $H^0(F) = 0$, for any nonzero section $s \in H^0(F(1))$, $\operatorname{coker}(s)$ is torsion-free and of the form $\mathcal{I}_C(1)$ for a l.c.m. curve $C \subset X$.

A Chern class computation leads to $P_C(n) = 2n + 1$ hence C is a plane conic and from the short exact sequence

$$0 \to \mathcal{O}_X \to F(1) \to \mathcal{I}_C(1) \to 0 \tag{45}$$

we compute that $h^0(F(1)) = 3$ hence $h^1(F(1)) = 0$. If $F^{\vee\vee} \simeq \mathcal{S}$ instead, from the short exact sequence

$$0 \to F \to \mathcal{S} \to \mathcal{O}_l(-1) \to 0,$$

we get $h^i(F(1)) = h^i(\mathcal{S}(1)) = 0$, $i = 2, 3$; as moreover $H^0(\mathcal{S}(1)) \to H^0(\mathcal{O}_l)$ can not be the zero map (for $l' \subset X$ general, there are no surjection $\mathcal{I}_{l'}(1) \twoheadrightarrow \mathcal{O}_l$) we conclude that $h^0(F(1)) = 3$ and $h^1(F(1)) = 0$. Since any global section of $\mathcal{S}$ has torsion-free cokernel, for any non-zero $s \in H^0(F(1))$, $\operatorname{coker}(s) \simeq \mathcal{I}_Z(1)$ fits in

$$0 \to \mathcal{I}_Z(1) \to \mathcal{I}_{l'}(1) \to \mathcal{O}_l \to 0. \tag{46}$$

with $l' = \operatorname{coker}(\iota(s))$, $\iota := H^0(F(1)) \hookrightarrow H^0(\mathcal{S}(1))$. Therefore Z is a reducible conic supported on $l \cup l'$ (note that for s general, as $\mathcal{I}_{l'}$ surjects onto $\mathcal{O}_l(-1)$, l' will meet l at a point). □

It is straightforward to check that this construction can be "reversed", leading to the following claim.

Lemma 67. *Serre correspondence establishes a 1-1 correspondence between*

- *Pairs* (F, s) *with* $[F] \in \mathcal{R}(2;-1,2,2)$, $s \in \mathbb{P}(H^0(F(1)))$ *(resp. pairs* (F, s) *with* $[F] \in \mathcal{M}_X(2;-1,2,2)\backslash\mathcal{R}(2;-1,2,2)$, $s \in \mathbb{P}(H^0(F(1)))$*)*
- *Pairs* (C, ξ) *with* $[C] \in \operatorname{Hilb}_{2t+1}(X)$, $\xi \in \mathbb{P}(H^0(\omega_C(2)))$ *vanishing along 2 points on* C *(resp.* (C, ξ), $[C] \in \operatorname{Hilb}_{2t+1}(X)$, *reducible* $\xi \in \mathbb{P}(H^0(\omega_C(2)))$ *vanishing along a component of* C*)*

Mimicking what we have done for sheaves in $\mathcal{L}(2)$, we are now going to describe in detail the linear system $\mathbb{P}(H^0(F(1))) \simeq \mathbb{P}^2$ of conics associated to $[F] \in \mathcal{M}_X(-2; -1, 2, 2)$. This will help us to better understand the geometry of the scheme $\mathcal{M}_X(2; -1, 2, 2)$.

Proposition 68. (1) *If* $[F] \in \mathcal{R}(2; -1, 2, 2)$ *and* $\mathrm{Sing}(F)$ *consists of two distinct points* $p_1,\ p_2$, $\mathbb{P}(H^0(F(1)))$ *identifies with the linear system of conics containing* $p_i,\ i = 1, 2$ *and its image under the isomorphism* $\mathrm{Hilb}_{2t+1}(X) \simeq \mathbb{G}(2, V^*)$ *is the Schubert variety of planes containing* $\overline{p_1 p_2}$.

(2) *If* $[F] \in \mathcal{R}(2; -1, 2, 2)$ *and* $\mathrm{Sing}(F)_{\mathrm{red}} = p$, *there exists a line* $l \subset \mathbb{P}^4$ *tangent at* p *to each conic in* $\mathbb{P}(H^0(F(1)))$ *and the image of* $\mathbb{P}(H^0(F(1)))$ *under the isomorphism* $\mathrm{Hilb}_{2t+1}(X) \simeq \mathbb{G}(2, V^*)$ *is the Schubert variety of planes containing* l.

(3) *If* $[F] \in \mathcal{M}_X(2; -1, 2, 2)\backslash\mathcal{R}(2; -1, 2, 2)$, $\mathbb{P}(H^0(F(1)))$ *identifies with the Schubert variety of planes containing the line* $\mathrm{Sing}(F)$.

Proof. (1) Suppose at first that F reflexive with $\mathrm{Sing}(F) = \{p_1,\ p_2\},\ p_1 \neq p_2$. It is easy then to compute that the conics in X passing through these points are parameterized by a plane: a conic $C \subset X$ passes indeed through the points $p_1,\ p_2$ if and only if $\overline{p_1 p_2} \subset \langle C\rangle$. This means that the image of the family of conics passing through the p_is, under the isomorphism $\mathrm{Hilb}_{2t+1}(X) \simeq \mathbb{G}(2, V^*),\ C \mapsto \langle C\rangle$ is the Schubert variety of planes containing $\overline{p_1 p_2}$ that is a plane in $\mathbb{G}(2, V^*)$. The proposition follows since every conic in $\mathbb{P}(H^0(F(1)))$ contains $p_i,\ i = 1, 2$.

(2) Whenever $\mathrm{Sing}(F)$ is supported on a single point p we can compute again that the linear system of conics containing $\mathrm{Sing}(F)$ is a plane in $\mathbb{G}(2, V^*)$. The scheme $\mathrm{Sing}(F)$ corresponds indeed to the data of the point p together with a tangent direction $v \in T_pX$ or equivalently, to a line l tangent to X at p. A conic C contains $\mathrm{Sing}(F)$ if and only if $\langle C\rangle$ contains l, hence $\mathbb{P}(H^0(F(1)))$ identifies with the Schubert variety $\mathbb{P}^2 \subset \mathbb{G}(2, V^*)$ of planes containing l.

(3) Finally if F is singular along a line l, consider the inclusion $\iota : H^0(F(1)) \hookrightarrow H^0(\mathcal{S}(1))$. Each $\iota(s)$ defines a line l' giving rise to a short exact sequence of the form (46). Since $\mathcal{I}_{l'}$ surjects onto $\mathcal{O}_l(-1)$ if and only if either $l = l'$ or $l \cap l'$ consists of a point, we deduce therefore that $\mathbb{P}(\iota(H^0(F(1)))$ identifies with the space of lines meeting l and that $\mathbb{P}(H^0(F(1))$ identifies therefore with the family of planes containing l. □

From now on the family of conics associated to $[F] \in \mathcal{M}_X(2; -1, 2, 2)$ will simply be denoted by $\mathbb{P}(H^0(F(1)))$ and the line contained in every plane $\langle C\rangle$, $C \in \mathbb{P}(H^0(F(1)))$ will be denoted by l_F (note that for $[F]$ belonging to the closed subscheme $\mathcal{M}_X(2; -1, 2, 2)\backslash\mathcal{R}(2; -1, 2, 2)$, $l_F = \mathrm{Sing}(F)$).

Lemma 69. $[F] \in \mathcal{R}(2; -1, 2, 2)$ *if and only if* $l_F \not\subset X$.

Proof. If $l_F \subset X$, all the conics $C \in \mathbb{P}(H^0(F(1)))$ contain l_F and $\mathrm{Sing}(F) \subset l_F$. This can not happen if F is reflexive, since if ever a section $\xi \in H^0(\omega_C(2))$, $[C] \in \mathbb{P}(H^0(F(1))$ vanishes along 2 points on $l_F \subset C$, it would vanish along the entire l_F contradicting the reflexivity of F. The converse implication is obvious since $l_F \not\subset X$ ensures that $\mathrm{Sing}(F) = l_F \cap X$ consists of two points hence, by Lemma 64, $[F] \in \mathcal{R}(2; -1, 2, 2)$. □

Let us now analyze the local behavior of the moduli space $\mathcal{M}_X(2; -1, 2, 2)$.

Proposition 70. *For each point* $[F] \in \mathcal{M}_X(2; -1, 2, 2)$ *we have* $\mathrm{ext}^2(F, F) = 0$ *and* $\mathrm{ext}^1(F, F) = 6$

Proof. A general section $s \in H^0(F(1))$ defines a short exact sequence of the form (45).

Applying the functor $\mathrm{Hom}(\,\cdot\,, F)$ we get a sequence:

$$\mathrm{Ext}^2(\mathcal{I}_C, F) \to \mathrm{Ext}^2(F, F) \to \mathrm{Ext}^2(\mathcal{O}_X(-1), F);$$

The right side term is zero since, from (44), $H^2(F(1)) = 0$; from (45) we compute the vanishing of $H^i(F) = 0,\ i = 2, 3$ yielding:

$$\mathrm{Ext}^2(\mathcal{I}_C, F) \simeq \mathrm{Ext}^3(\mathcal{O}_C, F) \simeq \mathrm{Hom}(F, \mathcal{O}_C(-3))^*.$$

Applying $- \otimes \mathcal{O}_C$ to (45), we obtain an exact sequence

$$0 \to \mathcal{T}or_1^{\mathcal{O}_X}(F, \mathcal{O}_C) \xrightarrow{a} \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{I}_C, \mathcal{O}_C) \xrightarrow{b} \mathcal{O}_C(-1) \xrightarrow{c} F|_C \xrightarrow{d} \mathcal{N}_{C/X}^{\vee} \to 0$$

from which we extract the short exact sequences:

$$0 \to \ker(d) \to F|_C \xrightarrow{d} \mathcal{N}_{C/X}^{\vee} \to 0 \quad (47)$$

$$0 \to \ker(c) \to \mathcal{O}_C(-1) \to \ker(d) \to 0 \quad (48)$$

Applying $\mathrm{Hom}(\,\cdot\,\mathcal{O}_C(-3))$ to (48), we get that $\mathrm{Hom}(\ker(d), \mathcal{O}_C(-3))$ injects into $\mathrm{Hom}(\mathcal{O}_C(-1), \mathcal{O}_C(-3)) \simeq H^0(\mathcal{O}_C(-2)) = 0$. Therefore, from (47), we get $\mathrm{Hom}(F|_C, \mathcal{O}_C(-3)) \simeq \mathrm{Hom}(\mathcal{N}_{C/X}^{\vee}, \mathcal{O}_C(-3)) \simeq H^0(\mathcal{N}_{C/X}(-3))$. C is a plane section of X, therefore $\mathcal{N}_{C/X} \simeq \mathcal{O}_C(1)^{\oplus 2}$ hence $H^0(\mathcal{N}_{C/X}(-3)) = 0$ implying $\mathrm{Hom}(F, \mathcal{O}_C(-3)) = 0$ and finally, $\mathrm{Ext}^2(F, F) = 0$. Let us now compute $\chi(F, F)$. Since this value is constant on the entire moduli $\mathcal{M}_X(2; -1, 2, 2)$ (the Euler bilinear is indeed defined on the Grothendieck group $K_0(X)$), it is sufficient to compute it for F reflexive. In this case we can then argue as in [14, Proposition 3.4], getting

$$\chi(F, F) = -5.$$

The stability of F implies that $\hom(F, F) = 1$ and that $\mathrm{ext}^3(F, F) = 0$; from our previous arguments $\mathrm{ext}^2(F, F) = 0$ hence $\mathrm{ext}^1(F, F) = 6$. □

We consider now $\mathrm{Hilb}_{t^2+tn+2}(\mathbb{G}(1, \mathbb{P}(V^*)))$, the Hilbert scheme of planes in $\mathbb{G}(1, \mathbb{P}(V^*))$. Recall that this scheme has two components: a component Ω parameterizing families of planes contained in the same hyperplane and a second component Λ parameterizing families of planes Λ_l containing a fixed line l. This latter is isomorphic to $\mathbb{G}(1, \mathbb{P}(V))$ via the morphism:

$$\mathbb{G}(1, \mathbb{P}(V)) \longrightarrow \Lambda, l \mapsto \Lambda_l \simeq \mathbb{G}(1, \mathbb{P}(H^0(\mathcal{I}_l(1))).$$

We consider now the map:

$$\mathcal{M}_X(2; -1, 2, 2) \xrightarrow{\alpha} \Lambda \simeq \mathbb{G}(1, \mathbb{P}(V)),\ [F] \mapsto \mathbb{P}(H^0(F(1))) \leftrightarrow l_F.$$

Proposition 71. *α is an isomorphism of scheme; it identifies $\mathcal{R}(2;-1,2,2)$ (resp. $\mathcal{M}_X(2;-1,2,2)\backslash\mathcal{R}(2;-1,2,2)$) with $\mathbb{G}(1,\mathbb{P}(V))\backslash F(X)$ (resp. $F(X)$).*

Proof. We apply verbatim the arguments used in the proof of Proposition 54. Doing so we show that the sheaf $\mathcal{G} := \mathcal{E}xt^1_{p_1}(\mathcal{I}_{\mathbf{C}}, \mathcal{O}_{X\times \mathrm{Gr}}(-1))$ on $\mathrm{Hilb}_{2t+1} \simeq \mathbb{G}(1,\mathbb{P}(V)^*)$, for $\mathbf{C} \subset \mathbb{G}(1,\mathbb{P}(V^*)) \times X$ the universal curve, is locally free of rank 3 and that the projective bundle $\mathbb{P}(\mathcal{G})$ carries a family $\hat{\mathbf{F}} \in Coh(\mathbb{P}(\mathcal{G}) \times X)$ such that $[\hat{\mathbf{F}}_{(C,e)}]$ is the sheaf constructed from $e \in \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-1))$. $p_{1*}(\hat{\mathbf{F}}(1))$ defines a family of linear systems of conics over $\mathbb{P}(\mathcal{G})$ inducing a morphism $\mathbb{P}(\mathcal{G}) \xrightarrow{\gamma} \Lambda$ such that $p_{1*}(\hat{\mathbf{F}}(1)) \simeq \gamma^*(\mathcal{T})$ for $\mathcal{T}$ the tautological rank 3 bundle over Λ (that is to say, $\mathcal{T}$ is the bundle whose fibre over $\Lambda_l \in \Lambda$ is the vector space $\bigwedge^2(H^0(\mathcal{I}_l(1))) \simeq \mathbb{C}^3$ of planes belonging to Λ_l). γ is the morphism mapping (C,e) to $\Lambda_{l_F} \simeq \mathbb{P}(H^0(F(1)))$ for F the sheaf arising from $e \in Ext^1(\mathcal{I}_C, \mathcal{O}_X(-1))$.

The family $\hat{\mathbf{F}}$ induces a morphism $\mathbb{P}(\mathcal{G}) \xrightarrow{\beta} \mathcal{M}_X(2;-1,2,2)$, and applying the argument used in the proof of Corollary 56 we show that β is, in the étale topology, a $\mathbb{P}^2$-bundle. In this way we also deduce that $\mathcal{M}_X(2;-1,2,2)$ is irreducible of dimension 6 hence, by Proposition 70, we get $\mathrm{ext}^1(F,F) = 6$, $\forall [F] \in \mathcal{M}_X(2;-1,2,2)$. Reasoning then as in Proposition 58, we show that due to the properties of β, α is well defined as a morphism of schemes. Since α maps bijectively $\mathcal{M}_X(2;-1,2,2)$ into $\Lambda \simeq \mathbb{G}(1,\mathbb{P}(V))$ and since both schemes are smooth, we conclude that α is an isomorphism. The fact that $\alpha(\mathcal{R}(2;-1,2,2)) = \mathbb{G}(1,\mathbb{P}(V)) \setminus F(X)$ is due to lemma 69. This ends the proof of the proposition. □

This completes the proof of Theorem 65.

6.2. Description of $\mathcal{C}$

We can finally come back to the description of $\mathcal{C}$.

Proposition 72. *For $[E] \in \mathcal{C}$, $h^0(E(1)) = 2$ and for all $s \in H^0(E(1))$, $s \neq 0$,* $\mathrm{coker}(s) \simeq \mathcal{I}_\Gamma(1)$*, for Γ a curve union of a conic and a point. More precisely, all the curves Γ in $\mathbb{P}(H^0(E(1)))$ are of the form:*

$$0 \to \mathcal{O}_p \to \mathcal{O}_\Gamma \to \mathcal{O}_C \to 0$$

with p fixed and with C varying in a pencil of conics contained in a fixed hyperplane.

Proof. From Proposition 61, E always fits in a short exact sequence:

$$0 \to E \to F \to \mathcal{O}_p \to 0 \tag{49}$$

with $[F] \in \mathcal{M}_X(2;-1,2,2)$. Twisting (49) and taking global section, we deduce that $h^0(E(1)) \neq 0$; moreover the fact that $\forall s \in H^0(F(1))$, $\mathrm{coker}(s)$ is torsion-free, ensures that the same holds for all non-zero $s \in H^0(E(1))$. As usual we denote by ι the inclusion $\iota : H^0(E(1)) \hookrightarrow H^0(F(1))$. For any non-zero $s \in H^0(E(1))$ we therefore have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow s & & \downarrow \iota(s) & & \downarrow & & \\
0 & \longrightarrow & E(1) & \longrightarrow & F(1) & \longrightarrow & \mathcal{O}_p(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow id & & \\
0 & \longrightarrow & \mathcal{I}_\Gamma(1) & \longrightarrow & \mathcal{I}_C(1) & \longrightarrow & \mathcal{O}_p(1) & \longrightarrow & 0
\end{array}$$

from which we deduce that $\operatorname{coker}(s) \simeq \mathcal{I}_\Gamma(1)$, for Γ a curve with Hilbert polynomial $2t+2$ and supported on $C \cup p$. Since we can have no plane section of X containing Γ, we have $h^0(\mathcal{I}_\Gamma(1)) = 1$ hence $h^0(E(1)) = 2$. Now, $\mathbb{P}(\iota(H^0(E(1)))$ is a pencil in $\mathbb{P}(H^0(F(1))) \simeq \Lambda_{l_F}$, therefore there exists a unique hyperplane section containing all the conics in $\mathbb{P}(\iota(H^0(E(1)))$. □

Remark 73. Suppose that $p \notin \operatorname{Sing}(F)$. From the proof of the previous proposition, we learn that each $f \in \mathbb{P}(\operatorname{Hom}(F, \mathcal{O}_p))$ locates a unique hyperplane $H \in \mathbb{P}(H^0(\mathcal{I}_{\langle l_F, p\rangle}(1)))$ containing the curves $\mathbb{P}(H^0(E(1)))$, $E := \ker(f)$. For $C \in \mathbb{P}(\iota(H^0(E(1))))$ general, this is the hyperplane generated by $\langle C\rangle$ and p. Notice also that for $f,\ f' \in \mathbb{P}(\operatorname{Hom}(F, \mathcal{O}_p))$, $f \neq f'$, denoting by $E := \ker(f)$, $E' := \ker(f')$, the pencils $\mathbb{P}(\iota(H^0(E(1))))$ and $\mathbb{P}(\iota(H^0(E'(1))))$ meet just at $\langle l_F, p\rangle \cap X$. If ever $\mathbb{P}(\iota(H^0(E(1)))) = \mathbb{P}(\iota(H^0(E'(1))))$, we would indeed have that for all conic $C \in \mathbb{P}(\iota(H^0(E(1)))$ such that $p \notin C$, $C \cup p \in \mathbb{P}(H^0(E(1))) \cap \mathbb{P}(H^0(E'(1)))$. Since $\operatorname{Ext}^1(\mathcal{I}_{C\cup p}, \mathcal{O}_X(-1)) \simeq \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-1))$, E and E' would then both arise from the unique extension class image of the element $\xi \in \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-1))$ defining F, which would lead to $E \simeq E'$, a contradiction.

We describe in this way a pencil (parameterized by $\mathbb{P}(\operatorname{Hom}(F, \mathcal{O}_p))$) of lines in $\mathbb{P}(H^0(F(1)))$ that identifies with the family of hyperplanes containing $\langle l_F, p\rangle$.

Now, since each point $[E] \in \mathcal{C}$ uniquely determines a pair $(F, p) \in \mathcal{M}_X(2; -1, 2, 2) \times X$, we have a well-defined map (for the moment just defined at the level of sets)

$$\delta : \mathcal{C} \longrightarrow \mathcal{M}_X(2; -1, 2, 2) \times X,\ [E] \mapsto (F, p)$$

where (F, p) are such that E is obtained by elementary transformation of F along p. Consider now the open subset $(\mathcal{M}_X(2; -1, 2, 2) \times X)_0$ parameterizing pairs (F, p) such that $p \notin \operatorname{Sing}(F)$ and denote by $\mathcal{C}_0 \subset \mathcal{C}$ its preimage under δ.

Proposition 74. *For each point* $[E] \in \mathcal{C}_0,\ \operatorname{ext}^1(E, E) = 10$.

Proof. We know that E always fits in a short exact sequence:

$$0 \to E \to F \to \mathcal{O}_p \to 0 \tag{50}$$

with $[F] \in \mathcal{M}_X(2; -1, 2, 2)$. We apply $\operatorname{Hom}(\,\cdot\,, E)$ to it. We can see immediately that the stability of E and of F imposes $\operatorname{Hom}(F, E) = 0$; therefore, we obtain the following exact sequence:

$$\begin{aligned} 0 &\to \operatorname{Hom}(E, E) \to \operatorname{Ext}^1(\mathcal{O}_p, E) \to \operatorname{Ext}^1(F, E) \to \\ &\to \operatorname{Ext}^1(E, E) \to \operatorname{Ext}^2(\mathcal{O}_p, E) \to \operatorname{Ext}^2(F, E) \end{aligned} \tag{51}$$

The term $\mathrm{Ext}^2(F, E)$ fits into:

$$\mathrm{Ext}^1(F, \mathcal{O}_p) \to \mathrm{Ext}^2(F, E) \to \mathrm{Ext}^2(F, F)$$

The right side term vanishes due to Proposition 70; since moreover we are assuming $[E] \in \mathcal{C}_0$, F is locally free at p so that $\mathrm{Ext}^1(F, \mathcal{O}_p) \simeq H^1(\mathcal{H}om(F, \mathcal{O}_p)) = 0$.

These computations lead to $\mathrm{Ext}^2(F, E) = 0$. Let us now compute the dimensions of the spaces $\mathrm{Ext}^{3-i}(\mathcal{O}_p, E) \simeq \mathrm{Ext}^i(E, \mathcal{O}_p(-3))^*,\ i = 1, 2$. Again, since $p \notin \mathrm{Sing}(F),\ \mathrm{Ext}^i(F, \mathcal{O}_p(-3)) = 0,\ i = 1, 2$, therefore

$$\mathrm{Ext}^i(E, \mathcal{O}_p(-3)) \simeq \mathrm{Ext}^{i+1}(\mathcal{O}_p, \mathcal{O}_p(-3)) \simeq \mathrm{Ext}^{2-i}(\mathcal{O}_p, \mathcal{O}_p)^*,\ i = 1, 2.$$

For $i = 2$ we thus get $\mathrm{ext}^1(\mathcal{O}_p, E) = 1 = \mathrm{hom}(\mathcal{O}_p, \mathcal{O}_p)$, which implies that $\mathrm{Hom}(E, E) \simeq \mathrm{Ext}^1(\mathcal{O}_p, E)$, whilst for $i = 1$ we obtain $\mathrm{ext}^2(\mathcal{O}_p, E) = \mathrm{ext}^1(\mathcal{O}_p \mathcal{O}_p) = h^0(\mathcal{N}_{p/X}) = 3$, so that $\mathrm{ext}^1(E, E) = \mathrm{ext}^1(F, E) + 3$. In order to determine $\mathrm{ext}^1(F, E)$, we apply this time $\mathrm{Hom}(F, \cdot)$ to (50) which leads to:

$$0 \to \mathrm{Hom}(F, F) \to \mathrm{Hom}(F, \mathcal{O}_p) \to \mathrm{Ext}^1(F, E) \to \mathrm{Ext}^1(F, F) \to 0 \quad (52)$$

($\mathrm{Hom}(F, E)$ vanishes due to the stability of E and F). Since F must be simple and as $p \notin \mathrm{Sing}(F)$, $\mathrm{hom}(F, F) = 1,\ \mathrm{hom}(F, \mathcal{O}_p) = 2$ so that, as $\mathrm{ext}^1(F, F) = 6$ (cf. Proposition 70), we obtain $\mathrm{ext}^1(F, E) = 7$. This allows us to conclude that $\mathrm{ext}^1(E, E) = 10$. □

Proposition 75. *$\mathcal{C}_0$ is a smooth 10-dimensional irreducible scheme.*

Proof. We will construct a $\mathbb{P}^1$ bundle $\mathbb{P}(\mathcal{A})$ over $(\mathcal{M}_X(2; -1, 2, 2) \times X)_0$ and show that this is endowed with a morphism $\mathbb{P}(\mathcal{A}) \to \mathcal{M}$ mapping $\mathbb{P}(\mathcal{A})$ bijectively into $\mathcal{C}_0$. We consider the Grassmanniann of lines $\mathbb{G}(1, \mathbb{P}(V))$ in $\mathbb{P}(V) \simeq \mathbb{P}^4$. For the ease of notations towards the rest of the proof this latter will always be denoted simply by Gr. We define $(\mathrm{Gr} \times X)_0$ as the open set:

$$(\mathrm{Gr} \times X)_0 := \{([l], p) \in \mathrm{Gr} \times X \mid p \notin l\}.$$

This scheme carries a family of planes $\tilde{\Pi} \subset (\mathrm{Gr} \times X)_0 \times X$ such that $\tilde{\Pi}_{(l,p)} \simeq \langle l, p\rangle$ and the sheaf $\mathcal{A} := p_{1*}(\mathcal{I}_{\tilde{\Pi}}(1))$ is a rank 2 vector bundle on $(\mathrm{Gr} \times X)_0$.

Now, the isomorphism $\beta : \mathcal{M}_X(2; -1, 2, 2) \to \mathrm{Gr}$ (see Proposition 65) induces an isomorphism $(\mathrm{Gr} \times X)_0 \xrightarrow{\simeq} (\mathcal{M}_X(2; -1, 2, 2) \times X)_0$, hence $\mathbb{P}(\mathcal{A})$ is a $\mathbb{P}^1$-bundle over $(\mathcal{M}_X(2; -1, 2, 2) \times X)_0$ as well (the fibre over a point (F, p) identifies with the pencil $\mathbb{P}(H^0(\mathcal{I}_{\langle l_F, p\rangle}(1)))$ of hyperplanes containing $\langle l_F, p\rangle$). In order to prove that $\mathbb{P}(\mathcal{A})$ admits a morphism to $\mathcal{M}$, we consider an étale cover $\mathbf{W}$ of $\mathcal{M}_X(2; -1, 2, 2)$ supporting a universal sheaf $\mathbf{F}_\mathbf{W}$. This induces an étale cover $\tilde{\mathbf{W}} \to (\mathcal{M}_X(2; -1, 2, 2) \times X)_0$, we denote by $\tilde{\mathcal{A}}_\mathbf{W}$ the pullback of $\mathcal{A}$ to $\tilde{\mathbf{W}}$ and by $\tilde{\mathbf{F}}_\mathbf{W}$ the pullback of $\mathbf{F}_\mathbf{W}$ to $\tilde{\mathbf{W}} \times X$. $\mathbb{P}(\tilde{\mathcal{A}}_\mathbf{W})$ identifies with the $\mathbb{P}^1$ subbundle of $G_2(p_{1*}(\tilde{\mathbf{F}}_\mathbf{W}(1)))$ whose fibre over a point $(w, p) \in \tilde{\mathbf{W}}$ is the 1-dimensional linear space:

$$\pi^{-1}_{\tilde{\mathcal{A}}_\mathbf{W}}(w, p) = \{\mathbb{P}^1 \subset \mathbb{P}(H^0(\mathbf{F}_w(1))) \simeq \Lambda_{l_{\mathbf{F}_w}} \mid [\langle l_{\mathbf{F}_w}, p\rangle] \in \mathbb{P}^1\}$$

Define now $\tilde{\Delta}_{\mathbf{W}}$ as the pullback to $\tilde{\mathbf{W}} \times X$ of the diagonal $\Delta \subset X \times X$ and consider the sheaf $\tau_{\mathbf{W}} := p_{1*}(\mathcal{H}om(\tilde{\mathbf{F}}_{\mathbf{W}}, \mathcal{O}_{\tilde{\Delta}_{\mathbf{W}}}))$. This is a rank 2 vector bundle over $\tilde{\mathbf{W}} \times X$ whose fibre over (w, p) identifies with $\mathrm{Hom}(\mathbf{F}_w, \mathcal{O}_p)$. We claim that we have an isomorphism: $\mathbb{P}(\tilde{\mathcal{A}}_{\mathbf{W}}) \simeq \mathbb{P}(\tau_{\mathbf{W}})$. Denote by $\hat{\mathbf{F}}_{\mathbf{W}}$, $\hat{\Delta}_{\mathbf{W}}$ the pullback to $\mathbb{P}(\tau_{\mathbf{W}})$ of $\tilde{\mathbf{F}}_{\mathbf{W}}$, $\tilde{\Delta}_{\mathbf{W}}$, respectively. The image of the identity $id_{\tau_{\mathbf{W}}} \in \mathrm{End}(\tau_{\mathbf{W}})$ through the isomorphism:

$$\begin{aligned}\mathrm{Hom}(\tau_{\mathbf{W}}, \tau_{\mathbf{W}}) &\simeq H^0(\tau_{\mathbf{W}} \otimes \tau_{\mathbf{W}}^{\vee}) \simeq H^0(\tau_{\mathbf{W}} \otimes \pi_{\tau_{\mathbf{W}}*}\mathcal{O}(1)) \simeq \\ &\simeq H^0(p_{1*}\mathcal{H}om(\hat{\mathbf{F}}_{\mathbf{W}}, \mathcal{O}_{\hat{\Delta}_{\mathbf{W}}}) \otimes \mathcal{O}_{\mathbb{P}(\tau_{\mathbf{W}})}(1)) \\ &\simeq \mathrm{Hom}(\hat{\mathbf{F}}_{\mathbf{W}}, \mathcal{O}_{\hat{\Delta}_{\mathbf{W}}} \otimes p_1{}^*\mathcal{O}_{\mathbb{P}(\tau_{\mathbf{W}})}(1))\end{aligned}$$

defines a morphism $\hat{\mathbf{F}}_{\mathbf{W}} \to \mathcal{O}_{\hat{\Delta}_{\mathbf{W}}} \otimes p_1{}^*\mathcal{O}(1)$ inducing a short exact sequence on $\mathbb{P}(\tau_{\mathbf{W}}) \times X$:

$$0 \to \hat{\mathbf{E}}_{\mathbf{W}} \to \hat{\mathbf{F}}_{\mathbf{W}} \to \mathcal{O}_{\hat{\Delta}_{\mathbf{W}}} \otimes p_1{}^*\mathcal{O}_{\mathbb{P}(\tau_{\mathbf{W}})}(1) \to 0.$$

Twisting and applying p_{1*}, we obtain a rank 2 vector bundle $p_{1*}(\hat{\mathbf{E}}_{\mathbf{W}}(1))$; this latter induces an embedding $\mathbb{P}(\tau_{\mathbf{W}}) \to G_2(p_{1*}\tilde{\mathbf{F}}_{\mathbf{W}}(1))$ that maps $\mathbb{P}(\tau_{\mathbf{W}})$ bijectively into $\mathbb{P}(\tilde{\mathcal{A}}_{\mathbf{W}})$. This induces an isomorphism $\mathbb{P}(\tau_{\mathbf{W}}) \simeq \mathbb{P}(\tilde{\mathcal{A}}_{\mathbf{W}})$ mapping a point $f \in \mathrm{Hom}(\mathbf{F}_w, \mathcal{O}_p)$ in $\pi_{\tau_{\mathbf{W}}}^{-1}$ to $\mathbb{P}(H^0(E_w(1)))$, $E_w = \ker f$. The sheaves $\hat{\mathbf{E}}_{\mathbf{W}}$ determine morphisms $\psi_{\tilde{\mathbf{W}}} : \mathbb{P}(\tilde{\mathcal{A}}_{\mathbf{W}}) \xrightarrow{\simeq} \mathbb{P}(\tau_{\mathbf{W}}) \to \mathcal{M}$ that descend to a morphism $\mathbb{P}(\mathcal{A}) \to \mathcal{M}$ which maps $\mathbb{P}(\mathcal{A})$ bijectively to $\mathcal{C}_0$. This means that $\mathcal{C}_0$ is irreducible and has dimension 10. Since by Proposition 74, the dimension of the Zarisky tangent space at each point $[E] \in \mathcal{C}_0$ is 10, we conclude that $\mathcal{C}_0$ inherits from $\mathbb{P}(\mathcal{A})$ the structure of a smooth 10-dimensional scheme. □

From these last results, we deduce that $\overline{\mathcal{C}_0}$ is an irreducible component of $\mathcal{C}$ and that this latter is smooth along $\mathcal{C}_0$. We finally want to show that actually, we have an equality $\mathcal{C} = \overline{\mathcal{C}_0}$, that is to say, that $\mathcal{C}$ is irreducible.

Proposition 76. *$\mathcal{C}$ is an irreducible 10-dimensional scheme that coincides with $\overline{\mathcal{C}_0}$.*

Proof. We show that $\mathcal{C}_0$ is dense in $\mathcal{C}$. From Proposition 72, we learn that pairs (E, s), $[E] \in \mathcal{C} \backslash \mathcal{C}_0$, $s \in \mathbb{P}(H^0(E(1))$ corresponds to pairs (Γ, ξ) with Γ a non l.cm. curve consisting of a conic C with an embedded point $p \in C$ and $\xi \in \mathrm{Ext}^1(\mathcal{I}_\Gamma, \mathcal{O}_X(-1)) \simeq \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X(-1)) \simeq H^0(\mathcal{O}_C(2 \cdot p))$ vanishing at the point p. For a point $[E] \in \mathcal{C} \backslash \mathcal{C}_0$, let then Γ be a curve defined by a global section $s \in H^0(E(1))$ and ξ the corresponding element in $\mathrm{Ext}^1(\mathcal{I}_\Gamma, \mathcal{O}_X(-1))$. The projective space $\mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_\Gamma, \mathcal{O}_X(-1))) \simeq \mathbb{P}^2$ determines a family $\mathbf{E}$, flat over $\mathbb{P}^2$ such that $\mathbf{E}_{[\xi]} \simeq E$. Now, the points $x \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_\Gamma, \mathcal{O}_X(-1)))$ such that $\mathbf{E}_x \in \mathcal{C} \setminus \mathcal{C}_0$ are parameterized by a divisor isomorphic to $\mathbb{P}^1$. We can therefore always deform $[\xi]$ in a direction normal to this divisor and produce in this way a family $\mathbf{E}'$ whose central fiber is isomorphic to $[E]$ and whose general fiber lies in $\mathcal{C}_0$. □

6.3. Intersection of $\mathcal{C}$ and $\overline{\mathcal{L}(2)}$

We end our description of the moduli space $\mathcal{M}$ addressing the issue of its connectedness. Since we have already proved that $\mathcal{M}$ is the union of two irreducible components, this reduces to proving that their intersection is non-empty. We present here how to construct a 5-dimensional irreducible family $\mathcal{P}$ contained in $\mathcal{C} \cap \overline{\mathcal{L}(2)}$. To begin with we consider $\text{Tan}(X) \subset \mathbb{G}(1, \mathbb{P}(V))$, the variety of tangent lines to X and we construct then the variety $\Sigma_T \subset \mathbb{G}(1, \mathbb{P}(V)) \times X$ defined as

$$\Sigma_T := \{([l], p) \in \text{Tan}(X) \times X \mid l \subset \mathbb{T}_p X\};$$

where $\mathbb{T}_p X$ denotes the projective tangent space to X at p, that is to say, the linear space defined by the linear form $d_p q = \sum_{i=0}^{4} \frac{\partial q}{\partial X_i}(p) X_i$, for $q(X_0, \ldots, X_4)$ the quadratic form defining X. The fibers of the projection on the second factor $\Sigma_T \to X$ are isomorphic to $\mathbb{P}^2$, hence Σ_T is a smooth and irreducible 5-dimensional variety (note that since the projection on the first factor $\Sigma_T \to \mathbb{G}(1, \mathbb{P}(V))$ is 1-1 onto $\text{Tan}(X)$, $\text{Tan}(X)$ has dimension 5 as well). We denote now by $\mathcal{R}_T \subset \mathcal{R}(2; -1, 2, 2) \times X$ the image of Σ_T under the isomorphism $\mathbb{G}(1, \mathbb{P}(V)) \times X \xrightarrow{\simeq} \mathcal{M}_X(2; -1, 2, 2) \times X$. By definition, for each $([F], p) \in \mathcal{R}_T$, l_F is tangent to X at $p = \text{Sing}(F)_{\text{red}}$. Now, reasoning exactly as in the proof of Proposition 75, starting from an étale cover $\mathbf{W}$ of $\mathcal{M}_X(2; -1, 2, 2)$ supporting a Poincare sheaf $\mathbf{F_W}$, we construct an étale cover $\mathbf{W}_T$ of $\mathcal{R}_T$ and we consider the sheaf $p_{1*}(\mathcal{H}om(\mathbf{F}_{\mathbf{W}_T}, \mathcal{O}_{\Delta_{\mathbf{W}_T}}))$ on $\mathbf{W}_T$, where $\mathbf{F}_{\mathbf{W}_T} \in Coh(\mathbf{W}_T \times X)$ and $\Delta_{\mathbf{W}_T} \subset \mathbf{W}_T \times X$ are, respectively, the pullback of the universal sheaf and the diagonal.

The sheaf $p_{1*}(\mathcal{H}om(\mathbf{F}_{\mathbf{W}_T}, \mathcal{O}_{\Delta_{\mathbf{W}_T}}))$ is a rank 3 vector bundle and replying the reasoning presented in the proof of Proposition 75, on $\mathbb{P}(p_{1*}(\mathcal{H}om(\mathbf{F}_{\mathbf{W}_T}, \mathcal{O}_{\Delta_{\mathbf{W}_T}})) \times X$, there exists a family $\hat{\mathbf{E}}_{\mathbf{W}_T} \hookrightarrow \hat{\mathbf{F}}_{\mathbf{W}_T}$, $\hat{\mathbf{F}}_{\mathbf{W}_T}$ being the pullback of $\mathbf{F}_{\mathbf{W}_T}$, whose direct image under the projection p_1 fits in:

$$0 \to p_{1*}(\hat{\mathbf{E}}_{\mathbf{W}_T}(1)) \to p_{1*}(\hat{\mathbf{F}}_{\mathbf{W}_T}(1)) \to p_{1*}(\mathcal{O}_{\hat{\Delta}_{\mathbf{W}_T}}) \otimes \mathcal{O}(1) \to 0.$$

Denote now by $\mathbf{l}_{\mathbf{W}_T} \subset \Sigma_T \times_{\mathcal{R}_T} \mathbf{W}_T$ the pullback of the universal line $\mathbf{l} \subset \Sigma_T \times X$. We observe that the projective bundle $\mathbb{P}(p_{1*}\mathcal{I}_{\mathbf{W}_T}(1))$ identifies with the Grassmann bundle $G_2(p_{1*}(\mathbf{F}_{\mathbf{W}_T}(1))$ (since for $F \in \mathcal{R}(2; -1, 2, 2)$ such that $p \in l_F$ $\mathbb{G}(1, \mathbb{P}(H^0(F(1)))) \simeq \mathbb{P}(H^0(\mathcal{I}_{l_F}(1))) \simeq \mathbb{P}^2$); as $p_{1*}(\hat{\mathbf{E}}_{\mathbf{W}_T}(1))$ is a rank-2 subbundle of $p_{1*}(\hat{\mathbf{F}}_{\mathbf{W}_T}(1))$ it induces therefore a morphism of Σ_T schemes:

$$\epsilon : \mathbb{P}(p_{1*}(\mathcal{H}om(\mathbf{F}_{\mathbf{W}_T}, \mathcal{O}_{\Delta_{\mathbf{W}_T}})) \to \mathbb{P}(p_{1*}\mathcal{I}_{\mathbf{l}_{\mathbf{W}_T}}(1))$$

(ϵ is the morphism mapping $f \in Hom(F, \mathcal{O}_p)$ to the unique hyperplane containing all curves in $\mathbb{P}(H^0(E(1)))$ for $E := \ker f$). This morphism is bijective hence, by the smoothness of both $\mathbb{P}(p_{1*}(\mathcal{H}om(\mathbf{F}_{\mathbf{W}_T}, \mathcal{O}_{\Delta_{\mathbf{W}_T}}))$ and $\mathbb{P}(p_{1*}\mathcal{I}_{\mathbf{l}_{\mathbf{W}_T}}(1))$ it is an isomorphism.

Also this time, the families of sheaves $\hat{\mathbf{E}}_{\mathbf{W}_T}$ induces morphisms $\mathbb{P}(p_{1*}\mathcal{I}_{\mathbf{l}_{\mathbf{W}_T}}(1)) \to \mathcal{C}$ that descend to a well defined morphism $\psi : \mathbb{P}(p_{1*}\mathcal{I}_{\mathbf{l}}(1)) \to \mathcal{C}$. Consider now the variety

$$\tilde{\Sigma}_T := \{([l], p, H) \in \Sigma_T \times X^* \mid H = \mathbb{T}_p X\}$$

$\tilde{\Sigma}_T$ is isomorphic to Σ_T and it identifies with a subset of $\mathbb{P}(p_{1*}(\mathcal{I}_{\mathbf{l}}(1))$; we finally define $\mathcal{P}$ as the scheme theoretic image of $\tilde{\Sigma}_T$ under the morphism ψ.

Proposition 77. *$\mathcal{P}$ is a 5 dimensional irreducible scheme contained in $\overline{\mathcal{L}(2)} \cap \mathcal{C}$.*

Proof. The dimension and the irreducibility of $\mathcal{P}$ are immediate consequences of the fact that $\tilde{\Sigma}_T$ is irreducible and of dimension 5. Let us now prove that $\mathcal{P}$ is indeed contained in both components of $\mathcal{M}$. For a general point $[E] \in \psi(\tilde{\Sigma}_T)$, $E^{\vee\vee} := F \in \mathcal{R}(2; -1, 2, 2)$ and $F/E \simeq \mathcal{O}_p$, $\mathrm{Sing}(F)_{\mathrm{red}} = p$. Let us give a geometric interpretation of the family of curves defined by $H^0(E(1))$. Consider the short exact sequence:

$$0 \to H^0(E(1)) \xrightarrow{\iota} H^0(F(1)) \to H^0(\mathcal{O}_p) \to 0.$$

By definition the pencil $\mathbb{P}(\iota(H^0(E(1))))$ identifies with the pencil of planes containing l_F and contained in $\mathbb{T}_p X$; accordingly it identifies with the locus of singular conics in $\mathbb{P}(H^0(F(1))$ (to see this just notice that the singular plane sections in $\Lambda_{l_F} \simeq \mathbb{P}(H^0(F(1)))$ identifies with the locus of tangents to $\mathbb{P}(H^0(\mathcal{I}_{l_F}(1))) \cap X^*$ i.e. with the pencils $\langle h, \mathbb{T}_p X \rangle$ with h a hyperplane containing l_F.)

Consider now the singular quadric surface $Q_p := \mathbb{T}_p X \cap X$. Q_p is the cone with vertex p over a smooth conic C and the pencil $\mathbb{P}(H^0(E(1)))$ uniquely determines a 1-dimensional linear system $\mathbb{P}^1_E \subset \mathbb{P}(H^0(\mathcal{O}_C(1))$ hence, a point in the projective bundle

$$\mathbb{P}(\mathrm{Sym}^2(\mathcal{T}^\vee)) \longrightarrow \mathbb{G}(1, F(X))$$

where, as usual, $\mathcal{T}$ is the tautological bundle. This projective bundle is a smooth 6-dimensional variety containing the variety $\mathcal{B} \simeq \mathcal{L}(2)$ (see Sect. 5.2) as an open subset. We can therefore always construct a regular affine curve with a marked point $(\mathrm{Spec}(A), 0)$ and a 1-parameter family of pencils $\mathbb{P}^1_t$, flat over $\mathrm{Spec}(A)$, such that $\mathbb{P}^1_t \in \mathcal{B}$ for t general and whose central fiber $\mathbb{P}^1_0$ is isomorphic to $\mathbb{P}^1_E$. Define $E_t := \zeta^{-1}(\mathbb{P}^1_t)$, $t \neq 0$. This is a family of instantons, flat over $\mathrm{Spec}(A) \setminus \{0\}$ such that the pencil of curves $\mathbb{P}(H^0(E_t(1)))$ coincides with $\mathbb{P}^1_t$. E_t admits a flat limit $E_0 \in \overline{\mathcal{L}(2)}$ and since for such a sheaf, the pencil of curves $\mathbb{P}(H^0(E_0(1)))$ must be the flat limit of $\mathbb{P}^1_t$, we conclude that $E_0 \simeq E$. This means that $\psi(\tilde{\Sigma}_T) \subset \overline{\mathcal{L}(2)} \cap \mathcal{C}$ therefore the same holds for $\mathcal{P}$. □

This shows the connectedness of $\mathcal{M}$, ending the proof of Theorem 63.

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Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Data availability I hereby declare that data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] Arrondo, E.: A home-made Hartshorne–Serre correspondence. Rev. Mat. Complut. **20**, 423–443 (2007)
[2] Atiyah, M., Drinfeld, V., Hitchin, N., Manin, Yu.: Construction of instantons. Phys. Lett. **A65**(3), 185–187 (1978)
[3] Antonelli, V., Casnati, G.: Instanton sheaves on projective schemes. J. Pure Appl. Algebra **227**, 107246 (2023). (**40 pp**)
[4] Antonelli, V., Malaspina, F.: H-instanton bundles on three-dimensional polarized projective varieties. J. Algebra **598**, 570–607 (2022)
[5] Banica, C., Forster, O.: Multiplicity structures on space curves. Contemp. Math. **58**, 47–64 (1986)
[6] Brambilla, M.C., Faenzi, D.: Vector bundles on Fano threefolds of genus 7 and Brill–Noether loci. Int. J. Math. **25**, 1450023 (2014)
[7] Casnati, G., Coskun, E., Genç, Ö., Malaspina, F.: Instanton bundles on the blow-up of the projective 3-space at a point. Mich. Math. J. **70**, 807–836 (2021)
[8] Costa, L., Ottaviani, G.: Nondegenerate multidimensional matrices and instanton bundles. Trans. Am. Math. Soc. **355**, 49–55 (2003)
[9] Druel, S.: Espace des modules des faisceaux de rang 2 semi-stables de classes de Chern $c_1 = 0$, $c_2 = 2$ et $c_3 = 0$ sur la cubique de $\mathbb{P}^4$. Int. Math. Res. Not. **19**, 985–1004 (2000)
[10] Ein, L., Sols, I.: Stable vector bundles on quadric hypersurfaces. Nagoya Math. J. **96**, 11–22 (1984)
[11] Faenzi, D.: Even and odd instanton bundles on Fano threefolds of Picard number one. Manuscr. Math. **144**, 199–239 (2014)
[12] Gargate, M., Jardim, M.: Singular loci of instanton sheaves on projective space. Int. J. Math. **27**, 1640006 (2016)
[13] Hartshorne, R.: Stable vector bundles of rank 2 on $\mathbb{P}^3$. Math. Ann. **238**, 229–280 (1978)
[14] Hartshorne, R.: Stable reflexive sheaves. Math. Ann. **254**, 121–176 (1980)
[15] Hauzer, M., Langer, A.: Moduli spaces of framed perverse instantons on $\mathbb{P}^3$. Glasg. Math. J. **53**, 51–96 (2011)
[16] Huybrechts, D., Lehn, M.: The Geometry of Moduli Spaces of Sheaves. Cambridge Mathematical Library, 2nd edn. Cambridge University Press, Cambridge (2010)
[17] Iskovskih, V.A., Prokhorov, Yu.G.: Algebraic Geometry V: Fano varieties. Encyclopaedia of Mathematical Sciences, vol. 47. Springer-Verlag, Berlin (1999)
[18] Jardim, M.: Instanton sheaves on complex projective spaces. Collect. Math. **57**(1), 69–91 (2006)
[19] Jardim, M., Verbitsky, M.: Trihyperkähler reduction and instanton bundles on $\mathbb{CP}^3$. Compos. Math. **150**, 1836–1868 (2014)
[20] Jardim, M., Maican, M., Tikhomirov, A.S.: Moduli spaces of rank 2 instanton sheaves on the projective space. Pac. J. Math. **291**, 399–424 (2017)
[21] Jardim, M., Markushevich, D., Tikhomirov, A.S.: Two infinite series of moduli spaces of rank 2 sheaves on $\mathbb{P}^3$. Ann. Mat. Pura Appl. **196**, 1573–1608 (2017)
[22] Jardim, M., Markushevich, D., Tikhomirov, A.S.: New divisors in the boundary of the instanton moduli space. Mosc. Math. J. **18**, 117–148 (2018)
[23] Kuznetsov, A.: Instanton bundles on Fano threefolds. Cent. Eur. J. Math. **10**, 1198–1231 (2012)
[24] Lange, H.: Universal families of extensions. J. Algebra **83**, 101–112 (1983)
[25] Mamone Capria, M., Salamon, S.M.: Yang-Mills fields on quaternionic spaces. Nonlinearity **1**, 517–530 (1988)

[26] Markushevich, D., Tikhomirov, A.: The Abel–Jacobi map of a moduli component of vector bundles on the cubic threefold. J. Algebraic Geom. **10**, 37–62 (2001)
[27] Maruyama, M.: Boundedness of semistable sheaves of small ranks. Nagoya Math. J. **78**, 65–94 (1980)
[28] Mukai, S.: Biregular classification of Fano 3-folds and Fano manifolds of coindex 3. Proc. Natl. Acad. Sci. **86**, 3000–3002 (1989)
[29] Okonek, C., Schneider, M., Spindler, H.: Vector Bundles on Complex Projective Spaces. Birkhauser, Basel (1980)
[30] Ottaviani, G.: Spinor bundles on quadrics. Trans. Am. Math. Soc. **307**, 301–316 (1988)
[31] Ottaviani, G., Szurek, M.: On moduli of stable 2-bundles with small Chern classes on Q3. Ann. Mat. Pura Appl. **167**, 191–241 (1994)
[32] Tikhomirov, A.S.: Moduli of mathematical instanton vector bundles with odd c_2 on projective space. Izv. Math. **76**, 991–1073 (2012)
[33] Tikhomirov, A.S.: Moduli of mathematical instanton vector bundles with even c_2 on projective space. Izv. Math. **77**, 1195–1223 (2013)
[34] Qin, X.: Compactification of the moduli space of minimal instantons on the Fano 3-fold V_5. J. Pure Appl. Algebra **225**, 106526 (2021)
[35] Qin, X.: Compactification of the moduli space of minimal instantons on the Fano threefold V_4. Eur. J. Math. **7**, 1502–1523 (2021)