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Oksana S. Yakimova

Poisson commutative subalgebras associated with a Cartan subalgebra

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Abstract. Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra. The $\mathfrak{t}$-stable decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ yields a bi-grading of the symmetric algebra $\mathcal{S}(\mathfrak{g})$. The subalgebra $\mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ generated by the bi-homogenous components of the symmetric invariants $F \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is known to be Poisson commutative. Furthermore the algebra $\tilde{\mathcal{Z}} = \mathsf{alg}\langle \mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}, \mathfrak{t}\rangle$ is also Poisson commutative. We investigate relations between $\tilde{\mathcal{Z}}$ and Mishchenko–Fomenko subalgebras. In type A, we construct a quantisation of $\tilde{\mathcal{Z}}$ making use of quantum Mishchenko–Fomenko algebras.

Introduction

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero. The symmetric algebra $\mathcal{S}(\mathfrak{g}) \cong \Bbbk[\mathfrak{g}^*]$ of $\mathfrak{g}$ is equipped with the standard Poisson structure, i.e., the Lie–Poisson bracket $\{\ ,\ \}$. A subalgebra $A \subset \mathcal{S}(\mathfrak{g})$ is *Poisson commutative* if $\{A, A\} = 0$. Poisson commutative subalgebras attract a great deal of attention, because of their relationship to integrable systems and geometric representation theory. If $\{A, A\} = 0$, then tr.deg $A \leqslant \boldsymbol{b}(\mathfrak{g})$, where $\boldsymbol{b}(\mathfrak{g}) := \frac{1}{2}(\dim \mathfrak{g} + \mathsf{rk}\, \mathfrak{g})$ is the dimension of a Borel subalgebra of $\mathfrak{g}$.

The celebrated "argument shift method", which goes back to Mishchenko–Fomenko [7], produces interesting Poisson commutative subalgebras. Namely, to any $\gamma \in \mathfrak{g}^*$, one associates the subalgebra $(\mathcal{MF})_\gamma \subset \mathcal{S}(\mathfrak{g})$. Following Vinberg [19], we say that $(\mathcal{MF})_\gamma$ is the *Mishchenko–Fomenko subalgebra* associated with γ. This algebra can be described as follows. Let $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ be the *Poisson centre* of $(\mathcal{S}(\mathfrak{g}), \{\ ,\ \})$, i.e.,

$$\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \{H \in \mathcal{S}(\mathfrak{g}) \mid \{H, x\} = 0 \ \ \forall x \in \mathfrak{g}\}.$$

For $F \in \mathcal{S}(\mathfrak{g})$, let $\partial_\gamma F$ be the directional derivative of F with respect to $\gamma \in \mathfrak{g}^*$, i.e.,

$$\partial_\gamma F(x) = \frac{d}{dt} F(x + t\gamma)\Big|_{t=0}.$$

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O. S. Yakimova (✉): Institut für Mathematik, Friedrich-Schiller-Universität Jena, 07737 Jena, Germany, e-mail: oksana.yakimova@uni-jena.de

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By definition, the algebra $(\mathcal{MF})_\gamma$ is generated by all $\partial_\gamma^k F$ with $k \geqslant 0$ and $F \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Clearly, $(\mathcal{MF})_\gamma$ is a graded subalgebra of $\mathcal{S}(\mathfrak{g})$. The importance of these subalgebras and their quantum counterparts is demonstrated e.g. in [3,4,8,19]. Suppose that γ is regular. Then $(\mathcal{MF})_\gamma$ is a maximal Poisson commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ [11]. For regular semisimple elements, this has been earlier proved by Tarasov [17]. Furthermore, $(\mathcal{MF})_\gamma$ is freely generated by $\boldsymbol{b}(\mathfrak{g})$ homogeneous elements, see e.g. [11].

Let $\mathfrak{f}$ be a subalgebra of $\mathfrak{g}$. Suppose that there is an $\mathfrak{f}$-stable decomposition $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{m}$, i.e., $[\mathfrak{f}, \mathfrak{m}] \subset \mathfrak{m}$. This yields a bi-homogeneous decomposition of $\mathcal{S}(\mathfrak{g})$:

$$\mathcal{S}(\mathfrak{g}) = \bigoplus_{i,j \geqslant 0} \mathcal{S}^i(\mathfrak{f}) \otimes \mathcal{S}^j(\mathfrak{m})$$

and for any $F \in \mathcal{S}(\mathfrak{g})$ we get the decomposition $F = \sum_{i,j} F_{i,j}$, where $F_{i,j} \in \mathcal{S}^i(\mathfrak{f}) \otimes \mathcal{S}^j(\mathfrak{m})$.

Let $\mathcal{Z}_{(\mathfrak{g},\mathfrak{f})}$ be the subalgebra of $\mathcal{S}(\mathfrak{g})$ generated by the bi-homogeneous components $H_{i,j}$ of all $H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Since H is $\mathfrak{g}$-invariant, each $H_{i,j}$ is $\mathfrak{f}$-invariant, hence $\mathcal{Z}_{(\mathfrak{g},\mathfrak{f})} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{f}}$. The subalgebras $\mathcal{Z}_{(\mathfrak{g},\mathfrak{f})}$ are not necessarily Poisson commutative [13, Example 2.3]. However, it is proved in loc. cit. that $\mathcal{Z}_{(\mathfrak{g},\mathfrak{f})}$ is Poisson commutative whenever $[\mathfrak{f}, \mathfrak{f}] = 0$. In particular, if $\mathfrak{f} = \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ is Poisson commutative. Since $\mathcal{Z}_{(\mathfrak{g},\mathfrak{t})} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$ and $\mathfrak{t}$ is commutative, the subalgebra generated by $\mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ and $\mathfrak{t}$ is still Poisson commutative. We denote it by $\tilde{\mathcal{Z}} = \mathsf{alg}\langle \mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}, \mathfrak{t}\rangle$. By [13, Theorem 3.2], $\tilde{\mathcal{Z}}$ is a polynomial algebra, it is a maximal Poisson commutative subalgebra and tr.deg $\tilde{\mathcal{Z}} = \boldsymbol{b}(\mathfrak{g})$. Results of Section 3 in [13] demonstrate that $\tilde{\mathcal{Z}}$ is closely related to Mishchenko–Fomenko subalgebras. One of the goals of this paper is to further elaborate on these relations.

Let $\mathfrak{m} \subset \mathfrak{g}$ be the $\mathfrak{t}$-stable complement of $\mathfrak{t}$ in $\mathfrak{g}$. Then $\mathfrak{t}^*$ (resp. $\mathfrak{m}^*$) is identified with the annihilator $\mathrm{Ann}(\mathfrak{m}) \subset \mathfrak{g}^*$ (resp. $\mathrm{Ann}(\mathfrak{t}) \subset \mathfrak{g}^*$). Our first result is that, for any $\gamma \in \mathfrak{t}^*$, the restrictions of $\tilde{\mathcal{Z}}$ and $(\mathcal{MF})_\gamma$ to $\gamma + \mathfrak{m}^*$ coincide, see Theorem 2.1.

There is a Poisson bracket $\{\ ,\ \}_{(\gamma)}$ on $\Bbbk[\gamma + \mathfrak{m}^*]^{\mathfrak{t}} \cong \mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$, inherited from $\mathcal{S}(\mathfrak{g})$. The rank of this bracket is equal to $\dim \mathfrak{g} - 3\,\mathsf{rk}\,\mathfrak{g}$ and if $\mathcal{B}$ is a Poisson commutative subalgebra of $(\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}, \{\ ,\ \}_{(\gamma)})$, then tr.deg $\mathcal{B} \leqslant \frac{1}{2}(\dim \mathfrak{g} - \mathsf{rk}\,\mathfrak{g})$, see Sect. 2.1. We show that if γ is a regular point of $\mathfrak{g}^*$, then $\tilde{\mathcal{Z}}|_{\gamma+\mathfrak{m}^*}$ is a maximal Poisson commutative subalgebra of $(\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}, \{\ ,\ \}_{(\gamma)})$ of transcendence degree $\frac{1}{2}(\dim \mathfrak{g} - \mathsf{rk}\,\mathfrak{g})$.

If $\mu \in \mathfrak{t}^*$ is regular in $\mathfrak{g}^*$, then $(\mathcal{MF})_\mu \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$ and the component of grade 2 in $(\mathcal{MF})_\mu$ equals $\mathcal{S}^2(\mathfrak{t}) \oplus \mathbb{V}(\mu)$, where $\mathbb{V}(\mu)$ is a certain subspace of dimension $\mathsf{rk}[\mathfrak{g}, \mathfrak{g}]$. There is a natural choice for such a $\mathbb{V}(\mu)$, which is explicitly described by Vinberg [19]. We recall it in Sect. 2. In [14], Rybnikov proved that if μ is generic enough, then $(\mathcal{MF})_\mu$ is equal to the Poisson centraliser of $\mathbb{V}(\mu)$ in $\mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$. His proof goes through for all simple types, but does not apply to all regular points. We prove that in type A, the centraliser description holds for **all** regular μ, see Proposition 2.8.

An interesting task is to produce a *quantisation* of $\tilde{\mathcal{Z}}$, i.e., a commutative subalgebra $\mathscr{Z}$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ such that $\mathrm{gr}(\mathscr{Z}) \subset \mathcal{S}(\mathfrak{g})$ coincides with $\tilde{\mathcal{Z}}$. In case of $(\mathcal{MF})_\mu$, the quantisation problem was raised in [19]. A solution, the *quantum Mishchenko–Fomenko subalgebra* $\mathcal{F}_\mu \subset \mathcal{U}(\mathfrak{g})$, is obtained in [15] and [4]. These subalgebras are studied in [3,4,8,16,18]. For $\mathfrak{g} = \mathfrak{gl}_n$, we quantise $\tilde{\mathcal{Z}}$ in Sect. 3, see Theorem 3.2. For any $\mathfrak{g}$, we can lift quadratic in $\mathfrak{m}$ elements of $\tilde{\mathcal{Z}}$ to commuting elements of $\mathcal{U}(\mathfrak{g})$. In the context of Sect. 3, we have found many similarities with quantum counterparts of Mishchenko–Fomenko subalgebras. This may be an indication that the algebra $\tilde{\mathcal{Z}}$ always has a quantisation.

1. Preliminaries on the coadjoint action and Poisson-commutativity

Let $\mathfrak{q}$ be a Lie algebra over $\Bbbk$. Let $\mathfrak{q}^\xi$ denote the stabiliser in $\mathfrak{q}$ of $\xi \in \mathfrak{q}^*$. The *index of* $\mathfrak{q}$, ind $\mathfrak{q}$, is defined by $\operatorname{ind}\mathfrak{q} = \min_{\xi\in\mathfrak{q}^*} \dim \mathfrak{q}^\xi$. The set of *regular* elements of $\mathfrak{q}^*$ is

$$\mathfrak{q}^*_{\mathsf{reg}} = \{\eta \in \mathfrak{q}^* \mid \dim \mathfrak{q}^\eta = \operatorname{ind}\mathfrak{q}\}. \tag{1.1}$$

Then $\mathfrak{q}^*_{\mathsf{sing}} = \mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathsf{reg}}$. Set further $\boldsymbol{b}(\mathfrak{q}) = (\dim\mathfrak{q} + \operatorname{ind}\mathfrak{q})/2$. If $\mathfrak{q} = \mathfrak{g}$ is reductive, then $\operatorname{ind}\mathfrak{g} = \mathsf{rk}\,\mathfrak{g}$ and $\boldsymbol{b}(\mathfrak{g})$ is the dimension of a Borel subalgebra of $\mathfrak{g}$.

For any $\gamma \in \mathfrak{q}^*$, one defines the Poisson bracket $\{\,,\,\}_\gamma$ on $\mathfrak{q}^*$ by $\{\xi,\eta\}_\gamma = \gamma([\xi,\eta])$ for $\xi,\eta \in \mathfrak{q}$. This new bracket is *compatible* with the standard Lie–Poisson bracket $\{\ ,\ \}$ on $\mathcal{S}(\mathfrak{q})$, i.e., any linear combination of $\{\ ,\ \}$ and $\{\ ,\ \}_\gamma$ is again a Poisson brackets. For more details, see [2, Sect. 1.8.3]. There is a well-known method, *the Lenard–Magri scheme*, for constructing "large" Poisson commutative subalgebras of $\mathcal{S}(\mathfrak{q})$, which is related to compatible brackets, see e.g. [5]. In this way, one obtains $(\mathcal{MF})_\gamma$ from the pair $(\{\ ,\ \}, \{\ ,\ \}_\gamma)$.

In [13], starting form an $\mathfrak{f}$-stable decomposition $\mathfrak{q} = \mathfrak{f} \oplus \mathfrak{m}$ with $[\mathfrak{f},\mathfrak{f}] = 0$, a Poisson commutative subalgebra $\mathcal{Z}_{(\mathfrak{q},\mathfrak{f})} \subset \mathcal{S}(\mathfrak{q})$ was constructed. From now on, we consider a particularly interesting case, where $\mathfrak{q} = \mathfrak{g} = \mathsf{Lie}\,G$ is a reductive Lie algebra and $\mathfrak{f} = \mathfrak{t}$ is a Cartan subalgebra. Set $l = \mathsf{rk}\,\mathfrak{g}$.

The algebra $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is a polynomial ring. Let $\{H_1,\ldots,H_l\}$ be a set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with $\deg H_j = d_j$. We have $\sum_{j=1}^{l} d_j = \boldsymbol{b}(\mathfrak{g})$. Let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$ be a fixed triangular decomposition. Set $\mathfrak{m} = \mathfrak{u} \oplus \mathfrak{u}^-$. The vector space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ provides the bi-homogeneous decomposition of each H_j:

$$H_j = \sum_{i=0}^{d_j} (H_j)_{(i,d_j-i)},$$

where $(H_j)_{(i,d_j-i)} \in \mathcal{S}^i(\mathfrak{t}) \otimes \mathcal{S}^{d_j-i}(\mathfrak{m}) \subset \mathcal{S}^{d_j}(\mathfrak{g})$. Then we say that $d_j - i$ is the $\mathfrak{m}$-*degree* of $(H_j)_{(i,d_j-i)}$. Now, $\mathcal{Z} := \mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}$ is the algebra generated by

$$\{(H_j)_{(i,d_j-i)} \mid j = 1,\ldots,l;\, i = 0,1,\ldots,d_j-3, d_j-2, d_j\},$$

see [13]. The total number of functions in this family equals $\sum_{j=1}^{l}(d_j+1) - l = \boldsymbol{b}(\mathfrak{g})$ and they are algebraically independent [13]. Replacing the elements $(H_j)_{(d_j,0)} \in \mathcal{S}^{d_j}(\mathfrak{t})$ with a basis of $\mathfrak{t}$, we obtain a larger subalgebra, denoted $\tilde{\mathcal{Z}}$, which is still polynomial and Poisson commutative.

1.1. Notation and conventions

For a subalgebra $A \subset \mathcal{S}(\mathfrak{g})$ and $\gamma \in \mathfrak{g}^*$, we set $d_\gamma A = \{d_\gamma F \mid F \in A\}$.

Given a Poisson algebra $\mathcal{A}$ and $a \in \mathcal{A}$, let $\mathcal{Z}_a\mathcal{A} = \{F \in \mathcal{A} \mid \{a, F\} = 0\}$ denote the *Poisson centraliser* of a in $\mathcal{A}$.

Let $\mathfrak{l} \subset \mathfrak{g}$ be a Lie subalgebra. Then $\mathcal{S}(\mathfrak{g})^{\mathfrak{l}} = \{F \in \mathcal{S}(\mathfrak{g}) \mid \{\xi, F\} = 0\ \forall \xi \in \mathfrak{l}\}$ and $\mathcal{U}(\mathfrak{g})^{\mathfrak{l}}$ stands for the centraliser $\{X \in \mathcal{U}(\mathfrak{g}) \mid [\xi, X] = 0\ \forall \xi \in \mathfrak{l}\}$ of $\mathfrak{l}$ in $\mathcal{U}(\mathfrak{g})$.

For an irreducible affine variety Y over $\Bbbk$, we let $\Bbbk[Y]$ stand for the ring of regular functions on Y and $\Bbbk(Y) = \operatorname{Quot} \Bbbk[Y]$ for the field of rational functions on Y. A statement that a certain assertion holds for *generic points* of Y (or for generic orbits on Y) means that this assertion holds for all points of a nonempty open subset $U \subset Y$ (for all orbits intersecting U).

If $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{m}$ and $\xi \in \mathfrak{g}$, then $\xi = \xi_{\mathfrak{f}} + \xi_{\mathfrak{m}}$, where $\xi_{\mathfrak{f}} \in \mathfrak{f}$ and $\xi_{\mathfrak{m}} \in \mathfrak{m}$.

Let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{t})$ and $\Delta^+ \subset \Delta$ the subset of positive roots corresponding to $\mathfrak{u}$. For $\alpha \in \Delta$, let $e_\alpha \in \mathfrak{g}_\alpha$ be a nonzero root vector. We let $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}$ be such that $\alpha(h_\alpha) = 2$.

We say that $\mathfrak{g}$ is of type A, if $\mathfrak{g} = \mathfrak{sl}_{l+1}$. In that case, we fix $\mathfrak{t}$ as the subspace of diagonal matrices and use the standard linear functions $\varepsilon_i \in \mathfrak{t}^*$ such that $\varepsilon_i(\operatorname{diag}(a_1, \ldots, a_{l+1})) = a_i$. We fix the standard triangular decomposition with $\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. For $1 \leqslant i, j \leqslant n$, let $E_{ij} \in \mathfrak{gl}_n$ be a matrix unit (elementary matrix).

If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ via the Killing form κ.

2. Partial localisations and reductions by the action of $\mathfrak{t}$

Results of [13] show that $\tilde{\mathcal{Z}} = \mathsf{alg}\langle \mathcal{Z}_{(\mathfrak{g},\mathfrak{t})}, \mathfrak{t}\rangle$ is closely related to Mishchenko–Fomenko subalgebras. Our goal now is to elaborate on this relation.

Recall that $\mathfrak{m} = \mathfrak{u} \oplus \mathfrak{u}^-$. Consider $\mathcal{A} = \mathcal{S}(\mathfrak{g})^{\mathfrak{t}} \otimes_{\mathcal{S}(\mathfrak{t})} \Bbbk(\mathfrak{t}^*) \subset \Bbbk(\mathfrak{g}^*)^{\mathfrak{t}}$ as a ring of $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$-valued rational functions on $\mathfrak{t}^*$; here $FM \otimes \tilde{F} = M \otimes F\tilde{F}$ for $M \in \mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$, $F \in \mathcal{S}(\mathfrak{t}) \cong \Bbbk[\mathfrak{t}^*]$, $\tilde{F} \in \Bbbk(\mathfrak{t}^*)$ and $(FM \otimes \tilde{F})(\mu) = F(\mu)\tilde{F}(\mu)M$, if $\mu \in \mathfrak{t}^*$. Since $\{\mathcal{S}(\mathfrak{g})^{\mathfrak{t}}, \mathcal{S}(\mathfrak{g})^{\mathfrak{t}}\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$ and $\{\mathcal{S}(\mathfrak{g})^{\mathfrak{t}}, \mathfrak{t}\} = 0$, the ring $\mathcal{A}$ inherits a Poisson structure from $\mathcal{S}(\mathfrak{g})$ and this Poisson structure is $\Bbbk(\mathfrak{t}^*)$-linear. For $\mu \in \mathfrak{t}^*$, set

$$\mathcal{A}_\mu = \{A \in \mathcal{A} \mid A(\mu) \text{ is well-defined}\}.$$

Let $\psi_\mu : \mathcal{A}_\mu \to \mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ be the evaluation homomorphism. Then $\mathcal{S}(\mathfrak{g})^{\mathfrak{t}} \subset \mathcal{A}_\mu$ for each μ and on $\mathcal{S}(\mathfrak{g})^{\mathfrak{t}} \cong \Bbbk[\mathfrak{g}^*]^{\mathfrak{t}}$, the map ψ_μ coincides with the restriction homomorphism

$$\Bbbk[\mathfrak{g}^*]^{\mathfrak{t}} \to \Bbbk[\mu + \mathfrak{m}^*]^{\mathfrak{t}} \cong \mathcal{S}(\mathfrak{m})^{\mathfrak{t}}.$$

We define a bi-linear map $\{\ ,\ \}_{(\mu)} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \oplus \Bbbk$ by

$$\{x, y\}_{(\mu)} := \psi_\mu([x, y]) = [x, y]_{\mathfrak{m}} + \mu([x, y]_{\mathfrak{t}}) \ \text{ for } \ x, y \in \mathfrak{m}$$

and then extend it to a bi-linear operation on $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ using the Leibniz rule. For X, Y in $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}} \subset \mathcal{A}$, we have

$$\psi_\mu(\{X, Y\}) = \{X, Y\}_{(\mu)} = \{\psi_\mu(X), \psi_\mu(Y)\}_{(\mu)}.$$

Using the $\Bbbk(\mathfrak{t}^*)$-linearity of the Poisson bracket on $\mathcal{A}$, we conclude that $\{\ ,\ \}_{(\mu)}$ is a Poisson structure on $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ and ψ_μ is a Poisson homomorphism with respect to $\{\ ,\ \}_{(\mu)}$.

The ring $\mathcal{A} = \bigoplus_{N\geqslant 0} \mathcal{A}_N$ is graded by the degree in $\mathfrak{m}$ with $\mathcal{A}_N = \mathcal{S}^N(\mathfrak{m})^{\mathfrak{t}} \otimes_{\Bbbk} \Bbbk(\mathfrak{t}^*)$ and $\psi_\mu(\mathcal{A}_\mu \cap \mathcal{A}_N) = \mathcal{S}^N(\mathfrak{m})^{\mathfrak{t}}$. Since the Poisson bracket on $\mathcal{A}$ is $\Bbbk(\mathfrak{t}^*)$-linear, for any $A \in \mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$ and any $N \geqslant 0$, the subset $\mathcal{Z}_A(\mathcal{A}_{\leqslant N}) = \mathcal{Z}_A\mathcal{A} \cap (\bigoplus_{i=0}^{N} \mathcal{A}_i)$ is a vector space over $\Bbbk(\mathfrak{t}^*)$. It is a subspace of a finite-dimensional $\Bbbk(\mathfrak{t}^*)$-space $\mathcal{A}_{\leqslant N}$. Evaluating the defining equations of the centraliser $\mathcal{Z}_A(\mathcal{A}_{\leqslant N})$ at $\mu \in \mathfrak{t}^*$, we obtain

$$\dim_{\Bbbk(\mathfrak{t}^*)} \mathcal{Z}_A(\mathcal{A}_{\leqslant N}) = \dim \left(\mathcal{Z}_{\psi_\mu(A)}(\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}, \{\ ,\ \}_{(\mu)}) \cap \mathcal{S}^{\leqslant N}(\mathfrak{m}) \right), \tag{2.1}$$

whenever μ is generic enough.

Recall that we identify $\mathfrak{t}^*$ with $\mathrm{Ann}(\mathfrak{m}) \subset \mathfrak{g}^*$.

Theorem 2.1. *If $\mu \in \mathfrak{t}^*$, then $\psi_\mu(\tilde{\mathcal{Z}}) = \psi_\mu((\mathcal{MF})_\mu)$.*

Proof. Suppose first that $\mu \neq 0$. We fix $h \in \mathfrak{t}$ such that $\mu(h) = 1$ and write $\mathfrak{g} = \Bbbk h \oplus \ker\mu$, where $\mathfrak{m} \subset \ker\mu$. Let $H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ be homogeneous with $\deg H = d$. We decompose H as a sum

$$H = H_0 h^d + H_1 h^{d-1} + \cdots + H_k h^k + \cdots + H_d,$$

where $H_k \in \mathcal{S}^k(\ker\mu)$. By the choice of h, we have

$$\partial_\mu^k H = \sum_{r=k}^{d} r(r-1)\ldots(r-k+1)h^{r-k} H_{d-r}$$

and

$$\psi_\mu(\partial_\mu^k H) = \sum_{r=k}^{d} r(r-1)\ldots(r-k+1)\psi_\mu(H_{d-r}).$$

Therefore $\psi_\mu((\mathcal{MF})_\mu) = \mathsf{alg}\langle \psi_\mu(H_{d-r}) \mid d \geqslant 1, H \in \mathcal{S}^d(\mathfrak{g})^{\mathfrak{g}}, 0 \leqslant r < d\rangle$.

Let $M_{d-i} \in \mathcal{S}^{d-i}(\mathfrak{m})$ be such that the bi-homogeneous component $H_{i,d-i} \in \mathcal{S}^i(\mathfrak{t}) \otimes \mathcal{S}^{d-i}(\mathfrak{m})$ of H lies in $h^i M_{d-i} + (\ker\mu \cap \mathfrak{t})\mathcal{S}(\mathfrak{g})$. Then

$$\psi_\mu(H_{i,d-i}) = M_{d-i} = \psi_\mu(h^i H_{d-i}) = \psi_\mu(H_{d-i})$$

for all i. Since $\tilde{\mathcal{Z}}$ is generated by $\mathfrak{t}$ and $H_{i,d-i}$ with $H \in \mathcal{S}^d(\mathfrak{g})^{\mathfrak{g}}$, we are done for $\mu \neq 0$.

If $\mu = 0$, then $(\mathcal{MF})_0 = \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and $\psi_0(H) = \psi_0(H_{0,d})$. For other generators of $\tilde{\mathcal{Z}}$, we have $\psi_0(\mathfrak{t}) = 0$ and $\psi_0(H_{i,d-i}) = 0$, whenever $i > 0$. □

A Mishchenko–Fomenko subalgebra is Poisson commutative and ψ_μ is a Poisson homomorphism; hence

$$0 = \psi_\mu(\{(\mathcal{MF})_\mu, (\mathcal{MF})_\mu\}) = \{\psi_\mu((\mathcal{MF})_\mu), \psi_\mu((\mathcal{MF})_\mu)\}_{(\mu)} \\ = \{\psi_\mu(\tilde{\mathcal{Z}}), \psi_\mu(\tilde{\mathcal{Z}})\}_{(\mu)} = \psi_\mu(\{\tilde{\mathcal{Z}}, \tilde{\mathcal{Z}}\})$$

for each $\mu \in \mathfrak{t}^*$. Since for any $A \in \mathcal{A} \setminus \{0\}$, the image $\psi_\mu(A)$ is well-defined and nonzero for almost all μ, Theorem 2.1 provides a new proof for the fact that $\{\tilde{\mathcal{Z}}, \tilde{\mathcal{Z}}\} = 0$.

Assume that $\mu \in \mathfrak{t}^* \cap \mathfrak{g}^*_{\mathsf{reg}}$. Then the polynomials $\partial_\mu^k H_j$ with $1 \leqslant j \leqslant l$ and $0 \leqslant k < d_j$ are algebraically independent, cf. [11]. In particular, $\{\partial_\mu^{d_j-1} H_j \mid 1 \leqslant j \leqslant l\}$ is a basis of $\mathfrak{t}$ and $\mathfrak{t} \subset (\mathcal{MF})_\mu$. This fact follows also from the *Kostant regularity criterion* for $\mathfrak{g}$ [6, Theorem 9],

$$\langle d_\xi H_j \mid 1 \leqslant j \leqslant l\rangle_{\Bbbk} = \mathfrak{g}^\xi \ \text{ if and only if } \ \xi \in \mathfrak{g}^*_{\mathsf{reg}}, \tag{2.2}$$

since $d_\mu H_j = \frac{1}{(d_j-1)!}\partial_\mu^{d_j-1} H_j$.

Quadratic elements of $(\mathcal{MF})_\mu$ are of particular importance. If not stated otherwise, assume that $\mathfrak{g}$ is semisimple. Set $\boldsymbol{f}_j := \boldsymbol{f}_j(\mu) := \psi_\mu((H_j)_{(d_j-2,2)})$. Then

$$\partial_\mu^{d_j-2} H_j \in (d_j-2)!\boldsymbol{f}_j + \mathcal{S}^2(\mathfrak{t}). \tag{2.3}$$

Since $\mathfrak{t} \subset (\mathcal{MF})_\mu$, we have $\boldsymbol{f}_j \in (\mathcal{MF})_\mu$ for all j. Furthermore, the component of grade 2 in $(\mathcal{MF})_\mu$ is equal to $\mathcal{S}^2(\mathfrak{t}) \oplus \mathbb{V}(\mu)$, where $\mathbb{V}(\mu) = \langle \boldsymbol{f}_j \mid 1 \leqslant j \leqslant l\rangle_{\Bbbk}$. If μ is generic enough, then $(\mathcal{MF})_\mu$ is equal to the Poisson centraliser

$$\mathcal{Z}_{\mathbb{V}(\mu)} := \{F \in \mathcal{S}(\mathfrak{g})^{\mathfrak{t}} \mid \{F, \boldsymbol{f}_j\} = 0\ \forall j\}$$

by [14, Theorem 1].

Remark 2.2. An explicit description of the elements $\boldsymbol{f}_j$ is crucial for the considerations in [14, 19] and many others. We present a quick elementary argument that produces such a description. Set $h_j = d_\mu H_j$. Similar to the proof of Theorem 2.1, write $\mathfrak{g} = \Bbbk h \oplus \ker\mu$, where $\mu(h) = 1$. Then

$$H_j = ch^{d_j} + h^{d_j-1}h' + h^{d_j-2}H' + \sum_{\alpha\in\Delta^+} C_\alpha h^{d_j-2} e_\alpha e_{-\alpha} + \sum_{k=3}^{d_j} h^{d_j-k} H_{j,k},$$

where c, C_α are scalars, $h', H', H_{j,k} \in \mathcal{S}(\ker\mu)$, and $h' \in \mathfrak{t}$, $H' \in \mathcal{S}^2(\mathfrak{t})$. In this notation, $h_j = cd_j h + h'$. Since $H_j \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, we have $\{e_\alpha, H_j\} = 0$ for each $\alpha \in \Delta^+$. Note that $\{e_\alpha, e_{-\alpha}\} = \mu(h_\alpha)h + h''$, where $h'' \in (\ker\mu \cap \mathfrak{t})$. Considering the terms of $\{e_\alpha, H_j\}$ that lie in $\Bbbk h^{d_j-1}\mathfrak{m}$, and then necessarily in $\Bbbk h^{d_j-1}e_\alpha$, we obtain

$$-\alpha(cd_j h + h') + C_\alpha \mu(h_\alpha) = 0$$

for each positive root α. Since $\mu \in \mathfrak{t}^* \cap \mathfrak{g}^*_{\mathsf{reg}}$, we have $\mu(h_\alpha) \neq 0$ for each $\alpha \in \Delta^+$. Hence $C_\alpha = \dfrac{\alpha(h_j)}{\mu(h_\alpha)}$. Here $\boldsymbol{f}_j = \sum_{\alpha\in\Delta^+} C_\alpha e_\alpha e_{-\alpha}$.

The discussion in Remark 2.2 confirms the description obtained by Vinberg in [19]:

$$\mathbb{V}(\mu) = \left\{ \sum_{\alpha \in \Delta^+} \frac{\alpha(h)}{\mu(h_\alpha)} e_\alpha e_{-\alpha} \mid h \in \mathfrak{t} \right\}, \tag{2.4}$$

whenever $\mu \in \mathfrak{t}^* \cap \mathfrak{g}^*_{\mathsf{reg}}$.

2.1. Bi-linear operations $\{\ ,\ \}_{(\gamma)}$ *and* $\{\ ,\ \}_{\mathfrak{m}}$

For any $\gamma \in \mathfrak{t}^*$, we have defined a Poisson bracket $\{\ ,\ \}_{(\gamma)} = \{\ ,\ \}_{\mathfrak{m}} + \{\ ,\ \}_\gamma$ on $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ as a sum of two bi-linear operations. The second summand is the restriction to $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ of the Poisson bracket "with frozen argument" $\{\ ,\ \}_\gamma$, which is defined on $\mathcal{S}(\mathfrak{g})$. Note that the operation $\{\ ,\ \}_{\mathfrak{m}}$ is not a Poisson bracket on $\mathcal{S}(\mathfrak{m})$, because it does not satisfy the Jacobi identity. However, in case $\gamma = 0$, we obtain $\{\ ,\ \}_{\mathfrak{m}} = \{\ ,\ \}_{(0)}$, which is a Poisson bracket on $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$.

Let $\widehat{x}$ be a skew-symmetric form on $\mathfrak{g}$ associated with $x \in \mathfrak{g}^*$, i.e., $\widehat{x}(\xi, \eta) = x([\xi, \eta])$ if $\xi, \eta \in \mathfrak{g}$. For a Poisson structure, one defines its *Poisson tensor (bivector)* π by the property that $\{F, H\} = \pi(dF \wedge dH)$ for functions F and H. In this terms, $\widehat{x} = \pi(x)$, if π is the Poisson tensor of $\{,\}$. In general, one says that the *rank of the Poisson structure* is equal to the rank $\mathsf{rk}\,\pi$ of its Poisson tensor. Then $\mathsf{rk}\,\pi$ is the maximal dimension of a symplectic leave of π, see e.g. [2, Chapter 1].

Proposition 2.3. *Let $\mathcal{B}$ be a Poisson commutative subalgebra of* $(\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}, \{\ ,\ \}_{(\gamma)})$. *Then for any γ, we have* $\text{tr.deg}\,\mathcal{B} \leqslant \dim \mathfrak{u}$.

Proof. We identify $\mathcal{S}(\mathfrak{m})$ with $\Bbbk[\gamma + \mathfrak{m}^*]$. Then

$$\{F, H\}_{(\gamma)}(x) = x([d_x F, d_x H]) = \widehat{x}(d_x F, d_x H) \quad \text{for all} \quad x \in \gamma + \mathfrak{m}^*,\ F, H \in \mathcal{S}(\mathfrak{m})^{\mathfrak{t}}.$$

Since $\{\mathcal{B}, \mathcal{B}\}_{(\gamma)} = 0$, the subspace $d_x\mathcal{B}$ is isotropic w.r.t. $\widehat{x}$ for any $x \in \gamma + \mathfrak{m}^*$. Furthermore $\widehat{x}(d_x\mathcal{S}(\mathfrak{g})^{\mathfrak{t}}, \mathfrak{t}) = 0$.

Let $T \subset G$ be the torus with $\mathsf{Lie}\,T = \mathfrak{t}$. Generic T-orbits on $\gamma + \mathfrak{m}^*$ are closed, hence they are separated by the regular T-invariants $\Bbbk[\gamma + \mathfrak{m}^*]^T \cong \mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$. Thus $d_x\mathcal{S}(\mathfrak{m})^{\mathfrak{t}} \subset T^*_x(\gamma + \mathfrak{m}^*)$ is the annihilator of the tangent space $T_x(Tx) = \mathrm{ad}^*(\mathfrak{t})x$ of the orbit Tx for a generic $x \in \gamma + \mathfrak{m}^*$. The orthogonal complement $\mathfrak{t}^{\perp\widehat{x}}$ of $\mathfrak{t}$ in $\mathfrak{g}$ w.r.t. $\widehat{x}$ is the subset

$$\{\xi \in \mathfrak{g} \mid x([\xi, \mathfrak{t}]) = 0\} = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*(\mathfrak{t})x \ \text{ annihilates } \ \xi\}.$$

Here we have $\mathfrak{t}^{\perp\widehat{x}} = \mathfrak{t} \oplus d_x\mathcal{S}(\mathfrak{m})^{\mathfrak{t}} = d_x\mathcal{S}(\mathfrak{g})^{\mathfrak{t}}$. Note that $\mathsf{rk}\,\widehat{x} = \dim \mathfrak{g} - l$.

Keeping the assumption that x is generic, we have $\mathfrak{t} \cap \ker \widehat{x} = 0$. Since $\widehat{x}(\mathfrak{t}, \mathfrak{t}) = 0$, the rank of $\widehat{x}|_{\mathfrak{t}^{\perp\widehat{x}}}$ is equal to $\mathsf{rk}\,\widehat{x} - 2\dim \mathfrak{t}$. Thus

$$\mathsf{rk}(\widehat{x}|_{d_x\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}}) = \mathsf{rk}(\widehat{x}|_{d_x\mathcal{S}(\mathfrak{g})^{\mathfrak{t}}}) = \dim \mathfrak{g} - 3l = 2(\dim \mathfrak{u} - l). \tag{2.5}$$

Since $\dim d_x\mathcal{S}(\mathfrak{m})^{\mathfrak{t}} = \dim \mathfrak{g} - 2l$, we obtain $\dim d_x\mathcal{B} \leqslant (\dim \mathfrak{u} - l) + (\dim \mathfrak{g} - 2l - (\dim \mathfrak{g} - 3l)) = \dim \mathfrak{u}$. Thus, $\text{tr.deg}\,\mathcal{B} \leqslant \dim \mathfrak{u}$. □

The equality (2.5) shows that the rank of the Poisson bracket $\{\ ,\ \}_{(\gamma)}$ on $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ is equal to $2(\dim\mathfrak{u}-l)=\dim\mathfrak{m}-2l$.

Remark 2.4. Set $Y_\gamma := \gamma+\mathfrak{m}^*$. The algebra $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}\cong\Bbbk[Y_\gamma]^T$ is the algebra of regular functions on the affine variety $Y_\gamma/\!\!/T$ and the bracket $\{\ ,\ \}_{(\gamma)}$ on $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ is obtained by the Hamiltonian reduction w.r.t. the restriction $\mathfrak{g}^*\to\mathfrak{t}^*$. We have $\dim Y_\gamma/\!\!/T=\operatorname{tr.deg}\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}=\dim\mathfrak{g}-2l$. The bound for $\operatorname{tr.deg}\mathcal{B}$ given by Proposition 2.3 is of the form $\dim Y_\gamma/\!\!/T-\frac{1}{2}(\dim\mathfrak{m}-2l)$, where $\dim\mathfrak{m}-2l$ is the rank of the Poisson structure in question. This is a general upper bound, existing for any Poisson algebra. The equality (2.5) can be deduced from the fact that generic symplectic leaves of $Y_\gamma/\!\!/T$ are of the form $(Gx\cap Y_\gamma)/\!\!/T$ with $\dim(Gx\cap Y_\gamma)/\!\!/T=\dim\mathfrak{g}-3\dim\mathfrak{t}$.

Theorem 2.5. *Suppose that $\mu\in\mathfrak{t}^*\cap\mathfrak{g}^*_{\mathsf{reg}}$. Then for any $\gamma\in\mathfrak{t}^*$, we have $\operatorname{tr.deg}\psi_\gamma((\mathcal{MF})_\mu)=\dim\mathfrak{u}$, and $\psi_\gamma((\mathcal{MF})_\mu)$ is a maximal Poisson commutative subalgebra of $(\mathcal{S}(\mathfrak{m})^{\mathfrak{t}},\{\ ,\ \}_{(\gamma)})$.*

Proof. Let $\{e,h,f\}\subset\mathfrak{g}$ be a principal $\mathfrak{sl}_2$-triple such that $h\in\mathfrak{t}$, $e\in\mathfrak{u}$, $f\in\mathfrak{u}^-$. Set $\chi_+=\kappa(e,.)$, $\chi_-=\kappa(f,.)\in\mathfrak{g}^*$. Since $e+(\mathfrak{t}\oplus\mathfrak{u}^-)$ and $f+(\mathfrak{t}\oplus\mathfrak{u})$ consist of regular elements [6], we have $(\Bbbk\chi_+\oplus\Bbbk\chi_-\oplus\Bbbk\mu)\cap\mathfrak{g}^*_{\mathsf{sing}}=0$. Therefore $\dim d_x(\mathcal{MF})_\mu=\boldsymbol{b}(\mathfrak{g})$ for any nonzero $x\in\Bbbk\chi_+\oplus\Bbbk\chi_-$, see e.g. [12, Cor. 1.6 & Lemma 2.1].

For any $F\in\mathcal{S}(\mathfrak{g})$ and $y\in\mathfrak{m}^*$, we have $d_yF\in d_y(\psi_0(F))+\mathfrak{t}$. Hence $\dim d_x\psi_0((\mathcal{MF})_\mu)$ is equal to $\dim\mathfrak{u}$. Each $\psi_\gamma((\mathcal{MF})_\mu)$ is a Poisson commutative subalgebra of $(\mathcal{S}(\mathfrak{m})^{\mathfrak{t}},\{\ ,\ \}_{(\gamma)})$, thereby $\operatorname{tr.deg}\psi_\gamma((\mathcal{MF})_\mu)\leqslant\dim\mathfrak{u}$ by Proposition 2.3. Thus, $\operatorname{tr.deg}\psi_0((\mathcal{MF})_\mu)=\dim\mathfrak{u}$.

The differentials $d_x(\psi_0(\partial^k_\mu H_j))$ with $k<d_j-1$ and $1\leqslant j\leqslant l$ are linearly independent for each $x\in(\Bbbk\chi_+\oplus\Bbbk\chi_-)\setminus\{0\}$. Thus, $\mathcal{J}\cap(\Bbbk\chi_+\oplus\Bbbk\chi_-)\subset\{0\}$ for the Jacobian subset

$$\mathcal{J}=\{y\in\mathfrak{m}^*\mid \bigwedge_{0\leqslant k<d_j-1,\ 1\leqslant j\leqslant l} d_y(\psi_0(\partial^k_\mu H_j))=0\}.$$

If $F\in\mathcal{S}(\mathfrak{g})$ is homogeneous, then $\psi_0(F)$ is also homogeneous. This applies to each $\partial^k_\mu H_j$ and leads to the conclusion that $\mathcal{J}$ does not contain divisors. By [10, Theorem 1.1], $\psi_0((\mathcal{MF})_\mu)$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{m})$, i.e., if $F\in\mathcal{S}(\mathfrak{m})$ is algebraic over the quotient field $\operatorname{Quot}\psi_0((\mathcal{MF})_\mu)$, then $F\in\psi_0((\mathcal{MF})_\mu)$.

Suppose $\psi_0((\mathcal{MF})_\mu)\subset\mathcal{B}\subset(\mathcal{S}(\mathfrak{m})^{\mathfrak{t}},\{\ ,\ \}_{(\gamma)})$, where $\mathcal{B}$ is a Poisson commutative subalgebra. Then $\operatorname{tr.deg}\mathcal{B}\leqslant\dim\mathfrak{u}$ by Proposition 2.3. Thereby the inclusion $\psi_0((\mathcal{MF})_\mu)\subset\mathcal{B}$ is an algebraic extension and $\psi_0((\mathcal{MF})_\mu)=\mathcal{B}$. The argument shows also that $\psi_0((\mathcal{MF})_\mu)$ coincides with its Poisson centraliser in $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ w.r.t. $\{\ ,\ \}_{\mathfrak{m}}$ and finishes the case $\gamma=0$.

For any homogeneous $F\in\mathcal{S}(\mathfrak{g})\setminus\mathfrak{t}\mathcal{S}(\mathfrak{g})$, the image $\psi_0(F)$ is the highest degree component of any $\psi_\gamma(F)$. In particular, the equality $\operatorname{tr.deg}\psi_0((\mathcal{MF})_\mu)=\dim\mathfrak{u}$ leads to $\operatorname{tr.deg}\psi_\gamma((\mathcal{MF})_\mu)\geqslant\dim\mathfrak{u}$, thereby $\operatorname{tr.deg}\psi_\gamma((\mathcal{MF})_\mu)=\dim\mathfrak{u}$. Assume that $F\in\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ commutes with $\psi_\gamma((\mathcal{MF})_\mu)$ w.r.t. $\{\ ,\ \}_{(\gamma)}$ and does not lie in $\psi_\gamma((\mathcal{MF})_\mu)$. Then the highest degree component of F commutes with $\psi_0((\mathcal{MF})_\mu)$ w.r.t. $\{\ ,\ \}_{\mathfrak{m}}$, which means that this component lies in $\psi_0((\mathcal{MF})_\mu)$. Then we can reduce the degree of F by subtracting a suitable element of $\psi_\gamma((\mathcal{MF})_\mu)$. This standard reduction argument proves that $\psi_\gamma((\mathcal{MF})_\mu)$ is a maximal Poisson commutative subalgebra of $(\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}, \{\ ,\ \}_{(\gamma)})$ for each $\gamma \in \mathfrak{t}^*$. □

Corollary 2.6. *Both, $\mathcal{B}_1 = (\mathcal{MF})_\mu \otimes_{\mathcal{S}(\mathfrak{t})} \Bbbk(\mathfrak{t}^*)$ and $\mathcal{B}_2 = \tilde{\mathcal{Z}} \otimes_{\mathcal{S}(\mathfrak{t})} \Bbbk(\mathfrak{t}^*)$, are maximal Poisson commutative subalgebras of $\mathcal{A}$.*

Proof. By the construction, $\{\mathcal{B}_i, \mathcal{B}_i\} = 0$ for both i. Assume that $\mathcal{B}_i$ is not maximal. Then there is $a \in \mathcal{A} \setminus \mathcal{B}_i$ such that $\{a, \mathcal{B}_i\} = 0$. For each $\gamma \in \mathfrak{t}^*$, we have $\psi_\gamma((\mathcal{MF})_\mu) \subset \psi_\gamma(\mathcal{B}_1)$ and $\psi_\gamma(\tilde{\mathcal{Z}}) \subset \psi_\gamma(\mathcal{B}_2)$. For any $\gamma \in \mathfrak{t}^*$ such that $a(\gamma)$ is well-defined, $\{\psi_\gamma(a), \psi_\gamma(\mathcal{B}_i)\}_{(\gamma)}$ is zero. If γ is regular in $\mathfrak{g}^*$, then $\psi_\gamma((\mathcal{MF})_\mu) = \psi_\gamma(\mathcal{B}_1)$, $\psi_\gamma(\tilde{\mathcal{Z}}) = \psi_\gamma((\mathcal{MF})_\gamma) = \psi_\gamma(\mathcal{B}_2)$, and $\psi_\gamma(a) \in \psi_\gamma(\mathcal{B}_i)$ by Theorems 2.5, 2.1.

There is $N \geqslant 0$ such that $a \in \mathcal{A}_{\leqslant N}$. Here $\mathcal{A}_{\leqslant N}$ is a finite-dimensional vector space over $\Bbbk(\mathfrak{t}^*)$ and $a \notin \mathcal{A}_{\leqslant N} \cap \mathcal{B}_i$. Then for almost all γ, we have $\psi_\gamma(a) \notin \psi_\gamma(\mathcal{A}_{\leqslant N} \cap \mathcal{B}_i)$. The algebra $\tilde{\mathcal{Z}}$ is generated by bi-homogeneous elements. Hence it is graded by the $\mathfrak{m}$-degree, $\tilde{\mathcal{Z}} = \bigoplus_{k\geqslant 0}(\mathcal{A}_k \cap \tilde{\mathcal{Z}})$. Recall that $\psi_\gamma(\mathcal{A}_k) = \mathcal{S}^k(\mathfrak{m})^{\mathfrak{t}}$. Thus, in case $i = 2$,

$$\psi_\gamma(\mathcal{B}_2) = \psi_\gamma(\tilde{\mathcal{Z}}) = \bigoplus_{k\geqslant 0} \psi_\gamma(\mathcal{A}_k \cap \tilde{\mathcal{Z}}),$$

and we can conclude that $\psi_\gamma(a) \notin \psi_\gamma(\mathcal{B}_2)$ for generic γ, which is a contradiction.

The algebra $(\mathcal{MF})_\mu$ is not homogeneous in $\mathfrak{m}$. However, the highest $\mathfrak{m}$-degree components of the generators $\partial_\mu^k H_j$ with $k < d_j - 1$ lie in $\mathcal{S}(\mathfrak{m})$ and are algebraically independent by Theorem 2.5. Therefore, if $\psi_\gamma(a) \notin \psi_\gamma(\mathcal{A}_{\leqslant N} \cap \mathcal{B}_1)$, then $\psi_\gamma(a) \notin \psi_\gamma(\mathcal{B}_1)$. □

2.2. Poisson centraliser of the quadratic part

For any $F \in \mathcal{S}(\mathfrak{g})$, let $F_\bullet$ be the component of F of the highest degree in $\mathfrak{t}$. If $F \in \mathcal{S}(\mathfrak{g})$ is homogeneous and $\psi_\gamma(F_\bullet) \neq 0$ for $\gamma \in \mathfrak{t}^*$, then $\psi_\gamma(F_\bullet)$ is the lowest degree component of $\psi_\gamma(F)$. Let $F \in \mathcal{Z}_{\mathbb{V}(\mu)}$ be homogeneous. Since $\boldsymbol{f}_j = \boldsymbol{f}_j(\mu) \in \mathcal{S}(\mathfrak{m})$ for each j, we may write $\psi_\gamma(\boldsymbol{f}_j) = \boldsymbol{f}_j$. Then $\{\boldsymbol{f}_j, \psi_\gamma(F)\}_{(\gamma)} = 0$ and $\{\boldsymbol{f}_j, \psi_\gamma(F_\bullet)\}_\gamma = 0$. A computation of $\{\boldsymbol{f}_j, \psi_\gamma(F_\bullet)\}_\gamma$ is not difficult, cf. [14], because

$$\{e_\beta e_{-\beta}, \prod_{\alpha\in\Delta^+} (e_\alpha^{r_\alpha} e_{-\alpha}^{r_\alpha^-})\}_\gamma = \gamma(h_\beta)(r_\beta^- - r_\beta) \prod_{\alpha\in\Delta^+} (e_\alpha^{r_\alpha} e_{-\alpha}^{r_\alpha^-}),$$

if $\beta \in \Delta^+$. Note that the centraliser $\mathcal{Z}_{\mathbb{V}(\mu)}$ is a homogeneous subalgebra of $\mathcal{S}(\mathfrak{g})$.

We write $\psi_\gamma(F_\bullet)$ in the basis $\{e_{\pm\alpha} \mid \alpha \in \Delta^+\}$. Let $M = c_{\bar{r},\gamma} \prod_{\alpha\in\Delta^+}(e_\alpha^{r_\alpha} e_{-\alpha}^{r_\alpha^-})$ be a summand of $\psi_\gamma(F_\bullet)$ with $c_{\bar{r},\gamma} \neq 0$. Then the explicit description of $\mathbb{V}(\mu)$, see (2.4), implies that

$$\sum_{\alpha\in\Delta^+} \frac{\alpha(\tilde{h})}{\mu(h_\alpha)}(r_\alpha - r_\alpha^-)\gamma(h_\alpha) = 0 \tag{2.6}$$

for each $\tilde{h} \in \mathfrak{t}$. The equality holds for all points γ such that $c_{\bar{r},\gamma} \neq 0$. These points form a dense open subset of $\mathfrak{t}^*$. Thereby

$$\sum_{\alpha\in\Delta^+} \frac{\alpha(\tilde{h})}{\mu(h_\alpha)}(r_\alpha - r_\alpha^-)h_\alpha = 0. \tag{2.7}$$

If the numbers $\frac{1}{\mu(h_\alpha)}$ are linearly independent over $\mathbb{Q}$, then $r_\alpha = r_\alpha^-$ for each positive root α [14]. In the same paper, it is shown that indeed the numbers $\frac{1}{\mu(h_\alpha)}$ are linearly independent over $\mathbb{Q}$ for generic $\mu \in \mathfrak{t}^*$. We keep the assumption $\mu \in \mathfrak{g}^*_{\mathsf{reg}}$.

Lemma 2.7. (cf. [14]) *Suppose that $\psi_\gamma(F_\bullet)$ lies in $\Bbbk[e_\alpha e_{-\alpha} \mid \alpha \in \Delta^+]$ for any $F \in \mathcal{Z}_{\mathbb{V}(\mu)}$ and any $\gamma \in \mathfrak{t}^*$. Then $\mathcal{Z}_{\mathbb{V}(\mu)} = (\mathcal{MF})_\mu$.*

Proof. If $F_\bullet$ does not lie in $\Bbbk[\mathfrak{t}, e_\alpha e_{-\alpha} \mid \alpha \in \Delta^+]$ for some $F \in \mathcal{Z}_{\mathbb{V}(\mu)}$, then there is $\gamma \in \mathfrak{t}^*$, such that $\psi_\gamma(F_\bullet) \notin \Bbbk[e_\alpha e_{-\alpha} \mid \alpha \in \Delta^+]$, a contradiction. Thus $F_\bullet \in \Bbbk[\mathfrak{t}, e_\alpha e_{-\alpha} \mid \alpha \in \Delta^+]$ for each $F \in \mathcal{Z}_{\mathbb{V}(\mu)}$ and tr.deg $\Bbbk[F_\bullet \mid F \in \mathcal{Z}_{\mathbb{V}(\mu)}] \leqslant |\Delta^+|+l = \boldsymbol{b}(\mathfrak{g})$. The algebras $\mathcal{Z}_{\mathbb{V}(\mu)}$ and $\Bbbk[F_\bullet \mid F \in \mathcal{Z}_{\mathbb{V}(\mu)}]$ are homogeneous and their Poincaré series coinside. Hence they have one and the same transcendence degree by [1, Satz 4.5]. Thus tr.deg $\mathcal{Z}_{\mathbb{V}(\mu)} \leqslant \boldsymbol{b}(\mathfrak{g})$. By the construction, $(\mathcal{MF})_\mu \subset \mathcal{Z}_{\mathbb{V}(\mu)}$. Moreover, tr.deg $(\mathcal{MF})_\mu = \boldsymbol{b}(\mathfrak{g})$ and the algebra $(\mathcal{MF})_\mu$ is algebraically closed in $\mathcal{S}(\mathfrak{g})$ [11, Sect. 3]. Since $(\mathcal{MF})_\mu \subset \mathcal{Z}_{\mathbb{V}(\mu)}$ is an algebraic extension, we have $(\mathcal{MF})_\mu = \mathcal{Z}_{\mathbb{V}(\mu)}$. □

Proposition 2.8. *Suppose that $\mathfrak{g}$ is of type A_{n-1}. Then $(\mathcal{MF})_\mu = \mathcal{Z}_{\mathbb{V}(\mu)}$ for any $\mu \in \mathfrak{t}^* \cap \mathfrak{g}^*_{\mathsf{reg}}$.*

Proof. Let $M = c_{\bar{r},\gamma} \prod_{\alpha\in\Delta^+}(e_\alpha^{r_\alpha} e_{-\alpha}^{r_\alpha^-})$ be a summand of $\psi_\gamma(F_\bullet)$ with $c_{\bar{r},\gamma} \neq 0$ for some homogeneous element $F \in \mathcal{Z}_{\mathbb{V}(\mu)}$ and some $\gamma \in \mathfrak{t}^*$. For $\tilde{h} = \frac{1}{n}\mathrm{diag}(n-1, -1, \ldots, -1)$, the equality (2.7) reads

$$\sum_{k=2}^{n} \frac{1}{\mu(E_{11} - E_{kk})}(r_{\varepsilon_1-\varepsilon_k} - r^-_{\varepsilon_1-\varepsilon_k})(E_{11} - E_{kk}) = 0.$$

Since the matrices $E_{11} - E_{kk}$ with $2 \leqslant k \leqslant n$ are linearly independent, we conclude that $r_{\varepsilon_1-\varepsilon_k} = r^-_{\varepsilon_1-\varepsilon_k}$ for each k. Then inserting $\frac{1}{n-1}\mathrm{diag}(0, n-2, -1, \ldots, -1)$ as $\tilde{h}$ brings $r_{\varepsilon_2-\varepsilon_k} = r^-_{\varepsilon_2-\varepsilon_k}$ for all $k \geqslant 3$. Continuing in this way, we prove that the assumptions of Lemma 2.7 are satisfied for F and hence also for all elements of $\mathcal{Z}_{\mathbb{V}(\mu)}$. □

3. Remarks on quantisation

A commutative subalgebra $\mathcal{Q} \subset \mathcal{U}(\mathfrak{g})$ is a *quantisation* of a Poisson commutative subalgebra $\mathcal{B} \subset \mathcal{S}(\mathfrak{g})$, if $\mathrm{gr}(\mathcal{Q}) = \langle \mathrm{gr}(a) \mid a \in \mathcal{Q} \rangle_{\Bbbk}$ coincides with $\mathcal{B}$.

Keep the assumption $\mu \in \mathfrak{t}^* \cap \mathfrak{g}^*_{\mathsf{reg}}$. Then $\mathfrak{t} \subset (\mathcal{MF})_\mu$. Assume that $\mathcal{Q} \subset \mathcal{U}(\mathfrak{g})$ is a quantisation of $(\mathcal{MF})_\mu$. Then $\mathfrak{t} \subset \mathcal{Q}$ and $\mathcal{Q} \subset \mathcal{U}(\mathfrak{g})^{\mathfrak{t}}$. We regard $\boldsymbol{f}_j = \sum_{\alpha\in\Delta^+} C_\alpha e_\alpha e_{-\alpha}$ as an element of $\mathcal{U}(\mathfrak{g})$ without adding new summands or changing the order of the factors. Since $\mathfrak{g}^{\mathfrak{t}} = \mathfrak{t}$, the subalgebra $\mathcal{Q}$ must contain all $\boldsymbol{f}_j \in \mathcal{U}(\mathfrak{g})$. In [19], it is shown that $[\boldsymbol{f}_j, \boldsymbol{f}_i] = 0$ in $\mathcal{U}(\mathfrak{g})$ for all i, j. Set $\widetilde{\mathbb{V}}(\mu) = \langle \boldsymbol{f}_j \mid 1 \leqslant j \leqslant l \rangle_{\Bbbk} \subset \mathcal{U}(\mathfrak{g})$.

If μ is generic enough, then $(\mathcal{MF})_\mu$ is equal to the Poisson centraliser $\mathcal{Z}_{\mathbb{V}(\mu)}$ [14]. Therefore $\mathcal{Q}$ is the centraliser of $\widetilde{\mathbb{V}}(\mu)$ in $\mathcal{U}(\mathfrak{g})^{\mathfrak{t}}$. The quantisation of this $(\mathcal{MF})_\mu$ is unique. In type A, it was shown earlier in [18], that the quantisation of $(\mathcal{MF})_\mu$ is unique for any $\mu \in \mathfrak{t}^* \cap \mathfrak{g}^*_{\mathsf{reg}}$. Proposition 2.8 provides a new proof for the uniqueness result in case $\mathfrak{g} = \mathfrak{sl}_n$.

The existence of $\mathcal{Q}$ is proven in [9] for classical $\mathfrak{g}$, in [17] for type A, and in [4,15] for any $\mathfrak{g}$. Let $\mathcal{F}_\gamma \subset \mathcal{U}(\mathfrak{g})$ with $\gamma \in \mathfrak{g}^*$ be the *quantum Mishchenko–Fomenko subalgebra*, which is a commutative algebra constructed in [15] and [4]. Then $\mathrm{gr}(\mathcal{F}_\gamma) = (\mathcal{MF})_\gamma$ for any $\gamma \in \mathfrak{g}^*_{\mathsf{reg}}$ [4,15] and for any $\gamma \in \mathfrak{g}^*$ in case $\mathfrak{g}$ is of type A or C [8].

Now we lift elements $F_j = (H_j)_{(d_j-2,2)} \in \tilde{\mathcal{Z}}$ to $\mathcal{U}(\mathfrak{g})$. Write $F_j = \sum_{\alpha\in\Delta^+} F_{j,\alpha} e_\alpha e_{-\alpha}$, where $F_{j,\alpha} \in \mathcal{S}^{d_j-2}(\mathfrak{t})$, and regard this sum as an element of $\mathcal{U}(\mathfrak{g})$. Recall that in $\mathcal{S}(\mathfrak{g})$, we have $\boldsymbol{f}_j \in (\mathcal{MF})_\mu$ and $\psi_\mu(F_j) = \psi_\mu(\boldsymbol{f}_j) = \boldsymbol{f}_j$.

We work with $\mathcal{C} = \mathcal{U}(\mathfrak{g})^{\mathfrak{t}} \otimes_{\mathcal{S}(\mathfrak{t})} \Bbbk(\mathfrak{t}^*)$ as with a non-commutative algebra over $\Bbbk(\mathfrak{t}^*)$ generated by the monomials

$$M = e_{\alpha_1}^{r_1} \dots e_{\alpha_N}^{r_N} e_{-\alpha_N}^{r_N^-} \dots e_{-\alpha_1}^{r_1^-}, \tag{3.1}$$

where $N = |\Delta^+|$, some numbering of the positive roots is fixed, and $\sum_{i=1}^N (r_i - r_i^-)\alpha_i = 0$. These monomials form a basis of $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$. We say that $\sum_{i=1}^N (r_i + r_i^-) =: \deg_{\mathfrak{m}} M$ is the degree (or the $\mathfrak{m}$-degree) of M. The algebra structure of $\mathcal{C}$ is given by the coefficients $Q_M^{M',M''} \in \Bbbk(\mathfrak{t}^*)$ of $M'M'' = \sum_M Q_M^{M',M''} M$, where M, M', M'' are of the form (3.1). In these terms, one can extend the map $\psi_\mu : \mathcal{A}_\mu \to \mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$ to a rational map from $\mathcal{C}$ by evaluating at μ the coefficients $Q_M^{M',M''}$. Formally, set $\mathcal{C}_\mu = \{A \in \mathcal{C} \mid A(\mu) \text{ is well-defined}\}$. As a vector space, the image $\psi_\mu(\mathcal{C}_\mu) =: \tilde{\mathcal{U}}(\mathfrak{m})^{\mathfrak{t}}$ is isomorphic to $\mathcal{S}(\mathfrak{m})^{\mathfrak{t}}$. We let $[\ ,\ ]_{(\mu)}$ stand for the Lie algebra structure on it. The algebra $(\tilde{\mathcal{U}}(\mathfrak{m})^{\mathfrak{t}}, [\ ,\ ]_{(\mu)})$ should not be regarded as a subset of $\mathcal{U}(\mathfrak{m})^{\mathfrak{t}}$ in whatever sense! Note that a similar construction exists for any $\Bbbk(\mathfrak{t}^*)$-basis of $\mathcal{C}$.

Example 3.1. We check that $[F_j, F_s] = 0$ in $\mathcal{U}(\mathfrak{g})$ for all j and s. By definition

$$\begin{aligned}[F_j, F_s] &= \sum_{\alpha,\beta\in\Delta^+} F_{j,\alpha} F_{s,\beta} e_\alpha e_{-\alpha} e_\beta e_{-\beta} - \sum_{\alpha,\beta\in\Delta^+} F_{j,\alpha} F_{s,\beta} e_\beta e_{-\beta} e_\alpha e_{-\alpha} \\ &= \sum_{\alpha\neq\beta} F_{j,\alpha} F_{s,\beta} (e_\alpha e_{-\alpha} e_\beta e_{-\beta} - e_\beta e_{-\beta} e_\alpha e_{-\alpha})\end{aligned}$$

$$= \sum_{\alpha\neq\beta} F_{j,\alpha}F_{s,\beta}([e_\alpha,e_\beta]e_{-\alpha}e_{-\beta}+e_\alpha[e_{-\alpha},e_\beta]e_{-\beta}+e_\beta[e_\alpha,e_{-\beta}]e_{-\alpha}$$
$$+e_\beta e_\alpha[e_{-\alpha},e_{-\beta}]).$$

In this particular case, further straightening of the sums in brackets does not involve elements of $\mathfrak{t}$. For each $\gamma\in\mathfrak{t}^*\cap\mathfrak{g}^*_{\mathsf{reg}}$, the elements $\psi_\gamma(F_j)$ and $\psi_\gamma(F_s)$ belong to the quantum Mishchenko–Fomenko subalgebra $\mathcal{F}_\gamma\subset\mathcal{U}(\mathfrak{g})$ associated with γ. If we replace each $F_{j,\alpha}F_{s,\beta}$ with $\psi_\gamma(F_{j,\alpha})$, $\psi_\gamma(F_{s,\beta})$, then the total sum is zero for each $\gamma\in\mathfrak{t}^*\cap\mathfrak{g}^*_{\mathsf{reg}}$. This implies that the initial sum is zero in $\mathcal{U}(\mathfrak{g})$.

The algebra $\mathcal{C}=\bigcup_{N\geqslant 0}\mathcal{C}_N$ is filtered by the degree in $\mathfrak{m}$. Here $\mathcal{C}_0=\mathcal{C}_1=\Bbbk(\mathfrak{t}^*)$ and the $\Bbbk(\mathfrak{t}^*)$-space $\mathcal{C}_2$ has a basis $\{1, e_\alpha e_{-\alpha}\mid\alpha\in\Delta^+\}$. More generally, any $\mathcal{C}_N$ has a monomial basis consisting of the monomials M, of the form (3.1), with $\deg_{\mathfrak{m}} M\leqslant N$. Since the commutator in $\mathcal{C}$ is $\Bbbk(\mathfrak{t}^*)$-linear, for any $A\in\mathcal{U}(\mathfrak{g})^{\mathfrak{t}}$ and any $N\geqslant 0$, the centraliser $\mathcal{Z}_A(\mathcal{C}_N)=\mathcal{Z}_A\mathcal{C}\cap\mathcal{C}_N$ is a vector space over $\Bbbk(\mathfrak{t}^*)$. It is a subspace of the finite-dimensional space $\mathcal{C}_N$. Evaluating the defining equations of $\mathcal{Z}_A(\mathcal{C}_N)$ at $\mu\in\mathfrak{t}^*$, we obtain

$$\dim_{\Bbbk(\mathfrak{t}^*)}\mathcal{Z}_A(\mathcal{C}_N)=\dim\left(\mathcal{Z}_{\psi_\mu(A)}(\tilde{\mathcal{U}}(\mathfrak{m})^{\mathfrak{t}},[\ ,\ ]_{(\mu)})\cap\tilde{\mathcal{U}}_N(\mathfrak{m})\right), \tag{3.2}$$

whenever μ is generic enough. This applies also to centralisers of finite subsets of elements. Since $\psi_\mu(\mathcal{F}_\mu)$ commutes with all $\psi_\mu(\boldsymbol{f}_j)=\psi_\mu(F_j)$ w.r.t. $[\ ,\ ]_{(\mu)}$, the equality (3.2) shows that the coefficients of the Poincaré series of the centraliser

$$\mathcal{Z}_{\widetilde{\mathbb{V}}(\mu)}\mathcal{U}(\mathfrak{g})^{\mathfrak{t}}=\{\Xi\in\mathcal{U}(\mathfrak{g})^{\mathfrak{t}}\mid[\Xi,F_j]=0\ \forall j\}$$

are large, $\dim_{\Bbbk(\mathfrak{t}^*)}\mathcal{Z}_{\widetilde{\mathbb{V}}(\mu)}(\mathcal{C}_N)\geqslant D_N$, where

$$D_N=\dim(\mathcal{S}^{\leqslant N}(\mathfrak{m})\mathcal{S}(\mathfrak{t})\cap\Bbbk[\partial^k_\mu H_j\mid 1\leqslant j\leqslant l, 0\leqslant k<d_j-1]).$$

In other words, enough elements of $\mathcal{U}(\mathfrak{g})^{\mathfrak{t}}$ commute with $\widetilde{\mathbb{V}}(\mu)$. It is not known, whether the centraliser $\mathcal{Z}_{\widetilde{\mathbb{V}}(\mu)}\mathcal{U}(\mathfrak{g})^{\mathfrak{t}}$ is commutative or not. In order to solve the quantisation problem for $\tilde{\mathcal{Z}}$, one may try to obtain upper bounds for dimensions related to $\mathcal{Z}_{\widetilde{\mathbb{V}}(\mu)}\mathcal{U}(\mathfrak{g})^{\mathfrak{t}}$ or to $\mathcal{Z}_{\widetilde{\mathbb{V}}(\mu)}\mathcal{C}$.

Let $\mathsf{symm}\colon\mathcal{S}(\mathfrak{g})\to\mathcal{U}(\mathfrak{g})$ be the canonical symmetrisation map.

Theorem 3.2. *Suppose that $\mathfrak{g}$ is of type A_l. Then there is a commutative subalgebra $\mathscr{Z}\subset\mathcal{U}(\mathfrak{g})^{\mathfrak{t}}$ such that $\mathrm{gr}(\mathscr{Z})=\tilde{\mathcal{Z}}$ and $F_j=\sum_{\alpha\in\Delta^+}F_{j,\alpha}e_\alpha e_{-\alpha}\in\mathscr{Z}$ for each $1\leqslant j\leqslant l$.*

Proof. For convenience, we work with $\mathfrak{g}=\mathfrak{gl}_n$ instead of $\mathfrak{sl}_{l+1}$. Let $H_j=\Delta_j$ be coefficients of the characteristic polynomial, here $\deg\Delta_j=j$. We write Δ_j in the basis $\{E_{ik}\mid i,k\leqslant n\}$. Let X be a monomial appearing in Δ_j with a nonzero coefficient. If E_{ii} is a factor of X, then for all other factors E_{sk} of X, we have $i\notin\{s,k\}$. For $\mathfrak{t}=\langle E_{ii}\mid 1\leqslant i\leqslant n\rangle_{\Bbbk}$, the $\mathfrak{t}$-factors of X commute with all factors of X. Another feature of the set $\{\Delta_j\mid 1\leqslant j\leqslant n\}$ is that

$$\mathcal{F}_\xi=\mathsf{alg}\langle\mathsf{symm}(\partial^k_\xi\Delta_j)\mid 1\leqslant j\leqslant n, 0\leqslant k<j\rangle\ \text{ for any }\ \xi\in\mathfrak{g}^*,$$

see [16,18] and [8, Theorem 3.1]. Set

$$\mathscr{Z} = \mathsf{alg}\langle \mathfrak{t}, \mathsf{symm}((\Delta_j)_{(i,j-i)}) \mid 1 \leqslant j \leqslant n, 0 \leqslant i \leqslant j-2\rangle \subset \mathcal{U}(\mathfrak{g})^{\mathfrak{t}}.$$

We have $\mathrm{gr}(\mathscr{Z}) = \tilde{\mathfrak{Z}}$, because a basis of $\mathfrak{t}$ and the elements $(\Delta_j)_{(i,j-i)}$ form an algebraically independent set of generators of $\tilde{\mathfrak{Z}}$. In order to prove that $\mathscr{Z}$ is commutative, we use maps $\psi_\mu\colon \mathcal{C}_\mu \to \tilde{\mathcal{U}}(\mathfrak{m})^{\mathfrak{t}}$ with $\mu \in \mathfrak{t}^*$, working now with the $\Bbbk(\mathfrak{t}^*)$-basis

$$\{\mathsf{symm}(M) \mid M \in \mathcal{S}(\mathfrak{m})^{\mathfrak{t}} \text{ monomial in } E_{ij} \text{ with } i, j \leqslant n\}$$

of $\mathcal{C}$.

We decompose $(\Delta_j)_{(i,j-i)} = \sum_s P^{(s)}_{j,i} M^{(s)}_{j,j-i}$, where $P^{(s)}_{j,i} \in \mathcal{S}^i(\mathfrak{t})$ are pairwise different monomials in elements E_{kk} and $M^{(s)}_{j,u} \in \mathcal{S}^u(\mathfrak{m})$ are nonzero. Since the factors of $P^{(s)}_{j,i}$ commute with $M^{(s)}_{j,j-i}$, we have

$$\mathsf{symm}(P^{(s)}_{j,i} M^{(s)}_{j,j-i}) = P^{(s)}_{j,i}\mathsf{symm}(M^{(s)}_{j,j-i})$$

for each s. Furthermore, $\partial^k_\mu(P^{(s)}_{j,i} M^{(s)}_{j,j-i}) = (\partial^k_\mu P^{(s)}_{j,i}) M^{(s)}_{j,j-i}$ and

$$\mathsf{symm}(\partial^k_\mu(P^{(s)}_{j,i} M^{(s)}_{j,j-i})) = (\partial^k_\mu P^{(s)}_{j,i})\mathsf{symm}(M^{(s)}_{j,j-i}).$$

Finally $\psi_\mu \circ \mathsf{symm}(\partial^k_\mu(P^{(s)}_{j,i} M^{(s)}_{j,j-i})) = i(i-1)\dots(i-k+1) P^{(s)}_{j,i}(\mu)\mathsf{symm}(M^{(s)}_{j,j-i})$. Therefore $\psi_\mu(\mathcal{F}_\mu)$ is generated by $\sum_s P^{(s)}_{i,j}(\mu)\mathsf{symm}(M^{(s)}_{j,j-i})$ with $1 \leqslant j \leqslant n$ and $0 \leqslant i \leqslant j-2$. Recall that $\psi_\mu(\mathcal{F}_\mu)$ is a commutative subalgebra of $(\tilde{\mathcal{U}}(\mathfrak{m})^{\mathfrak{t}}, [\ ,\ ]_{(\mu)})$ by the construction.

Next we observe that $\psi_\mu \circ \mathsf{symm}((\Delta_j)_{(i,j-i)}) = \sum_s P^{(s)}_{j,i}(\mu)\mathsf{symm}(M^{(s)}_{j,j-i}) \in \psi_\mu(\mathcal{F}_\mu)$. Thus, $\psi_\mu(\mathscr{Z})$ is commutative for any $\mu \in \mathfrak{t}^*$. By a general principle already used in Sect. 2, this implies $[\mathscr{Z}, \mathscr{Z}] = 0$. In case of

$$F_j = (\Delta_j)_{(j-2,2)} = \sum_{\alpha\in\Delta^+} F_{j,\alpha} e_\alpha e_{-\alpha} \quad \text{with} \quad j \geqslant 2,$$

the coefficients $F_{j,\alpha}$ are monomials in E_{kk} and $\mathsf{symm}(F_j) = \sum_{\alpha\in\Delta^+} F_{j,\alpha}(e_\alpha e_{-\alpha} - \frac{1}{2}h_\alpha) \in \mathscr{Z}$. Therefore also $\sum_{\alpha\in\Delta^+} F_{j,\alpha} e_\alpha e_{-\alpha} \in \mathscr{Z}$.

In order to return from $\mathfrak{gl}_n$ to $\mathfrak{sl}_{l+1}$ with $l+1 = n$, we restrict the invariants Δ_j to $\mathfrak{sl}_n^*$. This can be achieved by writing first $E_{ii} = \tilde{E}_{ii} + z$ for each i with $z = \frac{1}{n}\mathrm{diag}(1, \dots, 1)$ and then by setting $z = 0$. If $\tilde{\Delta}_j = \Delta_j|_{\mathfrak{sl}_n^*}$ and $\tilde{\mathscr{Z}} \subset \mathcal{U}(\mathfrak{sl}_n)$ is generated by $\mathfrak{t}$ together with the elements $\mathsf{symm}((\tilde{\Delta}_j)_{(i,j-i)})$, where $j \geqslant 2$, then $\tilde{\mathscr{Z}}$ is a required commutative subalgebra. □

The quantisation $\mathscr{Z} \subset \mathcal{U}(\mathfrak{g})^{\mathfrak{t}}$ of $\tilde{\mathfrak{Z}}$ described in the proof of Theorem 3.2 is a curious subalgebra. Let V_λ be an irreducible finite-dimensional $\mathfrak{g}$-module with $\mathfrak{g} = \mathfrak{sl}_n$ and $(V_\lambda)_\mu$ the subspace of V_λ corresponding to a $\mathfrak{t}$-weight μ. Then $\mathscr{Z}$ acts on $(V_\lambda)_\mu$ as $\mathcal{F}_\mu$. In particular, the action of $\mathscr{Z}$ on V_λ is diagonalisable, since μ takes real values on the standard real form of $\mathfrak{t}$, see [3]. Furthermore, if $\mu \in \mathfrak{t}^* \cap \mathfrak{g}^*_{\mathsf{reg}}$, then also by results of [3], $\mathcal{F}_\mu$, and hence $\mathscr{Z}$, acts on $(V_\lambda)_\mu$ with a simple spectrum. However, the action of $\mathscr{Z}$ on $(V_\lambda)_\mu$ may not have a simple spectrum if $\mu \notin \mathfrak{t}^* \cap \mathfrak{g}^*_{\mathsf{reg}}$. For instance, $\mathscr{Z}$ acts via scalars on the zero weight subspace $V_\lambda^{\mathfrak{t}}$.

Example 3.3. Suppose that $\mathfrak{g} = \mathfrak{sl}_3$. Then $\tilde{\mathfrak{Z}}$ is generated by a basis of $\mathfrak{t}$, two invariants H_1, H_2, and $F_2 = (H_2)_{(1,2)} = \sum_{\alpha\in\Delta^+} F_{2,\alpha} e_\alpha e_{-\alpha}$. Here $F_{2,\alpha} \in \mathfrak{t}$ for each α. We regard F_2 as an element of $\mathcal{U}(\mathfrak{g})$. Then $\mathscr{Z}$ is generated by $\mathfrak{t}$, F_2, and $\tilde{H}_1$, $\tilde{H}_2$, where $\tilde{H}_1, \tilde{H}_2 \in \mathcal{U}(\mathfrak{g})$ are independent central elements. On an irreducible finite-dimensional $\mathfrak{g}$-module V_λ, the last two generators act via scalar multiplication. The actions of $\mathfrak{t}$ and F_2 annihilate $V_\lambda^{\mathfrak{t}}$.

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