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On Riemannian polyhedra with non-obtuse dihedral angles in 3-manifolds with positive scalar curvature

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Abstract. We determine the combinatorial types of all the 3-dimensional simple convex polytopes in \mathbb{R}^3 that can be realized as mean curvature convex (or totally geodesic) Riemannian polyhedra with non-obtuse dihedral angles in Riemannian 3-manifolds with positive scalar curvature. This result can be considered as an analogue of Andreev’s theorem on 3-dimensional hyperbolic polyhedra with non-obtuse dihedral angles. In addition, we construct many examples of such kind of simple convex polytopes in higher dimensions.

1. Introduction

A *Riemannian polyhedron* of dimension n is a polyhedral domain W with faces in an n -dimensional Riemannian manifold (M, g) with the induced Riemannian metric $g(W) = g|_W$ (see Gromov [12, 1]). Moreover, each codimension-one face (or *facet*) F_i of W is contained in a smooth hypersurface Σ_i of M such that

- Whenever two facets F_i and F_j of W are adjacent, the corresponding hypersurfaces Σ_i and Σ_j intersect transversely in M .
- The boundary ∂F_i of each facet F_i consists of $F_i \cap F_j$ where F_j ranges over all the facets adjacent to F_i , and the decomposition $\partial F_i = \bigcup_j F_i \cap F_j$ gives an $(n - 1)$ -dimensional Riemannian polyhedron structure to F_i .

Note that the ambient manifold M here is not necessarily compact or closed. Typical examples of Riemannian polyhedra are the intersections of finitely many domains with smooth mutually transversal boundaries in a Riemannian manifold (e.g. convex polytopes in the Euclidean space \mathbb{R}^n).

Definition 1.1. Let W_i be a Riemannian polyhedron in (M_i, g_i) , $i = 1, 2$. We call W_1 and W_2 *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation. We call W_1 and W_2 *pseudo-diffeomorphic* if there exists a homeomorphism $\varphi : W_1 \rightarrow W_2$ which is the restriction of a

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diffeomorphism from an open neighborhood of W_1 in M_1 to an open neighborhood of W_2 in M_2 , and we call φ a *pseudo-diffeomorphism* from W_1 to W_2 . If, moreover, φ is face-preserving (i.e. φ maps every face of W_1 homeomorphically onto a face of W_2), we call φ a *diffeomorphism* from W_1 to W_2 . In this case, we also say that W_2 is a *realization* of W_1 in (M_2, g_2) . It is clear that a diffeomorphism from W_1 to W_2 induces a combinatorial equivalence.

By Wiemeler [24, Corollary 5.3] (also see Davis [8, Corollary 1.3]), two simple convex polytopes in \mathbb{R}^n are diffeomorphic (as Riemannian polyhedra) if and only if they are combinatorially equivalent. Recall that a convex polytope P in \mathbb{R}^n is called *simple* if every codimension- k face of P is contained in exactly k different facets of P . The reader is referred to [27] for the basic notions of convex polytopes.

Definition 1.2. Let W be a Riemannian polyhedron.

- We say that W has *acute* or *non-obtuse dihedral angles* if the dihedral angle function on every codimension-two face of W ranges in $(0, \pi/2)$ or $(0, \pi/2]$. Especially, we call W *right-angled* if the dihedral angle function on every codimension-two face of W is constantly $\pi/2$.
- We call W *mean curvature convex* if every facet of W has non-negative mean curvature in the ambient Riemannian manifold. Especially, we call W *totally geodesic* if every facet of W is a totally geodesic submanifold.

Convention In this paper, the mean curvature of a boundary point of a domain in a Riemannian manifold is always taken with respect to the inward unit normal vector.

The main purpose of this paper is to study the combinatorial types of all the 3-dimensional simple convex polytopes in \mathbb{R}^3 that can be realized as mean curvature convex (or totally geodesic) Riemannian polyhedra with non-obtuse dihedral angles in Riemannian 3-manifolds with positive scalar curvature.

For $n \geq 1$, let Δ^n and $[0, 1]^n$ denote the standard n -simplex and n -cube in \mathbb{R}^n , respectively. The following is the main theorem of this paper.

Theorem 1.3. *Suppose P is a 3-dimensional simple convex polytope in \mathbb{R}^3 . Then P can be realized as a mean curvature convex Riemannian polyhedron with non-obtuse dihedral angles in a Riemannian 3-manifold with positive scalar curvature if and only if P is combinatorially equivalent to a convex polytope that can be obtained from Δ^3 by a sequence of vertex-cuts.*

Definition 1.4. (*Vertex-Cut*) Let P be an n -dimensional simple convex polytope in \mathbb{R}^n and v a vertex of P . Choose a plane H in \mathbb{R}^n such that H separates v from the other vertices of P . Let H_{\geq} and H_{\leq} be the two half spaces determined by H and assume that v belongs to H_{\geq} . Then $P \cap H_{\geq}$ is an $(n-1)$ -simplex, and $P \cap H_{\leq}$ is a simple convex polytope which we refer to as a *vertex-cut* of P . For example, a vertex-cut of Δ^3 is combinatorially equivalent to $\Delta^2 \times [0, 1]$ (the triangular prism).

The simplicial polytope dual to a convex polytope that is obtained from Δ^3 by a sequence of vertex-cuts is known as a stacked 3-polytope. By definition, a *stacked n -polytope* is a polytope obtained from Δ^n by repeatedly gluing another n -simplex onto one of its facets (see [18]). One reason for the significance of stacked

polytopes is that, among all simplicial n -polytopes with a given number of vertices, the stacked polytopes have the fewest possible higher-dimensional faces.

A well known special case of Theorem 1.3 is: if a 3-dimensional simple convex polytope P can be realized as a totally geodesic polyhedron with non-obtuse dihedral angles in the standard unit 3-sphere S^3 , then combinatorially P must be a 3-simplex (see [7, Sec. 6.3]).

Corollary 1.5. *Suppose P is a 3-dimensional simple convex polytope in \mathbb{R}^3 . Then P can be realized as a right-angled totally geodesic Riemannian polyhedron in a Riemannian 3-manifold with positive scalar curvature if and only if P is combinatorially equivalent to a convex polytope that can be obtained from Δ^3 by a sequence of vertex-cuts.*

Note that Theorem 1.3 and Corollary 1.5 still hold if we assume the scalar curvature of the ambient Riemannian 3-manifold to be positive constant (see Corollary 3.2).

Theorem 1.3 can be thought of as an analogue of Andreev's theorem (see [1,2]) on totally geodesic polyhedra with non-obtuse dihedral angles in the 3-dimensional hyperbolic space \mathbb{H}^3 (see [20] for a new proof). Andreev's theorem is essential for proving Thurston's Hyperbolization theorem for Haken 3-manifolds. Especially, Andreev's theorem tells us that a simple convex 3-polytope P can be realized as a right-angled totally geodesic hyperbolic polyhedron in \mathbb{H}^3 if and only if P has no prismatic 3-circuits or prismatic 4-circuits (this result was also obtained by A. V. Pogorelov in an earlier paper [19]).

In addition, the following question proposed by Gromov in [12] is related to our study.

Question (Gromov [12, 1.7]): What are the possible combinatorial types of mean curvature convex Riemannian polyhedra W with acute dihedral angles in a Riemannian manifold with non-negative scalar curvature?

If W in the above question is the realization of 3-dimensional simple convex polytope in a Riemannian 3-manifold with positive scalar curvature, then the combinatorial type of W must belong to the cases described in Theorem 1.3. But conversely, it is not clear whether we can construct a Riemannian polyhedron with acute dihedral angles for every combinatorial type described in Theorem 1.3. Indeed, the Riemannian polyhedra constructed in the proof of Theorem 1.3 are all right-angled.

The paper is organized as follows. In Sect. 2, we review the definition of real moment-angle manifold associated to a simple convex polytope. Besides, we quote a result from Wu and Yu [25] on when a 3-dimensional real moment-angle manifold can admit a Riemannian metric with positive scalar curvature. In Sect. 3, we give a proof of Theorem 1.3 using the idea of "doubling-smoothing" of Riemannian manifolds described in [12, 2.1] along with the result from [25]. In Sect. 4, we construct some examples of totally geodesic non-obtuse Riemannian polyhedra with positive scalar curvature in higher dimensions and propose a question. '.

2. Real moment-angle manifolds

Suppose P is an n -dimensional simple convex polytope in Euclidean space \mathbb{R}^n . Let $\mathcal{F}(P) = \{F_1, \dots, F_m\}$ be the set of all facets of P . Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and let e_1, \dots, e_m be a basis of $(\mathbb{Z}_2)^m$. Define a function $\lambda_0 : \mathcal{F}(P) \rightarrow (\mathbb{Z}_2)^m$ by

$$\lambda_0(F_i) = e_i, \quad 1 \leq i \leq m. \tag{1}$$

For any proper face f of P , let G_f denote the subgroup of $(\mathbb{Z}_2)^m$ generated by the set $\{\lambda_0(F_i) \mid f \subset F_i\}$. For any point $p \in P$, let $f(p)$ denote the unique face of P that contains p in its relative interior. In [6, Construction 4.1], the *real moment-angle manifold* $\mathbb{R}\mathcal{Z}_P$ of P is a closed orientable n -manifold defined by the following quotient construction

$$\mathbb{R}\mathcal{Z}_P := P \times (\mathbb{Z}_2)^m / \sim \tag{2}$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in G_{f(p)}$. So at every vertex of P , 2^n copies of P are glued together so that locally they look like the 2^n cones of \mathbb{R}^n bounded by the n coordinate hyperplanes meeting at the origin. Let

$$\eta : P \times (\mathbb{Z}_2)^m \rightarrow \mathbb{R}\mathcal{Z}_P \tag{3}$$

be the quotient map. There is a *canonical action* of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_P$ defined by

$$g' \cdot [(p, g)] = [(p, g' + g)], \quad \forall p \in P, \forall g, g' \in (\mathbb{Z}_2)^m,$$

whose orbit space can be identified with P . Let

$$\Theta_P : \mathbb{R}\mathcal{Z}_P \rightarrow P$$

be the orbit map. Note that each facet F of P is also a simple convex polytope and, $\Theta_P^{-1}(F)$ is a disjoint union of several copies of $\mathbb{R}\mathcal{Z}_F$ embedded in $\mathbb{R}\mathcal{Z}_P$ with trivial normal bundles.

The study of real moment-angle manifolds is an important subject in toric topology. The reader is referred to Davis and Januszkiewicz [6], Buchstaber and Panov [5], Kuroki et al. [15] and Wu and Yu [25] for more information of the topological and geometric properties of real moment-angle manifolds. The construction of $\mathbb{R}\mathcal{Z}_P$ in (2) also makes sense for any smooth nice manifold with corners. The topology of such generalized spaces are studied in a recent paper Yu [26].

In addition, we can realize $\mathbb{R}\mathcal{Z}_P$ as a non-degenerate intersection of $m - n$ real quadrics (quadric hypersurfaces) in \mathbb{R}^m , which induces a $(\mathbb{Z}_2)^m$ -invariant smooth structure on $\mathbb{R}\mathcal{Z}_P$ (see [5, 6]). Consider a presentation of P as follows:

$$P = P(A, b) = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle + b_i \geq 0, \quad i = 1, \dots, m\} \tag{4}$$

where $A = (a_1, \dots, a_m)$ is an $n \times m$ real matrix. Since P has a vertex, the rank of A is equal to n . Define a map

$$i_{A,b} : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad i_{A,b}(x) = A^t x + b \tag{5}$$

where $b = (b_1, \dots, b_m)^t \in \mathbb{R}^m$. So the map $i_{A,b}$ embeds P into the positive cone $\mathbb{R}_{\geq 0}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_i \geq 0, i = 1, \dots, m\}$. We can define a space $\mathbb{R}\mathcal{Z}_{A,b}$ by the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}\mathcal{Z}_{A,b} & \xrightarrow{i_{\mathcal{Z}}} & \mathbb{R}^m \\ \downarrow & & \downarrow \mu \\ P & \xrightarrow{i_{A,b}} & \mathbb{R}_{\geq 0}^m \end{array}$$

where $\mu(x_1, \dots, x_m) = (x_1^2, \dots, x_m^2)$. Clearly $(\mathbb{Z}_2)^m$ acts on $\mathbb{R}\mathcal{Z}_{A,b}$ with quotient space P and $i_{\mathcal{Z}}$ is a $(\mathbb{Z}_2)^m$ -equivariant embedding. It is easy to see that $\mathbb{R}\mathcal{Z}_{A,b}$ is homeomorphic to $\mathbb{R}\mathcal{Z}_P$. In addition, the image of \mathbb{R}^n under $i_{A,b}$ is an affine plane of dimension n in \mathbb{R}^m , which we can specify by $m - n$ linear equations:

$$\begin{aligned} i_{A,b}(\mathbb{R}^n) &= \{y \in \mathbb{R}^m \mid y = A^t x + b, x \in \mathbb{R}^n\} \\ &= \{y \in \mathbb{R}^m \mid \Gamma y = \Gamma b\} \end{aligned}$$

where $\Gamma = (\gamma_{jk})$ is an $(m - n) \times m$ matrix of rank $m - n$ so that $\Gamma A^t = 0$. In other words, the rows of Γ form a basis of all the linear relations among a_1, \dots, a_m . Then we can write the image of $\mathbb{R}\mathcal{Z}_{A,b}$ under $i_{\mathcal{Z}}$ explicitly as the common zeros of $m - n$ real quadratic equations in \mathbb{R}^m :

$$i_{\mathcal{Z}}(\mathbb{R}\mathcal{Z}_{A,b}) = \left\{ (y_1, \dots, y_m)^t \in \mathbb{R}^m \mid \sum_{k=1}^m \gamma_{jk} y_k^2 = \sum_{k=1}^m \gamma_{jk} b_k, 1 \leq j \leq m - n \right\}. \tag{6}$$

The above intersection of real quadrics is non-degenerate (i.e. the gradients of these quadrics are linearly independent everywhere in their intersection). This implies that $\mathbb{R}\mathcal{Z}_{A,b}$ is embedded as an n -dimensional smooth submanifold in \mathbb{R}^m where $(\mathbb{Z}_2)^m$ acts smoothly. So P is embedded as a Riemannian polyhedron in $i_{\mathcal{Z}}(\mathbb{R}\mathcal{Z}_{A,b})$ with the induced metric from \mathbb{R}^m .

Example 2.1. The standard simplex $\Delta^n \subset \mathbb{R}^n$ is defined by

$$\Delta^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n \leq 1, x_i \geq 0, i = 1, \dots, n.\}$$

By the notation in (4), we have $\Delta^n = P(A, b)$ where

$$A_{n \times (n+1)} = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}, \quad b = (0, 0, \dots, 0, 1)^t \in \mathbb{R}^{n+1}.$$

So the image of the embedding $i_{A,b} : \Delta^n \hookrightarrow \mathbb{R}^{n+1}$ [see (5)] is given by

$$\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \dots + x_{n+1} = 1, x_i \geq 0, i = 1, \dots, n + 1\}.$$

Then by (6), $i_{\mathcal{Z}}(\mathbb{R}\mathcal{Z}_{A,b}) \subset \mathbb{R}^{n+1}$ is given by the following equation

$$y_1^2 + \cdots + y_{n+1}^2 = 1, (y_1, \dots, y_{n+1})^t \in \mathbb{R}^{n+1}$$

which is exactly the standard unit sphere S^n in \mathbb{R}^{n+1} . Moreover, the canonical $(\mathbb{Z}_2)^{n+1}$ -action on $\mathbb{R}\mathcal{Z}_{\Delta^n}$ is equivalent to the action of $(\mathbb{Z}_2)^{n+1}$ on S^n by the reflections about the coordinate hyperplanes of \mathbb{R}^{n+1} . More precisely, for each $1 \leq i \leq n + 1$, the i th generator of $(\mathbb{Z}_2)^{n+1}$ acts on S^n by

$$(y_1, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_{n+1}) \longrightarrow (y_1, \dots, y_{i-1}, -y_i, y_{i+1}, \dots, y_{n+1}).$$

In the following, we think of $\mathbb{R}\mathcal{Z}_{\Delta^n}$ as S^n equipped with the above $(\mathbb{Z}_2)^{n+1}$ -action.

The theorem below tells us what kind of 3-dimensional real moment-angle manifolds can admit a Riemannian metric with positive scalar curvature.

Theorem 2.2. (Proposition 4.8 and Corollary 4.10 in Wu-Yu [25]) *Let P be a 3-dimensional simple convex polytope in \mathbb{R}^3 with m facets. Then $\mathbb{R}\mathcal{Z}_P$ admits a Riemannian metric with positive scalar curvature if and only if P is combinatorially equivalent to a convex polytope that can be obtained from Δ^3 by a sequence of vertex-cuts. Moreover, we can choose the Riemannian metric with positive scalar curvature on $\mathbb{R}\mathcal{Z}_P$ to be invariant with respect to the canonical $(\mathbb{Z}_2)^m$ -action on $\mathbb{R}\mathcal{Z}_P$.*

3. Proof of Theorem 1.3

Our proof of Theorem 1.3 is inspired by Gromov’s proof of the dihedral rigidity of the n -cube in [12]. Gromov’s proof is based on the idea of doubling the cube n times and uses the well-known fact that the n -dimensional torus admits no metric with positive scalar curvature. Here we observe that for a simple convex polytope P with m facets, doubling P m times gives the real moment-angle manifold $\mathbb{R}\mathcal{Z}_P$. Then starting from a Riemannian metric with positive scalar curvature on P that satisfies certain conditions on the boundary of P as stated in Theorem 1.3, we can construct an invariant Riemannian metric on $\mathbb{R}\mathcal{Z}_P$ with positive scalar curvature by the procedure of “doubling-smoothing” of metrics described in [12]. So the combinatorial type of P is restricted by Theorem 2.2. But the description of the “doubling-smoothing” of Riemannian metrics in [12] is very sketchy, we will give more detailed exposition in our proof of Theorem 1.3 below. Our proof slightly generalizes the argument in Gromov and Lawson [10, Theorem 5.7].

Let A be a subspace of a topological space X . The *double of X along A* is the quotient space of the disjoint union of two copies of X by identifying each point of A in one copy of X to the same point in the other copy.

Suppose F and F' are two facets of a Riemannian polyhedron W which intersect transversely at a codimension-two face $F \cap F'$.

- Let $\angle(F, F')_x$ denote the dihedral angle of W at a point $x \in F \cap F'$.
- Let $\angle(F, F')$ denote the dihedral angle of W at an arbitrary point of $F \cap F'$.

Proof of Theorem 1.3. Let P be a simple convex 3-polytope in \mathbb{R}^3 whose facets are F_1, \dots, F_m . Let $\Theta_P : \mathbb{R}\mathcal{Z}_P \rightarrow P$ be the orbit map of the canonical $(\mathbb{Z}_2)^m$ -action on $\mathbb{R}\mathcal{Z}_P$. So each $\Theta_P^{-1}(F_i)$, $1 \leq i \leq m$, consists of some closed connected 2-manifolds that intersect transversely in $\mathbb{R}\mathcal{Z}_P$. Since we are working in dimension 3, we do not need to worry about the existence or uniqueness of the smooth structures in our manifolds.

We first prove the “if” part. Suppose P is combinatorially equivalent to a convex polytope that can be obtained from Δ^3 by a sequence of vertex-cuts. Then by Theorem 2.2, there exists a $(\mathbb{Z}_2)^m$ -invariant Riemannian metric g_0 with positive scalar curvature on $\mathbb{R}\mathcal{Z}_P$. Note that P is realized as the fundamental domain of the canonical $(\mathbb{Z}_2)^m$ -action on $\mathbb{R}\mathcal{Z}_P$ which is bounded by the submanifolds $\Theta_P^{-1}(F_i)$, $1 \leq i \leq m$. Moreover, by definition each $\Theta_P^{-1}(F_i)$ is the fixed point set of the generator $e_i \in (\mathbb{Z}_2)^m$ under the canonical $(\mathbb{Z}_2)^m$ -action [see (1)]. It is a standard fact that every connected component of the fixed point set (with the induced Riemannian metric) of an isometry on a Riemannian manifold is a totally geodesic submanifold (see [14, Theorem 1.10.15]). So each $\Theta_P^{-1}(F_i)$ consists of totally geodesic submanifolds of $(\mathbb{R}\mathcal{Z}_P, g_0)$. In addition, the dihedral angles between any components of $\Theta_P^{-1}(F_i)$ and $\Theta_P^{-1}(F_j)$ (whenever they intersect) are always equal to $\pi/2$ since the $(\mathbb{Z}_2)^m$ -action on $(\mathbb{R}\mathcal{Z}_P, g_0)$ is isometric. Therefore, P is realized as a right-angled totally geodesic Riemannian polyhedron in $(\mathbb{R}\mathcal{Z}_P, g_0)$. The “if” part is proved.

Next, we prove the “only if” part. We can obtain $\mathbb{R}\mathcal{Z}_P$ by iterated doubling of P as follows. By the notation in the definition of $\mathbb{R}\mathcal{Z}_P$ [see (1) and (3)], define

$$H_j := \text{the subgroup of } \mathbb{Z}_2^m \text{ generated by } e_1, \dots, e_j, \quad 1 \leq j \leq m, \text{ and } H_0 := \{0\};$$

Then let

$$Y^{(j)} := \eta(P \times H_j), \quad F_i^{(j)} := \eta(F_i \times H_j), \quad 1 \leq i, j \leq m.$$

So $Y^{(j)}$ is the gluing of 2^j copies of P under η whose boundary is

$$\partial Y^{(j)} = \bigcup_{i>j} F_i^{(j)}.$$

In addition, for any facet F_i of P and any element $g \in \mathbb{Z}_2^m$, let

$$F_{i,g} := \eta(F_i \times \{g\}), \quad 1 \leq i \leq m.$$

Then we have

$$F_i^{(j)} = \bigcup_{g \in H_j} F_{i,g}, \quad 1 \leq i, j \leq m.$$

So $Y^{(j)}$ is a Riemannian polyhedron in $\mathbb{R}\mathcal{Z}_P$ whose facets are $\{F_{i,g} \mid i > j, g \in H_j\}$.

By identifying P with $Y^{(0)}$, we have a filtration (see Fig. 1 for example)

$$P = Y^{(0)} \subset Y^{(1)} \subset \dots \subset Y^{(m)} = \mathbb{R}\mathcal{Z}_P.$$

Clearly, $Y^{(j+1)}$ is the double of $Y^{(j)}$ along $F_{j+1}^{(j)}$ for each $0 \leq j \leq m - 1$. Note that $F_{j+1}^{(j)}$ may not be connected.

In the following, we do induction on j and assume that:

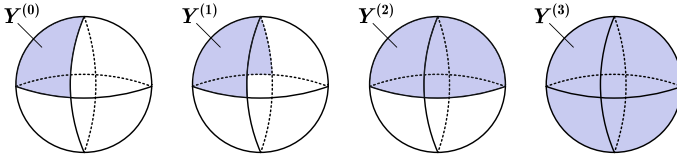


Fig. 1. The filtration of $\mathbb{R}Z_{\Delta_2} = S^2$

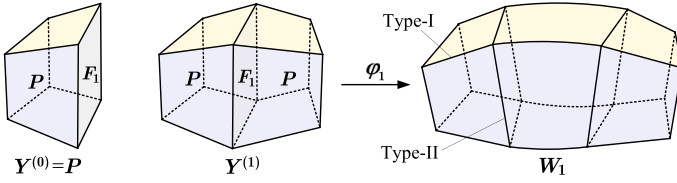


Fig. 2. Two types of edges on the boundary of W_1

- (a) There exists a pseudo-diffeomorphism $\varphi_j : Y^{(j)} \rightarrow W_j$ where W_j is a mean curvature convex Riemannian polyhedron in a Riemannian 3-manifold (M_j, g_j) with positive scalar curvature, and φ_j maps every $F_i^{(j)}$ on $\partial Y^{(j)}$ to be a union of facets of W_j .

Note that here we cannot require φ_j to be face-preserving when $j \geq 1$ since W_j may have more facets than $Y^{(j)}$ (see Fig. 2 for example).

We say that a facet E of W_j comes from the facet F_i of P if

$$\Theta_P(\varphi_j^{-1}(E)) \subset F_i \text{ or equivalently } E \subset \varphi_j(F_i^{(j)}) \text{ where } j < i \leq m.$$

There are two types of edges (codimension-two faces) on ∂W_j (see Fig. 2):

Type-I The intersection of two facets that come from different facets of P .

Type-II The intersection of two facets that come from the same facet of P .

Moreover, we assume W_j to satisfy the following two conditions in our induction hypothesis.

- (b) The dihedral angles of W_j are non-obtuse at every Type-I edge on ∂W_j .
- (c) The dihedral angles of W_j range in $(0, \pi]$ at every Type-II edge on ∂W_j .

From the above assumptions, we will construct a pseudo-diffeomorphism φ_{j+1} from $Y^{(j+1)}$ to a mean curvature convex Riemannian polyhedron in a Riemannian 3-manifold with positive scalar curvature that satisfies (a),(b) and (c). First of all, let

$$V_j := \{x \in M_j; \text{dist}(x, W_j) \leq \delta\}, \delta \ll 1;$$

By [9, Theorem 1.1], we can multiply a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ to some components of the metric $g_j|_{W_j}$ and obtain a new metric g'_j with positive scalar curvature on W_j so that the mean curvature of g'_j at every facet of W_j is positive. So without loss of generality, we can just assume that the mean curvature of g_j at every facet of W_j is positive at the beginning. Then for a sufficiently small δ , it is not hard to show that V_j is a mean curvature convex domain in (M_j, g_j) .

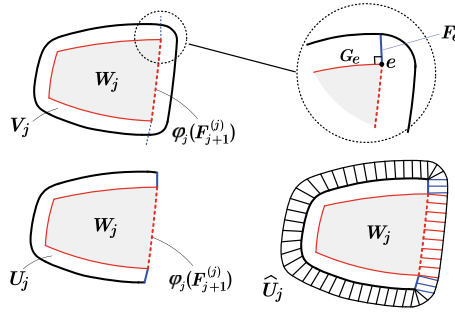


Fig. 3. A domain U_j containing W_j

Let e be an edge on boundary of $\varphi_j(F_{j+1}^{(j)})$. We can write $e = G_e \cap \varphi_j(F_{j+1}^{(j)})$ where G_e is a facet of W_j . Let F_e be the union of the geodesic segments in V_j emanating from the points of e that are orthogonal to G_e (see Fig. 3). Then the dihedral angles between F_e and the facets in $\varphi_j(F_{j+1}^{(j)})$ and ∂V_j are all less than π .

If we cut off an open subset from V_j along $\varphi_j(F_{j+1}^{(j)})$ and these F_e 's, we obtain a compact polyhedral domain U_j containing W_j where $\partial U_j \cap \partial W_j = \varphi_j(F_{j+1}^{(j)})$ (see Fig. 3). Clearly, U_j is also mean curvature convex. Moreover, define

$$\begin{aligned} \widehat{U}_j &:= \{x \in M_j; \text{dist}(x, U_j) \leq \delta\}, \delta \ll 1; \\ \widehat{W}_j &:= W_j \cup \{\text{geodesic segments in } \widehat{U}_j \text{ emanating orthogonally} \\ &\quad \text{from faces in } \varphi_j(F_{j+1}^{(j)})\}. \end{aligned}$$

So $\widehat{U}_j \setminus U_j$ is a thin collar of $\partial \widehat{U}_j$. Then consider

$$D(U_j) := \{p \in \widehat{U}_j \times [-1, 1]; \text{dist}(p, U_j \times \{0\}) = \varepsilon\}, \quad 0 < \varepsilon < \delta/2, \quad (7)$$

$$D(W_j) := \{p \in \widehat{W}_j \times [-1, 1]; \text{dist}(p, W_j \times \{0\}) = \varepsilon\}, \quad 0 < \varepsilon < \delta/2. \quad (8)$$

It is easy to see that $D(U_j)$ is homeomorphic to the double of U_j along ∂U_j , and $D(W_j)$ is homeomorphic to the double of W_j along $\varphi_j(F_{j+1}^{(j)})$ (see Fig. 4). Hence $D(W_j)$ is homeomorphic to $Y^{(j+1)}$. More precisely, there exists a homeomorphism

$$\varphi_{j+1} : Y^{(j+1)} \rightarrow D(W_j)$$

which maps a normal neighborhood $N(F_{j+1}^{(j+1)})$ of $F_{j+1}^{(j+1)}$ in $Y^{(j+1)}$ onto the bending region of $D(W_j)$ and maps the two components of $Y^{(j+1)} \setminus N(F_{j+1}^{(j+1)})$ onto the two copies of W_j in $D(W_j)$ that are parallel to $W_j \times \{0\}$. Moreover, by our assumption that $\varphi_j : Y^{(j)} \rightarrow W_j$ is a pseudo-diffeomorphism, we can extend φ_{j+1} to a diffeomorphism from an open neighborhood of $Y^{(j+1)}$ in \mathbb{RZ}_P to an open neighborhood of $D(W_j)$ in $D(U_j)$. So φ_{j+1} is also a pseudo-diffeomorphism.

We remark that here if the dihedral angles of W_j were greater than π at an edge in $\varphi_j(F_{j+1}^{(j)})$, then $D(U_j)$ and $D(W_j)$ would have some holes in the bending region. This is the reason why we need to assume W_j to satisfy the condition (c).

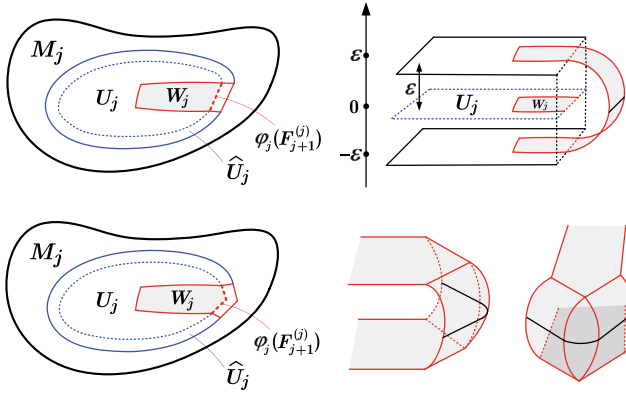


Fig. 4. $D(U_j)$ and $D(W_j)$

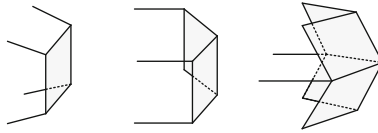


Fig. 5. Local configurations of facets of W_j in dimension 3

Geometrically, $D(U_j)$ and $D(W_j)$ will have some codimension-one creases where the induced metric from $\widehat{U}_j \times [0, 1]$ is not smooth. In addition, the facets in $\varphi_j(F_{j+1}^{(j)}) \subset \partial W_j$ may have different types of local configurations which will cause different shapes of $D(W_j)$ in the bending region (see Fig. 4). In particular in dimension 3, there are possibly three types of local configurations of facets in $\varphi_j(F_{j+1}^{(j)})$ and ∂U_j (see Fig. 5).

Next, we use the (slightly generalized) argument in [10, Theorem 5.7] to obtain a Riemannian metric on $D(U_j)$ with positive scalar curvature. Let

- $M_{j+1} = D(U_j)$ with the creases smoothed out;
- $g_{j+1} =$ the metric on M_{j+1} induced from the product metric $g_j \times g_{[-1,1]}$ on $U_j \times [-1, 1]$.

Using [9, Theorem 1.1] again, we can assume that the mean curvature of g_j at every facet of U_j is positive. Then similarly to the proof of [10, Theorem 5.7], we can compute the scalar curvature of g_{j+1} by estimating the principal curvatures of $D(U_j)$ in $U_j \times [-1, 1]$ as follows.

On the region parallel to $U_j \times \{0\}$ in $\widehat{U}_j \times [-1, 1]$, the scalar curvature of g_{j+1} on $D(U_j)$ is clearly positive. So the difficulty comes from the bending region of $D(U_j)$. The following argument is parallel to the argument in the proof of Gromov-Lawson [10, Theorem 5.7]. Let x be an arbitrary point on ∂U_j . We have the following three cases according to where x lies.

- (i) If x is the relative interior of a facet E on ∂U_j , then there is a unique normal direction of ∂U_j at x . Let σ_x be the geodesic segment in \widehat{U}_j emanating orthog-

onally from ∂U_j at x . Then $\sigma_x \times [-1, 1]$ is totally geodesic in $\widehat{U}_j \times [-1, 1]$. The intersection of $\sigma_x \times [-1, 1]$ with $D(U_j)$ is of the form shown in Fig. 6. Let μ_1, μ_2 be the principal curvatures of ∂U_j at x . Then at a point p corresponding to the angle $\theta \in (-\pi/2, \pi/2)$, the principal curvatures of $D(U_j)$ will be of the form

$$\begin{aligned} \lambda_0 &= \frac{1}{\varepsilon} \cos \theta + O(\varepsilon), \quad \lambda_1 = (\mu_1 + O(\varepsilon)) \cos \theta + O(\varepsilon^2), \\ \lambda_2 &= (\mu_2 + O(\varepsilon)) \cos \theta + O(\varepsilon^2). \end{aligned}$$

Let $\widehat{\kappa}$ be the scalar curvature function of \widehat{U}_j (and $\widehat{U}_j \times [-1, 1]$). The by the Gauss equation, the scalar curvature κ of g_{j+1} at p is of the form

$$\kappa = \widehat{\kappa} + \left(\frac{2}{\varepsilon} H + O(1)\right) \cos^2 \theta + O(\varepsilon)$$

where $H = \mu_1 + \mu_2$ is the mean curvature of ∂U_j at x . By our assumption of g_j , we have $\widehat{\kappa} > 0$ and $H > 0$. So we have $\kappa > 0$ when ε is sufficiently small.

- (ii) If x is the relative interior of an edge $E \cap E'$ where E and E' are two facets on ∂U_j , the normal direction of ∂U_j at x in \widehat{U}_j is not unique. Let σ_x and σ'_x be the geodesic segments orthogonal to E and E' at x , respectively. Let $\Sigma_x^{(E, E')}$ be the union of the geodesic segments in \widehat{U}_j orthogonal to $E \cap E'$ at x which is bounded σ_x and σ'_x . So $\Sigma_x^{(E, E')}$ looks like a fan with fan angle $\pi - \angle(E, E')_x$. Note that $\angle(E, E')_x \in (0, \pi]$ by the condition (b) and (c) of W_j . Moreover, x determines an oval-shaped patch (see Fig. 7) in the intersection of $\Sigma_x^{(E, E')} \times [-1, 1]$ and $D(U_j)$. Let μ_1, μ_2 be the principal curvatures of E at x . At a point p corresponding to the angles $\phi \in [0, \pi - \angle(E, E')_x]$ and $\theta \in (-a_\phi, a_\phi)$ where $0 < a_\phi \leq \pi/2$ is determined by ϕ , the principal curvatures of $D(U_j)$ will be of the form (see Fig. 7):

$$\begin{aligned} \lambda_0 &= \frac{1}{\varepsilon} \cos \phi \cos \theta + O(\varepsilon), \quad \lambda_1 = (\mu_1 + O(\varepsilon)) \cos \phi \cos \theta + O(\varepsilon^2), \\ \lambda_2 &= (\mu_2 + O(\varepsilon)) \cos \phi \cos \theta + O(\varepsilon^2). \end{aligned}$$

So we have

$$\kappa = \widehat{\kappa} + \left(\frac{2}{\varepsilon} H + O(1)\right) \cos^2 \phi \cos^2 \theta + O(\varepsilon).$$

By our assumption of $\widehat{\kappa}$ and H , we have $\kappa > 0$ when ε is sufficiently small.

- (iii) If $x = E \cap E' \cap E'' \cap E'''$ is a vertex where E, E', E'' and E''' are four facets on ∂U_j , the geodesic segments in \widehat{U}_j emanating from x determine a cone C_x which is bounded by the fans $\Sigma_x^{(E, E')}$, $\Sigma_x^{(E', E'')}$, $\Sigma_x^{(E'', E''')}$ and $\Sigma_x^{(E''', E)}$ (see Fig. 8). Moreover, x determines a football-shaped region in the intersection of $C_x \times [-1, 1]$ and $D(U_j)$ which is parametrized by three angles ϕ_1, ϕ_2 and θ , where $\phi_1 \in [0, \pi - \angle(E, E')_x]$, $\phi_2 \in [0, \pi - \angle(E, E'')_x]$ and $\theta \in (-a_{\phi_1, \phi_2}, a_{\phi_1, \phi_2})$,

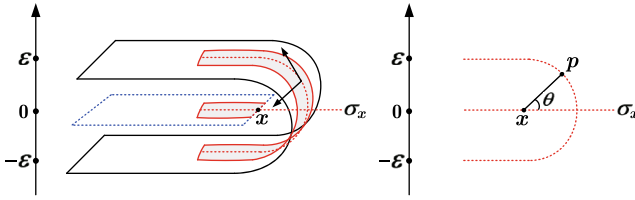


Fig. 6. The bending region of $D(U_j)$ parametrized by one angle

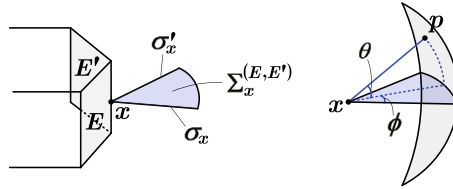


Fig. 7. The bending region of $D(U_j)$ parametrized by two angles

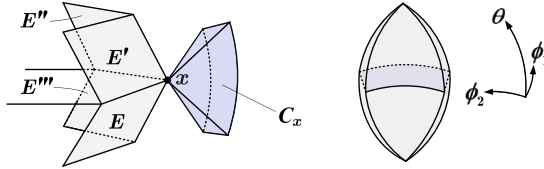


Fig. 8. The bending region of $D(U_j)$ parametrized by three angles

where $0 < a_{\phi_1, \phi_2} \leq \pi/2$ is determined by ϕ_1 and ϕ_2 . Similarly to the previous cases, for a point corresponding to ϕ_1 , ϕ_2 and θ , we obtain

$$\kappa = \widehat{\kappa} + \left(\frac{2}{\varepsilon} H + O(1)\right) \cos^2 \phi_1 \cos^2 \phi_2 \cos^2 \theta + O(\varepsilon).$$

By our assumption of $\widehat{\kappa}$ and H , we have $\kappa > 0$ when ε is sufficiently small.

So in all cases, we have $\kappa > 0$. By the above discussion, the bending region of $D(U_j)$ can be written as $A_1 \cup A_2 \cup A_3$ where A_1 , A_2 and A_3 consist of the points parametrized by one, two and three angles, respectively. Besides, let A_0 be the region of $D(U_j)$ that is parallel to $U_j \times \{0\}$. There are some obvious creases at the intersections of A_0 , A_1 , A_2 and A_3 where the metric g_{j+1} does not have continuous second derivatives. But by some small perturbations of $D(U_j)$, we can smooth out these creases so that the condition $\kappa > 0$ still holds. The local model of the smoothing is given by the Cartesian product of an open subset of the quadrant $\mathbb{R}_{\geq 0}^2$ with the curve γ shown in Fig. 9 where γ is smooth except at $t = 0$. For every point $x \in \mathbb{R}_{\geq 0}^2$, we deform $\{x\} \times \gamma$ to be $\{x\} \times \tilde{\gamma}$, where $\tilde{\gamma}$ is an everywhere smooth curve that differs from γ only in a small interval $0 < t < \delta$ for some $\delta \ll 1$ (see Fig. 9). After smoothing the creases, $D(W_j)$ becomes a Riemannian polyhedron in (M_{j+1}, g_{j+1}) that we desire.

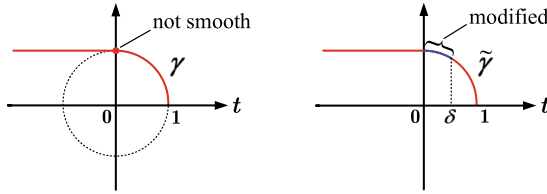


Fig. 9. Smoothing a curve γ near $t = 0$

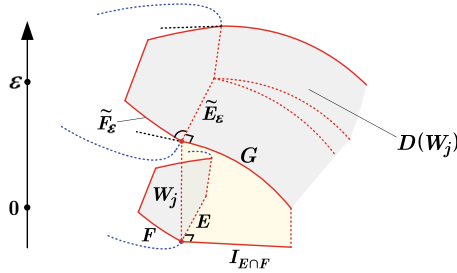


Fig. 10. Facets of $D(W_j)$ in the bending region

Next, let us verify that $D(W_j)$ satisfies the conditions (b) and (c). By the definition of \widehat{W}_j and $D(W_j)$, any facet G of $D(W_j)$ in the bending region is of the following form:

$$G = (I_{E \cap F} \times [-1, 1]) \cap D(W_j)$$

where E, F are two facets of W_j with $E \subseteq \varphi_j(F_{j+1}^{(j)})$ and $F \not\subseteq \varphi_j(F_{j+1}^{(j)})$, and $I_{E \cap F}$ is union of the geodesic segments in \widehat{U}_j orthogonal to E that emanate from points in $E \cap F$ (see Fig. 10). This implies:

- G is a totally geodesic facet of $D(W_j)$.
- $\angle(\widetilde{E}_{\pm\epsilon}, G) = \pi/2$ where $\widetilde{E}_{\pm\epsilon}$ is the parallel copy of E in $W_j \times \{\pm\epsilon\}$.

Let $\widetilde{F}_{\pm\epsilon}$ be the parallel copy of F in $W_j \times \{\pm\epsilon\}$. By the condition (b) of W_j , the dihedral angles $\angle(\widetilde{E}_{\epsilon}, \widetilde{F}_{\epsilon})$ and $\angle(\widetilde{E}_{-\epsilon}, \widetilde{F}_{-\epsilon})$ are both non-obtuse. So

$$\angle(G, \widetilde{F}_{\pm\epsilon}) \in (0, \pi]. \tag{9}$$

If an edge L of $D(W_j)$ is not contained in the two parallel copies of $\varphi_j(F_{j+1}^{(j)})$ in $W_j \times \{\pm 1\}$, the dihedral angles of $D(W_j)$ at L will agree with the dihedral angles of the counterpart of L in $W_j \times \{0\}$. These include all the Type-I edges of $D(W_j)$. Then since W_j satisfies the condition (b), so does $D(W_j)$.

All the Type-II edges of $D(W_j)$ that lie in the two parallel copies of $\varphi_j(F_{j+1}^{(j)})$ in $W_j \times \{\pm\epsilon\}$ are:

$$\{G \cap \widetilde{F}_{\pm\epsilon}; F \not\subseteq \varphi_j(F_{j+1}^{(j)}), G \text{ is a facet of } D(W_j) \text{ in the bending region}\}.$$

It follows from (9) that $D(W_j)$ satisfies the condition (c) at these edges. At other Type-II edges, $D(W_j)$ also satisfies the condition (c) by our assumption of W_j .

Moreover, the conditions (b) and (c) still hold for $D(W_j)$ after we smooth the creases of $D(U_j)$ since the smoothing does not change the tangent spaces of the facets of $D(U_j)$ incident to the creases.

In addition, $D(W_j)$ is mean curvature convex in (M_{j+1}, g_{j+1}) since its facets in the bending region are all totally geodesic while its facets outside the bending region have the same mean curvatures as their counterparts in $W_j \times \{0\}$.

So from the above arguments, we obtain a pseudo-diffeomorphism φ_{j+1} from $Y^{(j+1)}$ to $W_{j+1} := D(W_j)$ where W_{j+1} is a mean curvature convex Riemannian polyhedron in (M_{j+1}, g_{j+1}) with positive scalar curvature and W_{j+1} satisfies the conditions (b) and (c). This finishes the induction.

Observe that when $j = m - 1$, $\partial U_{m-1} = \partial W_{m-1} = \varphi_{m-1}(F_m^{(m-1)})$. This implies $W_{m-1} = U_{m-1}$. So by our doubling construction (8), the pseudo-diffeomorphism $\varphi_m : Y^{(m)} = \mathbb{R}\mathcal{Z}_P \rightarrow W_m = D(W_{m-1})$ is actually a diffeomorphism, where W_m is a compact Riemannian 3-manifold with positive scalar curvature. Then by Theorem 2.2, P must be combinatorially equivalent to a convex polytope that can be obtained from Δ^3 by a sequence of vertex cuts. So we finish the proof of the “only if” part and hence the whole theorem. \square

Remark 3.1. In the proof of the “only if” part of Theorem 1.3, there is a canonical action of $H_j \cong (\mathbb{Z}_2)^j$ on both $Y^{(j)}$ and W_j , $1 \leq j \leq m$. Indeed, we can define the action of H_j on W_j inductively through the doubling construction $D(W_{j-1})$. From our definition of the pseudo-diffeomorphism $\varphi_j : Y^{(j)} \rightarrow W_j$, it is easy to see that φ_j is equivariant with respect to the canonical H_j -actions for all $1 \leq j \leq m$.

Proof of Corollary 1.5. If P is combinatorially equivalent to a convex polytope obtained from Δ^3 by a sequence of vertex-cuts, it follows from the proof of the “if” part of Theorem 1.3 that P is realized as a right-angled totally geodesic Riemannian polyhedron in $(\mathbb{R}\mathcal{Z}_P, g_0)$. The reverse direction also follows from Theorem 1.3 immediately. \square

Corollary 3.2. *The statements of Theorem 1.3 and Corollary 1.5 still hold if we assume that the ambient Riemannian 3-manifold in these two theorems has positive constant scalar curvature.*

Proof. By the solution of the equivariant Yamabe problem in Hebey and Vaugon [13], if $\mathbb{R}\mathcal{Z}_P$ has a $(\mathbb{Z}_2)^m$ -invariant Riemannian metric g_0 with positive scalar curvature, then there exists a $(\mathbb{Z}_2)^m$ -invariant Riemannian metric \bar{g}_0 conformal to g_0 on $\mathbb{R}\mathcal{Z}_P$ which has positive constant scalar curvature. So we can prove this corollary by applying the same proof of the “if” part of Theorem 1.3 to $(\mathbb{R}\mathcal{Z}_P, \bar{g}_0)$. \square

4. Examples in higher dimensions

The argument of “doubling-smoothing” of Riemannian polyhedra in the proof of Theorem 1.3 can be generalized to any dimension $n \geq 2$ (also see [12, 2.1]). Indeed, the only new ingredient in the proof of the higher dimensions is: there are more types

of local configurations of facets on the boundary of the polyhedra in our iterated doubling construction. So if an n -dimensional simple convex polytope P in \mathbb{R}^n can be realized as a mean curvature convex Riemannian polyhedron with non-obtuse dihedral angles in a Riemannian n -manifold with positive scalar curvature, then we can construct a Riemannian metric on $\mathbb{R}\mathcal{Z}_P$ with positive scalar curvature. Note that the iterated doubling construction induces a smooth structure on $\mathbb{R}\mathcal{Z}_P$ from P which is equivariant with respect to the canonical $(\mathbb{Z}_2)^m$ -action. Then since the equivariant smooth structure on $\mathbb{R}\mathcal{Z}_P$ is known to be unique up to equivariant diffeomorphisms (see [15, Proposition 3.8]), this smooth structure on $\mathbb{R}\mathcal{Z}_P$ should agree with the smooth structure on $\mathbb{R}\mathcal{Z}_P$ determined by the embedding $i_{\mathcal{Z}} : \mathbb{R}\mathcal{Z}_P \rightarrow \mathbb{R}^m$ (see Sect. 2). But unfortunately, we do not have the analogue of Theorem 2.2 for $\mathbb{R}\mathcal{Z}_P$ in higher dimensions. So we cannot determine all possible combinatorial types of P in dimension greater than 3.

On the other hand, we can construct many examples of such kind of simple convex polytopes P in arbitrarily high dimensions as follows. Let us first quote some well known results on the existence of Riemannian metrics with positive scalar curvature.

Theorem 4.1. (see Gromov and Lawson [11] and Schoen and Yau [21]) *Let N be a closed manifold which admits a Riemannian metric with positive scalar curvature. If M is a manifold that is obtained from N by surgery in codimension ≥ 3 , then M also admits a Riemannian metric with positive scalar curvature.*

Moreover, there is an equivariant version of Theorem 4.1 for manifolds equipped with a compact Lie group action as follows.

Theorem 4.2. (Theorem 11.1 in [3]) *Let M and N be G -manifolds where G is a compact Lie group. Assume that N admits an G -invariant metric of positive scalar curvature. If M is obtained from N by equivariant surgeries of codimension at least three, then M admits a G -invariant metric of positive scalar curvature.*

It is shown in Bosio and Meersseman [4, Lemma 2.3] that up to combinatorial equivalence any n -dimensional simple convex polytope P can be obtained from the n -simplex by a finite number of flips at some proper faces. Let f be a proper face of P which is combinatorially equivalent to the k -simplex Δ^k . Roughly speaking, the flip of P at f gives us a new polytope, denoted by $\text{flip}_f(P)$, which is obtained by cutting off a small neighborhood $N(f)$ of f from P and a neighborhood $N(\Delta^k)$ of Δ^k from Δ^n , and then gluing $P \setminus N(f)$ and $\Delta^n \setminus N(\Delta^k)$ together along their cutting sections and merging the nearby facets (see [17, 23] or [4] for the precise definition). For example, Fig. 11 shows the flip of a 3-dimensional simple convex polytope at a vertex and at an edge, respectively. Note that doing a flip on P at a vertex v is equivalent to cutting off v from P , which will increase the number of facets by one. But whenever $\dim(f) > 0$, the number of facets of $\text{flip}_f(P)$ will be equal to that of P . In addition, a flip of P at a face corresponds to a bistellar move on ∂P^* where P^* is the dual simplicial polytope of P (see [4, 17]).

From the viewpoint of the construction of real moment-angle manifolds, a flip of P at a codimension- k face f corresponds to an equivariant surgery on $\mathbb{R}\mathcal{Z}_P$

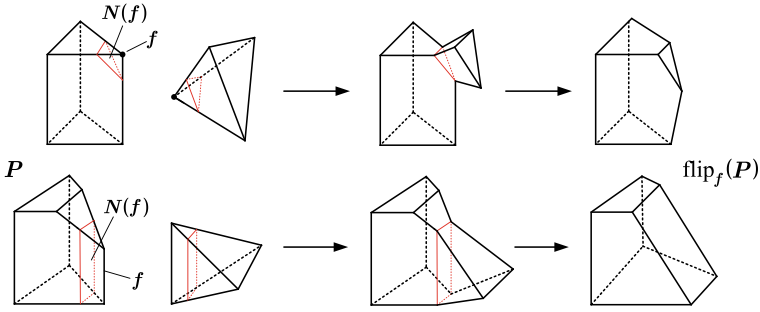


Fig. 11. Flip of a simple convex polytope P at a face f

at some codimension- k submanifolds. More specifically, it is a $(\mathbb{Z}_2)^m$ -equivariant surgery on $\mathbb{R}\mathcal{Z}_P$ if $\dim(f) > 0$ or a $(\mathbb{Z}_2)^{m+1}$ -equivariant surgery on $\mathbb{R}\mathcal{Z}_P \times \mathbb{Z}_2$ if $\dim(f) = 0$, where m is the number of facets of P (see [16, 4]). Then it follows from [4, Lemma 2.3] that we can obtain $\mathbb{R}\mathcal{Z}_P$ from $\mathbb{R}\mathcal{Z}_{\Delta^n} = S^n$ (see Example 2.1) by a sequence of equivariant surgeries for any n -dimensional simple convex polytope P . Moreover, the sequence of equivariant surgeries induce an (unique) equivariant smooth structure on $\mathbb{R}\mathcal{Z}_P$ step by step from the standard smooth structure on S^n .

Proposition 4.3. *Suppose P is an n -dimensional simple convex polytope in \mathbb{R}^n with $n \geq 3$. If P is combinatorially equivalent to a convex polytope that can be obtained from the n -simplex Δ^n by a sequence of flips at faces of codimension at least three, then P can be realized as a right-angled totally geodesic Riemannian polyhedron in a Riemannian n -manifold with positive scalar curvature.*

Proof. By our assumption of P , $\mathbb{R}\mathcal{Z}_P$ can be obtained from $\mathbb{R}\mathcal{Z}_{\Delta^n} = S^n$ by a sequence of equivariant surgeries at some submanifolds with codimension at least three. Note that the induced Riemannian metric g_0 on S^n from \mathbb{R}^{n+1} has positive constant scalar (sectional) curvature, and g_0 is invariant with respect to the canonical action of $(\mathbb{Z}_2)^{n+1}$ on S^n (see Example 2.1). Then it follows from Theorem 4.2 that $\mathbb{R}\mathcal{Z}_P$ has a $(\mathbb{Z}_2)^m$ -invariant Riemannian metric g_1 with positive scalar curvature where m is the number of facets of P . Finally, by applying the proof of the “if” part of Theorem 1.3 to $(\mathbb{R}\mathcal{Z}_P, g_1)$, we can deduce that P is realized as a right-angled totally geodesic Riemannian polyhedron in $(\mathbb{R}\mathcal{Z}_P, g_1)$. \square

To explore the generalization of Theorem 1.3 to higher dimensions, it is natural for us to ask the following question.

Question: Is it true that the examples given in Proposition 4.3 are exactly all the n -dimensional simple convex polytopes in \mathbb{R}^n ($n \geq 3$) that can be realized as mean curvature convex (or totally geodesic) Riemannian polyhedra with non-obtuse dihedral angles in Riemannian n -manifolds with positive scalar curvature?

To answer the above question, we need to understand for what simple convex polytope P the manifold $\mathbb{R}\mathcal{Z}_P$ can admit a Riemannian metric with positive scalar curvature. It is well known that in dimension ≥ 5 , such kind of problems are related

to the existence of spin structures, \widehat{A} -genus and α -invariant defined by index theory (see [22]). In dimension 4, the existence of Riemannian metrics with positive scalar curvature implies the vanishing of Seiberg-Witten invariants. But the calculations of these invariants for $\mathbb{R}\mathcal{Z}_P$ are very difficult in general.

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Declaration

Conflict of interest The author declares that he has no conflict of interest. Data sharing is not applicable to this article as no datasets were generated or analyzed in this paper.

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