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Masaaki Murakami

Surfaces with $c_1^2 = 9$ and $\chi = 5$ whose canonical classes are divisible by 3

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Abstract. We shall study minimal complex surfaces with $c^2 = 9$ and $\chi = 5$ whose canonical classes are divisible by 3 in the integral cohomology groups, where c_1^2 and χ denote the first Chern number of an algebraic surface and the Euler characteristic of the structure sheaf, respectively. The main results are a structure theorem for such surfaces, the unirationality of the moduli space, and a description of the behavior of the canonical map. As a byproduct, we shall also rule out a certain case mentioned in a paper by Ciliberto–Francia–Mendes Lopes. Since the irregularity q vanishes for our surfaces, our surfaces have geometric genus $p_g = 4$.

1. Introduction

When one wants to study the behavior of canonical maps of algebraic surfaces, surfaces of general type with $p_g = 4$ are in a sense the most primitive objects, since their canonical images are in most cases hypersurfaces of the 3-dimensional projective space $\mathbb{P}^3$. Partly for such reasons, these surfaces have attracted many algebraic geometers, even from the time of classical Italian school.

After Noether and Enriques studied the case $c_1^2 = 4$, surfaces with $p_g = 4$ have been studied from various view points (e.g, [6,9,15]). As for the classification, Horikawa and Bauer completed that for the surfaces of cases $4 \leq c_1^2 \leq 7$ ([1,12–14]). Complete classification of the surfaces of case $c_1^2 = 8$ seems not completely out of reach, but for the moment, only partial classifications and several examples are known (e.g.,[3,9,10]). We also notice that even though the surfaces have been classified for the case $c_1^2 = 6$, the number of the irreducible components of the moduli space remains unknown even after [2].

Among such works, the results most connected to the present paper are those on even surfaces for the case $c_1^2 = 8$. Recall that an algebraic surface is said to be even if its canonical class is divisible by 2. In [19], Oliverio studied regular even surfaces of case $c_1^2 = 8$, and showed that if S is a surface of this class with base point free

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M. Murakami (✉): Department of Mathematics and Computer Science, Kagoshima University, Korimoto 1-21-35, Kagoshima 890-0065, Japan
e-mail: murakami@sci.kagoshima-u.ac.jp

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canonical system, then its canonical model is a (6, 6)-complete intersection in the weighted projective space $\mathbb{P}(1, 1, 2, 3, 3)$. He also showed that these surfaces fill up an open dense subset of a 35-dimensional irreducible component $\mathcal{M}_{\mathcal{F}}$ of the moduli space $\mathcal{M}^{\text{ev}}_{8,4,0}$ of even regular surfaces of case $c_1^2 = 8$. Though this paper [19] studied these surfaces only under the condition that the canonical systems are base point free, Catanese, Liu, and Pignatelli later in [7] classified all even regular surfaces with $c_1^2 = 8$ and $p_g = 4$ and showed that $\mathcal{M}^{\text{ev}}_{8,4,0}$ consists exactly of two irreducible components $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{M}_{\mathcal{E}}$, both of dimension 35 and intersecting each other in codimension one locus.

In this paper, we go one step up, and study regular surfaces of case $c_1^2 = 9$ with canonical classes divisible by 3. We shall prove three theorems. Our first theorem asserts that any surface of this class has the canonical model isomorphic to a (6, 10)-complete intersection of the weighted projective space $\mathbb{P}(1, 2, 2, 3, 5)$ (Theorem 1). Our second theorem asserts that the moduli space of our surfaces is unirational of dimension 34, hence also the uniqueness of the diffeomorphic type of our surfaces (Theorem 2). Our third theorem asserts that the canonical map $\Phi_{|K|}$ of a surface of this class is either birational onto a singular sextic or generically two-to-one onto a cubic surface (Theorem 3). The surfaces with birational $\Phi_{|K|}$ and those with generically two-to-one $\Phi_{|K|}$ form an open dense subset and a 33-dimensional locus, respectively, in $\mathcal{M}$.

Possibility of the existence of surfaces with $c_1^2 = 9$ and $p_g = 4$ and with canonical classes divisible by 3 has already been mentioned in [10, (ii), Proposition 1.7], though for the case of canonical map composite with a pencil. In fact, the construction of examples of Case (ii) above was one of the motivations for our work. In the course of the proof of our Theorem 1, however, we shall show that this Case (ii) never occurs, even for the case of positive irregularity (Proposition 2). This sharpens their Proposition 1.7 slightly.

Let L be a divisor linearly equivalent to the canonical divisor of our surface. Our strategy of the first part is to study the map $\Phi_{|2L|}$ to compute the dimensions of some cohomology groups, where $\Phi_{|2L|}$ is the map associated to the linear system $|2L|$. Although the main tools for this part are classical ones, e.g., the double cover technique, a result by the author given in [18] on the torsion groups of surfaces with $c_1^2 = 2\chi - 1$ is also used to rule out some cases. Then we divide our argument into two cases depending on whether $\Phi_{|2L|}$ is composite with a pencil or not, and study each case. For the case where $\Phi_{|2L|}$ is composite with a pencil, it turns out that we are in Case (ii) of [10, Proposition 1.7]. Using results in [10] and applying to $\Phi_{|2L|}$ the structure theorem for genus 3 fibrations given in [8], we shall rule out this case. For the case where $\Phi_{|2L|}$ is not composite with a pencil, we shall study the semicanonical ring $R = \bigoplus_{n=0}^{\infty} H^0(\mathcal{O}(nL))$. Using arguments similar to those in [5], we shall find generators of the ring R and relations among them, which gives us the structure theorem. As for the results on the moduli space and the canonical maps, we shall prove them using this structure theorem. In addition to the theorems stated above, we shall also give a double cover description of our surfaces with $\deg \Phi_{|K|} = 2$ (Proposition 7).

After all the main results of the present paper were obtained, Kazuhiro Konno pointed out the normality of the canonical images of our surfaces of case $\deg \Phi_{|K|} =$

1 (Proposition 6). As informed to the author by him, our surfaces therefore give one of the missing examples of the list given in Konno's work [16] on normal canonical surfaces with $p_g = 4$.

NOTATION AND TERMINOLOGY

All varieties in this article are defined over the complex number field $\mathbb{C}$. Let V be a smooth variety. We denote by K_V, ω_V, Ω^1_V, and Θ_V, a canonical divisor, the dualizing sheaf, the cotangent sheaf, and the tangent sheaf, respectively, of V. For a divisor D, we denote by $\mathcal{O}(D)$ the coherent sheaf associated to D. For a coherent sheaf $\mathcal{F}$ on V, we denote by $H^i(\mathcal{F})$, $h^i(\mathcal{F})$, and $\chi(\mathcal{F})$, the i-th cohomology group of $\mathcal{F}$, its dimension $\dim_{\mathbb{C}} H^i(\mathcal{F})$, and the Euler characteristic $\sum(-1)^i h^i(\mathcal{F})$, respectively. We denote by $S^n(\mathcal{F})$ and $\bigwedge^n \mathcal{F}$ the n-th symmetric product and the n-th exterior product, respectively, of $\mathcal{F}$. Let $f : V \to W$ be a morphism to a smooth variety W, and D, a divisor on W. We denote by $f^*(D)$ the total transform of D.

The symbols $\sim$ and $\sim_{\mathrm{num}}$ mean the linear equivalence and the numerical equivalence, respectively, of two divisors. If D and D' are two divisors on V and $D - D'$ is a non-negative divisor, we write $D \succeq D'$.

For a smooth algebraic surface S, we denote by $c_1(S)$, $p_g(S)$, and $q(S)$, the first Chern class, the geometric genus, and the irregularity of S, respectively.

2. Some numerical restrictions

Let S be a minimal algebraic surface with $c_1^2 = 9$ and $\chi = 5$ whose canonical class is divisible by 3 in the cohomology group $H^2(S, \mathbb{Z})$. We take a divisor L such that $K = K_S \sim 3L$. In this section, as a preliminary, we shall find some restrictions to numerical invariants associated to the divisor L. Note that by the unbranched covering trick we have $q = 0$, hence $p_g = 4$. In what follows, we use the standard fact that if D and D' are two effective divisors the inequality $h^0(\mathcal{O}(D + D')) \geq h^0(\mathcal{O}(D)) + h^0(\mathcal{O}(D')) - 1$ holds.

Let us begin with the dimension $h^0(\mathcal{O}_S(2L))$.

Lemma 2.1. $3 \leq h^0(\mathcal{O}_S(2L)) \leq 5$.

Proof. By the Riemann–Roch theorem, we see that

$$h^0(\mathcal{O}_S(L)) + h^0(\mathcal{O}_S(2L)) \geq 4. \tag{1}$$

By this together with $h^0(\mathcal{O}_S(L)) \leq h^0(\mathcal{O}_S(2L))$, we obtain $2 \leq h^0(\mathcal{O}_S(2L))$. But if $h^0(\mathcal{O}_S(2L)) = 2$, then by (1) we must have $2 \leq h^0(\mathcal{O}_S(L))$, which contradicts $h^0(\mathcal{O}_S(2L)) \geq 2h^0(\mathcal{O}_S(L)) - 1$. Thus we obtain $3 \leq h^0(\mathcal{O}_S(2L))$. To obtain the remaining inequality, use $h^0(\mathcal{O}_S(6L)) = \chi(\mathcal{O}_S) + K^2 = 14$ and $h^0(\mathcal{O}_S(6L)) \geq 3h^0(\mathcal{O}_S(2L)) - 2$. □

Let $\Phi_{|2L|} : S - - \to \mathbb{P}^{l_2}$ be the rational map associated to the linear system $|2L|$, where $l_2 = h^0(\mathcal{O}_S(2L)) - 1$. We have two cases: the case where $\Phi_{|2L|}$ is composite with a pencil and the case where $\Phi_{|2L|}$ is not composite with a pencil. First, we study the former case.

Lemma 2.2. *Assume that the rational map $\Phi_{|2L|}$ is composite with a pencil $\mathcal{P}$. Then $h^0(\mathcal{O}_S(L)) = 2$ and $h^0(\mathcal{O}_S(2L)) = 3$ hold. Moreover $|L|$ has no fixed component, and the pencil $\mathcal{P}$ is given by $\Phi_{|L|} : S - - \to \mathbb{P}^1$.*

Proof. Assume that $\Phi_{|2L|}$ is composite with a pencil $\mathcal{P}$. Since S is regular and $|2L|$ is complete, there exists an effective divisor D_2 of S such that $h^0(\mathcal{O}_S(D_2)) \geq 2$ and $|2L| = |l_2 D_2| + F_2$, where F_2 is the fixed part of $|2L|$, and l_2 is as in the definition of $\Phi_{|2L|}$. Naturally, we have

$$2 = 2L^2 = l_2 D_2 L + F_2 L. \tag{2}$$

Assume that we have $D_2 L = 0$. Then we have $F_2 L = 2$, which together with $2LD_2 = l_2 D_2^2 + D_2 F_2$ and $2LF_2 = l_2 D_2 F_2 + F_2^2$ implies $F_2^2 = 4$ and $D_2^2 = D_2 F_2 = 0$. Then by Hodge's Index Theorem, we obtain $D_2 = 0$, which contradicts the definition of the divisor D_2.

Thus $D_2 L > 0$ holds. Since we have $l_2 \geq 2$ by Lemma 2.1, we see from this together with (2) that $l_2 = 2$, $D_2 L = 1$, and $F_2 L = 0$. In particular, we obtain $2 = 2LD_2 = l_2 D_2^2 + D_2 F_2$. But D_2^2 is odd, since $D_2 K = 3$. Thus this implies $D_2^2 = 1$ and $D_2 F_2 = F_2^2 = 0$, hence $F_2 = 0$. Thus we obtain $2L \sim l_2 D_2 + F_2 \sim 2D_2$. This however implies $L \sim D_2$, since by [18, Theorem 4] the surface S has no torsion. Since $l_2 = h^0(\mathcal{O}_S(2L)) - 1$, the assertion follows from this linear equivalence and $h^0(\mathcal{O}_S(2L)) \geq 2h^0(\mathcal{O}_S(L)) - 1$. □

Next, we study the latter case. In what follows, we denote by $|M_2|$ and F_2 the variable part and the fixed part, respectively, of the linear system $|2L|$. We also denote by $p_2 : \tilde{S}_2 \to S$ the shortest composite of quadric transformations such that the variable part of $p_2^*|M_2|$ is free from base points.

Lemma 2.3. *Assume that the rational map $\Phi_{|2L|}$ is not composite with a pencil. Then $h^0(\mathcal{O}_S(2L)) = 3$, $h^0(\mathcal{O}_S(L)) = 1$, and $h^1(\mathcal{O}_S(L)) = 0$ hold. Moreover, the inequality $2 \leq \tilde{M}_2^2 \leq 4$ holds, where $|\tilde{M}_2|$ is the variable part of the linear system $p_2^*|M_2|$.*

Proof. By Lemma 2.1 we have $2 \leq l_2 \leq 4$, where $l_2 = h^0(\mathcal{O}_S(2L)) - 1$. Since we have assumed that $\Phi_{|2L|}$ is not composite with a pencil, a general member $\tilde{M}_2$ of $|\tilde{M}_2|$ is a smooth irreducible curve on $\tilde{S}_2$. In what follows, we assume that $\tilde{M}_2$ is general, hence smooth, and define the divisors E_2 and ε_2 by $p_2^*|M_2| = |\tilde{M}_2| + E_2$ and $\tilde{K} = K_{\tilde{S}_2} \sim p_2^*(3L) + \varepsilon_2$, respectively.

First, let us show that $l_2 \leq 3$. By the Serre duality, we have $h^2(\mathcal{O}_{\tilde{S}_2}(\tilde{M}_2)) = h^0(\mathcal{O}_{\tilde{S}_2}(\tilde{K} - \tilde{M}_2)) < p_g(S) = 4$. From this together with the standard exact sequence

$$0 \to \mathcal{O}_{\tilde{S}_2} \to \mathcal{O}_{\tilde{S}_2}(\tilde{M}_2) \to \mathcal{O}_{\tilde{M}_2}(\tilde{M}_2) \to 0,$$

we see easily that $h^1(\mathcal{O}_{\tilde{M}_2}(\tilde{M}_2)) \geq 1$. Thus applying Clifford's theorem for $\tilde{M}_2|_{\tilde{M}_2}$, we obtain

$$2(l_2 - 1) \leq \tilde{M}_2^2 \leq \tilde{M}_2^2 + \tilde{M}_2 E_2 + M_2 F_2 + 2LF_2 = (2L)^2 = 4, \quad (3)$$

hence in particular $l_2 \leq 3$.

Assume that we have $l_2 = h^0(\mathcal{O}_S(2L)) - 1 = 2$. Then by the Riemann–Roch theorem, we have $h^0(\mathcal{O}_S(L)) = h^1(\mathcal{O}_S(L)) - h^0(\mathcal{O}_S(2L)) + 4 \geq 1$. And also, we have $3 = h^0(\mathcal{O}_S(2L)) \geq 2h^0(\mathcal{O}_S(L)) - 1$, hence $2 \geq h^0(\mathcal{O}_S(L))$. The case $h^0(\mathcal{O}_S(L)) = 2$ however is impossible, since we have assumed that $\Phi_{|2L|}$ is not composite with a pencil. Thus we obtain $h^0(\mathcal{O}_S(L)) = 1$ and $h^1(\mathcal{O}_S(L)) = 0$. Moreover, the inequality $2 \leq \tilde{M}_2^2 \leq 4$ follows from (3), hence as in the assertion. Therefore, we only need to rule out the case $l_2 = 3$.

So assume that we have $l_2 = 3$. In this case we obtain by (3) that $\tilde{M}_2 E_2 = M_2 F_2 = 2LF_2 = 0$, which implies the base point freeness of the linear system $|2L|$. Since S is of general type, we infer easily from this that $\deg \Phi_{|2L|} = \deg \Phi_{|2L|}(S) = 2$. Thus we have two cases:

Case A: the image $\Phi_{|2L|}(S) \subset \mathbb{P}^3$ is a smooth quadric;

Case B: the image $\Phi_{|2L|}(S) \subset \mathbb{P}^3$ is a quadric cone.

In what follows, we put $g = \Phi_{|2L|}$. We shall rule out the two cases separately.

Case A. Assume that $\Phi_{|2L|}(S)$ is a smooth quadric. Then the image $\Phi_{|2L|}(S)$ is the Hirzebruch surface Σ_0 of degree 0 embedded by $|\Delta_0 + \Gamma|$, where Δ_0 and Γ denote the minimal section and a fiber of the Hirzebruch surface Σ_0, respectively. Let R and $B = g_*(R)$ denote the ramification divisor and the branch divisor of the generically two-to-one morphism $g : S \to \Sigma_0$, respectively. Then since $2L \sim g^*(\Delta_0 + \Gamma)$, we see easily that $R \sim 7L$, hence $B\Delta_0 = B\Gamma = 7$. This however is impossible, because B needs to be linearly equivalent to twice a divisor on Σ_0. Thus Case A does not occur.

Case B. Assume that $\Phi_{|2L|}(S)$ is a quadric cone. Then the image $\Phi_{|2L|}(S)$ is the image of the morphism $\Phi_{|\Delta_0 + 2\Gamma|} : \Sigma_2 \to \mathbb{P}^3$, where Σ_2 is a Hirzebruch surface of degree 2, and Δ_0 and Γ are its minimal section and a fiber, respectively. Let $p_2' : S_2' \to S$ be the shortest composite of the quadric transformations such that $g \circ p_2'$ lifts to a morphism $g' : S_2' \to \Sigma_2$. We denote by $K' = K_{S_2'}$ a canonical divisor of S_2', and define the divisor ε_2' by $K' \sim p_2'^*(3L) + \varepsilon_2'$. We also denote by R and $B = g'_*(R)$ the ramification divisor and the branch divisor of the generically two-to-one morphism $g' : S_2' \to \Sigma_2$.

Since ε_2' is contracted by $g \circ p_2'$, there exists a natural number ν such that $g'_*(\varepsilon_2') = \nu\Delta_0$. Then from $p_2'^*(3L) + \varepsilon_2' \sim g'^*(-2\Delta_0 - 4\Gamma) + R$ and $p_2'(2L) \sim g'^*(\Delta_0 + 2\Gamma)$ we infer that $B\Delta_0 = -2\nu$ and $B\Gamma = 7 + \nu$. Since B is linearly equivalent to twice a divisor on Σ_2, this implies $\nu \geq 1$, hence $B\Delta_0 < 0$. Thus Δ_0 is a component of the branch divisor B. In particular, we have $\nu = 1$, from which we see that the multiplicity in ε_2' of the (-1)-curve appearing at the last quadric transformation in p_2' is equal to 1. Thus $p_2' : S_2' \to S$ is a blowing up at one point, and ε_2' is a (-1)-curve. Then by $p_2'^*(2L) \sim g'^*(\Delta_0 + 2\Gamma)$ we obtain $2(p_2'^*L - \varepsilon_2' - g'^*\Gamma)) \sim 0$. This implies the linear equivalence $p_2'^*L \sim \varepsilon_2' + g'^*\Gamma$, since by [18, Theorem 4] our surface S has no torsion. Thus we obtain $h^0(\mathcal{O}_S(L)) \geq h^0(\mathcal{O}_{\Sigma_2}(\Gamma)) = 2$. This however is impossible, since we have $h^0(\mathcal{O}_S(2L)) = 4$ and $4 = h^0(\mathcal{O}_S(3L)) \geq h^0(\mathcal{O}_S(2L)) + h^0(\mathcal{O}_S(L)) - 1$. Thus Case B does not occur. This concludes the proof of Lemma 2.3. □

3. Study of the map $\Phi_{|2L|}$

In this section, we shall study the map $\Phi_{|2L|}$, and rule out the case where $\Phi_{|2L|}$ is composite with a pencil. Assume that the rational map $\Phi_{|2L|}$ is composite with a pencil $\mathcal{P}$. Then by Lemma 2.2, we have $h^0(\mathcal{O}_S(2L)) = 3$ and $h^0(\mathcal{O}_S(L)) = 2$. The linear system $|L|$ has a unique base point, which is simple. Moreover, since $h^0(\mathcal{O}_L) = 1$ holds, $\mathcal{P}$ is a pencil of curves of genus 3, whose members correspond to fibers of $\Phi_{|L|} : S - - \to \mathbb{P}^1$. Let $p : \tilde{S} \to S$ be the blow up of S at the base point of $|L|$, and E, its exceptional curve. We denote by $f = \Phi_{|p^*L-E|} : \tilde{S} \to B = \mathbb{P}^1$ the morphism associated to the linear system $|p^*L - E|$.

Since the multiplication map $S^3(H^0(\mathcal{O}_S(L))) \to H^0(\mathcal{O}_S(3L))$ is surjective, the canonical map $\Phi_{|K|} : S - - \to \mathbb{P}^3$ is also composite with the pencil $\mathcal{P}$. Thus we are in Case (ii) of [10, Proposition 1.7]. In particular, any general member of $|L|$ is non-hyperelliptic, and all the fibers of $f : \tilde{S} \to B$ are 2-connected. Therefore, we can utilize the structure theorem given in [8] for 2-connected non-hyperelliptic fibrations of genus 3.

In what follows, we put $\tilde{L} = p^*L - E$ and $\tilde{K} = K_{\tilde{S}} = p^*(3L) + E$, and denote by $\omega_{S|B} = \mathcal{O}_S(\tilde{K} - f^*K_B)$ the relative canonical sheaf of the fibration $f : \tilde{S} \to B$. Moreover we denote by $V_n = f_*(\omega_{\tilde{S}|B}^{\otimes n})$ the direct image by f of the sheaf $\omega_{\tilde{S}|B}^{\otimes n}$. Recall that for any integer $n \geq 2$ we have

$$\operatorname{rk} V_n = 4n - 2, \qquad \deg V_n = 7 + 12n(n-1).$$

The latter equality on $\deg V_n$ is valid also for $n = 1$, but for the former equality on $\operatorname{rk} V_n$, we have instead $\operatorname{rk} V_1 = 3$ for $n = 1$.

Lemma 3.1. *The following hold*:
1) $V_1 \simeq \mathcal{O}_B(1)^{\oplus 2} \oplus \mathcal{O}_B(5)$,
2) $V_2 \simeq \left(\bigoplus_{k=2}^{4} \mathcal{O}_B(k)\right) \oplus \mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(10)$,
3) $V_4 \simeq \left(\bigoplus_{k=4}^{14} \mathcal{O}_B(k)\right) \oplus \mathcal{O}_B(16)^{\oplus 2} \oplus \mathcal{O}_B(20)$.

Proof. Recall that we have $\operatorname{rk} V_1 = 3$ and $\deg V_1 = 7$. Thus we can put $V_1 \simeq \bigoplus_{i=0}^{2} \mathcal{O}_B(a_i)$, where $a_0 \leq a_1 \leq a_2$ and $\sum_{i=0}^{2} a_i = 7$. Moreover we have $\omega_{\tilde{S}|B} \simeq \mathcal{O}_{\tilde{S}}(\tilde{K} - f^*K_B) \simeq \mathcal{O}_{\tilde{S}}(5\tilde{L} + 4E)$. Thus we obtain

$$h^0(V_1 \otimes \mathcal{O}_B(-k)) = h^0(\mathcal{O}_{\tilde{S}}((5-k)\tilde{L} + 4E)) = h^0(\mathcal{O}_S((5-k)L))$$

for any $k \geq 1$, from which we infer $h^0(V_1 \otimes \mathcal{O}_B(-1))) - h^0(V_1 \otimes \mathcal{O}_B(-2))) = 3$. This implies $a_i \geq 1$ for all $0 \leq i \leq 2$. Since $h^0(V_1 \otimes \mathcal{O}_B(-k))) - h^0(V_1 \otimes \mathcal{O}_B(-(k+1)))$ is equal to the numbers of i's satisfying $a_i \geq k$, using Lemma 2.2, we obtain the assertion 1). (See also the proof of [21, Lemma 3.7.].)

The assertions 2) and 3) can be proved exactly in the same way. For these two, use $\omega_{\tilde{S}|B}^{\otimes 2} \simeq \mathcal{O}_{\tilde{S}}(10\tilde{L} + 8E)$ and $\omega_{\tilde{S}|B}^{\otimes 4} \simeq \mathcal{O}_{\tilde{S}}(20\tilde{L} + 16E)$. □

In what follows, we denote by X_0, X_1, and X_2 local bases of the direct summands $\mathcal{O}_B(1)$, $\mathcal{O}_B(1)$, and $\mathcal{O}_B(5)$, respectively, of the sheaf V_1. We also denote by S_0, S_1, S_2, T_0, T_1, and U_0 local bases of the direct summands $\mathcal{O}_B(2)$, $\mathcal{O}_B(3)$, $\mathcal{O}_B(4)$, $\mathcal{O}_B(6)$, $\mathcal{O}_B(6)$, and $\mathcal{O}_B(10)$, respectively, of the sheaf V_2. By Lemma 3.1 we have

$$S^2(V_1) \simeq \mathcal{O}_B(2)^{\oplus 3} \oplus \mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(10),$$

where the local bases of the direct summands are given by X_0^2, X_0X_1, X_1^2, X_0X_2, X_1X_2, and X_2^2, respectively. With these local bases, the multiplication morphism $\sigma_2 : S^2(V_1) \to V_2$ is expressed by a 6×6 matrix A in the following form:

$$A = \begin{pmatrix} A' & O_3 \\ * & I_3 \end{pmatrix}, \quad \text{where} \quad A' = \begin{pmatrix} a_0 & a_1 & a_2 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \end{pmatrix}. \tag{4}$$

Here O_3 and I_3 denote the 3×3 zero matrix and the 3×3 identity matrix, respectively, and $a_i \in H^0(\mathcal{O}_B)$, $\alpha_j \in H^0(\mathcal{O}_B(1))$, and $\beta_k \in H^0(\mathcal{O}_B(2))$ are global sections for each $0 \leq i,\ j,\ k \leq 2$.

Let us describe the 5-tuple for our genus 3 fibration $f : \tilde{S} \to B$. For the notion of the 5-tuple, see [8]. Let τ be the effective divisor of degree $\deg \tau = 3$ on B determined by the short exact sequence

$$0 \to S^2(V_1) \to V_2 \to \mathcal{O}_\tau \to 0. \tag{5}$$

Let $\mathcal{C} : S^2(\bigwedge^2 V_1) \to S^2(S^2(V_1))$ be the morphism given by $(a \wedge b)(c \wedge d) \mapsto (ac)(bd) - (ad)(bc)$. Then the morphism $S^2(\sigma_2) \circ \mathcal{C} : S^2(\bigwedge^2 V_1) \to S^2(V_2)$ has a locally free cokernel of rank 15, which we shall denote by $\tilde{V}_4 = \mathrm{Cok}\,(S^2(\sigma_2) \circ \mathcal{C})$. We denote by $\mathcal{L}_4'$ and $\mathcal{L}_4$ the kernel of the natural surjection $\tilde{V}_4 \to V_4$ and that of the natural morphism $S^4(V_1) \to V_4$, respectively. Then we obtain the natural inclusion morphism

$$\mathcal{L}_4' \simeq (\det V_1) \otimes \mathcal{O}_B(\tau) \simeq \mathcal{O}_B(10) \to \tilde{V}_4. \tag{6}$$

With the notation above, B, V_1, τ, (5), and (6) form the admissible 5-tuple associated to our fibration $f : \tilde{S} \to B$.

By Lemma 3.1 we have $\bigwedge^2 V_1 \simeq \mathcal{O}_B(2) \oplus \mathcal{O}_B(6)^{\oplus 2}$ and $S^2(\bigwedge^2 V_1) \simeq \mathcal{O}_B(4) \oplus \mathcal{O}_B(8)^{\oplus 2} \oplus \mathcal{O}_B(12)^{\oplus 3}$. We decompose each of the five sheaves $S^2(\bigwedge^2 V_1)$, $S^2(V_1)$, $S^2(S^2(V_1))$, V_2, and $S^2(V_2)$ into the lower degree part (L) and the higher degree part (H) as follows:

$$\begin{aligned} S^2(\bigwedge^2 V_1) &= [\mathcal{O}_B(4)] \oplus \left[\mathcal{O}_B(8)^{\oplus 2} \oplus \mathcal{O}_B(12)^{\oplus 3}\right] \\ &= S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \oplus S^2(\bigwedge^2 V_1)^{(\mathrm{H})}, \\ S^2(V_1) &= \left[\mathcal{O}_B(2)^{\oplus 3}\right] \oplus \left[\mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(10)\right] \\ &= S^2(V_1)^{(\mathrm{L})} \oplus S^2(V_1)^{(\mathrm{H})}, \\ S^2(S^2(V_1)) &= \left[S^2(S^2(V_1)^{(\mathrm{L})})\right] \oplus \left[\left(S^2(V_1)^{(\mathrm{L})} \otimes S^2(V_1)^{(\mathrm{H})}\right) \oplus S^2(S^2(V_1)^{(\mathrm{H})})\right] \end{aligned}$$

$$
\begin{aligned}
&= S^2(S^2(V_1))^{(\mathrm{L})} \oplus S^2(S^2(V_1))^{(\mathrm{H})},\\
V_2 &= \left[\bigoplus_{k=2}^{4} \mathcal{O}_B(k)\right] \oplus \left[\mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(10)\right]\\
&= V_2^{(\mathrm{L})} \oplus V_2^{(\mathrm{H})},\\
S^2(V_2) &= \left[S^2(V_2^{(\mathrm{L})})\right] \oplus \left[\left(V_2^{(\mathrm{L})} \otimes V_2^{(\mathrm{H})}\right) \oplus S^2(V_2^{(\mathrm{H})})\right]\\
&= S^2(V_2)^{(\mathrm{L})} \oplus S^2(V_2)^{(\mathrm{H})},
\end{aligned}
$$

where in each expression the first [ ] term corresponds to the lower degree part (L), and the second [ ] term corresponds to the higher degree part (H).

Let $\gamma : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \simeq \mathcal{O}_B(4) \to S^2(V_2)^{(\mathrm{L})}$ be the composition of the morphism $\mathcal{C}|_{S^2(\bigwedge^2 V_1)^{(\mathrm{L})}} : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to S^2(S^2(V_1))^{(\mathrm{L})}$ and the morphism $S^2(A') : S^2(S^2(V_1))^{(\mathrm{L})} = S^2(S^2(V_1)^{(\mathrm{L})}) \to S^2(V_2)^{(\mathrm{L})} = S^2(V_2^{(\mathrm{L})})$, where A' is the 3×3 matrix given in (4).

Lemma 3.2. $\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{L}'_4, \mathrm{Cok}\,\gamma) \neq \{0\}$.

Proof. Note that by (4) we have $(S^2(\sigma_2) \circ \mathcal{C})(S^2(\bigwedge^2 V_1)^{(\mathrm{H})}) \subset S^2(V_2)^{(\mathrm{H})}$. Thus $(S^2(\sigma_2) \circ \mathcal{C}) : S^2(\bigwedge^2 V_1) \to S^2(V_2)$ induces a morphism of $\mathcal{O}_B$-modules

$$
\gamma' : \frac{S^2(\bigwedge^2 V_1)}{S^2(\bigwedge^2 V_1)^{(\mathrm{H})}} \simeq S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \frac{S^2(V_2)}{S^2(V_2)^{(\mathrm{H})}} \simeq S^2(V_2)^{(\mathrm{L})}.
$$

Our morphism γ coincides with this γ', when we view γ' as a morphism from $S^2(\bigwedge^2 V_1)^{(\mathrm{L})}$ to $S^2(V_2)^{(\mathrm{L})}$. Thus by the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S^2(\bigwedge^2 V_1)^{(\mathrm{H})} & \longrightarrow & S^2(V_2)^{(\mathrm{H})} & \longrightarrow & \frac{S^2(V_2)^{(\mathrm{H})}}{S^2(\bigwedge^2 V_1)^{(\mathrm{H})}} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^2(\bigwedge^2 V_1) & \xrightarrow{S^2(\sigma_2)\circ\mathcal{C}} & S^2(V_2) & \longrightarrow & \tilde{V}_4 & \longrightarrow & 0
\end{array}
\tag{7}
$$

and 3×3 Lemma, we obtain the following two short exact sequences:

$$
\begin{aligned}
0 \to \frac{S^2(\bigwedge^2 V_1)}{S^2(\bigwedge^2 V_1)^{(\mathrm{H})}} \to \frac{S^2(V_2)}{S^2(V_2)^{(\mathrm{H})}} \to \mathrm{Cok}\,\gamma' \simeq \mathrm{Cok}\,\gamma \to 0,\\
0 \to \frac{S^2(V_2)^{(\mathrm{H})}}{S^2(\bigwedge^2 V_1)^{(\mathrm{H})}} \to \tilde{V}_4 \to \mathrm{Cok}\,\gamma' \simeq \mathrm{Cok}\,\gamma \to 0.
\end{aligned}
\tag{8}
$$

Now, assume that we have $\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{L}'_4, \mathrm{Cok}\,\gamma) = \{0\}$. Then by the short exact sequence (8) above, we obtain the surjectivity of the morphism $\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{L}'_4, \frac{S^2(V_2)^{(\mathrm{H})}}{S^2(\bigwedge^2 V_1)^{(\mathrm{H})}}) \to \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{L}'_4, \tilde{V}_4)$. This implies that the morphism $\frac{S^2(V_2)^{(\mathrm{H})}}{S^2(\bigwedge^2 V_1)^{(\mathrm{H})}} \to \tilde{V}_4$ in (7) factors through the inclusion morphism (6). On the other hand, however, since $\sigma_2|_{S_2(V_1)^{(\mathrm{H})}} : S^2(V_1)^{(\mathrm{H})} \to V_2^{(\mathrm{H})}$ is an isomorphism by (4), we have also the surjectivity of the morphism $V_2 \otimes S^2(V_1)^{(\mathrm{H})} \to S^2(V_2)^{(\mathrm{H})} = V_2 \cdot V_2^{(\mathrm{H})}$. Then with the help of the commutative diagram (7), we find immediately a contradiction to the definition of an admissible 5-tuple. (See [8, Condition (iv), Definition 7.10].) Thus $\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{L}'_4, \mathrm{Cok}\,\gamma) = \{0\}$ is impossible. □

Note that by Lemma 3.1 we have

$$S^2(V_2)^{(\mathrm{L})} \simeq \mathcal{O}_B(4) \oplus \mathcal{O}_B(5) \oplus \mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(7) \oplus \mathcal{O}_B(8).$$

Local bases of the direct summands are given by S_0^2, S_0S_1, S_1^2, S_0S_2, S_1S_2, and S_2^2, respectively. In what follows, we shall compute the sheaf $\mathrm{Cok}\,\gamma$, and rule out the case where $\Phi_{|2L|}$ is composite with a pencil. For this we divide our argument into several cases, normalizing the matrix A'.

First, by replacing the bases X_0 and X_1 of the sheaf V_1, we may assume $a_1 = 1$. Then by replacing the bases S_0, S_1, and S_2 of the sheaf V_2, we may assume $\alpha_1 = 0$ and $\beta_1 = 0$. Then we obtain

$$A' = \begin{pmatrix} a_0 & 1 & a_2 \\ \alpha_0 & 0 & \alpha_2 \\ \beta_0 & 0 & \beta_2 \end{pmatrix}.$$

We have two cases:

Case 1: $a_0a_2 \neq 1$;

Case 2: $a_0a_2 = 1$.

Lemma 3.3. *Case* 1 *does not occur.*

Proof. The composite of the morphism γ and the natural projection $S^2(V_2)^{(\mathrm{L})} \to \mathcal{O}_B(4)$ coincides with $(a_0a_2 - 1)\times : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \simeq \mathcal{O}_B(4) \to \mathcal{O}_B(4)$. Thus if we are in Case 1, then the image $\mathrm{Im}\,\gamma$ is a direct summand of $S^2(V_2)^{(\mathrm{L})}$. Thus we obtain

$$\mathrm{Cok}\,\gamma \simeq \mathcal{O}_B(5) \oplus \mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(7) \oplus \mathcal{O}_B(8),$$

which contradicts Lemma 3.2. □

Let us study Case 2. In this case, the composite of the morphism γ and the natural projection $S^2(V_2)^{(\mathrm{L})} \to \mathcal{O}_B(4)$ is a zero morphism. Thus γ is a composite of a morphism

$$\gamma_0 : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \simeq \mathcal{O}_B(4) \to \mathcal{F}_0 = \mathcal{O}_B(5) \oplus \mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(7) \oplus \mathcal{O}_B(8)$$

and the natural inclusion $\mathcal{F}_0 \to S^2(V_2)^{(\mathrm{L})}$, and we find $\mathrm{Cok}\,\gamma \simeq \mathcal{O}_B(4) \oplus \mathrm{Cok}\,\gamma_0$. By replacing the bases X_0 and X_1 by their multiples by non-zero constants, we may assume $a_0 = a_2 = 1$. Then by the short exact sequence

$$0 \to S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{F}_0 \to \mathrm{Cok}\,\gamma_0 \to 0, \tag{9}$$

we obtain

$$h^0(\text{Cok}\,\gamma_0 \otimes \mathcal{O}_B(-6)) = 8$$
$$h^0(\text{Cok}\,\gamma_0 \otimes \mathcal{O}_B(-7)) = 3 + \dim \text{Ker}\,((\alpha_0 + \alpha_2)\times), \tag{10}$$

where $(\alpha_0 + \alpha_2)\times : H^1(\mathcal{O}_B(-3)) \to H^1(\mathcal{O}_B(-2))$ is the morphism induced by the multiplication morphism by $\alpha_0 + \alpha_2$ of sheaves.

Case 2 splits into two cases:

Case 2–1: $\alpha_0 + \alpha_2 \neq 0 \in H^0(\mathcal{O}_B(1))$;

Case 2–2: $\alpha_0 + \alpha_2 = 0 \in H^0(\mathcal{O}_B(1))$.

Case 2–1. Let us study Case 2–1. In this case the morphism $(\alpha_0 + \alpha_2)\times : H^1(\mathcal{O}_B(-3)) \to H^1(\mathcal{O}_B(-2))$ is surjective. Therefore by (9) and (10) we obtain

$$h^0(\text{Cok}\,\gamma_0 \otimes \mathcal{O}_B(-7)) = 4$$
$$h^0(\text{Cok}\,\gamma_0 \otimes \mathcal{O}_B(-8)) = 1 + \dim \text{Ker}\,({}^t(\alpha_0 + \alpha_2,\ \alpha_0\alpha_2,\ \beta_0 + \beta_2)\times), \tag{11}$$

where ${}^t(\alpha_0+\alpha_2,\ \alpha_0\alpha_2,\ \beta_0+\beta_2)\times : H^1(\mathcal{O}_B(-4)) \to H^1(\mathcal{O}_B(-3)\oplus\mathcal{O}_B(-2)^{\oplus 2})$ is the morphism induced by the multiplication morphism by ${}^t(\alpha_0+\alpha_2,\ \alpha_0\alpha_2,\ \beta_0+\beta_2)$ of sheaves.

Case 2–1 splits into two cases:

Case 2–1–1: $\alpha_0 + \alpha_2$, $\alpha_0\alpha_2$, and $\beta_0 + \beta_2$ have no common zero;

Case 2–1–2: $\alpha_0 + \alpha_2$, $\alpha_0\alpha_2$, and $\beta_0 + \beta_2$ have a common zero.

Lemma 3.4. *Case* 2–1–1 *does not occur.*

Proof. Assume that we are in Case 2–1–1. Let us denote by $\gamma_0^{(1)} : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \simeq \mathcal{O}_B(4) \to \mathcal{O}_B(5)\oplus\mathcal{O}_B(6)^{\oplus 2}$ the multiplication morphism by ${}^t(\alpha_0+\alpha_2,\ \alpha_0\alpha_2,\ \beta_0+\beta_2)$. Then both Cok γ_0 and Cok $\gamma_0^{(1)}$ are locally free, and we have rk Cok $\gamma_0 = 4$ and rk Cok $\gamma_0^{(1)} = 2$. Put Cok $\gamma_0 \simeq \bigoplus_{i=0}^3 \mathcal{O}_B(b_i)$ where $b_0 \leq b_1 \leq b_2$. Then by the short exact sequence

$$0 \to S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{O}_B(5) \oplus \mathcal{O}_B(6)^{\oplus 2} \to \text{Cok}\,\gamma_0^{(1)} \to 0,$$

we obtain $h^0(\text{Cok}\,\gamma_0^{(1)} \otimes \mathcal{O}_B(-6)) = 3$ and $h^0(\text{Cok}\,\gamma_0^{(1)} \otimes \mathcal{O}_B(-7)) = 1$. From these together with $\deg \text{Cok}\,\gamma_0^{(1)} = 13$, we infer Cok $\gamma_0^{(1)} \simeq \mathcal{O}_B(6) \oplus \mathcal{O}_B(7)$, which in tern together with (11) implies $h^0(\text{Cok}\,\gamma_0 \otimes \mathcal{O}_B(-8))) = 1$. This together with (10) and (11) implies $b_0 = 6$ and $b_i \geq 7$ for all $1 \leq i \leq 3$. Then since $\deg \text{Cok}\,\gamma_0 = 28$, we obtain

$$\text{Cok}\,\gamma \simeq \mathcal{O}_B(4) \oplus \text{Cok}\,\gamma_0 \simeq \mathcal{O}_B(4) \oplus \mathcal{O}_B(6) \oplus \mathcal{O}_B(7)^{\oplus 2} \oplus \mathcal{O}_B(8),$$

which contradicts Lemma 3.2. □

Lemma 3.5. *Case* 2–1–2 *does not occur.*

Proof. Assume that we are in Case 2–1–2. Without loss of generality we may assume $\alpha_0 \neq 0 \in H^0(\mathcal{O}_B(1))$. Since the three sections $\alpha_0 + \alpha_2$, $\alpha_0\alpha_2$, and $\beta_0 + \beta_2$ have a common zero, there exist a number $a \in \mathbb{C}$ and a section $\lambda \in H^0(\mathcal{O}_B(1))$ such that $\alpha_0 + \alpha_2 = (1+a)\alpha_0$ and $\beta_0 + \beta_2 = \lambda\alpha_0$ hold. Note that we have $1+a \neq 0$, since we are in Case 2–1. Since $\beta_0\beta_2 = \beta_0(\lambda\alpha_0 - \beta_0)$, if the two sections $\alpha_0 + \alpha_2$ and $\beta_0\beta_2$ have a common zero P, then at this point P, the rank of $\sigma_2 \otimes k(P)$ drops at least by 2, which is impossible. (See [8].) Thus $\operatorname{Cok}\gamma_0$ is locally free.

Put $\operatorname{Cok}\gamma_0 \simeq \bigoplus_{i=0}^{3} \mathcal{O}_B(b_i)$, where $b_0 \leq b_1 \leq b_2 \leq b_3$. Let us denote by $\gamma_0^{(1)} : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \simeq \mathcal{O}_B(4) \to \mathcal{O}_B(5) \oplus \mathcal{O}_B(6)^{\oplus 2}$ the multiplication morphism by ${}^t(\alpha_0 + \alpha_2,\ \alpha_0\alpha_2,\ \beta_0 + \beta_2) = {}^t((1+a)\alpha_0,\ a\alpha_0^2,\ \lambda\alpha_0)$. Then denoting by $\gamma_0^{(2)} : \mathcal{O}_B(5) \to \mathcal{O}_B(5) \oplus \mathcal{O}_B(6)^{\oplus 2}$ the multiplication morphism by ${}^t((1+a),\ a\alpha_0,\ \lambda)$, we have $\gamma_0^{(1)} = \gamma_0^{(2)} \circ (\alpha_0\times)$, where $\alpha_0\times : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{O}_B(5)$ is the multiplication morphism by α_0. From this we see easily that the morphism $\gamma_0^{(1)} \otimes \mathcal{O}_B(-8) : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \otimes \mathcal{O}_B(-8) \to \mathcal{O}_B(-3) \oplus \mathcal{O}_B(-2)^{\oplus 2}$ induces a morphism $H^1(S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \otimes \mathcal{O}_B(-8)) \to H^1(\mathcal{O}_B(-3) \oplus \mathcal{O}_B(-2)^{\oplus 2})$ of rank 4. Thus we obtain $h^0(\operatorname{Cok}\gamma_0 \otimes \mathcal{O}_B(-8)) = 2$, which together with (10) and (11) implies $b_0 = b_1 = 6$. Then since $\deg \operatorname{Cok}\gamma_0 = 28$, we obtain

$$\operatorname{Cok}\gamma \simeq \mathcal{O}_B(4) \oplus \operatorname{Cok}\gamma_0 \simeq \mathcal{O}_B(4) \oplus \mathcal{O}_B(6)^2 \oplus \mathcal{O}_B(b_2) \oplus \mathcal{O}_B(b_3),$$

where $(b_2, b_3) = (8, 8)$ or $(7, 9)$, which contradicts Lemma 3.2. □

Case 2–2. Let us study Case 2–2. In this case we have $\alpha_2 = -\alpha_0 \neq 0 \in H^0(\mathcal{O}_B(1))$. Moreover, γ_0 is a composite of a morphism

$$\gamma_1 : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \simeq \mathcal{O}_B(4) \to \mathcal{F}_1 = \mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(7) \oplus \mathcal{O}_B(8)$$

and the natural inclusion $\mathcal{F}_1 \to \mathcal{F}_0$, and we find $\operatorname{Cok}\gamma_0 \simeq \mathcal{O}_B(5) \oplus \operatorname{Cok}\gamma_1$.

Case 2–2 splits into two cases:

Case 2–2–1: $\alpha_0\alpha_2 = -\alpha_0^2$ and $\beta_0 + \beta_2$ have no common zero;

Case 2–2–2: $\alpha_0\alpha_2 = -\alpha_0^2$ and $\beta_0 + \beta_2$ have a common zero.

Lemma 3.6. *Case* 2–2–1 *does not occur.*

Proof. Assume that we are in Case 2–2–1. Then the sheaf $\operatorname{Cok}\gamma_1$ is locally free of rank 3. Put $\operatorname{Cok}\gamma_1 \simeq \bigoplus_{i=1}^{3} \mathcal{O}_B(b_i)$, where $b_1 \leq b_2 \leq b_3$. Then since the multiplication morphism ${}^t(\alpha_0\alpha_2,\ \beta_0 + \beta_0)\times : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{O}_B(6)^{\oplus 2}$ by ${}^t(\alpha_0\alpha_2,\ \beta_0 + \beta_0)$ has a cokernel isomorphic to $\mathcal{O}_B(8)$, we obtain by the short exact sequence

$$0 \to S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{F}_1 \to \operatorname{Cok}\gamma_1 \to 0$$

that $h^0(\operatorname{Cok}\gamma_1 \otimes \mathcal{O}_B(-7)) = 5$ and $h^0(\operatorname{Cok}\gamma_1 \otimes \mathcal{O}_B(-8)) = 2$, which imply $b_i \geq 7$ for all $1 \leq i \leq 3$. Since $\deg \operatorname{Cok}\gamma_1 = 23$, we obtain

$$\operatorname{Cok}\gamma \simeq \mathcal{O}_B(4) \oplus \mathcal{O}_B(5) \oplus \mathcal{O}_B(b_1) \oplus \mathcal{O}_B(b_2) \oplus \mathcal{O}_B(b_3),$$

where $(b_1, b_2, b_3) = (7, 7, 9)$ or $(7, 8, 8)$, which contradicts Lemma 3.2. □

Let us study Case 2–2–2. In this case there exists a section $\lambda \in H^0(\mathcal{O}_B(1))$ such that $\beta_0 + \beta_2 = \lambda\alpha_0$. Then since $\beta_0\beta_2 = \beta_0(\lambda\alpha_0 - \beta_0)$ holds, if the two sections $-\alpha_0^2$ and $\beta_0\beta_2$ have a common zero P, then at this point P, the rank of $\sigma_2 \otimes k(P)$ drops at least by 2, which is impossible. Thus Cok γ_1 is locally free of rank 3. Put Cok $\gamma_1 \simeq \bigoplus_{i=1}^3 \mathcal{O}_B(b_i)$, where $b_1 \le b_2 \le b_3$. Then by the same short exact sequence as in the proof of Lemma 3.6, we obtain

$$h^0(\operatorname{Cok}\gamma_1 \otimes \mathcal{O}_B(-6)) = 8, \qquad h^0(\operatorname{Cok}\gamma_1 \otimes \mathcal{O}_B(-7)) = 5. \tag{12}$$

Case 2–2–2 splits into two cases:
Case 2–2–2–1: α_0 and λ have no common zero;
Case 2–2–2–2: α_0 and λ have a common zero.

Lemma 3.7. *Case* 2–2–2–1 *does not occur.*

Proof. Assume that we are in Case 2–2–2–1. Then since the multiplication morphism ${}^t(\alpha_0\alpha_2,\ \beta_0 + \beta_2)\times : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{O}_B(6)^{\oplus 2}$ is the composite of the two morphisms $\alpha_0 \times : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{O}_B(5)$ and ${}^t(-\alpha_0,\ \lambda)\times : \mathcal{O}_B(5) \to \mathcal{O}_B(6)^{\oplus 2}$, we see by the same short exact sequence as in the proof of Lemma 3.6 that $h^0(\operatorname{Cok}\gamma_1 \otimes \mathcal{O}_B(-8)) = 2$, which together with (12) implies $b_i \ge 7$ for all $1 \le i \le 3$. Since $\deg \operatorname{Cok}\gamma_1 = 23$, we obtain

$$\operatorname{Cok}\gamma \simeq \mathcal{O}_B(4) \oplus \mathcal{O}_B(5) \oplus \mathcal{O}_B(b_1) \oplus \mathcal{O}_B(b_2) \oplus \mathcal{O}_B(b_3),$$

where $(b_1, b_2, b_3) = (7, 7, 9)$ or $(7, 8, 8)$, which contradicts Lemma 3.2. □

Lemma 3.8. *Case* 2–2–2–2 *does not occur.*

Proof. Assume that we are in Case 2–2–2–2. Then there exists a number $c \in \mathbb{C}$ such that $\lambda = c\alpha_0$. If we have $c = 0$, then we obtain $\beta_2 = -\beta_0$. This however is impossible since $\sigma_2 \otimes k(P)$ needs to have rank 6 at a general point P of B. Thus we obtain $c \ne 0$. Note that the multiplication morphism ${}^t(-\alpha_0^2,\ \beta_0 + \beta_2)\times : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{O}_B(6)^{\oplus 2}$ is the composite of the two morphisms $\alpha_0^2 \times : S^2(\bigwedge^2 V_1)^{(\mathrm{L})} \to \mathcal{O}_B(6)$ and ${}^t(-1,\ c)\times : \mathcal{O}_B(6) \to \mathcal{O}_B(6)^{\oplus 2}$. Since we have Cok $({}^t(-1,\ c)\times) \simeq \mathcal{O}_B(6)$, we see by the same short exact sequence as in the proof of Lemma 3.6 that

$$h^0(\operatorname{Cok}\gamma_1 \otimes \mathcal{O}_B(-8)) = 3. \tag{13}$$

Since $\alpha_0\beta_2 + \alpha_2\beta_0 = \alpha_0(c\alpha_0^2 - 2\beta_0)$, if the two sections α_0 and $c\alpha_0^2 - 2\beta_0$ have a common zero P, then at this point P, the rank of $\sigma_2 \otimes k(P)$ drops at least by 2, which is impossible. Thus Cok $\gamma_1^{(1)}$ is locally free of rank 2, where we denote by

$$\gamma_1^{(1)} : \mathcal{O}_B(5) \to \mathcal{O}_B(6)^{\oplus 2} \oplus \mathcal{O}_B(7) \tag{14}$$

the multiplication morphism by ${}^t(-\alpha_0,\ c\alpha_0,\ c\alpha_0^2 - 2\beta_0)$. Since the multiplication morphism ${}^t(-\alpha_0,\ c\alpha_0)\times : \mathcal{O}_B(5) \to \mathcal{O}_B(6)^{\oplus 2}$ is the composite of the two morphisms $\alpha_0 \times : \mathcal{O}_B(5) \to \mathcal{O}_B(6)$ and ${}^t(-1,\ c)\times : \mathcal{O}_B(6) \to \mathcal{O}_B(6)^{\oplus 2}$, we obtain by (14) that $h^0(\operatorname{Cok}\gamma_1^{(1)} \otimes \mathcal{O}_B(-6)) = 4$, $h^0(\operatorname{Cok}\gamma_1^{(1)} \otimes \mathcal{O}_B(-7)) = 2$, and $h^0(\mathrm{Cok}\,\gamma_1^{(1)} \otimes \mathcal{O}_B(-8)) = 1$. From these together with $\deg \mathrm{Cok}\,\gamma_1^{(1)} = 14$, we infer $\mathrm{Cok}\,\gamma_1^{(1)} \simeq \mathcal{O}_B(6) \oplus \mathcal{O}_B(8)$. Thus by the same short exact sequence as in the proof of Lemma 3.6, we see that $h^0(\mathrm{Cok}\,\gamma_1 \otimes \mathcal{O}_B(-9)) = 1$, which together with (12) and (13) implies that $b_1 = 6$ and $b_i \geq 8$ for all $2 \leq i \leq 3$. Since $\deg \mathrm{Cok}\,\gamma_1 = 23$, we obtain

$$\mathrm{Cok}\,\gamma \simeq \mathcal{O}_B(4) \oplus \mathcal{O}_B(5) \oplus \mathcal{O}_B(6) \oplus \mathcal{O}_B(8) \oplus \mathcal{O}_B(9),$$

which contradicts Lemma 3.2. □

By Lemmas 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, and 3.8, we obtain the following:

Proposition 1. *The map $\Phi_{|2L|}$ is not composite with a pencil.*

Digression. As we have already cited the result in our proof, Ciliberto–Francia–Mendes Lopes [10, Proposition 1.7] shows that if a minimal surface S of general type with $4 \leq p_g$ and $K^2 \leq 9$ has canonical map composite with a pencil $\mathcal{P}$, then either (i) $\mathcal{P}$ is a pencil of curves of genus 2, or (ii) $\mathcal{P}$ is a rational pencil of non-hyperelliptic curves of genus 3. Their proposition moreover shows that in the latter case $K^2 = 9$, $p_g = 4$, and $K \sim 3C$ hold, where C is a general member of the pencil $\mathcal{P}$. As a byproduct of our computation, we prove the following:

Proposition 2. *Case (ii) in [10, Proposition 1.7] never occurs. Thus if a minimal surface S with $4 \leq p_g$ and $K^2 \leq 9$ has canonical map composite with a pencil $\mathcal{P}$, then $\mathcal{P}$ is a pencil of curves of genus* 2.

Proof. Assume that the surface S has numerical invariants as in the assertion, and the canonical map $\Phi_{|K|}$ is composite with a pencil $\mathcal{P}$ of curves of genus 3. Then we have $K^2 = 9$, $p_g = 4$, and $K \sim 3C$, where C is a general member of the pencil $\mathcal{P}$, which is non-hyperelliptic. If S has irregularity $q = 0$, then it contradicts our Proposition 1. Thus it suffices to rule out the case $q > 0$, where q is the irregularity of our surface S. In what follows we assume $q > 0$.

Note that we have $q \leq 2$. Indeed, since the restriction map $H^0(\mathcal{O}_S(C)) \to H^0(\mathcal{O}_C(C))$ has rank at least 1, the map $H^0(\mathcal{O}_S(4C)) \to H^0(\mathcal{O}_C(4C))$ also has rank at least 1. This together with the short exact sequence

$$0 \to \mathcal{O}_S(3C) = \mathcal{O}_S(K) \to \mathcal{O}_S(4C) \to \mathcal{O}_C(4C) = \omega_C \to 0$$

implies the inequality. Thus we obtain $\chi = \chi(\mathcal{O}_S) \geq 3$. Then by so-called Severi inequality proved by Pardini [20], we see that S is not of Albanese general type. We denote by $\alpha : S \to B$ the Albanese fibration of our surface S. Naturally we have $g(B) = q$, where $g(B)$ is the genus of the base curve B.

First, let us rule out the case $q = 2$. Assume that we have $q = 2$. Then B is a non-singular curve of genus $g(B) = 2$. Since $\mathcal{P}$ has a unique base point x, the restriction $\alpha|_C : C \to B$ is surjective for any general $C \in \mathcal{P}$. Moreover, by Hurwitz formula, we see that $\alpha|_C$ is an étale double cover. This implies $g(F) = 4$ for a general Albanese fiber F, since we have $FC = \deg \alpha|_C = 2$. Let F_0 denote the fiber of α passing through the base point x. Then since $(F_0 \cdot C)_x = 1$, we see that x is a smooth point of F_0 and that for any $y \in F_0 \setminus \{x\}$ there exists a unique member $C_y \in \mathcal{P}$ passing through y. This however contradicts the rationality of our pencil $\mathcal{P}$; the map $y \mapsto C_y$ is birational since $CF_0 = 2$, while F_0 is a non-singular curve of genus 4.

Next, let us rule out the case $q = 1$. We use the method used in the proof of [10, Proposition 2.4]. For the reader's convenience, we include the outline of our proof. Assume that we have $q = 1$. Then B is an elliptic curve. We take a point $o \in B$, and use this point for the zero of the additive structure of the elliptic curve B. For any closed point $b \in B$, we put $\xi_b = \mathcal{O}_B(b - o)$.

We first claim $h^0(\mathcal{O}_C(C) \otimes \alpha^*\xi_b) = 1$ for any $b \neq o \in B$. Indeed, assume otherwise. Then by the same method as in [10, Proposition 2.4] and the upper-semicontinuity, we see that $h^0(\mathcal{O}_S(C) \otimes \alpha^*\xi_{b'}) = 0$ holds for any general $b' \in B$. Then again by the same method as in [10, Proposition 2.4], we obtain $h^0(\mathcal{O}_C(2C) \otimes \alpha^*\xi_{b'}^{\vee}) \geq 3$ for any general $b' \neq o \in C$. We however have $g(C) = 3$ and $\deg \mathcal{O}_C(2C) \otimes \alpha^*\xi_{b'}^{\vee} = 2$. Thus this is impossible, since the curve $\Phi_{|\mathcal{O}_C(2C)\otimes\alpha^*\xi_{b'}^{\vee}|}(C)$ needs to be non-degenerate.

Now put $U = B \setminus \{o\}$, and denote by $\pi : C \times U \to U$ the second projection. Let $\varXi$ be an invertible sheaf on $C \times U$ such that $\varXi|_{C\times\{b\}} \simeq \mathcal{O}_C(C) \otimes \alpha^*\xi_b$ holds for all $b \in U$. Then by what we have shown in the preceding paragraph, we see that the direct image $\pi_*\varXi$ is an invertible sheaf on U and that the natural morphism $\pi^*\pi_*\varXi \to \varXi$ is non-trivial. Thus replacing U by smaller one if necessary, we obtain an effective divisor Z on $C \times U$ such that $Z \cap (C \times \{b\}) = \operatorname{div} s_b$ holds for all $b \in U$, where $s_b \neq 0 \in H^0(\mathcal{O}_C(C) \otimes \alpha^*\xi_b)$ is the unique non-zero global section to $\mathcal{O}_C(C) \otimes \alpha^*\xi_b$. Then the restriction $\pi|_Z : Z \to U$ is birational, since $\deg \mathcal{O}_C(C) \otimes \alpha^*\xi_b = 1$. By [10, Proposition 1.6], however, the first projection $Z \to C$ is dominant. Thus this contradicts the inequality $g(C) = 3 > g(B) = 1$. □

4. Structure theorem

Let us go back to the study of our surface S with $c_1^2 = 9$, $\chi = 5$, and $K \sim 3L$. By Proposition 1 and Lemma 2.3, we have $h^0(\mathcal{O}_S(L)) = 1$, $h^0(\mathcal{O}_S(2L)) = 3$, and $2 \leq \deg \Phi_{|2L|} \leq 4$, where $\deg \Phi_{|2L|}$ is the degree of the rational map $\Phi_{|2L|} : S - - \to \mathbb{P}^2$. In this section, we shall give a structure theorem for our surface S, by studying the structure of the graded ring $\bigoplus_{i=0}^{\infty} H^0(\mathcal{O}_S(nL))$.

Lemma 4.1. *Let $|K| = |3L| = |M_3| + F_3$ be the decomposition of the canonical system $|K|$ into the variable part $|M_3|$ and the fixed part F_3. Then $KF_3 = 0$ holds. In particular F_3 is at most a sum of fundamental cycles of rational double points.*

Proof. Let M_3 and F_3 be divisors as above. Let $p_3 : \tilde{S}_3 \to S$ be the shortest composite of quadric transformations such that the variable part $|\tilde{M}_3|$ of $p_3^*|M_3|$ is free from base points. Then we have $M_3^2 \geq \tilde{M}_3^2 \geq 4$ and $K^2 = M_3^2 + M_3F_3 + KF_3$, where $M_3F_3 \geq 0$ and $KF_3 \geq 0$ hold. Since we have $M_3^2 + M_3F_3 = KM_3 \equiv 0 \mod 3$, this implies $M_3^2 + M_3F_3 = 6$ or 9.

Assume that $M_3^2 + M_3F_3 = 6$. Then by $M_3^2 \geq 4$, we have $0 \leq M_3F_3 \leq 2$. From this together with Hodge's Index Theorem $M_3^2F_3^2 = M_3^2(KF_3 - M_3F_3) \leq$

$(M_3F_3)^2$, we see that $M_3F_3 = 2$, $M_3^2 = 4$, and $F_3^2 = 1$, hence $M_3 \sim_{\text{num}} 2F_3$ and $K \sim_{\text{num}} 3L \sim_{\text{num}} 3F_3$. Then by [18, Theorem 4], we obtain $L \sim F_3$ and $M_3 \sim 2L$, which contradicts $h^0(\mathcal{O}_S(2L)) = 3$ in Lemma 2.3, since we have $h^0(\mathcal{O}_S(M_3)) = p_g(S) = 4$. Thus we obtain $M_3^2 + M_3F_3 = 9$, hence the assertion. □

Take a base x_0 of the space of global sections $H^0(\mathcal{O}_S(L))$. The following lemma is trivial.

Lemma 4.2. (1) *There exist two elements y_0, $y_1 \in H^0(\mathcal{O}_S(2L))$ such that x_0^2, y_0, and y_1 form a base of $H^0(\mathcal{O}_S(2L))$.*

(2) *There exists an element $z_0 \in H^0(\mathcal{O}_S(3L))$ such that x_0^3, x_0y_0, x_0y_1, and z_0 form a base of $H^0(\mathcal{O}_S(3L))$.*

Take three elements x_0, y_1, and y_2 as in the lemma above. In what follows. we denote by C the unique member of the linear system $|L|$, and by C_0, its unique irreducible component such that $LC_0 = 1$. For the proof of the following lemma, see [5, Lemma 1.2]:

Lemma 4.3. *If a member $D \in |2L|$ satisfies $D \succeq C_0$, then $D = 2C$.*

Let us study higher homogeneous parts of the ring $\bigoplus_{n=0}^{\infty} H^0(\mathcal{O}_S(nL))$.

Lemma 4.4. *The space $H^0(\mathcal{O}_S(4L))$ has the decomposition $H^0(\mathcal{O}_S(4L)) = x_0H^0(\mathcal{O}_S(3L)) \oplus \bigoplus_{i=0}^{2} \mathbb{C}\, y_0^i y_1^{2-i}$.*

Proof. By the Riemann–Roch theorem, we have $h^0(\mathcal{O}_S(4L)) = 7$. Thus it suffices to prove that seven elements x_0^4, $x_0^2y_0$, $x_0^2y_1$, x_0z_0, y_0^2, y_0y_1, and y_1^2 are linearly independent over $\mathbb{C}$. Assume that these seven elements has a nontrivial linear relation. Then there exist (α_0, α_1) and $(\beta_0, \beta_1) \in \mathbb{C}^2 \setminus \{0\}$ such that $(\alpha_0y_0+\alpha_1y_1)(\beta_0y_0+\beta_1y_1) \in x_0H^0(\mathcal{O}_S(3L))$. This contradicts Lemma 4.3, since x_0^2, y_0, and y_1 are linearly independent. □

Lemma 4.5. *There exists an element $u_0 \in H^0(\mathcal{O}_S(5L))$ such that the equality $H^0(\mathcal{O}_S(5L)) = x_0H^0(\mathcal{O}_S(4L)) \oplus \bigoplus_{i=0}^{1} \mathbb{C}\, y_0^i y_1^{1-i} z_0 \oplus \mathbb{C}\, u_0$ holds.*

Proof. By the Riemann–Roch theorem, we have $h^0(\mathcal{O}_S(5L)) = 10$. Thus it suffices to prove that a base of $x_0H^0(\mathcal{O}_S(4L))$ together with y_0z_0 and y_1z_0 forms a set of linearly independent nine elements of $H^0(\mathcal{O}_S(5L))$. Assume that these nine elements are not linearly independent over $\mathbb{C}$. Then there exists an element $(\alpha_0, \alpha_1) \in \mathbb{C}^2 \setminus \{0\}$ such that $(\alpha_0y_0 + \alpha_1y_1)z_0 \in x_0H^0(\mathcal{O}_S(4L))$. The same argument as in the proof of Lemma 4.4 however shows that $\operatorname{div}(\alpha_0y_0 + \alpha_1y_1) \not\succeq C_0$. Thus we obtain $\operatorname{div} z_0 \succeq C_0$, which contradicts Lemma 4.1. □

Take an element u_0 as in the lemma above. In what follows, we denote by $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ the weighted polynomial ring with $\deg X_0 = 1$, $\deg Y_0 = \deg Y_1 = 2$, $\deg Z_0 = 3$, and $\deg U_0 = 5$.

Lemma 4.6. *There exists a homogeneous element $f_6 \in \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ of degree 6, unique up to multiplication by a non-zero constant, such that the equality $f_6(x_0, y_0, y_1, z_0, u_0) = 0 \in H^0(\mathcal{O}_S(6L))$ holds. The coefficient of Z_0^2 in f_6 is non-vanishing. Therefore the space of global sections to $6L$ decomposes as $H^0(\mathcal{O}_S(6L)) = x_0 H^0(\mathcal{O}_S(5L)) \oplus \bigoplus_{i=0}^{3} \mathbb{C}\, y_0^i y_1^{3-i}$. Moreover, by a proper choice of z_0, the polynomial f_6 can be set in such a way that it includes no term linear with respect to Z_0.*

Proof. The space $H^0(\mathcal{O}_S(6L))$ contains 15 monomials of x_0, y_0, y_1, z_0, and u_0; ten belonging to $x_0 H^0(\mathcal{O}_S(5L))$, four of the form $y_0^i y_1^{3-i}$ $(0 \le i \le 3)$, and z_0^2. Meanwhile we have $h^0(\mathcal{O}_S(6L)) = 14$. Thus there exists at least one non-trivial linear relation $f_6(x_0, y_0, y_1, z_0, u_0) = 0$ among these 15 monomials. Assume that the coefficient of Z_0^2 in f_6 vanishes. Then by Lemma 4.5, there exist three elements $(\alpha_0, \alpha_1), (\beta_0, \beta_1), (\gamma_0, \gamma_1) \in \mathbb{C}^2 \setminus \{0\}$ such that $(\alpha_0 y_0 + \alpha_1 y_1)(\beta_0 y_0 + \beta_1 y_1)(\gamma_0 y_0 + \gamma_1 y_1) \in x_0 H^0(\mathcal{O}_S(5L))$, from which we infer a contradiction by the same argument as in Lemma 4.5. Thus we obtain the non-vanishing of the coefficient of Z_0^2, hence also the uniqueness of f_6. The irreducibility of f_6 follows from Lemmas 4.2, 4.4, and 4.5. □

In what follows, in view of the lemma above, we assume that f_6 includes no term linear with respect to Z_0.

Lemma 4.7. *The linear system $|2L|$ has no base point. Thus the map $\Phi_{|2L|} : S \to \mathbb{P}^2$ associated to $|2L|$ is a morphism of degree 4.*

Proof. Assume that the linear system $|2L|$ has a base point $b \in S$. Then this point b is a common zero of x_0, y_0, and y_1. This together with Lemma 4.6 however implies that b is a base point of $|6L| = |2K|$, which contradicts the base point freeness of the bicanonical system. (See [4, Theorem 2].) □

Now let $\varphi_S : S \to \mathbb{P}(1, 2, 2, 3, 5) = \operatorname{Proj} \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ denote the morphism induced by $X_0 \mapsto x_0$, $Y_i \mapsto y_i$ $(i = 0, 1)$, $Z_0 \mapsto z_0$, and $U_0 \mapsto u_0$.

Lemma 4.8. *The morphism $\varphi_S : S \to \mathbb{P}(1, 2, 2, 3, 5)$ is birational onto its image.*

Proof. Since the morphism φ_S factors through the bicanonical map $\Phi_{|2K|}$, it suffices to prove the birationality of $\Phi_{|2K|} : S \to \mathbb{P}^{13}$. Assume that $\Phi_{|2K|}$ is non-birational. Then by [10, Theorem 1.8, Theorem 2.1], the surface S has a pencil $\mathcal{P}$ of curves of genus 2. Moreover, by their proof, we see that $\mathcal{P}$ can be chosen in such a way that a general member $D \in \mathcal{P}$ satisfies $D^2 = 0$ and $DK = 2$. This however contradicts the equivalence $K \sim 3L$. □

Lemma 4.9. *The space $H^0(\mathcal{O}_S(7L))$ has the decomposition $H^0(\mathcal{O}_S(7L)) = x_0 H^0(\mathcal{O}_S(6L)) \oplus \bigoplus_{i=0}^{1} \mathbb{C}\, y_0^i y_1^{1-i} u_0 \oplus \bigoplus_{i=0}^{2} \mathbb{C}\, y_0^i y_1^{2-i} z_0$.*

Proof. Assume otherwise. Then, since $h^0(\mathcal{O}_S(7L)) = 19$ and $h^0(\mathcal{O}_S(6L)) = 14$, there exists a non-zero homogeneous element $g_7 \in \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ of degree 7 satisfying $g_7(x_0, y_0, y_1, z_0, u_0) = 0 \in H^0(\mathcal{O}_S(7L))$ in which at least one of the five monomials $Y_0 U_0$, $Y_1 U_0$, $Y_0^2 Z_0$, $Y_0 Y_1 Z_0$, and $Y_1^2 Z_0$ has non-vanishing coefficient. By subtracting a multiple of f_6, we may assume that the coefficient of $X_0 Z_0^2$ in g_7 vanishes. Moreover, $Y_0 U_0$ or $Y_1 U_0$ has non-vanishing coefficient in g_7. Indeed, if both $Y_0 U_0$ and $Y_1 U_0$ have vanishing coefficient, then the same argument as in Lemma 4.5 shows that $\operatorname{div} z_0 \succeq C_0$, which contradicts Lemma 4.1.

Now let $\mathcal{Q}_7 \subset \mathbb{P}(1,2,2,3,5)$ be the subvariety defined by $f_6 = g_7 = 0$, and $\pi_{\mathcal{Q}_7} : \mathcal{Q}_7 \dashrightarrow \mathbb{P}(1,2,2)$, the restriction to $\mathcal{Q}_7$ of the natural dominant map $\mathbb{P}(1,2,2,3,5) \dashrightarrow \mathbb{P}(1,2,2)$. Since g_7 is not a multiple of f_6 in $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$, and since f_6 is irreducible, we have $\dim \mathcal{Q}_7 = 2$. Moreover, since Z_0 appears quadratically and U_0 appears at most linearly in f_6, and since U_0 appears linearly and Z_0 appears at most linearly in g_7, we see that $\deg \pi_{\mathcal{Q}_7} \le 2$. Meanwhile, since $\pi_{\mathcal{Q}_7} \circ \varphi_S : S \to \mathbb{P}(1,2,2)$ coincides with $\Phi_{|2L|}$ via the natural isomorphism $\mathbb{P}(1,2,2) \simeq \mathbb{P}^2 = \mathbb{P}(H^0(\mathcal{O}_S(2L)))$, we see by Lemma 4.7 that $\deg \pi_{\mathcal{Q}_7} \circ \varphi_S = 4$. Thus we obtain $\deg \varphi_S \ge 2$, which contradicts Lemma 4.8. □

By the same method, we can prove the following two lemmas:

Lemma 4.10. *The space* $H^0(\mathcal{O}_S(8L))$ *has the decomposition* $H^0(\mathcal{O}_S(8L)) = x_0 H^0(\mathcal{O}_S(7L)) \oplus \mathbb{C}\, z_0 u_0 \oplus \bigoplus_{i=0}^{4} \mathbb{C}\, y_0^i y_1^{4-i}$.

Lemma 4.11. *The space* $H^0(\mathcal{O}_S(9L))$ *has the decomposition* $H^0(\mathcal{O}_S(9L)) = x_0 H^0(\mathcal{O}_S(8L)) \oplus \bigoplus_{i=0}^{2} \mathbb{C}\, y_0^i y_1^{2-i} u_0 \oplus \bigoplus_{i=0}^{3} \mathbb{C}\, y_0^i y_1^{3-i} z_0$.

Indeed, we just need to consider $\mathcal{Q}_k = \{f_6 = g_k = 0\} \subset \mathbb{P}(1,2,2,3,5)$ and $\pi_{\mathcal{Q}_k} : \mathcal{Q}_k \dashrightarrow \mathbb{P}(1,2,2)$ for $k = 8, 9$: we obtain easily $\deg \pi_{\mathcal{Q}_k} \le 3$ for $k = 8$, 9, which leads us to a contradiction to Lemma 4.8.

Corollary 4.1. *The linear system* $|5L|$ *has no base point.*

Proof. Note that by Lemma 4.7 the three sections $x_0^2, y_0, y_1 \in H^0(\mathcal{O}_S(2L))$ have no common zero. Since we have $x_0^5, x_0^2 z_0, y_0 z_0, y_1 z_0, u_0 \in H^0(\mathcal{O}_S(5L))$, this implies that the base locus of $|5L|$ is contained in the subset $\{x_0 = z_0 = u_0 = 0\} \subset S$. Thus, by Lemma 4.11, if the linear system $|5L|$ has a base point $b \in S$, then this point b is also a base point of $|9L| = |3K|$, which contradicts the base point freeness of the tricanonical system. (See [4, Theorem 2].) □

Lemma 4.12. *There exists a homogeneous element* $g_{10} \in \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ *of degree* 10 *not multiple of* f_6 *such that* $g_{10}(x_0, y_0, y_1, z_0, u_0) = 0$ *holds in* $H^0(\mathcal{O}_S(10L))$. *The coefficient of* U_0^2 *in* g_{10} *is non-vanishing. The polynomial* g_{10} *can be chosen in such a way that it includes no monomial divisible by* Z_0^2, *and with this last condition imposed, the polynomial* g_{10} *is unique up to multiplication by a non-zero constant. Moreover the space* $H^0(\mathcal{O}_S(10L))$ *decomposes as* $H^0(\mathcal{O}_S(10L)) = x_0 H^0(\mathcal{O}_S(9L)) \oplus \bigoplus_{i=0}^{1} \mathbb{C}\, y_0^i y_1^{1-i} z_0 u_0 \oplus \bigoplus_{i=0}^{5} \mathbb{C}\, y_0^i y_1^{5-i}$.

Proof. The space $H^0(\mathcal{O}_S(10L))$ includes 41 monomials not divisible by z_0^2 of x_0, y_0, y_1, z_0, and u_0. Since $h^0(\mathcal{O}_S(10L)) = 40$, this implies that there exists at least one homogeneous element $g_{10} \in \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]_{10}$ as in the first assertion. Since g_{10} includes no monomial divisible by Z_0^2, it is not a multiple of f_6. Assume that the coefficient of U_0^2 in g_{10} vanishes. Then by the same argument as in the proof of Lemma 4.5, we see that ether the coefficient of $Y_0Z_0U_0$ or that of $Y_1Z_0U_0$ is non-vanishing. This however together with the same argument as in the proof of Lemma 4.9 leads us to a contradiction to Lemma 4.8. Thus the coefficient of U_0^2 is non-vanishing, from which the last assertion and the uniqueness of g_{10} follow. □

Let $\mathcal{Q}$ denote the subvariety of $\mathbb{P}(1, 2, 2, 3, 5)$ defined by the ideal (f_6, g_{10}). We define the subvarieties $\mathcal{Z}_0$, $\mathcal{Z}_1$, and $\mathcal{Z}_2$ of $\mathbb{P}(1, 2, 2, 3, 5)$ by

$$\begin{aligned}\mathcal{Z}_0 &= \{X_0 = Z_0 = U_0 = 0\},\\ \mathcal{Z}_1 &= \{X_0 = Y_0 = Y_1 = U_0 = 0\},\\ \mathcal{Z}_2 &= \{X_0 = Y_0 = Y_1 = Z_0 = 0\}.\end{aligned}$$

Note that outside $\bigcup_{i=0}^{2} \mathcal{Z}_i$ the weighted projective space $\mathbb{P}(1, 2, 2, 3, 5)$ has no singularity. The restriction of $\mathcal{O}(1)$ to $\mathbb{P}(1, 2, 2, 3, 5) \setminus \bigcup_{i=0}^{2} \mathcal{Z}_i$ is invertible.

Proposition 3. (1) *The morphism $\varphi_S : S \to \mathbb{P}(1, 2, 2, 3, 5)$ surjects to $\mathcal{Q}$.*

(2) *The variety $\mathcal{Q}$ does not intersect the locus $\bigcup_{i=0}^{2} \mathcal{Z}_i$.*

(3) *The inclusion map $\varphi_S^* : \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]/(f_6, g_{10}) \to R(S, L)$ is an isomorphism of graded $\mathbb{C}$-algebra, where $R(S, L) := \bigoplus_{n=0}^{\infty} H^0(\mathcal{O}_S(nL))$. The variety $\mathcal{Q}$ has at most rational double points as its singularities.*

Proof. Since $\deg \mathcal{Q} = 1 = L^2$, the assertion 1) follows from Lemma 4.8. Then the assertion 2) follows from the non-vanishing of the coefficient of Z_0^2 in f_6, that of the the coefficient of U_0^2 in g_{10}, Lemma 4.5, and Corollary 4.1. It only remains to prove the assertion 3). By the assertion 2), we see that $\mathcal{Q}$ is Gorenstein. Moreover we have $\omega_{\mathcal{Q}} \simeq \mathcal{O}_{\mathcal{Q}}(3)$, hence $\omega_S \simeq \varphi_S^* \omega_{\mathcal{Q}}$. Thus $\mathcal{Q}$ has at most rational double points as its singularities. Since $\varphi_S : S \to \mathcal{Q}$ gives the minimal desingularization of $\mathcal{Q}$, we obtain the assertion 3). □

Naturally, $R(S, L)^{(3)} = \bigoplus_{n=0}^{\infty} H^0(\mathcal{O}_S(3nL))$ is the canonical ring of the surface S. Thus, we obtain the following:

Theorem 1. *If a minimal surface S has $c_1^2 = 9$ and $\chi = 5$, and its canonical class is divisible by 3 in its integral cohomology group, then its canonical model is a $(6, 10)$-complete intersection of the weighted projective space $\mathbb{P}(1, 2, 2, 3, 5)$ that does not intersect the locus $\bigcup_{i=0}^{2} \mathcal{Z}_i$. Conversely, if a $(6, 10)$-complete intersection $\mathcal{Q} \subset \mathbb{P}(1, 2, 2, 3, 5)$ satisfying $\mathcal{Q} \cap \bigcup_{i=0}^{2} \mathcal{Z}_i = \emptyset$ has at most rational double points as its singularities, then its minimal desingularization S is a minimal surface with $c_1^2 = 9$ and $\chi = 5$ whose canonical class is divisible by 3.*

Note that for general f_6 and $g_{10} \in \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ of degree 6 and 10, respectively, the subvariety $\mathcal{Q} = \{f_6 = g_{10} = 0\} \subset \mathbb{P}(1, 2, 2, 3, 5)$ is non-singular. This can be verified with $X_0^6, Y_0^3, Y_1^3, Z_0^2 \in \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]_6$, $X_0^{10}, Y_0^5, Y_1^5, U_0^2 \in \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]_{10}$, and Bertini's Theorem.

Remark 1. Let S and S' be two minimal algebraic surfaces with invariants as in Theorem 1. Then as one can see from the proof of Theorem 1, the surfaces S and S' are isomorphic to each other, if and only if the varieties $\mathcal{Q}$ and $\mathcal{Q}'$ are projectively equivalent in the weighted projective space $\mathbb{P}(1, 2, 2, 3, 5)$, where $\mathcal{Q}$ and $\mathcal{Q}'$ are the $(6, 10)$-complete intersections corresponding to S and S', respectively.

Remark 2. Let f_6 and $g_{10} \in \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ be homogeneous polynomials of weighted degree 6 and 10, respectively. Assume that the coefficient of Z_0^2 in f_6 and that of U_0^2 in g_{10} are non-vanishing. Let $\mathcal{Q} \subset \mathbb{P}(1, 2, 2, 3, 5)$ denote the subvariety defined by the polynomials f_6 and g_{10}. Then $\mathcal{Q} \cap \bigcup_{i=0}^{2} \mathcal{Z}_i = \emptyset$ holds, if and only if the two sections $f_6(0, Y_0, Y_1, 0, 0) \in H^0(\mathcal{O}_{\mathbb{P}^1}(3))$ and $g_{10}(0, Y_0, Y_1, 0, 0) \in H^0(\mathcal{O}_{\mathbb{P}^1}(5))$ have no common zero on the projective line $\mathbb{P}^1 = \operatorname{Proj} \mathbb{C}[Y_0, Y_1]$.

5. Moduli space and the canonical maps

In this section, we study the moduli space. We also study the behavior of the canonical map of our surface S. Let us begin with the normal form of the defining polynomials f_6 and g_{10}.

Proposition 4. *Let S be a minimal surface as in Theorem 1. Then the defining polynomials f_6 and g_{10} in $\mathbb{P}(1, 2, 2, 3, 5)$ of its canonical model $\mathcal{Q}$ can be taken in the form*

$$f_6 = Z_0^2 + \alpha_0 X_0 U_0 + \alpha_3(X_0^2, Y_0, Y_1),$$
$$g_{10} = U_0^2 + \beta_3(X_0^2, Y_0, Y_1) X_0 Z_0 + \beta_5(X_0^2, Y_0, Y_1),$$

where $\alpha_0 \in \mathbb{C}$ is a constant, α_3, a homogeneous polynomial of degree 3, and β_i, a homogeneous polynomial of degree i for $i = 3, 5$.

Proof. By completing the square with respect to Z_0, we can take f_6 and g_{10} in the form

$$f_6 = Z_0^2 + \alpha_0 X_0 U_0 + \alpha_3(X_0^2, Y_0, Y_1),$$
$$g_{10} = U_0^2 + \beta_1(X_0^2, Y_0, Y_1) Z_0 U_0 + \beta_3(X_0^2, Y_0, Y_1) X_0 Z_0 + \beta_5(X_0^2, Y_0, Y_1).$$

Putting $X_0 = X_0'$, $Y_0 = Y_0'$, $Y_1 = Y_1'$, $Z_0 = Z_0' + \alpha_0 \beta_1 X_0'/4$, and $U_0 = U_0' - \beta_1 Z_0'/2 - \alpha_0 \beta_1^2 X_0'/4$, and employing $X_0', Y_0', Y_1', Z_0', U_0'$ as new X_0, Y_0, Y_1, Z_0, U_0, respectively, we easily obtain new f_6 and g_{10} in which the term $\beta_1 Z_0 U_0$ vanishes. □

Using this proposition, we prove the following theorem:

Theorem 2. *The coarse moduli space $\mathcal{M}$ of surfaces as in Theorem 1 is a unirational variety of dimension 34. In particular, any two surfaces S's as in Theorem 1 are deformation equivalent to each other.*

Proof. In what follows, for two weighted homogeneous polynomials f_6 and g_{10} as in Proposition 4, we denote by $S_{(f_6, g_{10})}$ the minimal desingularization of the variety $\mathcal{Q}_{(f_6, g_{10})} = \{f_6 = g_{10} = 0\} \subset \mathbb{P}(1, 2, 2, 3, 5)$. Note that the pair (f_6, g_{10}) in the normal form as in Proposition 4 has 42 linear parameters. Denote by V the Zariski open subset of $\mathbb{A}^{42}$ consisting of all (f_6, g_{10})'s such that 1) $\mathcal{Q}_{(f_6, g_{10})}$ has at most rational double points as its singularities, and 2) $\mathcal{Q}_{(f_6, g_{10})} \cap \bigcup_{i=0}^{2} \mathcal{Z}_i = \emptyset$ holds.

Then by the existence of the natural family of the canonical models $\mathcal{Q}_{(f_6,\, g_{10})}$'s over the space of parameters V, we obtain the irreducibility of the moduli space $\mathcal{M}$.

Let us compute the dimension of the moduli space $\mathcal{M}$. Note that for two points $(f_6,\, g_{10})$ and $(f_6',\, g_{10}')$ of V, the corresponding surfaces $S_{(f_6,\, g_{10})}$ and $S_{(f_6',\, g_{10}')}$ are isomorphic to each other if and only if the ideals $(f_6,\, g_{10})$ and $(f_6',\, g_{10}')$ are equivalent under the action by the group of homogeneous transformations on the graded algebra $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$. Since no monomial divisible by Z_0^2 appears in g_{10} and g_{10}', and since $10 - \deg U_0 < \deg Z_0^2$ holds, this is equivalent to the condition that the points $(f_6,\, g_{10})$ and $(f_6',\, g_{10}')$ of V are equivalent under the action by the group of homogeneous transformations of $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$. Moreover, we see easily that if a point $v \in V$ corresponding to a surface S is sufficiently general, then there exists a point $(f_6,\, g_{10}) \in V$ that gives the same isomorphism class of S and such that f_6 and g_{10} are in the form

$$f_6 = Z_0^2 + X_0U_0 + Y_0^3 + Y_1^3 + a_0X_0^2Y_0Y_1 + X_0^4(a_1Y_0 + a_2Y_1) + X_0^6,$$
$$g_{10} = U_0^2 + \beta_3(X_0^2, Y_0, Y_1)X_0Z_0 + \beta_5(X_0^2, Y_0, Y_1).$$

We denote by V' the 34-dimensional subvariety of V consisting of all $(f_6,\, g_{10})$'s in this form. Then the restriction $V' \to \mathcal{M}$ of the natural morphism $V \to \mathcal{M}$ is dominant.

Let us study fibers of the morphism $V' \to \mathcal{M}$. Let G be the group of homogeneous transformations of $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ that preserve the subvariety $V' \subset V$. We denote by ω the third root of unity, and define the two transformations $\sigma, \tau \in G$ by

$$\begin{array}{llllll} \sigma: & U_0 \mapsto U_0, & Z_0 \mapsto Z_0 & Y_0 \mapsto \omega Y_0, & Y_1 \mapsto \omega^2 Y_1 & X_0 \mapsto X_0 \\ \tau: & U_0 \mapsto U_0, & Z_0 \mapsto Z_0 & Y_0 \mapsto Y_1, & Y_1 \mapsto Y_0 & X_0 \mapsto X_0. \end{array}$$

Then σ and τ generate a subgroup $\langle \sigma, \tau \rangle \simeq \mathfrak{S}_3 \subset G$, where $\mathfrak{S}_3$ is the symmetric group of degree 3. Since each element of G induces a permutation of three prime divisors of $Y_0^3 + Y_1^3$, we have a natural group homomorphism $G \to \mathfrak{S}_3$, whose restriction $\langle \sigma, \tau \rangle \to \mathfrak{S}_3$ to $\langle \sigma, \tau \rangle \subset G$ is an isomorphism. Thus if we define $\Psi_{(\lambda_0, \lambda_1, \mu_0, a)} \in G$ by $U_0 \mapsto a^5U_0$, $Z_0 \mapsto (-1)^{\mu_0}a^3Z_0$, $Y_1 \mapsto \omega^{\lambda_1}a^2Y_1$, $Y_0 \mapsto \omega^{\lambda_0}a^2Y_0$, and $X_0 \mapsto aX_0$ for each $(\lambda_0, \lambda_1, \mu_0, a) \in (\mathbb{Z}/3)^{\oplus 2} \oplus \mathbb{Z}/2 \oplus \mathbb{C}^{\times}$, then each element of G can be written as $\rho \circ \Psi_{(\lambda_0, \lambda_1, \mu_0, a)}$ for an element $\rho \in \langle \sigma, \tau \rangle$ and an element $(\lambda_0, \lambda_1, \mu_0, a) \in (\mathbb{Z}/3)^{\oplus 2} \oplus \mathbb{Z}/2 \oplus \mathbb{C}^{\times}$. This implies that for a general point of $\mathcal{M}$ the fiber of $V' \to \mathcal{M}$ over this point consists of at most 108 points. Now since V' is a Zariski open subset of the affine space $\mathbb{A}^{34}$, we see that $\mathcal{M}$ is unirational of dimension 34. □

For the verification of the computations above, let us compute the dimensions of the cohomology groups of the tangent sheaf of our surface S.

Proposition 5. *Let S be a surface as in Theorem 1, and Θ_S, its tangent sheaf. Suppose that the canonical model of S is smooth. Then $h^1(\Theta_S) = 34$ and $h^2(\Theta_S) = 2$ hold. The Kuranishi space of S is non-singular of dimension $h^1(\Theta_S) = 34$.*

Proof. In what follows, we put $\mathbb{P} = \mathbb{P}(1, 2, 2, 3, 5) \setminus \bigcup_{i=0}^{2} \mathcal{Z}_i$, where $\mathcal{Z}_i$'s are as in Theorem 1. Since we have assumed that the canonical model $\mathcal{Q}$ of our surface S is smooth, we may assume that S is a subvariety of $\mathbb{P}$ and that the natural morphism $\varphi_S : S \to \mathbb{P}$ is the inclusion map. By [17, Remark 2.4.], we have a natural exact sequence

$$0 \to \mathcal{O}_S \to \bigoplus_{i=1}^{5} \mathcal{O}_S(e_i L) \to \varphi_S^* \Theta_{\mathbb{P}} \to 0, \tag{15}$$

where $e_1 = 1$, $e_2 = e_3 = 2$, $e_4 = 3$, and $e_5 = 5$. From the long exact sequence associated to the above, wee infer easily the equalities $h^0(\varphi_S^* \Theta_{\mathbb{P}}) = 20$ and $h^1(\varphi_S^* \Theta_{\mathbb{P}}) = 0$, provided that $h^2(\varphi_S^* \Theta_{\mathbb{P}}) = 2$. Note that these two equalities together with the standard exact sequence

$$0 \to \Theta_S \to \varphi_S^* \Theta_{\mathbb{P}} \to \mathcal{O}_S(6L) \oplus \mathcal{O}_S(10L) \to 0$$

imply the first assertion. Thus, to obtain the first assertion, we only need to show that $h^2(\varphi_S^* \Theta_{\mathbb{P}}) = 2$. To show this last equality, let us consider the exact sequence

$$0 \to \mathcal{O}_S(3L) \otimes \varphi_S^* \Omega_{\mathbb{P}}^1 \to \bigoplus_{i=1}^{5} \mathcal{O}_S((3 - e_i)L) \to \mathcal{O}_S(3L) \to 0,$$

the short exact sequence obtained by operating $\mathcal{O}_S(3L) \otimes \cdot$ to the dual of (15). Since the morphism $\mathcal{O}_S \to \bigoplus_{i=1}^5 \mathcal{O}_S(e_i L)$ in (15) is given by the transpose of the matrix $(x_0, 2y_0, 2y_1, 3z_0, 5u_0)$, we see from Lemma 4.2 that the induced morphism $H^0(\bigoplus_{i=1}^5 \mathcal{O}_S((3 - e_i)L)) \to H^0(\mathcal{O}_S(3L))$ has rank 4. Thus, by the Serre duality, we obtain $h^2(\varphi_S^* \Theta_{\mathbb{P}}) = h^0(\mathcal{O}_S(3L) \otimes \varphi_S^* \Omega_{\mathbb{P}}^1) = 2$, hence the first assertion. Now let us prove the second assertion. By the computations above, we have the surjectivity of $H^1(\Theta_S) \to H^1(\varphi_S^* \Theta_{\mathbb{P}})$ and the injectivity of $H^2(\Theta_S) \to H^2(\varphi_S^* \Theta_{\mathbb{P}})$. Thus by [11, Theorem 4.4.], their exists a family $(\mathcal{S}, \Phi, \varpi, M)$ of deformations of the holomorphic map $\Phi_o = \varphi_S : \varpi^{-1}(o) = S \to \mathbb{P} = \mathbb{P} \times \{o\}$, such that the parameter space M is non-singular at o and such that the characteristic map $T_o(M) \to D_{S/\mathbb{P}}$ is bijective, where $T_o(M)$ is the tangent space of M at o. (For the definition of $D_{S/\mathbb{P}}$, see [11].) Moreover, by [11, Lemma 4.2.] and $h^1(\varphi_S^* \Theta_{\mathbb{P}}) = 0$ shown above, we obtain the surjectivity of the natural morphism $D_{S/\mathbb{P}} \to H^1(\Theta_S)$, hence that of the Kodaira-Spencer map $T_o(M) \to H^1(\Theta_S)$ at o of the analytic family $(\mathcal{S}, \varpi, M)$. Thus we can take a non-singular analytic subspace $N \subset M$ passing through o such that $T_o(N) \to H^1(\Theta_S)$ is bijective, hence the second assertion. □

Finally, we study the behavior of the canonical map of our surface S. Let $\mathcal{Q} \simeq \mathrm{Proj}\, \mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]/(f_6, g_{10})$ be the canonical model of our surface S, where f_6 and g_{10} are in the normal form as in Proposition 4. Since the birational morphism $\varphi_S : S \to \mathcal{Q}$ factors through the canonical map $\Phi_{|K|} : S - - \to \mathbb{P}^3$, the study of the behavior of $\Phi_{|K|}$ is reduced to that of the behavior of the rational map $\Phi_{|\mathcal{O}_{\mathcal{Q}}(3)|} : \mathcal{Q} - - \to \mathbb{P}^3$. Let ξ_0, η_0, η_1, and ζ_0 be the homogeneous coordinates of $\mathbb{P}^3$ corresponding to the base X_0^3, $X_0 Y_0$, $X_0 Y_1$, Z_0 of $H^0(\mathcal{O}_{\mathcal{Q}}(3))$. Note that for an integer $d \geq 1$, an equation of $\Phi_{|K|}(S)$ in $\mathbb{P}^3$ of degree d corresponds to a relation among X_0^3, $X_0 Y_0$, $X_0 Y_1$, and Z_0 in the homogeneous part of degree $3d$ of $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]/(f_6, g_{10})$.

Theorem 3. *Let S be a minimal surface as in Theorem 1, and f_6 and g_{10}, the defining polynomials in $\mathbb{P}(1,2,2,3,5)$ of its canonical model $\mathcal{Q}$. Assume that f_6 and g_{10} are in the normal form as in Proposition 4.*

1) *If $\alpha_0 \neq 0$, then the canonical map $\Phi_{|K|}$ of S is birational onto its image, and the canonical image $\Phi_{|K|}(S)$ is a sextic surface in $\mathbb{P}^3$ defined by*

$$\left[\xi_0\zeta_0^2 + \alpha_3(\xi_0, \eta_0, \eta_1)\right]^2 + \alpha_0^2\left[\beta_3(\xi_0, \eta_0, \eta_1)\xi_0^2\zeta_0 + \beta_5(\xi_0, \eta_0, \eta_1)\xi_0\right] = 0.$$

Surfaces S's with birational $\Phi_{|K|}$ form an open dense subset of $\mathcal{M}$.

2) *If $\alpha_0 = 0$, then the canonical map $\Phi_{|K|}$ of S is generically two-to-one onto its image, and the canonical image $\Phi_{|K|}(S)$ is a cubic surface in $\mathbb{P}^3$ defined by*

$$\xi_0\zeta_0^2 + \alpha_3(\xi_0, \eta_0, \eta_1) = 0.$$

Surfaces S's with non-birational $\Phi_{|K|}$ form a 33-dimensional locus in $\mathcal{M}$.

Proof. The only non-trivial relation in the homogeneous part of degree 6 of $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]/(f_6, g_{10})$ is given by $f_6 = Z_0^2 + \alpha_0 X_0 U_0 + \alpha_3(X_0^2, Y_0, Y_1) = 0$. Assume that f_6 is a polynomial of X_0^3, X_0Y_0, X_0Y_1 and Z_0. Then $\alpha_3(0, Y_0, Y_1)$ must be zero in $\mathbb{C}[Y_0, Y_1]$. In this case, however, we have $f_6(0, Y_0, Y_1, 0, 0) = 0 \in \mathbb{C}[Y_0, Y_1]$, which contradicts the condition $\mathcal{Q} \cap \bigcup_{i=0}^{2} \mathcal{Z}_i = \emptyset$. (See Remark 2.) Thus $\Phi_{|K|}(S) \subset \mathbb{P}^3$ satisfies no equation of degree 2.

Assume that X_0^3, X_0Y_0, X_0Y_1, and Z_0 have a non-trivial relation in the homogeneous part of degree 9 of $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]/(f_6, g_{10})$. Then this relation must be written as $\gamma_1(X_0^3, X_0Y_0, X_0Y_1, Z_0) f_6 = 0$, where γ_1 is a linear form with coefficients in $\mathbb{C}$. Since this left hand is a polynomial of X_0^3, X_0Y_0, X_0Y_1, and Z_0, we see with the help of Remark 2 that $\alpha_0 = 0$ holds and that $\gamma_1(X_0^3, X_0Y_0, X_0Y_1, Z_0)$ is a multiple of X_0^3. Thus if $\alpha_0 = 0$, then $\Phi_{|K|}(S) \subset \mathbb{P}^3$ is a cubic surface as in the assertion, and if $\alpha_0 \neq 0$, then $\Phi_{|K|}(S) \subset \mathbb{P}^3$ satisfies no equation of degree 3.

Now that we have shown the assertion for the case $\alpha_0 = 0$, we assume in what follows that $\alpha_0 \neq 0$. By an argument similar to that in the preceding paragraph, we can prove the absence of equations of degree d of $\Phi_{|K|}(S) \subset \mathbb{P}^3$ for $d = 4, 5$. On the other hand, we can easily find an equation of degree 6 that is satisfied by $\Phi_{|K|}(S) \subset \mathbb{P}^3$. Note that in $\mathbb{C}[X_0, Y_0, Y_1, Z_0, U_0]$ we have $-\alpha_0 X_0 U_0 \equiv Z_0^2 + \alpha_3(X_0^2, Y_0, Y_1)$ and $-U_0^2 \equiv \beta_3(X_0^2, Y_0, Y_1) X_0 Z_0 + \beta_5(X_0^2, Y_0, Y_1)$ modulo the ideal (f_6, g_{10}). Eliminating U_0 from these two and then multiplying it by X_0^6, we obtain

$$\begin{aligned}&\left[X_0^3 Z_0^2 + \alpha_3(X_0^3, X_0Y_0, X_0Y_1)\right]^2 \\ &\quad + \alpha_0 X_0^3\left[\beta_3(X_0^3, X_0Y_0, X_0Y_1) X_0^3 Z_0 + \beta_5(X_0^3, X_0Y_0, X_0Y_1)\right] \equiv 0\end{aligned}$$

modulo the ideal (f_6, g_{10}). From this together with the absence of equation of lower degree, we see that if $\alpha_0 \neq 0$ then $\Phi_{|K|}(S) \subset \mathbb{P}^3$ is a sextic surface defined by the equation as in the assertion.

Now let us compute the mapping degree of the canonical map $\Phi_{|K|} : S \dashrightarrow \mathbb{P}^3$. Let $|K| = |3L| = |M_3| + F_3$ be the decomposition as in Lemma 4.1, and

$p_3 : \tilde{S}_3 \to S$, the shortest composite of quadric transformations such that the variable part $|\tilde{M}_3|$ of $p_3^*|M_3|$ is free from base points. Then we have

$$\deg \Phi_{|K|} \deg \Phi_{|K|}(S) = \tilde{M}_3^2 \leq M_3^2 \leq K^2 = 9. \quad (16)$$

Assume that $\alpha_0 \neq 0$. Then since $\deg \Phi_{|K|}(S) = 6$, we infer from the inequalities above that $\deg \Phi_{|K|} = 1$. Assume that $\alpha_0 = 0$. Then since $\deg \Phi_{|K|}(S) = 3$, we infer in the same way that $\deg \Phi_{|K|} \leq 3$. If $\deg \Phi_{|K|} = 3$ holds, however, we see from (16) that the linear system $|K|$ is base point free. This is impossible, because by Lemma 4.2 the canonical system $|K|$ needs to have a base point. Thus we obtain $\deg \Phi_{|K|} = 2$.

It is trivial that the surfaces S's with birational $\Phi_{|K|}$ form an open dense subset in $\mathcal{M}$. To show that the surfaces S's with non-birattional $\Phi_{|K|}$ form a 33-dimensional locus in $\mathcal{M}$, we just need to use the same method as in the computation of dim $\mathcal{M}$ in Theorem 2. □

Let us conclude this article by giving some more details on the canonical map and its image of our surface S. In what follows, we denote by W the canonical image $\Phi_{|K|}(S)$. Moreover, we denote by $p_3 : \tilde{S}_3 \to S$ the shortest composite of quadric transformations such that the variable part of $p_3^*|K|$ is free from base points, and by $\varphi : \tilde{S}_3 \to W$, the unique morphism such that $\Phi_{|K|} = \varphi \circ p_3^{-1}$.

First we study the case $\deg \Phi_{|K|} = 1$. In this case, the canonical image $W \subset \mathbb{P}^3$ is a sextic surface. Recall that for a singularity (W, x) of our surface W, the fundamental genus of (W, x) is the arithmetic genus of its fundamental cycle. Moreover, since $\varphi : \tilde{S}_3 \to W$ gives the minimal desingularization of the canonical image W, the geometric genus of (W, x) is the dimension of the vector space $(R^1\varphi_*\mathcal{O}_{\tilde{S}_3})_x$, where $R^1\varphi_*\mathcal{O}_{\tilde{S}_3}$ is the first higher direct image of the structure sheaf $\mathcal{O}_{\tilde{S}_3}$. The following proposition is a comment given to the author by Kazuhiro Konno:

Proposition 6. *Let S be a minimal algebraic surface as in Theorem 1. Suppose that* $\deg \Phi_{|K|} = 1$ *and that the canonical system $|K|$ has no fixed component. Then the canonical image $W = \Phi_{|K|}(S) \subset \mathbb{P}^3$ is normal. Moreover, if the surface S is sufficiently general, then the singularity (W, x) of W is a double point with fundamental genus* 3 *and geometric genus* 6, *where $x \in W$ is a point given by* $(\xi_0 : \eta_0 : \eta_1 : \zeta_0) = (0 : 0 : 0 : 1)$.

Proof. Assume that $|K|$ has no fixed component, as is indeed the case for our general S by Proposition 4. Then $p_3 : \tilde{S}_3 \to S$ is a blowing up at three simple base points of $|K|$. Thus for any hyperplane $H \subset \mathbb{P}^3$, the arithmetic genus of $W \cap H$ equals that of the pullback $\varphi^*(W \cap H) \in |p_3^*(K) - \varepsilon|$, where ε is the sum of the total transforms of the three (-1)-curves appearing by $p_3 : \tilde{S}_3 \to S$. Since the variable part $|p_3^*(K) - \varepsilon|$ of $p_3^*|K|$ is free from base points, this together with Bertini's Theorem implies that W has at most isolated singularities. This however implies that W is normal, since the canonical image $W \subset \mathbb{P}^3$ is a hypersurface. Note that the local equation at x of W in $\mathbb{P}^3$ is analytically in the form $w^2 - f_8(u, v) = 0$. Thus the invariants of the double point (W, x) can be computed by the canonical resolution. □

Remark 3. From Proposition 4, we see easily that the point x is the only singularity of W for a sufficiently general S. Thus one can compute the geometric genus of (W, x) also by writing down the Leray spectral sequence of $\varphi : \tilde{S}_3 \to W$ and comparing the invariants of W and those of $\tilde{S}_3$.

Next, we study the case $\deg \Phi_{|K|} = 2$. In this case, the canonical image $W \subset \mathbb{P}^3$ is a cubic surface. We shall describe the branch divisor of the canonical map $\Phi_{|K|}$. For simplicity, we shall do this only for the case where S satisfies the following three generality conditions:

i) the canonical image $W = \Phi_{|K|}(S)$ is smooth;
ii) the unique member $L \in |L|$ is irreducible;
iii) the base locus of $|K|$ consists of three distinct points.

Proposition 7. *Let S be a minimal algebraic surface as in Theorem 1, and $\varphi : \tilde{S}_3 \to W = \Phi_{|K|}(S)$, the morphism such that $\Phi_{|K|} = \varphi \circ p_3^{-1}$. Suppose that* $\deg \Phi_{|K|} = 2$ *and that S satisfies the three conditions above. Then the branch divisor B of φ splits as $B = \sum_{i=1}^{3} \Gamma_i + B'$, where Γ_i's are three coplanar lines in $\mathbb{P}^3$ meeting at one point $x \in W$, and B' is a member of $|-5K_W|$ that has an ordinary* 5*-tuple point at x and such that all other singularities if any are negligible ones.*

Proof. By the generality conditions, the three base points of the canonical system $|K|$ are non-singular points of the unique member L. Thus if we denote by ε the divisor such that $|K_{\tilde{S}_3}| = p_3^*|K| + \varepsilon$, then we have $p_3^*(L) = p_{3*}^{-1}(L) + \varepsilon$. Moreover, the divisor ε is a sum of three (-1)-curves. Thus from this together with $\varphi^*(-K_W) \sim p_3^*(3L) - \varepsilon$, we see that $\varphi_*\varepsilon = \sum_{i=1}^{3} \Gamma_i$, where Γ_1, Γ_2, and Γ_3 are the three lines in $\mathbb{P}^3$ corresponding to the irreducible components of the divisor ε.

Let R and $B = \varphi_*(R)$ be the ramification divisor and the branch divisor of $\varphi : \tilde{S}_3 \to W$, respectively. Then by

$$R \sim p_3^*(3L) + \varepsilon - \varphi^*(K_W) \sim 2\varphi^*(-K_W) + 2\varepsilon, \tag{17}$$

we have $BD = (-4K_W + 2\sum_{i=1}^{3} \Gamma_i)D$ for any divisor D on W, which implies $B \in |-4K_W + 2\sum_{i=1}^{3} \Gamma_i|$. Now let us denote by $\tilde{L}_3$ the strict transform by p_3 of the divisor L. By $\tilde{L}_3\varphi^*(-K_W) = 0$, we see that φ contracts $\tilde{L}_3$ to a single point $x \in W$, where we have $x \in \bigcap_{i=i}^{3} \Gamma_i$. Moreover, since $p_g(S) = h^0(\mathcal{O}_W(-K_W)) = 4$ and hence $3\tilde{L}_3 + 2\varepsilon \in |p_3^*(3L) - \varepsilon| = \varphi^*|-K_W|$, we obtain a member $\Gamma \in |-K_W|$ such that $\varphi^*(\Gamma) = 3\tilde{L}_3 + 2\varepsilon$ holds. Since we have $\Gamma = \sum_{i=1}^{3} \Gamma_i$ for this Γ, we see that the three lines Γ_1, Γ_2, and Γ_3 are coplanar, and that $\varepsilon + \tilde{L}_3 \preceq R$, since we have $\varphi(\tilde{L}_3) = \{x\}$. We therefore can put $R = \varepsilon + \tilde{L}_3 + R'$, where R' is a non-negative divisor on $\tilde{S}_3$. We put $B' = \varphi_*(R') \in |-5K_W|$.

Now let $\hat{q} : \hat{W} \to W$ be the blowing up at x, and Δ, its exceptional divisor. Then by $\varphi^*(\Gamma) = 3\tilde{L}_3 + 2\varepsilon$, we obtain $\varphi^*(\Gamma_i) = 2\bar{\Gamma}_i + \tilde{L}_3$ for each integer $1 \le i \le 3$, where $\bar{\Gamma}_i$'s are three (-1)-curves appearing by p_3. This implies the liftability of $\varphi : \tilde{S}_3 \to W$ to a morphism $\hat{\varphi} : \tilde{S}_3 \to \hat{W}$. Moreover, we obtain $\hat{\varphi}^*(\Delta) = \tilde{L}_3$. Thus Δ is not a component of the branch divisor of $\hat{\varphi}$, from which we infer $\hat{\varphi}_*(R') = \hat{q}_*^{-1}(B')$. Since we have $\hat{\varphi}_*(R')\Delta = R'\varepsilon = 5$ by (17), we see from this $\mathrm{ord}_x B' = 5$. But the standard double cover argument implies that $\sum_{i=1}^{3} \hat{q}_*^{-1}(\Gamma_i) + \hat{q}_*^{-1}(B')$

has at most negligible singularities. Thus the point x is an ordinary 5-tuple point of B', and all other singularities of B' are negligible ones. Finally, the equality $(\sum_{i=1}^{3} \Gamma_i \cdot B')_x = 15$ follows from $\hat{\varphi}^*(\Delta) = \tilde{L}_3$, since this latter implies the absence of singularities lying on Δ of the divisor $\sum_{i=1}^{3} \hat{q}_*^{-1}(\Gamma_i) + \hat{q}_*^{-1}(B')$. □

Remark 4. Conversely, a non-singular cubic surface $W \subset \mathbb{P}^3$ and a member $B \in |-6K_W|$ having the same properties as in Proposition 7 yields a minimal algebraic surface S as in Theorem 1 with $\deg \Phi_{|K|} = 2$. Naturally, one easily finds the divisor L, guided by the proof of the proposition above.

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Declarations

Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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