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From Milnor number to the Euler obstruction of a map on isolated determinantal singularities

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Abstract. Determinantal singularities, a generalization of complete intersections, have been extensively studied in algebraic geometry and singularity theory. In this work, we establish connections between various local invariants, including the Milnor number, the polar multiplicity, the local Euler obstruction, and the Euler obstruction of a function and a map, specifically focusing on the case of isolated determinantal singularities.

1. Introduction

An important direction of investigation in Singularity Theory is the search for local invariants associated with singular varieties. A central invariant is the Milnor number defined for hypersurfaces with an isolated singularity and Isolated Complete Intersection Singularities (ICIS) [1–4].

Let $(X, 0) = (f^{-1}(0), 0)$, $f : (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$, be a germ of a hypersurface with an isolated singularity at the origin. Milnor proves in [1] that the Milnor fiber $\overline{X}_s = f^{-1}(s) \cap B\varepsilon(0)$, where $0 < |s| < \varepsilon \ll 1$, and $B_\varepsilon(0)$ is the ball centered at 0 and radius ε, has the homotopy type of a bouquet of spheres with real dimension $N = dim(X)$. The number of these spheres, i.e., the middle Betti number of $\overline{X}_s$, is equal to the complex dimension of $\frac{\mathcal{O}_n}{Jf}$; $\mathcal{O}_n$ denotes the ring of analytic function-germs at the origin, and $J(f)$ the Jacobian ideal of f.

There are many equivalent ways to calculate the Milnor number. By the geometric approach, the Milnor number is equal to the number of Morse points of a Morsification of f, as well as the Poincaré-Hopf index of the complex conjugate of the gradient vector field of f.

The results of Hamm, Lê, and Greuel generalize that definition for isolated complete intersection. Hamm [3] shows that the Milnor fiber of an isolated complete

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intersection singularity $(X, 0)$, of codimension k, is homotopic to a bouquet of spheres of real dimension $N - k = dim(X)$.

A natural step to advance this investigation is to consider Determinantal Singularities, since they appear as a natural generalization of complete intersections.

Pereira and Ruas and Nuño-Ballesteros, Oréfice-Okamoto, and Tomazella in independent works define the Milnor number for classes of determinantal varieties with isolated singularities ([5], [6]).

In a more general setting, we have other invariants that can be seen as generalizations of the Milnor number. In this article, we relate these invariants for isolated determinantal singularities (IDS), described in Sect. 2.2 and Sect. 2.3.

One of them is the local Euler obstruction, defined by MacPherson [7]. In [8], Brasselet, Lê, and Seade relate this invariant to the topology of the Milnor fiber of a generic linear function defined on an analytic variety, and as a Corollary, we recall an easy way to compute the local Euler obstruction of an IDS.

For a holomorphic function f with an isolated singularity at the origin, the local Euler obstruction of the function, denoted by $Eu_{f,X}(0)$, considers the gradient vector field for its construction. Seade, Tibăr, and Verjovsky relate in [9] the number of Morse critical points of a stratified Morsification of f and the local Euler obstruction of f, showing that $Eu_{f,X}(0)$ can be seen as a generalization of the Milnor number. In our case, this invariant is related to the vanishing Euler characteristic, as defined by Ament, Nuño-Ballesteros, Oréfice-Okamoto, and Tomazella [10].

The last invariant described in Sect. 3 is the Euler obstruction of a map defined by Grulha [11] as a generalization of the Euler obstruction of a function. The Euler obstruction of a map is expressed in terms of the number of critical points in an appropriate Morsification of the coordinate functions of the map. This statement is presented by Grulha, Ruiz, and Santana in [12]. Still, in this section, it is shown relations among the local invariants discussed before.

In Sect. 4, we compute these invariants for simple Cohen-Macaulay codimension 2 singularities.

2. Preliminaries

This section is dedicated to reviewing results on the topology of an essential isolated determinantal singularity, and invariants such as the Milnor Number, the Euler Obstruction, and its generalizations.

2.1. *Topology of an EIDS*

Let $M_{m,n}$ be the set of $m \times n$ matrices with complex entries, and $0 < t \leq min\{m, n\}$. The subset of matrices with rank less than t, $M^t_{m,n}$, is called a generic determinantal variety.

Definition 1. Let $F : U \subset \mathbb{C}^N \to M_{m,n}$ represent an analytic mapping, where U is an open neighborhood containing the origin, and $F(0) = 0$. The set $X = F^{-1}(M^t_{m,n})$, where $0 < t \leq min\{m, n\}$, in $\mathbb{C}^N$ is called a determinantal variety of type (m, n, t), within the domain U, if $\operatorname{codim}(X) = \operatorname{codim} M^t_{m,n}$, where codim refers to the codimension of the variety within the encompassing space. The germ $(X, 0) \subset (\mathbb{C}^N, 0)$ of a determinantal variety is called a determinantal singularity.

Determinantal singularities are a natural generalization of isolated complete intersection singularities. In fact, ICIS are determinantal singularities of type $(1, n, 1)$. A determinantal variety typically exhibits non-isolated singularities. However, when the mapping F intersects the strata of the rank stratification transversally outside the origin, it results in a controlled behavior of the singular set. This principle forms the foundation of Ebeling and Gusein-Zade's approach, as presented in their work [13]. To serve their purpose, they introduced the concept of an *essentially isolated determinantal singularity* (EIDS).

Definition 2. ([13]) A determinantal variety $X \subset U$, where U is an open neighborhood of the origin in $\mathbb{C}^N$, defined by $X = F^{-1}(M^t_{m,n}), 0 < t \leq min\{m, n\}$, and where $F : U \subset \mathbb{C}^N \to M_{m,n}$ is an analytic mapping, is said to possess an essentially isolated determinantal singularity (EIDS) if F intersects the rank stratification of $M_{m,n}$ transversally, except potentially at the origin. An EIDS with an isolated singularity at the origin is called an isolated determinantal singularity (IDS).

If X is an EIDS within the domain U of type (n, m, t), the singular set of X is characterized by $F^{-1}(M^{t-1}_{m,n})$. The smooth part of X is represented by $F^{-1}(M^t_{m,n} \backslash M^{t-1}_{m,n})$ and denoted as X_{reg}. As emphasized by Ebeling and Gusein–Zade, an EIDS X has a isolated singularity at the origin only when $N \leq (n - t + 2)(m - t + 2)$.

A key approach in singularity theory is to construct a flat deformation of a singular variety $(X, 0)$ such that the nearby fibers are smooth. We call this construction a smoothing of $(X, 0)$. An important question in this subject is to determine in which situations we have the existence and uniqueness of the smoothing. For instance, a determinantal singularity has no uniqueness of the smoothing (Example 5.5 in [14]).

In this case, we have to choose allowable deformations that include the singularity and its generic perturbation. This is the notion of a landscape, introduced by Gaffney and Ruas [15]. In this work, we choose to perturb the presentation matrix that defines the determinantal singularity, i.e., we embed $(X, 0)$ in a family $\mathfrak{X} = \mathcal{F}^{-1}(M^t_{m,n} \times \mathbb{C})$, where

$$\mathcal{F} : (\mathbb{C}^N, 0) \times (\mathbb{C}, 0) \longrightarrow (M_{m,n}, 0) \times (\mathbb{C}, 0)$$

is a stabilization of the map F, that is, $\mathcal{F} = \mathcal{F}(x, s) = (F_s(x), s)$ with $F_0 = F$ and F_s is transverse to $M^t_{m,n}$, for all $s \neq 0$ sufficiently small. According to Thom's Transversality Theorem, F always admits a stabilization.

A subvariety X_s lying in a neighbourhood U of the origin in $\mathbb{C}^N$, defined by a perturbation $F_s : U \to M_{m,n}$ is called an *essential smoothing* of the EIDS $(X, 0)$. An essential smoothing of an IDS $(X, 0) \subset (\mathbb{C}^N, 0)$ of type (m, n, t) is a genuine smoothing if and only if $N < (m - t + 2)(n - t + 2)$ [13]. This condition is applicable to isolated surface and threefold determinantal singularities studied in [16,17].

The Milnor fiber is the main ingredient in [1] to study the topology of a complex germ of an analytic variety with an isolated singularity at the origin. Some authors have been investigating the topology of that fiber to recover information on the determinantal singularity. For the smoothable case of an isolated determinantal singularity, Pereira and Ruas [6] and Nuño-Ballesteros, Oréfice-Okamoto, and Tomazella [5] consider a generic constant perturbation of the defining matrix to define the determinantal Milnor fiber.

We collect some results about the topology of the fiber for smoothable determinantal singularities, which are applied in the next sections.

Definition 3. ([18]) Let $(X, 0) = (F^{-1}(M^t_{m,n}), 0)$ be an EIDS, and let $\mathcal{F}(x, s) = (F_s(x), s)$ be a stabilization of F. The determinantal Milnor fiber of $(X, 0)$ is

$$\overline{X}_s = F_s^{-1}(M^t_{m,n}) \cap B_M,$$

where $B_M \subset \mathbb{C}^N$ is the Milnor ball for $(X, 0)$.

Zach [18] proves the existence of a determinantal Milnor fiber for EIDS and an isomorphism (as stratified spaces) among the fibers. Zach [19] also determines the homotopy type of the determinantal Milnor fiber. The space $L^{t,N}_{m,n}$ used by Zach is a specific case of the complex link in [20], where the author studies the homotopy type of the Milnor fiber for an arbitrary singularity.

Proposition 2.1. *([19]) Let $(X, 0)$ be an EIDS given by a holomorphic map germ $F : (\mathbb{C}^N, 0) \to (M_{m,n}, 0)$ such that $X = F^{-1}(M^t_{m,n})$ is smoothable. If $\overline{X}_s$ is the determinantal Milnor fiber of $(X, 0)$, then*

$$\overline{X}_s \simeq_{ht} L^{t,N}_{m,n} \vee \bigvee_{i=1}^{\mu} \mathbb{S}^d, \tag{1}$$

where $d = N - (m - t + 1)(n - t + 1) = dim(X)$.

In [19], Zach computes the homotopy type of $L^{2,N}_{2,n}$ for many cases. For instance, if $(X, 0) \subset (\mathbb{C}^N, 0)$ is a smoothable EIDS of type $(2, n, 2)$, i.e., $n < N < 2n$, then $L^{2,N}_{2,n} \simeq_{ht} \mathbb{S}^2$. Consequently, Proposition 2.1 implies the following.

Corollary 2.2. *Let $(X, 0) \subset (\mathbb{C}^5, 0)$ be a determinantal threefold with an isolated singularity at the origin, defined by the 2×2-minors of a 2×3 matrix. Then, $b_2(X) = 1$.*

2.2. Milnor number and generalizations

The Lê-Greuel formula, [2,4], allows us to calculate inductively the Milnor number of an ICIS. The Theorems of Milnor [1], and Hamm [3] imply that $\mu(X) = (-1)^d(\chi(\overline{X}_s) - 1)$, for hypersurface and ICIS.

Another important invariant in this study is the polar multiplicity, $m_d(X)$, defined by Gaffney [21]. If X has a unique smoothing, then the polar multiplicity depends only on X, and it is an invariant of the analytic variety X.

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a codimension 2 determinantal variety with an isolated singularity at the origin, and $dim(X) = d = 2, 3$. Pereira and Ruas in [6] define the Milnor number of X by $\mu_{PR}(X) := b_d(X)$, where $b_d(X)$ is the d-th Betti number of a generic fiber of a smoothing of X and also prove a Lê-Greuel formula for determinantal surfaces of codimension 2.

Theorem 2.3. *([6]) Let $(X, 0) \subset (\mathbb{C}^4, 0)$ be the germ of a determinantal surface with an isolated singularity at the origin. Then,*

$$m_2(X, l) = \mu_{PR}(X) + \mu(X \cap l^{-1}(0)),$$

where $l : \mathbb{C}^4 \longrightarrow \mathbb{C}$ is a linear function such that the restriction to X has an isolated singularity at the origin.

Remark 1. If $(X, 0)$ is a determinantal threefold with an isolated singularity at the origin, in $(\mathbb{C}^5, 0)$ this result does not hold since $b_2(X) = 1$ (Corollary 2.2).

Let $\overline{X}_s$ be the determinantal Milnor fiber of a d-dimensional EIDS $(X, 0)$. The vanishing Euler characteristic of $(X, 0)$ is defined by $\nu(X, 0) := (-1)^d(\chi(\overline{X}_s) - 1)$, and it is used as a generalization of the Milnor number in [5] for smoothable isolated determinantal singularity. In the EIDS case, this invariant is related to extensions of the Poincaré-Hopf index ([13]). As in our setting we deal only with isolated singularity, we define the following for this context.

Let $(X, 0)$ be a smoothable reduced complex analytic germ with an isolated singularity at the origin such that $X \subset U$, where U is an open set of $\mathbb{C}^N$. The Poincaré-Hopf-Nash index of the 1-form ω on $(X, 0)$, $ind_{PHN}(\omega, X, 0)$, is the number of nondegenerate singular points of a generic perturbation $\tilde{\omega}$ on a smoothing $\tilde{X}$ of the singularity $(X, 0)$.

Remark 2. In the construction of $ind_{PHN}(\omega, X, 0)$, it is used a resolution given by the Nash transform. There are more two indices defined by Ebeling and Gusein-Zade in [13], using different resolutions, but in the case of smoothable IDS, these three indices coincide.

The following proposition is a combination of results proved by Ebeling and Gusein-Zade ([13,22]).

Proposition 2.4. *Let $(X, 0)$ be a smoothable IDS, and $f : (X, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated singularity at the origin. Then,*

$$ind_{PHN}(df, X, 0) = \nu(X, 0) + \nu(X \cap f^{-1}(0), 0).$$

It is remarked by Zach in [18] that for $f = l$ generic linear function, the Poincaré-Hopf-Nash index could be expressed in terms of the Euler characteristic of the Milnor fiber $\overline{X}_s$ and its complex link $\overline{Y}_\delta$.

2.3. Local Euler obstruction

The conjecture on the existence and uniqueness of Chern classes for singular algebraic varieties was initially proposed by Deligne and Grothendieck. It was MacPherson who managed to prove this conjecture in his work [7]. Later on, Brasselet and Schwartz showed the equivalence of MacPherson's approach with the characteristic class introduced by Schwartz [23], who constructs the classes even before Deligne and Grothendieck's discussion on the topic.

The construction of the MacPherson classes is based on the concept of the local Euler obstruction, which originally utilizes differential 1-forms. However, in their work [24], Brasselet and Schwartz provided an alternative definition of the local Euler obstruction using vector fields. Here, we will introduce and present methods for calculating this invariant.

Lemma 2.5. ([24]) *Let* $(X, 0) \subset (\mathbb{C}^N, 0)$ *be a* d*-equidimensional variety. Every stratified vector field* v*, non-null on a subset* $A \subset X$*, has a canonical lifting to a non-null section* $\tilde{v}$ *of the Nash bundle* $Nash(\mathcal{T})$ *over* $\nu^{-1}(A) \subset Nash(X)$ *with* $\nu : Nash(X) \to X$ *denoting the Nash transform of* $(X, 0)$*, and* $\mathcal{T}$ *the extension of the tautological bundle of* $Gr(d, N)$ *over* $U \times Gr(d, N)$*.*

Let X be a sufficiently small representative of the d-equidimensional analytic germ $(X, 0)$, and let $f : X \to \mathbb{C}$ be a holomorphic function with an isolated singularity at the origin. Consider $B_\varepsilon(p)$ a neighborhood of p in X, and its boundary $S_\varepsilon(p) := \partial B_\varepsilon(p)$. Let $\tilde{v}$ be the canonical lifting of v on $\nu^{-1}(X \cap S_\varepsilon(p))$ to a section of the Nash bundle, guaranteed by the Lemma 2.5. Denote by $\mathcal{O}(\tilde{v}) \in H^{2d}(\nu^{-1}(X \cap B_\varepsilon(p)), \nu^{-1}(X \cap S_\varepsilon(p)), \mathbb{Z})$ the obstruction cocycle to extending $\tilde{v}$ as a nowhere zero section of $Nash(\mathcal{T})$ inside $\nu^{-1}(X \cap B_\varepsilon(p))$.

Definition 4. The local Euler obstruction $Eu_X(v, p)$ of a stratified vector field v at an isolated singularity p is defined as the evaluation of the cocycle $\mathcal{O}(\tilde{v})$ on the fundamental class of the pair $[\nu^{-1}(X \cap B_\varepsilon(p)), \nu^{-1}(X \cap S_\varepsilon(p))]$.

Let us consider a stratified radial vector field v_{rad} and the stratified vector field $\overline{\nabla}_X f$ on X, which is homotopic to the gradient vector field on $S_\varepsilon(0) \cap X$ ([25]).

Definition 5. Let us consider the stratified vector fields described above. The integer $Eu_X(v_{rad}, p) = Eu_X(p)$ is called the local Euler obstruction of X at p. The integer $Eu_X(\overline{\nabla}_X f, 0) = Eu_{f,X}(0)$ is called the Euler obstruction of the function f.

The local Euler obstruction of a function is, in some sense, a generalization of the Milnor number. Indeed, Seade, Tibăr, and Verjovsky prove in [9] that $Eu_{f,X}(0)$ is, up to sign, the number of Morse critical points in a Morsification of f.

In [8], Brasselet, Lê and Seade prove a formula to make the calculation of the Euler obstruction easier. Let $\mathcal{V} = \{V_i\}$ be a complex analytic Whitney stratification of U adapted to X such that $\{0\}$ is a stratum.

Theorem 2.6. ([8]) *Let* $(X, 0)$ *and* $\mathcal{V}$ *be given as above, then for each generic linear form* l*, there exists* ε_0 *such that for any* ε *with* $0 < \varepsilon < \varepsilon_0$ *and* $\delta \neq 0$ *sufficiently small, such that the Euler obstruction of* $(X, 0)$ *is equal to*

$$Eu_X(0) = \sum_{i=1}^{q} \chi(V_i \cap B_\varepsilon(0) \cap l^{-1}(\delta)).Eu_X(V_i),$$

where $Eu_X(V_i)$ *is the Euler obstruction of* X *at a point of* V_i*,* $i = 1, \dots, q$ *and* $0 < |\delta| \ll \varepsilon \ll 1$.

Applying the theorem above, a direct connection between vanishing Euler characteristic and the local Euler obstruction is given in the next result.

Corollary 2.7. *Let* $(X, 0)$ *be an IDS* d*-dimensional. Denote by* $Y_\delta = X \cap l^{-1}(\delta)$ *the generic fiber of* $Y = X \cap l^{-1}(0)$*, where* l *is a generic linear form. Then,* $Eu_X(0) = (-1)^{d-1}\nu(Y, 0) + 1$.

Proof. Consider the strata $V_0 = \{0\}$ and $V_1 = X_{reg}$. Hence, the formula of the Theorem 2.6 implies

$$\begin{aligned} Eu_X(0) &= \chi(V_0 \cap l^{-1}(\delta) \cap B_\varepsilon).Eu_X(V_0) + \chi(V_1 \cap l^{-1}(\delta) \cap B_\varepsilon).Eu_X(V_1) \\ &= \chi(X \cap l^{-1}(\delta) \cap B_\varepsilon) = \chi(Y_\delta) \\ &= (-1)^{d-1}\nu(Y, 0) + 1. \end{aligned}$$

□

Theorem 2.6 ensures that the local Euler obstruction, considered as a constructible function on X, always satisfies the local Euler condition. In [25] the authors explore cases where the local Euler obstruction deviates from satisfying the local Euler condition, particularly in the context of functions with isolated singularities at $0 \in X$.

Theorem 2.8. ([25]) *Let* $(X, 0)$ *and* $\mathcal{V}$ *be given as above and let* $f : (X, 0) \to (\mathbb{C}, 0)$ *be a holomorphic function germ with an isolated singularity at* 0*. For* $0 < |\delta| \ll \varepsilon \ll 1$*, we have*

$$Eu_{f,X}(0) = Eu_X(0) - \sum_{i=1}^{q} \chi(V_i \cap B_\varepsilon(0) \cap f^{-1}(\delta)).Eu_X(V_i).$$

For a smoothable isolated determinantal singularity, there is a relation between the polar multiplicity, the local Euler obstruction, and the vanishing Euler characteristic, presented in the following result.

Theorem 2.9. ([5]) *Let* $(X, 0) \subset (\mathbb{C}^N, 0)$ *be a smoothable isolated determinantal singularity of dimension* d*. Then,*

$$Eu_X(0) + (-1)^d \# \Sigma(l_s|_{X_s}) = 1 + (-1)^d \nu(X, 0),$$

where $l : \mathbb{C}^N \to \mathbb{C}$ *is a generic linear function and* X_s *is generic fiber of a smoothing of* $(X, 0)$.

It is possible to relate the vanishing Euler characteristic of $(X, 0)$ to the Euler obstruction of a holomorphic function $f : (X, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 as follows.

Corollary 2.10. ([10]) *Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a smoothable IDS of dimension d and let $l : \mathbb{C}^N \to \mathbb{C}$ be a generic linear function. Then,*

$$Eu_{f,X}(0) = (-1)^d(\nu(X \cap f^{-1}(0), 0) - \nu(X \cap l^{-1}(0), 0)).$$

As consequence of this result and the Lê-Greuel formula, we have:

Corollary 2.11. ([10]) *Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0 over a smoothable d-dimensional IDS $(X, 0)$. Consider $l : \mathbb{C}^N \to \mathbb{C}$ a generic linear function. Then,*

$$Eu_{f,X}(0) = (-1)^d(\#\Sigma(f_s) - (\#\Sigma(l|_{X_s}))),$$

where $f_s : X_s \to \mathbb{C}$ is a Morsification of f over a smooth fiber X_s.

Brasselet, Massey, Parameswaran, and Seade [25] introduced a concept known as the *defect* for the case where f may have a non-isolated singularity at the origin. The defect of f coincides with the Euler obstruction of f when f has an isolated singularity.

The defect of f is closely related to another important invariant, which was independently defined by Dutertre and Grulha [26]. This invariant, known as the *Brasselet number* for a holomorphic function, is denoted by $B_{f,X}(0)$.

In particular, when f has an isolated singularity, the Brasselet number is given by $B_{f,X}(0) = Eu_X(0) - Eu_{f,X}(0)$. Combining with Corollaries 2.7 and 2.10, we obtain:

Proposition 2.12. ([10]) *Let $(X, 0)$ be a smoothable d-dimensional isolated determinantal singularity and let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity. Then,*

$$B_{f,X}(0) = (-1)^{d-1}\nu(X \cap f^{-1}(0), 0) + 1.$$

3. The Euler obstruction of a map

A natural generalization of the local Euler obstruction $Eu_X(v, p)$ of a stratified vector field v at an isolated singularity $p \in X$ could be obtained taking a stratified k-field $v^{(k)} = (v_1, \dots, v_k)$, instead of the vector field v. The following construction is present in details in [27].

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$ endowed by a complex analytic Whitney stratification of U adapted to X.

Denote by $|K|$ a triangulation of U compatible with the stratification, and let (D) be the cellular decomposition dual to $|K|$. Let σ be a cell of dimension $2(N-k+1)$, and $v^{(k)}$ a stratified k-field on $\sigma \cap X$ with an isolated singularity at the barycenter p of σ. Each component v_i of $v^{(k)}$ has a lift $\tilde{v}_i$ as a section of $Nash(\mathcal{T})$ over $\nu^{-1}(\partial\sigma \cap X)$ (see Lemma 2.5).

The class of the obstruction cocycle to extend $\tilde{v}^{(k)}$ to a k-field, without singularity, over $\nu^{-1}(\sigma \cap X)$ is denoted by $\mathcal{O}(\tilde{v}^{(k)}) \in H^{2(d-k+1)}(\nu^{-1}(\sigma \cap X), \nu^{-1}(\partial\sigma \cap X))$. The local Euler obstruction $Eu_X(v^{(k)}, p)$ of the stratified k-field $v^{(k)}$ at an isolated singularity p is the integer obtained by evaluating $\mathcal{O}(\tilde{v}^{(k)})$ on the fundamental class $[(\nu^{-1}(\sigma \cap X), \nu^{-1}(\partial\sigma \cap X))]$.

Consider $f : X \to \mathbb{C}^k$ a holomorphic map, with singular set $Sing(f)$. Grulha [11] constructs a stratified k-field $\overline{\nabla}_X^{(k)} f$ over $S_\varepsilon(0) \cap X \setminus Sing(f)$, without singularity, where $S_\varepsilon(p) = \partial B_\varepsilon(p)$ is centered at p. If σ is a cell with barycenter p such that $\sigma \cap Sing(f) = \emptyset$, then the k-field admits a lifting $\tilde{\overline{\nabla}}_X^{(k)} f$ as a section of $Nash(\mathcal{T})$ over $\nu^{-1}(\partial\sigma \cap X)$. The Euler obstruction of the map f is defined, *a priori*, as $Eu_{f,X}(\sigma, p) = Eu_X(\overline{\nabla}_X^{(k)} f, p)$. This definition depends on the cellular decomposition.

Using [28] the authors in [29] prove that the Euler obstruction of a map does not depend on a generic choice of σ. We define the Euler obstruction of a map as follows.

Definition 6. Let σ be a generic cell, and let $\overline{\nabla}_X^{(k)} f$ be as above. The Euler obstruction of the map f at a point p is $Eu_{f,X}(p) = Eu_X(\overline{\nabla}_X^{(k)} f, p)$.

Grulha, Santana, and Ruiz [12] applied the theory developed by Dutertre and Grulha [26] to express the Euler obstruction of the holomorphic map-germ $f : (X, 0) \to (\mathbb{C}^2, 0)$ in terms of the number of critical points of an appropriate Morsification. If the coordinate functions f_i and $(X, 0)$ have an isolated singularity at the origin, as a consequence of [12], we have the following result.

Theorem 3.1. *Let $X \subset U \subset \mathbb{C}^{N+1}$ be a $(d+1)$-dimensional variety, $d > 1$, with an isolated singularity at the origin and $f : (X, 0) \to (\mathbb{C}^2, 0)$, then*

$$Eu_{f, X \cap f_2^{-1}(\delta)}(x_0) = (-1)^d (\chi(X \cap f_2^{-1}(\delta)) - \chi(X \cap f_2^{-1}(\delta) \cap f_1^{-1}(\delta))).$$

In the case of $(X, 0)$ is isolated determinantal singularity, we are able to relate the invariants presented previously.

Proposition 3.2. *Let $(X, 0) \subset (\mathbb{C}^{N+1}, 0)$ be an IDS of dimension $d+1$, $d > 1$, and let $f : (X, 0) \to (\mathbb{C}^2, 0)$, $f(x) = (f_1(x), f_2(x))$, such that f_i are holomorphic functions with an isolated singularity at the origin, and f_2 is a generic linear function relative to X. Then, the Euler obstruction of the map f at a point $x_0 \in Y_\delta = X \cap f_2^{-1}(\delta)$ is equal to the following equivalent expressions*

(a) $\nu(Y, 0) + \nu(Y \cap f_1^{-1}(0), 0)$;
(b) $ind_{PHN}(df_1, Y, 0)$;

(c) $\nu(Y,0)+\nu(Y\cap l^{-1}(0),0)+(-1)^d Eu_{f_1,Y}(0)$*, where* $l:\mathbb{C}^N\to\mathbb{C}$ *is a generic linear function;*
(d) $\nu(Y,0)+(-1)^{d-1}(B_{f_1,Y}(0)-1)$.

Proof. Consider $(X,0)\subset(\mathbb{C}^{N+1},0)$ is an IDS of type (m,n,t). Realize that $(Y,0)\subset(\mathbb{C}^N,0)$, $Y=X\cap f_2^{-1}(0)$, is a smoothable IDS, since $N<(m-t+2)(n-t+2)$. The Theorem 3.1 implies

$$\begin{aligned}Eu_{f,X\cap f_2^{-1}(\delta)}(x_0)&=(-1)^d(\chi(Y_\delta)-1+1-\chi(Y_\delta\cap f_1^{-1}(\delta)))\\&=(-1)^d(\chi(Y_\delta)-1)+(-1)^{d-1}(\chi(Y_\delta\cap f_1^{-1}(\delta))-1)\\&=\nu(Y,0)+\nu(Y\cap f_1^{-1}(0),0).\end{aligned}$$

The Proposition 2.4 and the Corollary 2.10 give the equivalence between (a) and (b), and between (a) and (c), respectively.

Finally, using the description of the Brasselet number in Proposition 2.12 we have (d). □

Corollary 3.3. *Under the hypotheses of Proposition 3.2, suppose* $(Y,0)$ *has a unique smoothing. Then, the Euler obstruction* $Eu_{f,X\cap f_2^{-1}(\delta)}(x_0)$ *of* f *at a point* $x_0\in X\cap f_2^{-1}(\delta)$ *is equal to the equivalent expressions*

(i) $\nu(Y\cap f_1^{-1}(0),0)+m_d(Y)+(-1)^d(Eu_Y(0)-1)$*;*
(ii) $m_d(Y)+(-1)^d Eu_{f_1,Y}(0)$.
(iii) $\nu(Y\cap f_1^{-1}(0),0)+(-1)^d(Eu_X(0)+1)$.

Proof. If $l:\mathbb{C}^N\to\mathbb{C}$ is a generic linear projection it is well known that $\#\Sigma(\tilde{l}|_{Y_s})=m_d(Y)$, when $(Y,0)$ has a unique smoothing. The item (i) follows from Theorem 2.9 and Proposition 3.2 (a).

The item (ii) is a consequence of the Corollary 2.11. Using the Corollary 2.7 combined with (a) of Proposition 3.2 we obtain (iii). □

4. Applications on ICMC2

In this section, we apply the previous results to the class of simple isolated Cohen-Macaulay codimension 2 singularities (ICMC2), which were classified by Frühbis-Krüger in [30] and Frühbis-Krüger and Neumer in [17]. For computations, we use the invariants obtained in [6,16,18,31] for singular surfaces in $\mathbb{C}^4$ and singular threefolds in $\mathbb{C}^5$. In the case of determinantal surfaces in $\mathbb{C}^4$, we use the Milnor number obtained in [16] in Corollary 2.7 to find $Eu_X(0)$.

Corollary 4.1. *Let* $(X,0)=(F^{-1}(M_{2,3}^2),0)\subset(\mathbb{C}^4,0)$ *be one of the normal forms of simple determinantal surface singularities classified in [17]. Then, the local Euler obstruction* $Eu_X(0)$ *and the polar multiplicity* $m_2(X)$ *are given in the table.*

Presentation matrix		$\mu_{PR}(X)$	$Eu_X(0)$	$m_2(X)$
$\begin{pmatrix} w & y & x \\ z & w & y \end{pmatrix}$		1	-1	3
$\begin{pmatrix} w & y & x \\ z & w & y^k \end{pmatrix}$	$k \geq 2$	k	$-k$	$2k+1$
$\begin{pmatrix} w^r & y & x \\ z & w & y^k \end{pmatrix}$	$k \geq r \geq 2$	$k+r-1$	$-k$	$2k+r$
$\begin{pmatrix} z & y & x \\ x & w & y^2+z^k \end{pmatrix}$	$k \geq 2$	$k+2$	$-k$	$2k+3$
$\begin{pmatrix} z & y & x \\ x & w & z^2+yw \end{pmatrix}$		6	-2	9
$\begin{pmatrix} z & y & x \\ x & w & z^2+y^3 \end{pmatrix}$		7	-2	10
$\begin{pmatrix} z & y & x^k+w^2 \\ w^r & x & y \end{pmatrix}$	$k, r \geq 2$	$k+r+1$	$-k$	$2k+r+2$
$\begin{pmatrix} z & y+w^2 & x^2 \\ w^k & x & y \end{pmatrix}$	$k \geq 2$	$k+5$	-2	$k+8$
$\begin{pmatrix} z & y & x^2+w^3 \\ w^k & x & y \end{pmatrix}$	$k \geq 2$	$k+6$	-2	$k+9$
$\begin{pmatrix} z & y & w^2 \\ y & x & z+x^2 \end{pmatrix}$		7	-4	12
$\begin{pmatrix} z & y & x^3+w^2 \\ y & x & z \end{pmatrix}$		8	-5	14
$\begin{pmatrix} z & y & x^2 \\ y & x & z+w^2 \end{pmatrix}$		8	-3	12

Proof. Let $(X, 0)$ be the simple determinantal surface singularity given by the second normal form of the table. By a row operation, and a change of coordinates, the germ $(X, 0)$ is equivalent to the singularity given by the presentation matrix

$$\begin{pmatrix} z & w & x^k \\ w & x & y \end{pmatrix}.$$

Choosing $l_2 : \mathbb{C}^4 \to \mathbb{C}$ by $l_2(x, y, z, w) = w - y$, the determinantal curve $C = X \cap l_2^{-1}(0)$ has presentation matrix

$$\begin{pmatrix} z & y & x^k \\ 0 & x & y \end{pmatrix}.$$

The Milnor number $\mu(C) = k + 1$ is calculated in [30]. The Corollary 2.7 assures us that $Eu_X(0) = 1 - \mu(C) = -k$. The Lê-Greuel Formula 2.3 implies $m_2(X) = \mu_{PR}(X) + \mu(C) = 2k + 1$.

Similar arguments are used for the other normal forms of the table. □

To obtain a result in the case of isolated determinantal threefold singularity, we need to express the Euler obstruction of a map in terms of Milnor numbers.

Corollary 4.2. *Let $(X,0)\subset(\mathbb{C}^5,0)$ be an isolated determinantal threefold singularity and $l:(\mathbb{C}^5,0)\to(\mathbb{C}^2,0)$ such that l_2 is a generic linear projection relative to X, and $l_1|_H$ is a generic linear projection relative to $Y=X\cap l_2^{-1}(0)$, where $H=l_2^{-1}(0)$. Then,*

$$Eu_{l,X\cap l_2^{-1}(\delta)}(x_0)=\mu_{PR}(Y)+\mu(C),$$

where $C=Y\cap l_1^{-1}(0)$.

Proof. In fact, by Proposition 3.2 (a), we have:

$$Eu_{l,X\cap l_2^{-1}(\delta)}(x_0)=\nu(Y,0)+\nu(C,0).$$

Considering l_2 and l_1 generic linear functions, we have that Y is a determinantal surface in $\mathbb{C}^4$ and C is a determinantal curve in $\mathbb{C}^3$. Then,

$$\nu(Y,0)=\mu_{PR}(Y) \text{ and } \nu(C,0)=\mu(C).$$

□

The following result is obtained using the Corollary 2.7 and the Theorem 2.9.

Proposition 4.3. *Let $(X,0)=(F^{-1}(M_{2,3}^2),0)\subset(\mathbb{C}^5,0)$ be a simple determinantal threefold singularity classified in [17]. If $l:(\mathbb{C}^5,0)\to(\mathbb{C}^2,0)$ is a map such that l_2 is a generic linear projection relative to X, and $l_1|_H$ is a generic linear projection relative to $Y=X\cap l_2^{-1}(0)$, where $H=l_2^{-1}(0)$, then the local Euler obstruction, the polar multiplicity, and the Euler obstruction of the map l are given in the table.*

Presentation matrix		$Eu_X(0)$	$m_3(X)$	$Eu_{l,X\cap l_2^{-1}(\delta)}(x_0)$
$\begin{pmatrix} x & y & z \\ w & v & x \end{pmatrix}$		2	0	3
$\begin{pmatrix} x & y & z \\ w & v & x^{k+1}+y^2 \end{pmatrix}$	$k\geq 1$	3	$k+1$	5
$\begin{pmatrix} x & y & z \\ w & v & xy^2+x^{k-1} \end{pmatrix}$	$k\geq 4$	k	$2k-2$	$2k-1$
$\begin{pmatrix} x & y & z \\ w & v & x^3+y^4 \end{pmatrix}$		4	8	7
$\begin{pmatrix} x & y & z \\ w & v & x^3+xy^3 \end{pmatrix}$		4	9	7
$\begin{pmatrix} x & y & z \\ w & v & x^3+y^5 \end{pmatrix}$		4	10	7
$\begin{pmatrix} w & y & x \\ z & w & y+v^k \end{pmatrix}$	$k\geq 2$	2	0	3
$\begin{pmatrix} w & y & x \\ z & w & y^k+v^2 \end{pmatrix}$	$k\geq 2$	$k+1$	$2k-2$	$2k+1$

Presentation Matrix		$Eu_X(0)$	$m_3(X)$	$Eu_{l,X\cap l_2^{-1}(\delta)}(x_0)$
$\begin{pmatrix} w & y & x \\ z & w & yv+v^k \end{pmatrix}$	$k\geq 1$	$3k+1$	$3k$	
$\begin{pmatrix} w+v^k & y & x \\ z & w & yv \end{pmatrix}$	$k\geq 1$	$3k+3$	$3k+2$	
$\begin{pmatrix} w+v^2 & y & x \\ z & w & y^2+v^k \end{pmatrix}$	$k\geq 1$	3	k	5
$\begin{pmatrix} w & y & x \\ z & w & y^2+v^3 \end{pmatrix}$		3	3	5
$\begin{pmatrix} v^2+w^r & y & x \\ z & w & v^2+y^k \end{pmatrix}$	$k\geq r\geq 2$	$k+r$	$2k+2r-4$	$2k+r$
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & yv \end{pmatrix}$	$k\geq 2$	$k+2$	$2k$	$k+4$
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & y^2+v^r \end{pmatrix}$	$k\geq 2, r\geq 3$	$k+2$	$2k+r-2$	$k+4$
$\begin{pmatrix} wv+v^k & y & x \\ z & w & yv+v^k \end{pmatrix}$	$k\geq 3$	$3k$	$3k$	
$\begin{pmatrix} wv & y & x \\ z & w & y^2+v^3 \end{pmatrix}$		4	5	6
$\begin{pmatrix} w^2+v^3 & y & x \\ z & w & y^2+v^3 \end{pmatrix}$		4	6	6
$\begin{pmatrix} z & y & x \\ x & w & v^2+y^2+z^k \end{pmatrix}$	$k\geq 2$	$k+3$	$2k+2$	$2k+3$
$\begin{pmatrix} z & y & x \\ x & w & v^2+yz+y^kw \end{pmatrix}$	$k\geq 1$	$2k+4$	$4k+4$	
$\begin{pmatrix} z & y & x \\ x & w & v^2+yz+y^{k+1} \end{pmatrix}$	$k\geq 2$	$2k+6$	$4k+5$	
$\begin{pmatrix} z & y & x \\ x & w & v^2+yw+z^2 \end{pmatrix}$		7	10	9
$\begin{pmatrix} z & y & x \\ x & w & v^2+y^3+z^2 \end{pmatrix}$		8	12	10
$\begin{pmatrix} z & y & x+v^2 \\ x & w & y^2+z^2 \end{pmatrix}$		5	8	7

Proof. Let $(X,0)$ be the simple determinantal threefold singularity given by the eighth normal form of the table, and let $l_2:\mathbb{C}^5\to\mathbb{C}$, $l_2(x,y,z,w,v)=v$. Then, the surface $Y=X\cap l_2^{-1}(0)$ is given by the presentation matrix

$$\begin{pmatrix} w & y & x \\ z & w & y^k \end{pmatrix},$$

which has $\mu_{PR}(Y)=k$ [16].

For this case, the Corollary 2.7 is just $Eu_X(0)=\mu_{PR}(Y)+1=k+1$. The next step is using Theorem 2.9 to obtain $m_3(X)=Eu_X(0)+\mu_{PR}(X)-2=2k-2$, where $\mu_{PR}(X)=b_3(X)=k-1$ is computed in [18].

In the proof of Corollary 4.1 we could find $C = Y \cap l_1^{-1}(0)$ with presentation matrix

$$\begin{pmatrix} z & y & x^k \\ 0 & x & y \end{pmatrix},$$

and $\mu(C) = k + 1$.

Therefore, the Corollary 4.2 guaranties $Eu_{l,X\cap l_2^{-1}(\delta)}(x_0) = \mu_{PR}(Y) + \mu(C) = 2k + 1$. Following analogous arguments we are able to compute the remaining singularities of the table. □

According Frühbis-Krüger and Neumer the last case of simple ICMC2 are the fourfolds ([17]). In this case, we have $b_2(Y) = 1$ (Corollary 2.2) and the local Euler obstruction is computed in [32].

The following corollary is obtained since, by definition, $\nu(Y, 0) = \mu_{PR}(Y) - b_2(Y)$, with $Y = X \cap l_2^{-1}(0)$.

Corollary 4.4. *Let* $(X, 0) \subset (\mathbb{C}^6, 0)$ *be an isolated determinantal fourfold singularity and let* $l : (\mathbb{C}^6, 0) \to (\mathbb{C}^2, 0)$ *be a map germ such that* l_2 *is a generic linear projection relative to* X*, and* $l_1|_H$ *is a generic linear projection relative to* $Y = X \cap l_2^{-1}(0)$*, where* $H = l_2^{-1}(0)$*. Then,*

$$Eu_{l,X\cap l_2^{-1}(\delta)}(x_0) = \mu_{PR}(Y) + \mu_{PR}(S) - 1,$$

where $S = Y \cap l_1^{-1}(0)$.

Our main contribution in the next result is the computation of the Euler obstruction of a map.

Proposition 4.5. *Let* $(X, 0) = (F^{-1}(M_{2,3}^2), 0) \subset (\mathbb{C}^6, 0)$ *be a simple determinantal fourfold singularity classified in* [17]*. Let* $l : (\mathbb{C}^6, 0) \to (\mathbb{C}^2, 0)$ *be a map such that* l_2 *is a generic linear projection relative to* X*, and* $l_1|_H$ *is a generic linear projection relative to* $Y = X \cap l_2^{-1}(0)$*, where* $H = l_2^{-1}(0)$*. Then, the Euler obstruction of the map* l*,* $Eu_{l,X\cap l_2^{-1}(\delta)}(x_0)$*, is given in the table.*

Presentation matrix		$Eu_X(0)$	$Eu_{l,X\cap l_2^{-1}(\delta)}(x_0)$
$\begin{pmatrix} x & y & v \\ z & w & u \end{pmatrix}$		2	0
$\begin{pmatrix} x & y & v \\ z & w & x + u^k \end{pmatrix}$	$k \geq 2$	2	0
$\begin{pmatrix} x & y & z \\ w & v & u^2 + x^{k+1} + y^2 \end{pmatrix}$	$k \geq 1$	1	2
$\begin{pmatrix} x & y & z \\ w & v & u^2 + xy^2 + x^{k-1} \end{pmatrix}$	$k \geq 4$	-1	

Presentation matrix		$Eu_X(0)$	$Eu_{l,X\cap l_2^{-1}(\delta)}(x_0)$
$\begin{pmatrix} x & y & z \\ w & v & u^2+x^3+y^4 \end{pmatrix}$		0	4
$\begin{pmatrix} x & y & z \\ w & v & u^2+x^3+xy^3 \end{pmatrix}$		0	4
$\begin{pmatrix} x & y & z \\ w & v & u^2+x^3+y^5 \end{pmatrix}$		0	4
$\begin{pmatrix} x & y & z \\ w & v & x^2+y^2+u^3 \end{pmatrix}$		1	2
$\begin{pmatrix} w & y & x \\ z & w+vu & y+v^k+u^r \end{pmatrix}$	$k, r \geq 2$	2	0
$\begin{pmatrix} w & y & x \\ z & w+v^2 & y+u^3 \end{pmatrix}$		2	0
$\begin{pmatrix} w & y & x \\ z & w+v^2 & y+u^4 \end{pmatrix}$		2	0
$\begin{pmatrix} w & y & x \\ z & w+v^2+u^k & y+vu^2 \end{pmatrix}$	$k \geq 3$	2	0
$\begin{pmatrix} w & y & x \\ z & w+v^2+u^3 & y+u^k \end{pmatrix}$	$k \geq 4$	2	0
$\begin{pmatrix} w & y & x \\ z & w+v^2+u^3 & y+vu^k \end{pmatrix}$	$k \geq 3$	2	0
$\begin{pmatrix} w & y & x \\ z & w+v^2 & u^2+yv \end{pmatrix}$		1	8
$\begin{pmatrix} w & y & x \\ z & w+uv & u^2+yv+v^k \end{pmatrix}$	$k \geq 3$	1	$3k$
$\begin{pmatrix} w & y & x \\ z & w+uv^k & u^2+yv+v^3 \end{pmatrix}$	$k \geq 2$	1	9
$\begin{pmatrix} w & y & x \\ z & w+v^3 & u^2+yv \end{pmatrix}$		1	11
$\begin{pmatrix} w & y & x \\ z & w+v^k & u^2+y^2+v^3 \end{pmatrix}$	$k \geq 3$	1	2
$\begin{pmatrix} w & y & x \\ z & w+uv^k & u^2+y^2+v^3 \end{pmatrix}$	$k \geq 2$	1	2

Proof. Let $(X, 0)$ be the simple determinantal fourfold singularity obtained by the third normal form of the table. Consider $l_2 : \mathbb{C}^6 \to \mathbb{C}$, $l_2(x, y, z, w, v, u) = x - v$.

By a column operation, and a change of coordinates, the threefold $Y = X \cap l_2^{-1}(0)$ has presentation matrix

$$\begin{pmatrix} x & y & z \\ w & x & v^2+y^2 \end{pmatrix}.$$

Realize that Y is the eighth normal form in Proposition 4.3, for $k = 2$, which has $\mu_{PR}(Y) = 1$. As found in the proof of the Proposition 4.3, the surface $S =$

$Y \cap l_1^{-1}(0)$ is given by the presentation matrix

$$\begin{pmatrix} w & y & x \\ z & w & y^2 \end{pmatrix}.$$

The local Euler obstruction $Eu_X(0) = 1 - \nu(Y, 0) = 2 - \mu_{PR}(Y) = 1$ is computed using Corollary 2.7, and $b_3(Y) = \mu_{PR}(Y) = 1$ is calculated in [18]. Applying the Corollary 4.4 we obtain $Eu_{l, X \cap l_2^{-1}(\delta)}(x_0) = \mu_{PR}(Y) + \mu_{PR}(S) - 1 = 2$.

A similar construction is used in order to compute the invariants for the other normal forms of the table. □

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References

[1] Milnor, J.: Singular points of complex hypersurfaces. Ann. Math. St **61**, 591–648 (1968)
[2] Greuel, G.-M.: Der Gauss-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten. Math. Ann. **214**(3), 235–266 (1975)
[3] Hamm, H.: Lokale topologische Eigenschaften komplexer Räume. Math. Ann. **191**(3), 235–252 (1971)
[4] Lê, D.T.: Calculation of Milnor number of isolated singularity of complete intersection. Funct. Anal. Appl. **8**(2), 127–131 (1974)
[5] Nuño-Ballesteros, J.J., Oréfice-Okamoto, B., Tomazella, J.: The vanishing Euler characteristic of an isolated determinantal singularity. Israel J. Math. **197**(1), 475–495 (2013)
[6] Ruas, M.A.S., Pereira, M.S.: Codimension two determinantal varieties with isolated singularities. Math. Scand. 161–172 (2014)
[7] MacPherson, R.D.: Chern classes for singular algebraic varieties. Ann. Math. **100**(2), 423–432 (1974)
[8] Brasselet, J.-P., Lê, D.T., Seade, J.: Euler obstruction and indices of vector fields. Topology **39**(6), 1193–1208 (2000)
[9] Seade, J., Tibăr, M., Verjovsky, A.: Milnor numbers and Euler obstruction. Bull. Braz. Math. Soc. **36**(2), 275–283 (2005)
[10] Ament, D., Nuño-Ballesteros, J., Oréfice-Okamoto, B., Tomazella, J.: The Euler obstruction of a function on a determinantal variety and on a curve. Bull. Brazilian Math. Soc. New Ser. **47**(3), 955–970 (2016)
[11] Grulha Jr., N.G.: Obstrução de Euler de aplicações analíticas. PhD thesis, Universidade de São Paulo (2007)
[12] Grulha, N.G., Jr., Ruiz, C.M., Santana, H.: The geometrical information encoded by the Euler obstruction of a map. Int. J. Math. **33**(04), 2250029 (2022)
[13] Ebeling, W., Gusein-Zade, S.: On indices of 1-forms on determinantal singularities. In: Proceedings of the Steklov institute of mathematics **267**(1), 113–124 (2009)
[14] Frühbis-Krüger, A., Zach, M.: Determinantal Singularities. Preprint at https://arxiv.org/abs/2106.04855 (2021)
[15] Gaffney, T., Ruas, M.A.: Equisingularity and eids. Proc. Am. Math. Soc. **149**(4), 1593–1608 (2021)

[16] Pereira, M.S.: Variedades determinantais e singularidades de matrizes. PhD thesis, Universidade de São Paulo (2010)
[17] Frühbis-Krüger, A., Neumer, A.: Simple Cohen-Macaulay codimension 2 singularities. Comm. Algebra **38**(2), 454–495 (2010)
[18] Zach, M.: Topological invariants of isolated determinantal singularities. PhD thesis, Hannover: Gottfried Wilhelm Leibniz Universität Hannover (2017)
[19] Zach, M.: Bouquet decomposition for determinantal Milnor fibers. J. Singularities **22**, 190–204 (2020)
[20] Tibăr, M.: Bouquet decomposition of the Milnor fibre. Topology **35**(1), 227–241 (1996)
[21] Gaffney, T.: Polar multiplicities and equisingularity of map germs. Topology **32**(1), 185–223 (1993)
[22] Ebeling, W., Gusein-Zade, S.: Radial index and Euler obstruction of a 1-form on a singular variety. Geom. Dedicata. **113**(1), 231–241 (2005)
[23] Schwartz, M.-H.: Classes caractéristiques définies par une stratification d'une variété analytique complexe. CR Acad. Sci. Paris **260**, 3262–3264 (1965)
[24] Brasselet, J.-P., Schwartz, M.-H.: Sur les classes de Chern d'un ensemble analytique complexe, caracteristique d'Euler-Poincare. Astérisque **82**, 93–147 (1981)
[25] Brasselet, J.-P., Massey, D., Parameswaran, A., Seade, J.: Euler obstruction and defects of functions on singular varieties. J. Lond. Math. Soc. **70**(1), 59–76 (2004)
[26] Dutertre, N., Grulha, N.G., Jr.: Lê-greuel type formula for the Euler obstruction and applications. Adv. Math. **251**, 127–146 (2014)
[27] Brasselet, J.-P., Seade, J., Suwa, T.: Vector Fields on Singular Varieties, vol. 1987. Springer, Berlin (2009)
[28] Ebeling, W., Gusein-Zade, S.M.: Chern obstructions for collections of 1-forms on singular varieties. In: Singularity theory: dedicated to Jean-Paul Brasselet on His 60th Birthday, pp. 557–564. World Scientific, Marseille (2007)
[29] Brasselet, J.-P., Grulha, N.G., Jr., Ruas, M.A.: The Euler obstruction and the Chern obstruction. Bull. Lond. Math. Soc. **42**(6), 1035–1043 (2010)
[30] Frühbis-Krüger, A.: Classification of simple space curve singularities. Comm. Algebra **27**(8), 3993–4013 (1999)
[31] Damon, J., Pike, B.: Solvable groups, free divisors and nonisolated matrix singularities ii: Vanishing topology. Geom. Topol. **18**(2), 911–962 (2014)
[32] Siesquén, N.C.C.: Euler obstruction of essentially isolated determinantal singularities. Topol. Appl. **234**, 166–177 (2018)