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Weak approximation on Châtelet surfaces

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Abstract. We study weak approximation for Châtelet surfaces over number fields when all singular fibers are defined over rational points. We consider Châtelet surfaces which satisfy weak approximation over every finite extension of the ground field. We prove many of these results by showing that the Brauer–Manin obstruction vanishes, then apply results of Colliot-Thélène, Sansuc, and Swinnerton-Dyer.

1. Introduction

A Châtelet surface over a number field k is a smooth projective model of the affine surface given by the equation

$$y^2 - az^2 = P(x) \tag{1.1}$$

where $a \in k^\times \setminus k^{\times 2}$ and $P(x)$ is a separable polynomial of degree 3 or 4. The arithmetic of these surfaces have been studied extensively by Colliot-Thélène, Sansuc, and Swinnerton-Dyer in [3,4], where it was proven that over a number field, rational points on these surfaces are controlled by the Brauer–Manin obstruction. More precisely, if X is a Châtelet surface over a number field k, then X may fail weak approximation, i.e., $X(k)$ is not dense in $X(\mathbb{A})$, but it is always dense in the Brauer–Manin set $X(\mathbb{A})^{\mathrm{Br}}$.

In this paper, we focus on weak approximation on Châtelet surfaces. Since the Brauer group is largest when $P(x)$ splits into linear factors (see Sect. 2), one might expect that weak approximation fails in this case. When $P(x)$ splits linearly, we obtain a complete characterization for when X fails weak approximation.

Theorem 1.1. *Let X be a Châtelet surface over a number field k with $X(k) \neq \emptyset$. Assume that $P(x)$ splits into linear factors. Then X fails weak approximation if and only if either*

(1) *X has a place v of bad reduction where $a \notin k_v^{\times 2}$ or,*
(2) *k has a real embedding where $a < 0$.*

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See Sect. 4 for what we mean by bad reduction. Roughly, this translates to either $k_v(\sqrt{a})/k_v$ being ramified or $P(x)$ being nonseparable modulo v.

These conditions suggest that we expect weak approximation to fail when $P(x)$ splits linearly. This is in contrast to the case where $P(x)$ is irreducible or is a product of a linear and irreducible cubic where X always satisfies weak approximation [3, Theorem B]. We also collect some partial results and show that the situation is not as simple in the remaining case when $P(x)$ is a product of two irreducible quadratics, see Examples 5.3, 5.4.

For Châtelet surfaces, behavior of the Hasse principle or weak approximation under finite extensions have been studied in [8,9,11]. In [8,11], the authors give a Châtelet surface over any arbitrary number field k, which satisfies weak approximation over k but fails over some finite extension. On the other hand, there exist Châtelet surfaces that satisfy weak approximation over every finite extension K/k (e.g. when X is rational), in which case we say that it satisfies perpetual weak approximation. Using Theorem 1.1, we give a criterion for checking whether or not a Châtelet surface satisfies perpetual weak approximation

Theorem 1.2. *Let X be a Châtelet surface over a number field k. Let L be the splitting field of $P(x)$.*

(1) *X fails weak approximation over some finite field extension of k if either (1) or (2) in Theorem 1.1 is satisfied for X_L.*
(2) *X satisfies weak approximation over all finite field extensions of k if any of the following conditions are satisfied, (Here D_{2n} denotes the dihedral group of order $2n$)*
   (a) *if both (1) and (2) of Theorem 1.1 fail for X_L and $\mathrm{Gal}(L/k) \simeq \mathbb{Z}/3\mathbb{Z}, \{1\}$,*
   (b) *if $\mathrm{Gal}(L/k) \simeq D_8, (\mathbb{Z}/2\mathbb{Z})^2$ and $P(x)$ factors further over $k(\sqrt{a})/k$,*
   (c) *if $\mathrm{Gal}(L/k) \simeq D_6, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ and $\sqrt{a} \in L$.*

If neither of these conditions hold, then there is a subextension K/k inside L such that $P(x)$ factors into a product of two irreducible quadratics over K and $\sqrt{a} \in L$. In this case, X satisfies perpetual weak approximation if and only if X_K satisfies weak approximation.

We prove (1) and (2)(a) as a direct corollary of Theorem 1.1 in §4 and the proof for (2)(b)(c) is given in §2 where we discuss the Brauer group of Châtelet surfaces.

For (2)(b)(c), the conditions in fact imply that $\mathrm{Br}\, X_K/\mathrm{Br}_0\, X_K = 0$ for all finite extensions K/k. This is enough to conclude that X_K satisfies weak approximation. In fact, when $\mathrm{Gal}(L/k) \simeq D_6$ and L contains $\sqrt{a}$, then X is stably rational [1, Theorem 1], from which it follows in particular that X satisfies perpetual weak approximation. In the instances when $\sqrt{a}$ is contained in the residue field of one of the factors of $P(x)$, X is rational.

In (2)(a), $\mathrm{Br}\, X_L/\mathrm{Br}_0\, X_L \simeq (\mathbb{Z}/2\mathbb{Z})^2$ so in particular X is not rational. See Example 4.15 for a surface in this category.

2. Cyclic algebras over local fields

In this section let k be a nonarchimedian local field containing an nth roots of unity ζ. Let $\mathcal{O}$ be its ring of integers with uniformizer $\pi \in \mathcal{O}$ and $\mathbb{F}_q = \mathcal{O}/(\pi)$. Recall that the invariant map gives an isomorphism $\operatorname{inv}\colon \operatorname{Br} k \to \mathbb{Q}/\mathbb{Z}$.

For $a, b \in k^\times$, let $(a, b) = (a, b)_\zeta \in \operatorname{Br} k$ be the class of the cyclic algebra as defined in [6, §2.5]. When $n = 2$, we will simply write $(a, b) = (a, b)_{-1}$.

Lemma 2.1. *If $k(\sqrt[n]{a})/k$ is unramified, then for any $b \in k^\times$, $\operatorname{inv}(a, b)_\zeta = v_\pi(b)/n$.*

Proof. [10, §2.5 Proposition 2] □

Lemma 2.2. *Assume the characterisitic of $\mathbb{F}_q$ is even. If $k(\sqrt{a})/k$ is ramified, then there exists $u \in \mathcal{O}$ such that $\operatorname{inv}(a, 1 - \pi u) = 1/2$.*

Proof. By local class field theory, there exists some $b \in \mathcal{O}^\times$ such that $b \notin \mathrm{N}_{k(\sqrt{a})/k}\, k(\sqrt{a})^\times$ and so $\operatorname{inv}(a, b) = 1/2$. Since the squaring map $x \mapsto x^2$ is an isomorphism on $\mathbb{F}_q^\times$, there exists $d \in \mathcal{O}^\times$ such that $b - d^2 \equiv 0 \bmod \pi$. Write $b - d^2 = -\pi e$ for some $e \in \mathcal{O}$. Then,

$$(a, b) = (a, d^2 + b - d^2) = (a, d^2 - \pi e) = (a, 1 - \pi(e/d^2)).$$

Setting $u = e/d^2 \in \mathcal{O}$ gives our result. □

3. Generators for the Brauer group

Throughout the rest of this paper, X will denote a Châtelet surface over a field k of characteristic $\neq 2$ with affine model given by the equation

$$y^2 - az^2 = P(x)$$

where $a \in k^\times \setminus k^{\times 2}$ and $P(x)$ is a separable polynomial of degree 3 or 4. The morphism $X \to \mathbb{P}^1$ given by the x-coordinate gives X the structure of a conic bundle. The singular fibers are precisely the roots of $P(x)$ together with $\infty \in \mathbb{P}^1$ if $\deg(P(x)) = 3$.

Let $\operatorname{Br} X = \mathrm{H}^2_{\text{et}}(X, \mathbb{G}_m)$ denote the cohomological Brauer group of X. Let $\operatorname{Br}_0 X$ be the image of the natural map $\operatorname{Br} k \to \operatorname{Br} X$. The Brauer group of Châtelet surfaces, or more generally conic bundles, have been extensively studied, see [2, §11.3] for a detailed exposition. We summarize the necessary results for our purposes here.

Proposition 3.1. *The Brauer group of X depends on the factorization of $P(x)$ over k. Then $\operatorname{Br}(X)/\operatorname{Br}_0(X)$ is isomorphic to*

(1) $(\mathbb{Z}/2\mathbb{Z})^2$ *if $P(x)$ splits completely into linear factors,*
(2) $\mathbb{Z}/2\mathbb{Z}$ *if $P(x)$ has an irreducible quadratic factor and $\sqrt{a} \notin k(\alpha)$ for all roots α of $P(x)$,*
(3) $\{0\}$ *otherwise.*

Proof. [2, §11.3 Exercise 11.3.7] □

The points on $\mathbb{P}^1_k$ corresponding to the singular fibers determine the generators for the quotient $\operatorname{Br} X/\operatorname{Br}_0 X$. In particular, if $P(x)$ splits completely, we can map three of the roots of $P(x)$ to 0, 1, and $\infty \in \mathbb{P}^1$ via an automorphism of $\mathbb{P}^1$. In doing so, we may assume that $P(x) = cx(x-1)(x-\lambda)$ for some $\lambda \in k \smallsetminus \{0, 1\}$ and $c \in k$.

The main proofs of this paper work with fixed generators for the Brauer groups, and for the cases above where $\operatorname{Br} X/\operatorname{Br}_0 X \neq 0$, our generating algebras can be explicitly computed (see [2, §11.3 Corollary 11.3.5]):

(1) If $P(x)$ splits completely, then $\operatorname{Br} X/\operatorname{Br}_0 X$ is generated by quaternion algebras of the form

$$\mathcal{A} = (a, x(x-1)) \quad \text{and} \quad \mathcal{B} = (a, x(x-\lambda)).$$

(2) If $P(x) = cf(x)g(x)$ where both f and g are monic and at least one of $f(x)$ or $g(x)$ is an irreducible quadratic, then $\operatorname{Br} X/\operatorname{Br}_0 X$ is generated by the quaternion algebra

$$\mathcal{C} = (a, f(x)) = (a, cg(x)).$$

We now prove Theorem 1.2(2)(b)(c) from the introduction. It suffices to show that the Brauer group consists of only constant algebras, in which case there is no Brauer–Manin obstruction to weak approximation.

Theorem 3.2. *Let X be a Châtelet surface over k and L/k the splitting field of $P(x)$. Then $\operatorname{Br} X_K/\operatorname{Br}_0 X_K = 0$ for every finite extension K/k if and only if one of the following holds*

(1) $\operatorname{Gal}(L/k) \simeq D_8, (\mathbb{Z}/2\mathbb{Z})^2$ *and $P(x)$ factors further over $k(\sqrt{a})/k$.*
(2) $\operatorname{Gal}(L/k) \simeq D_6, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ *and $\sqrt{a} \in L$.*

Proof. Assume that either condition (1) or (2) listed above holds. Let K/k be a finite extension. By Proposition 3.1, we only need to consider the case when $P(x)$ splits into quadratic or smaller factors over K. Since X_K is rational if $\sqrt{a} \in K$, we also assume $\sqrt{a} \notin K$. In particular this means $L \not\subseteq K$. We break into cases based on the factorization of $P(x)$ over K.

(1) (Product of two quadratics) In this case $\operatorname{Gal}(LK/K)$ is either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. In the former case, $LK = K(\sqrt{a})$ and $\sqrt{a}$ is in the residue field of one of the quadratic factors, so $\operatorname{Br} X_K/\operatorname{Br}_0 X_K = 0$ by Proposition 3.1. In the latter case, we must have $\operatorname{Gal}(L/k)$ either D_8 or $(\mathbb{Z}/2\mathbb{Z})^2$. By assumption, over $K(\sqrt{a})$, $P(x)$ must contain a linear factor. This implies that over K, one of the quadratic factors on $P(x)$ has residue field $K(\sqrt{a})$.
(2) (Product of two linear and one quadratic) In this case $LK = K(\sqrt{a})$ and so the quadratic factor of $P(x)$ has residue field containing $\sqrt{a}$.

Conversely assume $\operatorname{Br} X_K/\operatorname{Br}_0 X_K = 0$ for every finite extension K/k. If $\sqrt{a} \notin L$ then $\operatorname{Br} X_L/\operatorname{Br}_0 X_L \simeq (\mathbb{Z}/2\mathbb{Z})^2$ by Proposition 3.1, so we may assume that $\sqrt{a} \in L$. We eliminate some possibilities by dividing into cases based on the isomorphism class of $\operatorname{Gal}(L/k)$.

(S_4) There is a unique quadratic extension contained in L/k corresponding to $A_4 \subset S_4$ which must be $k(\sqrt{a})$. Let K/k be an extension such that $\mathrm{Gal}(LK/K) \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Then $P(x)$ factors over K as a product of two quadratics. Since $\mathrm{Gal}(LK/K(\sqrt{a}))$ must move both roots, it follows the residue fields of either factors does not contain $\sqrt{a}$. Hence $\mathrm{Br}\, X_K / \mathrm{Br}_0\, X_K \simeq \mathbb{Z}/2\mathbb{Z}$.

(A_4) There are no index 2 subgroups in this case.

(D_8) If $P(x)$ remains irreducible over $k(\sqrt{a})$, then let K/k be any distinct quadratic extension so that $P(x)$ factors into two quadratics over K. It follows neither of the residue fields contain $\sqrt{a}$ by the initial assumption so that $\mathrm{Br}\, X_K / \mathrm{Br}_0\, X_K \simeq \mathbb{Z}/2\mathbb{Z}$.

(K_4) If $P(x)$ is irreducible over k then it must factor over $k(\sqrt{a})$. Otherwise, assume $P(x)$ is a product of two quadratics and remains so over $k(\sqrt{a})$. Then the same conclusion as the case above applies.

This leaves us with either (1) or (2) highlighted in the theorem. □

Corollary 3.3. *Let $P(x)$ satisfy either conditions (1) or (2) of Theorem 3.2. Then the Châtelet surface*

$$y^2 - az^2 = P(x)$$

satisfies weak approximation over all finite extensions K/k.

Proof. By Theorem 3.2, there is no Brauer–Manin obstruction to weak approximation over all such extensions K/k. So by [3, Theorem B(ii)(a)] they satisfy weak approximation. □

4. Failure of weak approximation when $P(x)$ is split

Let k be a number field. In this section, we consider the case when $P(x)$ factors into linear factors. Our goal is to prove Theorem 1.1. As discussed in Sect. 3, after moving one of the singular fibers to $\infty \in \mathbb{P}^1$, we may assume

$$P(x) = cx(x-1)(x-\lambda) \tag{4.1}$$

for some $\lambda \in k$ not equal to 0, 1 and $c \in k^\times$.

Let v be a finite place of k where $a \notin k_v^{\times 2}$, and $\pi \in \mathcal{O}_k$ be a uniformizer. After a change of coordinates, we may assume that $v(c) = 0, 1$. By considering the six cross-ratios on $\mathbb{P}^1$, we may also assume that $v(\lambda) \geq 0$. This allows us to define a model over $\mathcal{O}_v$ using the same equation. We say that X has bad reduction at v if the special fiber of this model is singular. Concretely, this means that at least one of $v(c)$, $v(\lambda)$, $v(\lambda - 1)$ is nonzero or that $k(\sqrt{a})/k$ is ramified at v.

Suppose the Châtelet surface X has bad reduction at v. We analyze the surjectivity of the evaluation map $\mathrm{ev}_{\mathcal{A}}\colon X(k_v) \to \mathrm{Br}\, k_v[2]$ for the algebra $\mathcal{A} \in \mathrm{Br}(X)$ listed in Sect. 3. We first consider the unramified case, and then deal with the ramified case separately for odd and even primes.

4.1. Unramified Case

Assume that $k_v(\sqrt{a})/k_v$ is unramified. For $x \in k_v$, there exists a point in $X(k_v)$ lying over the corresponding point in $\mathbb{P}^1$ if $v(P(x))$ is even. By Lemma 2.1, for any point $Q = (x, y, z) \in X(k_v)$,

$$\mathrm{inv}_v(\mathcal{A}(Q)) = \mathrm{inv}_v(a, x(x-1)) = v(x(x-1))/2 \in \mathbb{Q}/\mathbb{Z}.$$

The map $\mathrm{ev}_{\mathcal{A}} \colon X(k_v) \to \mathrm{Br}\, k_v[2] \simeq \mathbb{Z}/2\mathbb{Z}$ is surjective if and only if it is nonconstant, and for $\mathcal{A}(Q)$ to be nontrivial it is equivalent to $v(x(x-1))$ being odd.

Proposition 4.1. *If* $k_v(\sqrt{a})/k_v$ *is unramified, then the evaluation map* $\mathrm{ev}_{\mathcal{A}} \colon X(k_v) \to \mathrm{Br}\, k_v[2]$ *is surjective.*

Proof. Observe that for any $x \in k_v$, there exists $Q \in X(k_v)$ lying over x if and only if $v(cx(x-1)(x-\lambda))$ is even. So it suffices to prove there exists $x_1, x_2 \in k_v$ such that $v(P(x_1)), v(P(x_2))$ are even and $v(x_1(x_1-1)) \not\equiv v(x_2(x_2-1)) \bmod 2$.

First assume that $v(\lambda) = v(\lambda - 1) = 0$. This implies we must have $v(c) \neq 0$ so that $v(c) = 1$ by our initial setup. Take $x = 1/\pi$, then $v(P(x)) = -2$ and $v((1/\pi)(1/\pi - 1)) = -2$ is even. On the other hand, if we take $x = \pi$, then $v(P(x)) = 2$ and $v((\pi)(\pi - 1)) = 1$ is odd.

To finish, it suffices to consider when $v(\lambda) > 0$ (If $v(\lambda - 1) > 0$, then we may reduce to this case after a change of coordinates). We divide further into two cases, namely $v(c) = 0$ and $v(c) = 1$. Recall that

$$v(P(x)) = v(c) + v(x) + v(x-1) + v(x-\lambda).$$

If $v(c) = 0$ then first set $x = 1/\pi^2$ so that $v(P(x)) = -6$ and $v(x(x-1)) = -4$. Next, pick x such that $0 < v(x)$ and $v(x) \equiv v(x-\lambda) \equiv 1 \bmod 2$. Then $v(P(x))$ is even and $v(x(x-1)) = v(x)$ is odd.

If $v(c) = 1$, then first set $x = 1/\pi$ so that $v(P(x)) = -2$ and $v(x(x-1)) = -2$. Furthermore, if we set $x = 1+\pi$, then $v(P(x)) = 2$ and $v(x(x-1)) = 1$ as desired. □

4.2. Ramified case odd

If v is an odd place and $k_v(\sqrt{a})/k_v$ is ramified, then we may assume after a change of coordinates that $v(a) = 1$.

Proposition 4.2. *Assume that* $v(a) = 1$ *and* $P(x)$ *has the form* (4.1)*. Then the evaluation map* $\mathrm{ev}_\alpha \colon X(k_v) \to \mathrm{Br}\, k_v[2]$ *is surjective for some* $\alpha \in \{\mathcal{A}, \mathcal{B}\}$.

Proof. Dividing the Eq. (1.1) for X by a power of $-a$ and exchanging y, z if necessary, we can also assume that $v(c) = 0$. Let $R \in X(k)$ be the unique rational point over $\infty \in \mathbb{P}^1$. By using the following alternative formula for $\mathcal{A}, \mathcal{B}$ defined at R,

$$\mathcal{A} = (a, w(1-w)), \mathcal{B} = (a, w(1-\lambda w)),$$

where $w = 1/x$, we see that $\mathrm{ev}_{\mathcal{A}}(R) = \mathrm{ev}_{\mathcal{B}}(R) = 0$. To finish the proof, it suffices to exhibit a point where the evaluation is nontrivial.

Assume first that $\lambda \not\equiv 0, 1 \bmod \pi$. Let $E \subset X$ be the closed subset given by $z = 0$. Then E is an elliptic curve given by

$$y^2 = cx(x-1)(x-\lambda).$$

For $Q = (x, y) \in E(k_v)$, the quantities

$$cx,\ cx - c,\ cx - c\lambda \tag{4.2}$$

are all squares in k_v if and only if $Q \in 2E(k_v)$ (use [7, §1.4 Theorem 4.1] after suitable coordinate change).

Lemma 4.3. *$E(k_v) \setminus 2E(k_v)$ is nonempty*

Proof. Since $v \nmid 2$, E has good reduction at v and $E(k_v)[2^\infty] = \widetilde{E}(\mathbb{F}_v)[2^\infty]$, where $\widetilde{E}$ is the reduction modulo v. In particular, this means $E(k_v)[2^\infty]$ is finite and moreover nontrivial as $E(k_v)[2] \simeq (\mathbb{Z}/2\mathbb{Z})^2$. If $E(k_v) = 2E(k_v)$, then $E(k_v)$ is a 2-divisible group which must be either trivial or of infinite order. Hence it follows that $E(k_v) \neq 2E(k_v)$ and so $E(k_v) \setminus 2E(k_v)$ is nonempty. □

Remark 4.4. Although we will not need it, the previous lemma is true for $v \mid 2$ as well. Then there is a subgroup $E^* \subset E(k_v)$ of finite index such that $\mathcal{O}_v \simeq E^*$. Take any $u \in \mathcal{O}_v$ with $v(u) = 0$. Then u is not divisible by 2 in E^*, and iteratively dividing by 2 in $E(k_v)$ would produce infinitely many points in the quotient $E(k_v)/E^*$, a contradiction. Hence u is not 2-divisible in $E(k_v)$.

Let $Q \in E(k_v) \setminus 2E(k_v)$ so then not all of (4.2) are squares. Note also that at most one of $v(x), v(x-1), v(x-\lambda)$ can be nonzero since $\lambda \not\equiv 0, 1 \bmod \pi$. Moreover $v(x) \geq 0$ since otherwise (4.2) will all be squares as their product is a square. Lastly, they must all be even since $v(x(x-1)(x-\lambda))$ is even.

If none or exactly two of (4.2) are squares then the product $cx(x-1)(x-\lambda)$ cannot be a square. Hence exactly one of (4.2) is a square in k_v. Similarly we also obtain that at least one of $x(x-1), x(x-\lambda)$ is not a square. Hence, at least on of those products is not a square with even valuation which implies it is not a norm from $k_v(\sqrt{a})/k_v$. Hence, $\mathrm{inv}_v\, \alpha(Q) = 1/2$ for some $\alpha \in \{\mathcal{A}, \mathcal{B}\}$.

Now assume that $\lambda \equiv 0 \bmod \pi$. Choose $\overline{x} \in \mathbb{F}_v^\times \setminus \mathbb{F}_v^{\times 2}$ such that $\overline{x} - 1 \in \mathbb{F}_v^{\times 2}$. Lift $\overline{x}$ to $x \in \mathcal{O}_v$. Then $(a, x) = (a, x-\lambda) \neq 0$ but $(a, x-1) = 0$. Hence, there exists some point $Q \in X(k_v)$ lying over $x \in \mathbb{P}^1$ where

$$\mathrm{inv}_v\, \mathcal{A}(Q) = (a, x(x-1)) = 1/2.$$

The case $\lambda \equiv 1 \bmod \pi$ is very similar. □

4.3. Ramified case even

Proposition 4.5. *Let v be a place lying over 2. Assume that $k_v(\sqrt{a})/k_v$ is ramified and $P(x)$ has the form (4.1). Then the evaluation map $\mathrm{ev}_{\mathcal{A}}\colon X(k_v) \to \mathrm{Br}\, k_v[2]$ is surjective.*

We give the proof of this result after establishing some basic facts on the distribution of norms inside $\mathcal{O}_v$. For the remainder of this section, v will denote a place lying over 2 where $k_v(\sqrt{a})/k_v$ is ramified. Let w be the place lying over v and $L_w{:=}k_v(\sqrt{a})$. Let $\mathrm{N}\colon L_w \to k_v$ denote the norm map.

4.3.1. Equidistribution of norms among residues The subgroup of norms $\{x \in \mathcal{O}_v^\times \mid x \in \mathrm{N}(\mathcal{O}_w^\times)\}$ has index 2 inside $\mathcal{O}_v^\times$. For any subset $H \subset k_v$, let $H \bmod \pi^n$ denote the set of equivalence classes $H/\sim$ where $h_1 \sim h_2$ if $h_1 - h_2 \in \pi^n \mathcal{O}_v$.

Lemma 4.6. *Let $r \in \mathcal{O}_v$. Then*

$$\lim_{n\to\infty} \frac{\#\{x \in \mathcal{O}_v \mid x \in \mathrm{N}(\mathcal{O}_w), x \equiv r \bmod \pi\} \bmod \pi^n}{\#\{x \in \mathcal{O}_v \mid x \equiv r \bmod \pi\} \bmod \pi^n} = \frac{1}{2}.$$

Proof. Let $\mathcal{O}_v^{(r)}{:=}\{x \in \mathcal{O}_v \mid x \equiv r \bmod \pi\}$. We first prove the case when $r = 1$. Note that $\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)} \subset \mathcal{O}_v^{(1)}$ is a subgroup of index at most 2 under the multiplicative structure. By Lemma 2.2, there exists $u \in \mathcal{O}_v$ such that $1 - \pi u \notin \mathrm{N}(\mathcal{O}_w)$, and so $1 - \pi u \in \mathcal{O}_v^{(1)} \backslash (\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)})$. Hence it follows $\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)}$ has index 2. Consider the quotient map

$$q_n\colon \mathcal{O}_v^{(1)} \to \mathcal{O}_v^{(1)}/\pi^n.$$

The image $q_n(\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)})$ has either index 1 or 2. Since $N(\mathcal{O}_v)$ is closed in $\mathcal{O}_v$, it follows that for n large enough, $1 - \pi u + \pi^n \mathcal{O}_v$ consists of non-norms. Hence for such n, the image of q_n has index 2. The statement about the limit follows immediately.

Now assume $r \not\equiv 0 \bmod \pi$. Take any $x \in \mathcal{O}_v$ such that $x \equiv r \bmod \pi$. Multiplication by $1/x$ gives a bijection $\mathcal{O}_v^{(r)} \to \mathcal{O}_v^{(1)}$. Depending on x, this map sends $\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(r)}$ to either $\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)}$ or $\mathcal{O}_v^{(1)} \backslash \mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)}$. Hence, the limit then follows from what we proved for $\mathcal{O}_v^{(1)}$ above.

Finally assume $r \equiv 0 \bmod \pi$. Then noting that $-\pi \in \mathrm{N}(\mathcal{O}_w)$ and setting $x' = x/(-\pi)$ gives

$$\{x \in \mathcal{O}_v \mid x \in \mathrm{N}(\mathcal{O}_w), x \equiv 0 \bmod \pi\} = -\pi\{x' \in \mathcal{O}_v \mid x' \in \mathrm{N}(\mathcal{O}_w)\}.$$

Hence the limit in question is

$$\lim_{n\to\infty} \frac{\#\{x' \in \mathcal{O}_v \mid x' \in \mathrm{N}(\mathcal{O}_w)\} \bmod \pi^{n-1}}{\#\mathcal{O}_v \bmod \pi^{n-1}}.$$

Now, we can divide this limit according to the image of x' in $\mathcal{O}_v/\pi\mathcal{O}_v$. We can apply our previous result for $x' \not\equiv 0 \bmod \pi$ and argue inductively for those $x' \equiv 0 \bmod \pi$. Hence, we obtain that the above limit is $1/2$. □

Remark 4.7. As one can see from the proof, when $r \neq 0 \bmod \pi$, the quantity in question is in fact equal to $1/2$ for n sufficiently large enough.

Lemma 4.8. *Let a_n, b_n, c_n, d_n be sequences of real numbers. Let $A_n = \sum_{i=1}^{n} a_i$ and define B_n, C_n, D_n similarly. Assume that $B_n, D_n > 0$ for all n. Then*

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{C_n}{D_n} = L \implies \lim_{n\to\infty} \frac{A_n + C_n}{B_n + D_n} = L$$

Proof. Let $\epsilon > 0$. There exist some large N such that for all $n > N$ we have

$$L - \epsilon < \frac{A_n}{B_n}, \frac{C_n}{D_n} < L + \epsilon.$$

Hence

$$L - \epsilon = \frac{(L-\epsilon)B_n + (L-\epsilon)D_n}{B_n + D_n} < \frac{A_n + C_n}{B_n + D_n} < \frac{(L+\epsilon)B_n + (L+\epsilon)D_n}{B_n + D_n} = L + \epsilon$$

for all $n > N$. The limit follows. □

Applying the previous lemma inductively gives the following result

Lemma 4.9. *Let $a_n^{(i)}, b_n^{(i)}$ be sequences of real numbers for each $i = 1, \ldots, m$. Let $A_n^{(i)} = \sum_{j=1}^{n} a_j^{(i)}$ and define $B_n^{(i)}$ similarly. Assume that $B_n^{(i)} > 0$ for all i, n. Then*

$$\lim_{n\to\infty} \frac{A_n^{(i)}}{B_n^{(i)}} = L\ \forall i \implies \lim_{n\to\infty} \frac{\sum_{i=1}^{m} A_n^{(i)}}{\sum_{i=1}^{m} B_n^{(i)}} = L.$$

The following limit follows immediately from Lemmas 4.6 and 4.9.

$$\lim_{n\to\infty} \frac{\#\{x \in \mathcal{O}_v^\times \mid x \in \mathrm{N}(\mathcal{O}_w)\} \bmod \pi^n}{\#\mathcal{O}_v^\times \bmod \pi^n} = \frac{1}{2}.$$

Remark 4.10. Note that Lemma 4.6 fails when v does not lie over 2. Indeed whether a unit x is a norm or not cannot be determined by looking modulo π.

Let $\mathcal{O}_v^* = \mathcal{O}_v \setminus \{0\}$ be the nonzero elements (not to be confused with $\mathcal{O}_v^\times$, the nonzero units). Define the sets

$$A = \{x \in \mathcal{O}_v \mid x(x-1) \in \mathrm{N}(\mathcal{O}_w^*)\}, \quad B = \{x \in \mathcal{O}_v \mid x - \lambda \in \mathrm{N}(\mathcal{O}_w^*)\}.$$

Since $x \mapsto x(x-1)$ and $x \mapsto x - \lambda$ are continuous endomorphisms on $\mathcal{O}_v$, both A, B, being inverse images of the open and closed subset $\mathrm{N}(\mathcal{O}_w^*)$, are open and closed inside $\mathcal{O}_v \setminus \{0, 1\}$ and $\mathcal{O}_v \setminus \{\lambda\}$ respectively. Our first goal is to establish the following.

Proposition 4.11.

$$\lim_{n\to\infty} \frac{\#A \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} = \lim_{n\to\infty} \frac{\#B \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} = \frac{1}{2}$$

Proof. Since $\{x - \lambda \mid x \in \mathcal{O}_v\} = \mathcal{O}_v$, the limit for B follows immediately in view of Lemma 4.6. To prove the limit for A, we first divide $\mathcal{O}_v$ in the following way

$$\mathcal{O}_v^{(0)} = \{x \in \mathcal{O}_v \mid x \equiv 0 \bmod \pi\}, \mathcal{O}_v^{(1)} = \{x \in \mathcal{O}_v \mid x \equiv 1 \bmod \pi\},$$
$$\mathcal{O}'_v = \mathcal{O}_v \setminus (\mathcal{O}_v^{(0)} \cup \mathcal{O}_v^{(1)}).$$

We divide the set A in the analogous way

$$A^{(0)} = \{x \in A \mid x \equiv 0 \bmod \pi\}, A^{(1)} = \{x \in A \mid x \equiv 1 \bmod \pi\},$$
$$A' = A \setminus (A^{(0)} \cup A^{(1)}).$$

By Lemma 4.9, it suffices to prove each of the following limits,

$$\lim_{n\to\infty} \frac{\#A^{(0)} \bmod \pi^n}{\#\mathcal{O}_v^{(0)} \bmod \pi^n} = \lim_{n\to\infty} \frac{\#A^{(1)} \bmod \pi^n}{\#\mathcal{O}_v^{(1)} \bmod \pi^n} = \lim_{n\to\infty} \frac{\#A' \bmod \pi^n}{\#\mathcal{O}'_v \bmod \pi^n} = \frac{1}{2}.$$

Define the map $f : k_v^\times \to k_v \setminus \{1\}$ given by $f(x) = 1 - 1/x$. Observe that f is a bijection. Moreover, for any $x \in \mathcal{O}_v^* \setminus \{1\}$, $x(x-1)/f(x) = x^2$. Hence, $x(x-1) \in \mathrm{N}(\mathcal{O}_w^*)$ if and only if $f(x) \in \mathrm{N}(L_w^\times)$. In particular, this means that $x \in A$ if and only if $f(x) \in \mathrm{N}(L_w^\times)$.

Let n be a positive integer and $x \in \mathcal{O}_v^\times$. Then

$$f(x + \pi^n\mathcal{O}_v) = 1 - 1/x + \pi^n\mathcal{O}_v = y + \pi^n\mathcal{O}_v$$

where $y = 1 - 1/x \in \mathcal{O}_v$. Observe that $y \not\equiv 1 \bmod \pi$. Hence, f induces a bijection between the following two sets,

$$\{x + \pi^n\mathcal{O}_v \mid x \in \mathcal{O}_v^\times\} \overset{f}{\longleftrightarrow} \{y + \pi^n\mathcal{O}_v \mid y \in \mathcal{O}_v, y \not\equiv 1 \bmod \pi\}.$$

We may further decompose into the following bijections

$$\{x + \pi^n\mathcal{O}_v \mid x \in \mathcal{O}_v, x \not\equiv 0, 1 \bmod \pi\} \overset{f}{\longleftrightarrow} \{y + \pi^n\mathcal{O}_v \mid y \in \mathcal{O}_v, y \not\equiv 0, 1 \bmod \pi\},$$
$$\{x + \pi^n\mathcal{O}_v \mid x \in \mathcal{O}_v, x \equiv 1 \bmod \pi\} \overset{f}{\longleftrightarrow} \{y + \pi^n\mathcal{O}_v \mid y \in \mathcal{O}_v, y \equiv 0 \bmod \pi\}.$$

Written another way, f induces bijections

$$\mathcal{O}'_v \bmod \pi^n \overset{f}{\longleftrightarrow} \mathcal{O}'_v \bmod \pi^n,$$
$$\mathcal{O}_v^{(1)} \bmod \pi^n \overset{f}{\longleftrightarrow} \mathcal{O}_v^{(0)} \bmod \pi^n.$$

Moreover, under this bijection, $A \bmod \pi^n$ maps to

$$A' \bmod \pi^n \overset{f}{\longleftrightarrow} \{y \in \mathcal{O}_v \mid y \in \mathrm{N}(\mathcal{O}_w), y \not\equiv 0, 1 \bmod \pi\} \bmod \pi^n,$$
$$A^{(1)} \bmod \pi^n \overset{f}{\longleftrightarrow} \{y \in \mathcal{O}_v \mid y \in \mathrm{N}(\mathcal{O}_w), y \equiv 0 \bmod \pi\} \bmod \pi^n.$$

Therefore,

$$\lim_{n\to\infty} \frac{\#A' \bmod \pi^n}{\#\mathcal{O}'_v \bmod \pi^n} = \lim_{n\to\infty} \frac{\#\{y \in \mathcal{O}_v \mid y \in \mathrm{N}(\mathcal{O}_w),\, y \not\equiv 0, 1 \bmod \pi\} \bmod \pi^n}{\#\mathcal{O}'_v \bmod \pi^n} = \frac{1}{2},$$

$$\lim_{n\to\infty} \frac{\#A^{(1)} \bmod \pi^n}{\#\mathcal{O}_v^{(1)} \bmod \pi^n} = \lim_{n\to\infty} \frac{\#\{y \in \mathcal{O}_v \mid y \in \mathrm{N}(\mathcal{O}_w),\, y \equiv 0 \bmod \pi\} \bmod \pi^n}{\#\mathcal{O}_v^{(0)} \bmod \pi^n} = \frac{1}{2}$$

by Lemma 4.6. It remains to prove the limit for $A^{(0)}$. For this, consider the map $g\colon \mathcal{O}_v \to \mathcal{O}_v$ given by $g(x) = 1 - x$. This is clearly a bijection and sends $\mathcal{O}_v^{(0)}$ to $\mathcal{O}_v^{(1)}$. Moreover, since $x(x-1) = (g(x))(g(x)-1)$, g sends $A^{(0)}$ bijectively to $A^{(1)}$. Hence,

$$\lim_{n\to\infty} \frac{\#A^{(0)} \bmod \pi^n}{\#\mathcal{O}_v^{(0)} \bmod \pi^n} = \lim_{n\to\infty} \frac{\#A^{(1)} \bmod \pi^n}{\#\mathcal{O}_v^{(1)} \bmod \pi^n} = \frac{1}{2}.$$

□

4.3.2. Applying the equidistribution results Finally, we return to the proof of Proposition 4.5.

Proof of Proposition 4.5. We first consider the case when $(a, c) = 0$. This means in particular that the fiber over $\infty \in \mathbb{P}^1$ has a point $Q \in X_\infty(k_v)$ and $\mathrm{inv}_v\, \mathcal{A}(Q) = \mathrm{inv}_v\, \mathcal{B}(Q) = 0$. So it suffices to find another point with invariant $1/2$.

Lemma 4.12. *Let $U \subseteq \mathcal{O}_v$ be a non-empty open subset. Then*

$$\lim_{n\to\infty} \frac{\#U \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} > 0.$$

Proof. First, the limit exists since the fraction inside the limit is nonincreasing and nonnegative in n. As U is open, there exists some positive integer N and $u \in \mathcal{O}_v$ such that $u + \pi^N \mathcal{O}_v \subseteq U$. Thus

$$\frac{\#U \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} \geq \frac{\#\{u + \pi^N \mathcal{O}_v\} \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} \geq \frac{1}{\#\mathcal{O}_v \bmod \pi^N}.$$

The limit then follows as the right side is independent of n. □

Lemma 4.13.

$$\lim_{n\to\infty} \frac{\#\mathcal{O}_v \setminus (A \cup B) \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} > 0.$$

.

Proof. Using a smooth v-adic analytic neighborhood around any point in $X(k_v) \neq \emptyset$ (which is nonempty since there is a point on the fiber above $0 \in \mathbb{P}^1$), there exists $x \in \mathcal{O}_v$ with $x \neq 0, 1, \lambda$ such that $x(x-1)(x-\lambda) \in \mathrm{N}(\mathcal{O}_w)$. This means either $x \in \mathcal{O}_v \setminus (A \cup B)$ or $x \in A \cap B$. In the former case, since A, B are closed in $\mathcal{O}_v$, it follows $\mathcal{O}_v \setminus (A \cup B)$ is a nonempty open set. The positivity of the limit then follows from Lemma 4.12. In the latter case, we apply Lemma 4.12 again to obtain $\lim_{n\to\infty}(\#A \cap B \bmod \pi^n)/(\#\mathcal{O}_v \bmod \pi^n) > 0$. We may write

$$\frac{\#A \cup B \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} = \frac{\#A \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} + \frac{\#B \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} - \frac{\#A \cap B \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n}$$

Taking limit as $n \to \infty$ on both sides show that RHS is < 1 by Proposition 4.11. The lemma then follows. □

Let $x \in \mathcal{O}_v \backslash (A \cup B)$ and $x \neq 0, 1, \lambda$. This means $x(x-1)(x-\lambda) \in \mathrm{N}(L_w)$ but $x(x-1) \notin \mathrm{N}(L_w)$. Let $Q \in X(k_v)$ be a point with x-coordinate is x. Then

$$\mathrm{inv}_v\, \mathcal{A}(Q) = (a, x(x-1)) = 1/2.$$

For the case $(a, c) \neq 0$, we have the following

Lemma 4.14.

$$\lim_{n\to\infty} \frac{\#(A \setminus B) \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} > 0, \quad \lim_{n\to\infty} \frac{\#(B \setminus A) \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} > 0.$$

Proof. By Proposition 4.11, it suffices to show at least one of the limit is positive since that will imply the other is positive as well. Since $X(k_v) \neq \emptyset$ by assumption, there exists $x \in \mathcal{O}_v$ with $x \neq 0, 1, \lambda$ such that one of the following two cases happen

Case 1, $x(x-1) \in \mathrm{N}(\mathcal{O}_w)$ and $x - \lambda \notin \mathrm{N}(\mathcal{O}_w)$ *OR*

Case 2, $x(x-1) \notin \mathrm{N}(\mathcal{O}_w)$ and $x - \lambda \in \mathrm{N}(\mathcal{O}_w)$

Either case, at least one of $A \setminus B$ or $B \setminus A$ is nonempty, and the limit must also be positive since both are open in $\mathcal{O}_v$. □

To finish the proof, we choose $x_1 \in A \backslash B$ and $x_2 \in B \backslash A$. Then there exists $Q_1, Q_2 \in X(k_v)$ with x coordinate corresponding to x_1, x_2 respectively. It follows that

$$\mathrm{inv}_v\, \mathcal{A}(Q_1) = 0, \quad \mathrm{inv}_v\, \mathcal{A}(Q_2) = 1/2.$$

□

4.4. Proof of Theorem 1.1

Proof of Theorem 1.1. Let X be a Châtelet surface where $P(x)$ has the form (4.1). First assume that for every nonarchimedean place v, either X has good reduction or $\sqrt{a} \in k_v$. In the first case, $k_v(\sqrt{a})/k_v$ must be unramified, so that any generator α listed in Sect. 3 is in the kernel of $\mathrm{Br}\, X \to \mathrm{Br}\, X_{k_v^{\mathrm{nr}}}$ where k_v^{nr} is the maximal unramified extension of k_v. Then by [5, Lemma 2.2]the evaluation map $\mathrm{ev}_\alpha \colon X(k_v) \to \mathrm{Br}\, k_v[2]$ must be constant for any $\alpha \in \mathrm{Br}\, X$. In the latter case, ev_α is also constant since the Brauer classes listed in §3 are trivial over k_v. Moreover, if $a > 0$ for all real embeddings or k does not have a real embedding, then for any archimedean place v, $\sqrt{a} \in k_v$, so the evaluation map is constant again.

Since $X(k)$ is clearly nonempty (take any root of $P(x)$), it follows X satisfies weak approximation.

Conversely, assume either X has a place v of bad reduction with $\sqrt{a} \notin k_v$ or v is a real place and $a < 0$. To show failure of weak approximation, it suffices to show that there is a Brauer–Manin obstruction given by the surjectivity of the evaluation map $\mathrm{ev}_{\mathcal{A}}\colon X(k_v) \to \mathrm{Br}\, k_v[2]$.

If $a < 0$, then the evaluation map is surjective at v since taking x such that exactly two of $x, x-1, x-\lambda$ is negative gives rise to a real point Q where either $\mathrm{ev}_{\mathcal{A}}(Q)$ or $\mathrm{ev}_{\mathcal{B}}(Q)$ is nontrivial. On the other hand, taking x so that all $x, x-1, x-\lambda$ are positive gives rise to a real point Q such that $\mathrm{ev}_{\mathcal{A}}(Q) = \mathrm{ev}_{\mathcal{B}}(Q) = 0$.

Now assume v is a place of bad reduction. If $k(\sqrt{a})/k$ is unramified at v, then one of $v(c), v(\lambda), v(\lambda-1)$ must be nonzero. Then Proposition 4.1 implies the result. If $k(\sqrt{a})/k$ is ramified then Proposition 4.2 for odd v or Proposition 4.5 for even v gives the desired result.

Proof of Theorem 1.2(1). Let L be the splitting field of $P(x)$ as stated in the theorem. If v is a place of bad reduction as given in the hypothesis, then there exists a place w of L lying over v such that X_L has bad reduction at w and $a \notin L_w^{\times 2}$. Theorem 1.1 then implies that X_L fails weak approximation. □

Proof of Theorem 1.2(2)(a). Let K/k be a finite extension. By our assumption, either $P(x)$ has an irreducible factor of degree 3 or splits completely over K. In the first case, there is no obstruction to weak approximation by Proposition 3.1. In the latter case, if $\sqrt{a} \in K$, then X_K is rational so weak approximation holds as well. Hence, assume $\sqrt{a} \notin K$. Then Theorem 1.2 implies that X_K satisfies weak approximation. □

Example 4.15. Let us give an example of a surface satisfying the conditions of Theorem 1.2(2)(a). Let ω be a primitive cube root of unity and let $k = \mathbb{Q}(\omega, \sqrt{97})$. Then k has class number 2 and so the Hilbert class field $L = k(\sqrt{a})$ is an everywhere unramified quadratic extension. Let X be the Châtelet surface

$$y^2 - az^2 = x(x-1)(x+\omega).$$

Observe that X has everywhere good reduction and satisfies the conditions given in Theorem 1.2(2)(a). Hence X satisfies weak approximation over all finite field extensions.

5. Weak approximation in the quadratic case

In this section, we consider the case when $P(x)$ factors as

$$y^2 - az^2 = cP_1(x)P_2(x)$$

where P_1, P_2 are irreducible monic quadratic polynomials. By §2, the Brauer group modulo $\mathrm{Br}_0\, X$ is generated by the quaternion algebra

$$\mathcal{C} = (a, P_1(x)) = (a, cP_2(x)).$$

If the above Brauer class is constant (meaning it comes from $\operatorname{Br} k$), then X satisfies weak approximation. Hence, for the rest of this section, we assume that the class above is nonconstant. This is equivalent to the fact that $\sqrt{a}$ is not in the splitting field of $P_1(x)$ or $P_2(x)$. Moreover, after a change of coordinates, we assume the coefficients of $P_1(x), P_2(x)$ are in $\mathcal{O}_k$.

Let v be a nonarchimedian place of k with odd residue characteristic and $\pi \in \mathcal{O}_k$ a uniformizer.

Lemma 5.1. *Let $R(x) \in \mathbb{F}_v[x]$ be a monic irreducible quadratic polynomial. Then for exactly $(q-1)/2$ many of the values $x \in \mathbb{F}_v$, $R(x)$ is a square in $\mathbb{F}_v$.*

Proof. It suffices to show that $R(x) = y^2$ has $q-1$ solutions in $(x, y) \in \mathbb{F}_v^2$ (since $R(x)$ is irreducible, y is never 0). We may homogenize to define a smooth conic in $\mathbb{P}^2_{\mathbb{F}_v}$. Since this conic has two points at infinity, it is isomorphic to $\mathbb{P}^1_{\mathbb{F}_v}$ and thus has $q+1$ points. Removing the two points at infinity gives $q-1$ solutions to the original equation. □

Proposition 5.2. *Assume that $v(a) = 1$. If $P_1(x), P_2(x)$ are irreducible modulo π, then there is an obstruction to weak approximation.*

Proof. Write

$$P_i(x) = x^2 + d_i x + r_i$$

where $d_i, r_i \in \mathcal{O}_v$. Since $P_i(x)$ is irreducible modulo π, we must have $r_i \in \mathcal{O}_v^\times$. Suppose X has a k_v point on a smooth fiber $x = x_0 \in k_v$. If $v(x_0) < 0$, then the fiber over $\infty \in \mathbb{P}^1$ also has a k_v point. Applying the automorphism $x \mapsto 1/x$ on $\mathbb{P}^1$, we may rewrite the equation for X as

$$y^2 - az^2 = cr_1r_2(x^2 + d_1x/r_1 + 1/r_1)(x^2 + d_2x/r_2 + 1/r_2)$$

which has a k_v point over the smooth fiber $x = 0$. Hence, we may reduce to the case where there exists a point $Q_0 = [y_0, z_0, x_0] \in X(k_v)$ on a smooth fiber where $v(x_0) \geq 0$.

It suffices to find a point $Q_1 \in X(k_v)$ such that $\operatorname{inv}_v \mathcal{A}(Q_0) \neq \operatorname{inv}_v \mathcal{A}(Q_1)$. Let $\alpha = x_0 \bmod \pi \in \mathbb{F}_v$. By Lemma 5.1, there must be another $\beta \in \mathbb{P}^1(\mathbb{F}_v)$ such that

(1) $P_1(\alpha)$ is a square if and only if $P_1(\beta)$ is a nonsquare, and
(2) $P_2(\alpha)$ is a square if and only if $P_2(\beta)$ is a nonsquare.

Here we take the convention that $P_1(\infty) = P_2(\infty) = 1$ are squares. This is only needed if both $P_1(\alpha), P_2(\alpha)$ are nonsquares. Then $P_1(\beta)P_2(\beta)$ is nonzero and in the same square class as $P_1(\alpha)P_2(\alpha)$. Therefore, we may use Hensel's lemma to lift to a point $Q_1 = (y_1, z_1, x_1) \in X(k_v)$ where $x_1 \equiv \beta \bmod \pi$. But then

$$\operatorname{inv}_v \mathcal{C}(Q_0) = (a, x_0) \neq (a, x_1) = \operatorname{inv}_v \mathcal{C}(Q_1).$$

The inequality of the two Hilbert symbols is due to (1) and (2) above. □

If the hypothesis of Proposition 5.2 does not hold, the existence of an obstruction to weak approximation is much more intrinsic to the surface in question. In particular, one cannot expect a uniform result similar to the case when $P(x)$ splits completely or is irreducible. We illustrate this subtlety by considering two Châtelet surfaces whose defining equation differs by one coefficient.

Example 5.3. Let $X/\mathbb{Q}$ be the Châtelet surface given by

$$y^2 - 17z^2 = 3(x^2 - 7)(17x^2 - 43).$$

By Proposition 3.1, $\operatorname{Br} X/\operatorname{Br}_0 X$ is generated by $\mathcal{C} = (17, x^2-7) = (17, 3(17x^2-43))$. We show that X satisfies weak approximation.

We begin by showing $X(\mathbb{Q}_p) \neq \emptyset$ for all primes p. First, observe that for $p \neq 3, 17$ we have $X_\infty(\mathbb{Q}_p) \neq \emptyset$ because X_∞ is the conic $y^2 - 17z^2 = 51w^2$, which has $\mathbb{Q}_p$ points for $p \neq 3, 17$. Next, we observe that the fiber X_1 is the conic $y^2 - 17z^2 = 468w^2$ which has $\mathbb{Q}_3$ and $\mathbb{Q}_{17}$-points since 468 is a square in $\mathbb{Q}_3$ and $\mathbb{Q}_{17}$.

We show that the map $\operatorname{ev}_{\mathcal{C}}\colon X(\mathbb{Q}_p) \to \operatorname{Br}\mathbb{Q}_p$ is constant at all primes p. Since the evaluation map is constant at all primes of good reduction [5, Lemma 2.2], it remains to check $\operatorname{ev}_{\mathcal{C}}\left(X(\mathbb{Q}_p)\right)$ for the primes $p = 2, 3, 7, 17$ and 43. Let $(x_0, y_0, z_0) \in X(\mathbb{Q}_p)$ and $P(x) = 3(x^2-7)(17x^2-43)$.

($p = 2$) Note that $\operatorname{inv}_2(\operatorname{ev}_{\mathcal{C}}(X(\mathbb{Q}_2))) = 0$ because $17 \in \mathbb{Q}_2^{\times 2}$.

($p = 17$) Assume $v_{17}(x_0) < 0$ so then $v_{17}(P(x_0)) = 4v_{17}(x_0) + 1$. Furthermore, we can see that

$$17^{-(4v_{17}(x_0)+1)} P(x_0) \equiv 3 \pmod{17}$$

which is not a square in $\mathbb{Q}_{17}$. We can now conclude that in this case, $P(x)$ is never a norm from the ramified extension $\mathbb{Q}_{17}(\sqrt{17})$. Hence we must have $v_{17}(x_0) \geq 0$. For these values of x_0, $3(17x_0^2 - 43) \equiv 7 \pmod{17}$ is not a square so $\operatorname{inv}_{17}(\operatorname{ev}_{\mathcal{C}}(X(\mathbb{Q}_{17}))) = 1/2$.

($p = 7$) Since $3(17x^2 - 43)$ is irreducible over $\mathbb{Q}_7$, $v_7(3(17x_0^2 - 43))$ must be even. Hence $\operatorname{inv}_7(\operatorname{ev}_{\mathcal{C}}(X(\mathbb{Q}_7))) = 0$.

($p = 43$) Since $x^2 - 7$ is irreducible over $\mathbb{Q}_{43}$, we have $v_{43}(x_0^2 - 7)$ must be even. Hence $\operatorname{inv}_{43}(\operatorname{ev}_{\mathcal{C}}(X(\mathbb{Q}_{43}))) = 0$.

($p = 3$) Since $17x^2 - 43$ is irreducible over $\mathbb{Q}_3$, $v_3(17x_0^2 - 43)$ must be even. Hence $v_3(3(17x_0^2 - 43))$ is odd and so $\operatorname{inv}_3(\operatorname{ev}_{\mathcal{C}}(X(\mathbb{Q}_3))) = 1/2$.

Combining the above calculations, we obtain that the sum of invariants is always 0, which means weak approximation holds.

After a minor adjustments to the surface of Example 5.3, we obtain another Châtelet surface which fails weak approximation.

Example 5.4. Let $X/\mathbb{Q}$ be the Châtelet surface given by

$$y^2 - 17z^2 = 3(x^2 - 7)(17x^2 - 7 \cdot 43).$$

We first show that $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$. In a similar fashion to Example 5.3 we have $X(\mathbb{Q}_p) \neq \emptyset$ for all $p \neq 3, 17$ hence only non-emptiness of $X(\mathbb{Q}_3)$ and $X(\mathbb{Q}_{17})$

must be checked directly. One can compute the fibers X_1 and X_3 and see that the conic X_1 has $\mathbb{Q}_3$-points and the conic X_3 has $\mathbb{Q}_{17}$-points.

We claim that X fails weak approximation and to prove this, it suffices to verify that the map $\mathrm{ev}_{\mathcal{C}}\colon X(\mathbb{Q}_7) \to \mathrm{Br}\,\mathbb{Q}_7[2]$ is surjective. To see this, first note that $X_\infty(\mathbb{Q}_7) \neq \emptyset$ and $\mathrm{ev}_{\mathcal{C}}$ is trivial over such a point. On the other hand, as $P(0) = 3{\cdot}7^2{\cdot}43$ has even valuation, there exists $Q \in X_0(\mathbb{Q}_7)$ lying over $0 \in \mathbb{P}^1(\mathbb{Q}_7)$ where

$$\mathrm{ev}_{\mathcal{C}}(Q) = (17, -7) \neq 0.$$

This proves our claim.

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Declarations

Data availibility The authors declare that the data supporting the findings of this study are available within the paper and the included references.

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