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Global existence and blow-up of solutions to porous medium equation and pseudo-parabolic equation, I. Stratified groups

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Abstract. In this paper, we prove a global existence and blow-up of the positive solutions to the initial-boundary value problem of the nonlinear porous medium equation and the nonlinear pseudo-parabolic equation on the stratified Lie groups. Our proof is based on the concavity argument and the Poincaré inequality, established in Ruzhansky and Suragan (J Differ Eq 262:1799–1821, 2017) for stratified groups.

1. Introduction

The main purpose of this paper is to study the global existence and blow-up of the positive solutions to the initial-boundary problem of the nonlinear porous medium equation

$$\begin{cases} u_t(x,t) - \mathcal{L}_p(u^m(x,t)) = f(u(x,t)), & x \in D,\ t > 0, \\ u(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \overline{D}, \end{cases} \tag{1.1}$$

and the nonlinear pseudo-parabolic equation

$$\begin{cases} u_t(x,t) - \nabla_H \cdot (|\nabla_H u(x,t)|^{p-2} \nabla_H u_t(x,t) - \mathcal{L}_p u(x,t) = f(u(x,t)), & x \in D,\ t > 0, \\ u(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \overline{D}, \end{cases} \tag{1.2}$$

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where $m \geq 1$ and $p \geq 2$, f is locally Lipschitz continuous on $\mathbb{R}$, $f(0) = 0$, and such that $f(u) > 0$ for $u > 0$. Furthermore, we suppose that u_0 is a non-negative and non-trivial function in $C^1(\overline{D})$ with $u_0(x) = 0$ on the boundary ∂D for $p = 2$ and in $L^\infty(D) \cap \mathring{S}^{1,p}(D)$ for $p > 2$, respectively.

Definition 1.1. Let $\mathbb{G}$ be a stratified group. We say that an open set $D \subset \mathbb{G}$ is an admissible domain if it is bounded and if its boundary ∂D is piecewise smooth and simple, that is, it has no self-intersections.

Let $\mathbb{G}$ be a stratified group. Let $D \subset \mathbb{G}$ be an open set, then we define the functional spaces

$$S^{1,p}(D) = \{u : D \to \mathbb{R}; u, |\nabla_H u| \in L^p(D)\}. \tag{1.3}$$

We consider the following functional

$$\mathcal{J}_p(u) := \left(\int_D |\nabla_H u(x)|^p dx \right)^{\frac{1}{p}}.$$

Thus, the functional class $\mathring{S}^{1,p}(D)$ can be defined as the completion of $C_0^1(D)$ in the norm generated by $\mathcal{J}_p$, see e.g. [2].

A Lie group $\mathbb{G} = (\mathbb{R}^n, \circ)$ is called a stratified (Lie) group if it satisfies the following conditions:

(a) For some integer numbers $N_1 + N_2 + ... + N_r = n$, the decomposition $\mathbb{R}^n = \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_r}$ is valid, and for any $\lambda > 0$ the dilation

$$\delta_\lambda(x) := (\lambda x', \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)})$$

is an automorphism of $\mathbb{G}$. Here $x' \equiv x^{(1)} \in \mathbb{R}^{N_1}$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \ldots, r$.

(b) Let N_1 be as in (a) and let $X_1, \ldots, X_{N_1}$ be the left-invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, \ldots, N_1$. Then the Hörmander rank condition must be satisfied, that is,

$$\text{rank}\left(\text{Lie}\{X_1, \ldots, X_{N_1}\}\right) = n,$$

for every $x \in \mathbb{R}^n$.

Then, we say that the triple $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ is a stratified (Lie) group.

Recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see e.g. [3], [4]). The left-invariant vector field X_j has an explicit form:

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)} \left(x', ..., x^{(l-1)}\right) \frac{\partial}{\partial x_m^{(l)}}, \tag{1.4}$$

see e.g. [4]. The following notations are used throughout this paper:

$$\nabla_H := (X_1, \ldots, X_{N_1})$$

for the horizontal gradient, and

$$\mathcal{L}_p f := \nabla_H \cdot \left(|\nabla_H f|^{p-2} \nabla_H f \right), \quad 1 < p < \infty, \tag{1.5}$$

for the p-sub-Laplacian. When $p = 2$, that is, the second order differential operator

$$\mathcal{L} = \sum_{k=1}^{N_1} X_k^2, \tag{1.6}$$

is called the sub-Laplacian on $\mathbb{G}$. The sub-Laplacian $\mathcal{L}$ is a left-invariant homogeneous hypoelliptic differential operator and it is known that $\mathcal{L}$ is elliptic if and only if the step of $\mathbb{G}$ is equal to 1.

One of the important examples of the nonlinear parabolic equations is the porous medium equation, which describes widely processes involving fluid flow, heat transfer or diffusion, and its other applications in different fields such as mathematical biology, lubrication, boundary layer theory, and etc. Existence and nonexistence of solutions to problem (1.1) for the reaction term u^m in the case $m = 1$ and $m > 1$ have been actively investigated by many authors, for example, [5–19], Grillo, Muratori and Punzo considered fractional porous medium equation [20,21], and it was also considered in the setting of Cartan-Hadamard manifolds [22]. By using the concavity method, Schaefer [23] established a condition on the initial data of a Dirichlet type initial-boundary value problem for the porous medium equation with a power function reaction term when blow-up of the solution in finite time occurs and a global existence of the solution holds. We refer for more details to Vazquez's book [24] which provides a systematic presentation of the mathematical theory of the porous medium equation.

The energy for the isotropic material can be modeled by a pseudo-parabolic equation [25]. Some wave processes [26], filtration of the two-phase flow in porous media with the dynamic capillary pressure [27] are also modeled by pseudo-parabolic equations. The global existence and finite-time blow-up for the solutions to pseudo-parabolic equations in bounded and unbounded domains have been studied by many researchers, for example, see [28–35] and the references therein.

In [36], Veron and Pohozaev have obtained blow-up results for the following semi-linear diffusion equation on the Heisenberg groups

$$\frac{\partial u(x,t)}{\partial t} - \mathcal{L}u(x,t) = |u(x,t)|^p, \quad (x,t) \in \mathbb{H} \times (0, +\infty).$$

Also, blow-up of the solutions to the semi-linear diffusion and pseudo-parabolic equations on the Heisenberg groups was derived in [37–41]. In addition, in [42] the authors found the Fujita exponent on general unimodular Lie groups.

In some of our considerations a crucial role is played by

- The condition

$$\alpha F(u) \le u^m f(u) + \beta u^{pm} + \alpha\gamma, \quad u > 0, \tag{1.7}$$

where

$$F(u)=\frac{pm}{m+1}\int_0^u s^{m-1}f(s)ds,\quad m\geq 1,$$

introduced by Chung-Choi [43] for a parabolic equation. We will deal with several variants of such condition.

- The Poincaré inequality established by the first author and Suragan in [1] for stratified groups:

Lemma 1.2. *Let $D\subset\mathbb{G}$ be an admissible domain with N_1 being the dimension of the first stratum. Let $1<p<\infty$ with $p\neq N_1$. For every function $u\in C_0^\infty(D\backslash\{x'=0\})$ we have*

$$\int_D |\nabla_H u|^p dx \geq \frac{|N_1-p|^p}{(pR)^p}\int_D |u|^p dx, \tag{1.8}$$

where $R=\sup_{x\in D}|x'|$.

Note that condition on nonlinearity (2.1) includes the following cases:

1. Philippin and Proytcheva [44] used the condition
$$(2+\epsilon)F(u)\leq uf(u),\quad u>0, \tag{1.9}$$
where $\epsilon>0$. It is a special case of an abstract condition by Levine and Payne [45].
2. Bandle and Brunner [6] relaxed this condition as follows
$$(2+\epsilon)F(u)\leq uf(u)+\gamma,\quad u>0, \tag{1.10}$$
where $\epsilon>0$ and $\gamma>0$.

These cases were established on the bounded domains of the Euclidean space, and it is a new result on the stratified groups.

Also, the condition (1.7) depends on a domain D, due to the term βu^p where β is related to constant $\frac{|N_1-p|^p}{(pR)^p}$, which can be interpreted as a measure of the size of the domain D. Then β in (1.7) is dependent on the size of the domain D. If we choose β as arbitrary small in (2.1), then it gets closer to condition (1.10). For small β and $\gamma=0$, condition (2.1) gets closer to (1.9) in the case $p=2$ and $m=1$. Since the case $m>1$ is equivalent to $m=1$ we refer to Sect. 4 in [43] for more detailed discussion to condition (2.1).

Our paper is organised so that we discuss the existence and nonexistence of positive solutions to the nonlinear porous medium equation in Sect. 2 and the nonlinear pseudo-parabolic equation in Sect. 3.

2. Nonlinear porous medium equation

In this section, we prove the global solutions and blow-up phenomena of the initial-boundary value problem (1.1).

2.1. Blow-up solutions of the nonlinear porous medium equation

We start with the blow-up properly.

Theorem 2.1. *Let $\mathbb{G}$ be a stratified group with N_1 being the dimension of the first stratum. Let $D \subset \mathbb{G}$ be an admissible domain. Let $2 \le p < \infty$ with $p \neq N_1$.*

Assume that function f satisfies

$$\alpha F(u) \le u^m f(u) + \beta u^{pm} + \alpha\gamma, \quad u > 0, \tag{2.1}$$

where

$$F(u) = \frac{pm}{m+1} \int_0^u s^{m-1} f(s) ds, \quad m \ge 1,$$

for some

$$\gamma > 0, \; 0 < \beta \le \frac{|N_1 - p|^p}{(pR)^p} \frac{(\alpha - m - 1)}{m+1} \quad \text{and} \quad \alpha > m+1,$$

where $R = \sup_{x \in D} |x'|$ and $x = (x', x'')$ with x' being in the first stratum. Let $u_0 \in L^\infty(D) \cap \mathring{S}^{1,p}(D)$ satisfy the inequality

$$J(0) := -\frac{1}{m+1} \int_D |\nabla_H u_0^m(x)|^p dx + \int_D (F(u_0(x)) - \gamma) \, dx > 0. \tag{2.2}$$

Then any positive solution u of (1.1) blows up in finite time T^*, i.e., there exists

$$0 < T^* \le \frac{M}{\sigma \int_D u_0^{m+1}(x) dx}, \tag{2.3}$$

such that

$$\lim_{t \to T^*} \int_0^t \int_D u^{m+1}(x, \tau) dx d\tau = +\infty, \tag{2.4}$$

where $M > 0$ and $\sigma = \frac{\sqrt{pm\alpha}}{m+1} - 1 > 0$. In fact, in (2.3), we can take

$$M = \frac{(1+\sigma)(1+1/\sigma) \left(\int_D u_0^{m+1}(x) dx \right)^2}{\alpha(m+1) J_0}.$$

Proof of Theorem 2.1. Assume that $u(x,t)$ is a positive solution of (1.1). We use the concavity method for showing the blow-up phenomena introduced by Levine [46]. We introduce the functional

$$J(t) := -\frac{1}{m+1} \int_D |\nabla_H u^m(x,t)|^p dx + \int_D (F(u(x,t)) - \gamma) \, dx, \tag{2.5}$$

and by (2.2) we have

$$J(0) = -\frac{1}{m+1} \int_D |\nabla_H u_0^m(x)|^p dx + \int_D (F(u_0(x)) - \gamma) dx > 0. \tag{2.6}$$

Moreover, $J(t)$ can be written in the following form

$$J(t) = J(0) + \int_0^t \frac{dJ(\tau)}{d\tau} d\tau, \tag{2.7}$$

where

$$\begin{aligned}
\int_0^t \frac{dJ(\tau)}{d\tau} d\tau &= -\frac{1}{m+1} \int_0^t \int_D \frac{d}{d\tau} |\nabla_H u^m(x,\tau)|^p dx d\tau \\
&\quad + \int_0^t \int_D \frac{d}{d\tau} (F(u(x,\tau)) - \gamma) dx d\tau \\
&= -\frac{p}{m+1} \int_0^t \int_D |\nabla_H u^m(x,\tau)|^{p-2} \nabla_H u^m \cdot \nabla_H (u^m(x,\tau))_\tau dx d\tau \\
&\quad + \int_0^t \int_D F_u\left(u(x,\tau)\right) u_\tau(x,\tau) dx d\tau \\
&= \frac{p}{m+1} \int_0^t \int_D \left[\mathcal{L}_p(u^m) + f(u)\right] (u^m(x,\tau))_\tau dx d\tau \\
&= \frac{pm}{m+1} \int_0^t \int_D u^{m-1}(x,\tau) u_\tau^2(x,\tau) dx d\tau.
\end{aligned}$$

Define

$$E(t) = \int_0^t \int_D u^{m+1}(x,\tau) dx d\tau + M, \ \ t \geq 0,$$

with $M > 0$ to be chosen later. Then the first derivative with respect t of $E(t)$ gives

$$\begin{aligned}
E'(t) = \int_D u^{m+1}(x,t) dx &= (m+1) \int_D \int_0^t u^m(x,\tau) u_\tau(x,\tau) d\tau dx \\
&\quad + \int_D u_0^{m+1}(x) dx.
\end{aligned}$$

By applying (2.1), Lemma 1.2 and $0 < \beta \leq \frac{|N_1 - p|^p}{(pR)^p} \frac{(\alpha - m - 1)}{m+1}$, we estimate the second derivative of $E(t)$ as follows

$$\begin{aligned}
E''(t) &= (m+1) \int_D u^m(x,t) u_t(x,t) dx \\
&= -(m+1) \int_D |\nabla_H u^m(x,t)|^p dx + (m+1) \int_D u^m(x,t) f(u(x,t)) dx \\
&\geq -(m+1) \int_D |\nabla_H u^m(x,t)|^p dx \\
&\quad + (m+1) \int_D \left[\alpha F(u(x,t)) - \beta u^{pm}(x,t) - \alpha\gamma\right] dx \\
&= \alpha(m+1) \left[-\frac{1}{m+1} \int_D |\nabla_H u^m(x,t)|^p dx + \int_D (F(u(x,t)) - \gamma) dx\right] \\
&\quad + (\alpha - m - 1) \int_D |\nabla_H u^m(x,t)|^p dx - \beta(m+1) \int_D u^{pm}(x,t) dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \alpha(m+1)\left[-\frac{1}{m+1}\int_D |\nabla_H u^m(x,t)|^p dx + \int_D (F(u(x,t))-\gamma)dx\right] \\
&\quad + \left[\frac{|N_1-p|^p}{(pR)^p}(\alpha-m-1)-\beta(m+1)\right]\int_D u^{pm}(x,t)dx \\
&\geq \alpha(m+1)\left[-\frac{1}{m+1}\int_D |\nabla_H u^m(x,t)|^p dx + \int_D (F(u(x,t))-\gamma)dx\right] \\
&= \alpha(m+1)J(t) \\
&= \alpha(m+1)J(0) + p\alpha m\int_0^t\int_D u^{m-1}(x,\tau)u_\tau^2(x,\tau)dxd\tau.
\end{aligned}
$$

By employing the Hölder and Cauchy-Schwarz inequalities, we obtain the estimate for $[E'(t)]^2$ as follows

$$
\begin{aligned}
[E'(t)]^2 &\leq (1+\delta)\left(\int_D\int_0^t (u^{m+1}(x,\tau))_\tau d\tau dx\right)^2 + \left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}(x)dx\right)^2 \\
&= (m+1)^2(1+\delta)\left(\int_D\int_0^t u^m(x,\tau)u_\tau(x,\tau)dxd\tau\right)^2 \\
&\quad + \left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}(x)dx\right)^2 \\
&= (m+1)^2(1+\delta)\left(\int_D\int_0^t u^{(m+1)/2+(m-1)/2}(x,\tau)u_\tau(x,\tau)dxd\tau\right)^2 \\
&\quad + \left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}(x)dx\right)^2 \\
&\leq (m+1)^2(1+\delta)\left(\int_D\left(\int_0^t u^{m+1}d\tau\right)^{1/2}\left(\int_0^t u^{m-1}u_\tau^2(x,\tau)d\tau\right)^{1/2}dx\right)^2 \\
&\quad + \left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}(x)dx\right)^2 \\
&\leq (m+1)^2(1+\delta)\left(\int_0^t\int_D u^{m+1}dxd\tau\right)\left(\int_0^t\int_D u^{m-1}u_\tau^2(x,\tau)dxd\tau\right) \\
&\quad + \left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}(x)dx\right)^2,
\end{aligned}
$$

for arbitrary $\delta > 0$. So we have

$$
\begin{aligned}
[E'(t)]^2 &\leq (m+1)^2(1+\delta)\left(\int_0^t\int_D u^{m+1}dxd\tau\right)\left(\int_0^t\int_D u^{m-1}u_\tau^2 dxd\tau\right) \\
&\quad + \left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}dx\right)^2. \qquad (2.8)
\end{aligned}
$$

The previous estimates together with $\sigma = \delta = \frac{\sqrt{pm\alpha}}{m+1} - 1 > 0$ where positivity comes from $\alpha > m+1$, imply

$$
E''(t)E(t) - (1+\sigma)[E'(t)]^2
$$

$$\geq \alpha M(m+1)\left[-\frac{1}{m+1}\int_D |\nabla_H u_0^m|^p dx + \int_D (F(u_0)-\gamma)dx\right]$$
$$+ pm\alpha \left(\int_0^t \int_D u^{m+1}(x,\tau)dxd\tau\right)\left(\int_0^t \int_D u_\tau^2(x,\tau)u^{m-1}(x,\tau)dxd\tau\right)$$
$$-(m+1)^2(1+\sigma)(1+\delta)\left(\int_0^t \int_D u^{m+1}dxd\tau\right)\left(\int_0^t \int_D u^{m-1}u_\tau^2(x,\tau)dxd\tau\right)$$
$$-(1+\sigma)\left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}(x)dx\right)^2$$
$$\geq \alpha M(m+1)J(0) - (1+\sigma)\left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}(x)dx\right)^2.$$

By assumption $J(0) > 0$, thus if we select

$$M = \frac{(1+\sigma)\left(1+\frac{1}{\delta}\right)\left(\int_D u_0^{m+1}(x)dx\right)^2}{\alpha(m+1)J(0)},$$

that gives

$$E''(t)E(t) - (1+\sigma)(E'(t))^2 \geq 0. \tag{2.9}$$

We can see that the above expression for $t \geq 0$ implies

$$\frac{d}{dt}\left[\frac{E'(t)}{E^{\sigma+1}(t)}\right] \geq 0 \Rightarrow \begin{cases} E'(t) \geq \left[\frac{E'(0)}{E^{\sigma+1}(0)}\right] E^{1+\sigma}(t), \\ E(0) = M. \end{cases}$$

Then for $\sigma = \frac{\sqrt{pm\alpha}}{m+1} - 1 > 0$, we arrive at

$$-\frac{1}{\sigma}\left[E^{-\sigma}(t) - E^{-\sigma}(0)\right] \geq \frac{E'(0)}{E^{\sigma+1}(0)}t,$$

and some rearrangements with $E(0) = M$ give

$$E(t) \geq \left(\frac{1}{M^\sigma} - \frac{\sigma \int_D u_0^{m+1}(x)dx}{M^{\sigma+1}}t\right)^{-\frac{1}{\sigma}}.$$

Then the blow-up time T^* satisfies

$$0 < T^* \leq \frac{M}{\sigma \int_D u_0^{m+1} dx}.$$

That completes the proof. □

2.2. Global existence for the nonlinear porous medium equation

We now show that under some assumptions, if a positive solution to (1.1) exists, its norm is globally controlled.

Theorem 2.2. *Let $\mathbb{G}$ be a stratified group with N_1 being the dimension of the first stratum. Let $D \subset \mathbb{G}$ be an admissible domain. Let $2 \leq p < \infty$ with $p \neq N_1$.*

Assume that

$$\alpha F(u) \geq u^m f(u) + \beta u^{pm} + \alpha\gamma, \quad u > 0, \tag{2.10}$$

where

$$F(u) = \frac{pm}{m+1}\int_0^u s^{m-1} f(s)ds, \quad m \geq 1,$$

for some

$$\gamma \geq 0, \ \alpha \leq 0 \quad and \quad \beta \geq \frac{|N_1 - p|^p}{(pR)^p}\frac{(\alpha - m - 1)}{m+1},$$

where $R = \sup_{x\in D}|x'|$ and $x = (x', x'')$ with x' being in the first stratum.

Assume also that $u_0 \in L^\infty(D) \cap \mathring{S}^{1,p}(D)$ satisfies inequality

$$J(0) := \int_D (F(u_0(x)) - \gamma)dx - \frac{1}{m+1}\int_D |\nabla_H u_0^m(x)|^p dx > 0. \tag{2.11}$$

If u is a positive local solution of problem (1.1), then it is global and satisfies the following estimate:

$$\int_D u^{m+1}(x,t)dx \leq \int_D u_0^{m+1}(x)dx.$$

Proof of Theorem 2.2. Recall from the proof of Theorem 2.1, the functional

$$\begin{aligned} J(t) &:= -\frac{1}{m+1}\int_D |\nabla_H u^m(x,t)|^p dx + \int_D (F(u(x,t)) - \gamma)dx \\ &= J_0 + \frac{pm}{m+1}\int_0^t \int_D u^{m-1}(x,\tau)u_\tau^2(x,\tau)dxd\tau. \end{aligned}$$

Let us define

$$\mathcal{E}(t) = \int_D u^{m+1}(x,t)dx.$$

By applying (2.10), Lemma 1.2 and $\beta \geq \frac{|N_1-p|^p}{(pR)^p}\frac{(\alpha-m-1)}{m+1}$, respectively, one finds

$$\begin{aligned} \mathcal{E}'(t) &= (m+1)\int_D u^m(x,t)u_t(x,t)dx \\ &= (m+1)\Big[\int_D u^m(x,t)\nabla_H \cdot \big(|\nabla_H u^m(x,t)|^{p-2}\nabla_H u^m(x,t)\big) \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_D u^m(x,t) f(u(x,t))dx \Big] \\
&= (m+1)\left[-\int_D |\nabla_H u^m(x,t)|^p dx + \int_D u^m(x,t) f(u(x,t))dx \right] \\
&\leq (m+1)\left[-\int_D |\nabla_H u^m(x,t)|^p dx + \int_D \left[\alpha F(u(x,t)) - \beta u^{pm}(x,t) - \alpha\gamma\right] dx \right] \\
&= \alpha(m+1)\left[-\frac{1}{m+1}\int_D |\nabla_H u^m(x,t)|^p dx + \int_D (F(u(x,t)) - \gamma) dx \right] \\
&\quad - (m+1-\alpha)\int_D |\nabla_H u^m(x,t)|^p dx - \beta(m+1)\int_D u^{pm}(x,t)dx \\
&\leq \alpha(m+1)\left[-\frac{1}{m+1}\int_D |\nabla_H u^m(x,t)|^p dx + \int_D (F(u(x,t)) - \gamma) dx \right] \\
&\quad - \left[\frac{|N_1 - p|^p}{(pR)^p}(m+1-\alpha) + \beta(m+1) \right] \int_D u^{pm}(x,t)dx \\
&\leq \alpha(m+1)\left[-\frac{1}{m+1}\int_D |\nabla_H u^m(x,t)|^2 dx + \int_D (F(u(x,t)) - \gamma) dx \right] \\
&= \alpha(m+1)J(t).
\end{aligned}
$$

We can rewrite $\mathcal{E}'(t)$ by using (2.7) and $\alpha \leq 0$ as follows

$$
\mathcal{E}'(t) \leq \alpha(m+1)J(0) + p\alpha m \int_0^t \int_D u^{m-1}(x,\tau) u_\tau^2(x,\tau) dx d\tau \leq 0. \quad (2.12)
$$

That gives

$$
\mathcal{E}(t) \leq \mathcal{E}(0).
$$

This completes the proof of Theorem 2.2. □

3. Nonlinear pseudo-parabolic equation

In this section, we prove the global solutions and blow-up phenomena of the initial-boundary value problem (1.2).

3.1. Blow-up phenomena for the pseudo-parabolic equation

We start with conditions ensuring the blow-up of solutions in finite time.

Theorem 3.1. *Let $\mathbb{G}$ be a stratified group with N_1 being the dimension of the first stratum. Let $D \subset \mathbb{G}$ be an admissible domain. Let $2 \leq p < \infty$ with $p \neq N_1$.*

Assume that

$$
\alpha F(u) \leq u f(u) + \beta u^p + \alpha\gamma, \quad u > 0, \quad (3.1)
$$

where

$$
F(u) = \int_0^u f(s)ds,
$$

for some

$$\alpha > p \ \ and \ \ 0 < \beta \leq \frac{|N_1 - p|^p}{(pR)^p}\frac{(\alpha - p)}{p},$$
$$\gamma > 0 \ \ and \ \ R = \sup_{x \in D} |x'|. \tag{3.2}$$

Assume also that $u_0 \in L^\infty(D) \cap \mathring{S}^{1,p}(D)$ *satisfies*

$$\mathcal{F}_0 := -\frac{1}{p}\int_D |\nabla_H u_0(x)|^p dx + \int_D (F(u_0(x)) - \gamma)dx > 0. \tag{3.3}$$

Then any positive solution u of (1.2) blows up in finite time T^*, *i.e., there exists*

$$0 < T^* \leq \frac{M}{\sigma \int_D u_0^2 + \frac{2}{p}|\nabla_H u_0|^p dx}, \tag{3.4}$$

such that

$$\lim_{t \to T^*} \int_0^t \int_D \left[u^2 + \frac{2}{p}|\nabla_H u|^p\right] dx d\tau = +\infty, \tag{3.5}$$

where $\sigma = \sqrt{\frac{\alpha}{2}} - 1 > 0$ *and*

$$M = \frac{(1+\sigma)\left(1 + \frac{1}{\sigma}\right)\left(\int_D u_0^2 + \frac{2}{p}|\nabla_H u_0|^p dx\right)^2}{2\alpha \mathcal{F}_0}.$$

Proof of Theorem 3.1. The proof is based on a concavity method. The main idea is to show that $[E_p^{-\sigma}(t)]'' \leq 0$ which means that $E_p^{-\sigma}(t)$ is a concave function, for $E_p(t)$ defined below.

Let us introduce some notations:

$$\mathcal{F}(t) := -\frac{1}{p}\int_D |\nabla_H u(x,t)|^p dx + \int_D (F(u(x,t)) - \gamma)\, dx,$$

and

$$\mathcal{F}(0) := -\frac{1}{p}\int_D |\nabla_H u_0(x)|^p dx + \int_D (F(u_0(x)) - \gamma) dx,$$

with

$$F(u) = \int_0^u f(s) ds.$$

We know that

$$\mathcal{F}(t) = \mathcal{F}(0) + \int_0^t \frac{d\mathcal{F}(\tau)}{d\tau} d\tau, \tag{3.6}$$

where

$$\begin{aligned}\int_0^t \frac{d\mathcal{F}(\tau)}{d\tau}d\tau &= -\frac{1}{p}\int_0^t\int_D \frac{d}{d\tau}|\nabla_H u|^p dxd\tau + \int_0^t\int_D \frac{d}{d\tau}(F(u)-\gamma)dxd\tau\\ &= -\int_0^t\int_D |\nabla_H u|^{p-2}\nabla u\cdot\nabla_H u_\tau dxd\tau + \int_0^t\int_D F_u(u)u_\tau dxd\tau\\ &= \int_0^t\int_D \left[\mathcal{L}_p u + f(u)\right]u_\tau dxd\tau\\ &= \int_0^t\int_D u_\tau^2 - u_\tau\nabla_H\cdot\left(|\nabla_H u|^{p-2}\nabla_H u_\tau\right)dxd\tau\\ &= \int_0^t\int_D u_\tau^2 + |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 dxd\tau.\end{aligned}$$

Let us define

$$E_p(t) := \int_0^t\int_D\left[u^2+\frac{2}{p}|\nabla_H u|^p\right]dxd\tau + M,\ \ t\ge 0,$$

with a positive constant $M>0$ to be chosen later. Then

$$\begin{aligned}E_p'(t) = \int_D\left[u^2+\frac{2}{p}|\nabla_H u|^p\right]dx &= \int_0^t\frac{d}{d\tau}\int_D\left[u^2+\frac{2}{p}|\nabla_H u|^p\right]dxd\tau\\ &+\int_D u_0^2+\frac{2}{p}|\nabla_H u_0|^p dx. \end{aligned}\tag{3.7}$$

Now we estimate $E_p''(t)$ by using assumption (3.1) and integration by parts, that gives

$$\begin{aligned}E_p''(t) &= 2\int_D uu_t dx + \frac{2}{p}\int_D (|\nabla_H u|^p)_t dx\\ &= 2\int_D\left[u\mathcal{L}_p u + u\nabla_H\cdot\left(|\nabla_H u|^{p-2}\nabla_H u_t\right)+uf(u)\right]dx\\ &+\frac{2}{p}\int_D(|\nabla_H u|^p)_t dx\\ &= -2\int_D\left[|\nabla_H u|^p + |\nabla_H u|^{p-2}\nabla_H u\cdot\nabla_H u_t\right]dx + 2\int_D uf(u)dx\\ &+\frac{2}{p}\int_D(|\nabla_H u|^p)_t dx\\ &\ge -2\int_D|\nabla_H u|^p dx + 2\int_D\left[\alpha F(u)-\beta u^p-\alpha\gamma\right]dx\\ &= 2\alpha\left[-\frac{1}{p}\int_D|\nabla_H u|^p dx+\int_D(F(u)-\gamma)dx\right]\\ &+\frac{2(\alpha-p)}{p}\int_D|\nabla_H u|^p dx - 2\beta\int_D u^p dx.\end{aligned}$$

Next we apply Lemma 1.2, which gives

$$\begin{aligned}
&\geq 2\alpha\left[-\frac{1}{p}\int_D |\nabla_H u|^p dx + \int_D (F(u)-\gamma)dx\right] \\
&\quad + 2\left[\frac{|N_1-p|^p}{(pR)^p}\frac{(\alpha-p)}{p} - \beta\right]\int_D u^p dx \\
&\geq 2\alpha\left[-\frac{1}{p}\int_D |\nabla_H u|^p dx + \int_D (F(u)-\gamma)dx\right] \\
&= 2\alpha\mathcal{F}(t),
\end{aligned}$$

with $\mathcal{F}(t)$ as in (3.6), then $E_p''(t)$ can be rewritten in the following form

$$E_p''(t) \geq 2\alpha\mathcal{F}(0) + 2\alpha\int_0^t\int_D \left[u_\tau^2 + |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2\right]dxd\tau. \tag{3.8}$$

Also we have for arbitrary $\delta > 0$, in view of (3.7),

$$\begin{aligned}
[E_p'(t)]^2 &\leq (1+\delta)\left(\int_0^t \frac{d}{d\tau}\int_D \left[u^2 + \frac{2}{p}|\nabla_H u|^p\right]dxd\tau\right)^2 \\
&\quad + \left(1+\frac{1}{\delta}\right)\left(\int_D \left[u_0^2 + \frac{2}{p}|\nabla_H u_0|^p\right]dx\right)^2.
\end{aligned}$$

Then by taking $\sigma = \delta = \sqrt{\frac{\alpha}{2}} - 1 > 0$, we arrive at

$$\begin{aligned}
&E_p''(t)E_p(t) - (1+\sigma)[E_p'(t)]^2 \\
&\quad \geq 2\alpha M\mathcal{F}(0) + 2\alpha\left(\int_0^t\int_D \left[u_\tau^2 + |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2\right]dxd\tau\right) \\
&\quad \times \left(\int_0^t\int_D \left[u^2 + \frac{2}{p}|\nabla_H u|^p dx\right]d\tau\right) \\
&\quad\quad - (1+\sigma)(1+\delta)\left(\int_0^t \frac{d}{d\tau}\int_D \left[u^2 + \frac{2}{p}|\nabla_H u|^p\right]dxd\tau\right)^2 \\
&\quad\quad - (1+\sigma)\left(1+\frac{1}{\delta}\right)\left(\int_D \left[u_0^2 + \frac{2}{p}|\nabla_H u_0|^p\right]dx\right)^2 \\
&\quad = 2\alpha M\mathcal{F}(0) - (1+\sigma)\left(1+\frac{1}{\delta}\right)\left(\int_D \left[u_0^2 + \frac{2}{p}|\nabla_H u_0|^p\right]dx\right)^2 \\
&\quad\quad + 2\alpha\left[\left(\int_0^t\int_D \left[u_\tau^2 + |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2\right]dxd\tau\right)\right. \\
&\quad\quad \times \left(\int_0^t\int_D \left[u^2 + \frac{2}{p}|\nabla_H u|^p dx\right]d\tau\right) \\
&\quad\quad \left. - \left(\int_0^t\int_D \left[uu_\tau + |\nabla_H u|^{p-2}\nabla_H u\cdot\nabla_H u_\tau\right]dxd\tau\right)^2\right]
\end{aligned}$$

$$\geq 2\alpha M\mathcal{F}(0) - (1+\sigma)\left(1+\frac{1}{\delta}\right)\left(\int_D u_0^2 + \frac{2}{p}|\nabla_H u_0|^p dx\right)^2.$$

Note that in the last line we have used the following inequality

$$\begin{aligned}
&\left(\int_0^t \int_D \left[u^2 + |\nabla_H u|^p\right] dx d\tau\right)\left(\int_0^t \int_D \left[u_\tau^2 + |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2\right] dx d\tau\right) \\
&\quad - \left(\int_0^t \int_D \left[u u_\tau + |\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H u_\tau\right] dx d\tau\right)^2 \\
&\geq \left[\left(\int_D \int_0^t u^2 d\tau dx\right)^{\frac{1}{2}}\left(\int_D \int_0^t |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 d\tau dx\right)^{\frac{1}{2}}\right. \\
&\quad \left. - \left(\int_D \int_0^t |\nabla_H u|^p d\tau dx\right)^{\frac{1}{2}}\left(\int_D \int_0^t u_\tau^2 d\tau dx\right)^{\frac{1}{2}}\right]^2 \geq 0,
\end{aligned}$$

where making use of the Hölder inequality and Cauchy-Schawrz inequality we have

$$\begin{aligned}
&\left(\int_0^t \int_D [u u_\tau + |\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H u_\tau] dx d\tau\right)^2 \\
&\leq \left(\int_D \left(\int_0^t u^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t u_\tau^2 d\tau\right)^{\frac{1}{2}} dx\right. \\
&\quad \left. + \int_D \left(\int_0^t |\nabla_H u|^p d\tau\right)^{\frac{1}{2}}\left(\int_0^t |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 d\tau\right)^{\frac{1}{2}} dx\right)^2 \\
&= \left(\int_D \left(\int_0^t u^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t u_\tau^2 d\tau\right)^{\frac{1}{2}} dx\right)^2 \\
&\quad + \left(\int_D \left(\int_0^t |\nabla_H u|^p d\tau\right)^{\frac{1}{2}}\left(\int_0^t |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 d\tau\right)^{\frac{1}{2}} dx\right)^2 \\
&\quad + 2\left(\int_D \left(\int_0^t u^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t u_\tau^2 d\tau\right)^{\frac{1}{2}} dx\right) \\
&\times \left(\int_D \left(\int_0^t |\nabla_H u|^p d\tau\right)^{\frac{1}{2}}\left(\int_0^t |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 d\tau\right)^{\frac{1}{2}} dx\right) \\
&\leq \left(\int_D \int_0^t u^2 d\tau dx\right)\left(\int_D \int_0^t u_\tau^2 d\tau dx\right) + \left(\int_D \int_0^t |\nabla_H u|^p d\tau dx\right) \\
&\quad \times \left(\int_D \int_0^t |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 d\tau dx\right) \\
&\quad + 2\left[\left(\int_D \int_0^t u^2 d\tau dx\right)\left(\int_D \int_0^t u_\tau^2 d\tau dx\right)\right.
\end{aligned}$$

$$\times \left(\int_D \int_0^t |\nabla_H u|^p d\tau dx\right)\left(\int_D \int_0^t |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 d\tau dx\right)\Bigg]^{\frac{1}{2}}.$$

By assumption $\mathcal{F}(0) > 0$, thus we can select

$$M = \frac{(1+\sigma)\left(1+\frac{1}{\delta}\right)\left(\int_D u_0^2 + \frac{2}{p}|\nabla_H u_0|^p dx\right)^2}{2\alpha\mathcal{F}(0)},$$

that gives

$$E_p''(t)E_p(t) - (1+\sigma)[E_p'(t)]^2 \geq 0. \tag{3.9}$$

We can see that the above expression for $t \geq 0$ implies

$$\frac{d}{dt}\left[\frac{E_p'(t)}{E_p^{\sigma+1}(t)}\right] \geq 0 \Rightarrow \begin{cases} E_p'(t) \geq \left[\frac{E_p'(0)}{E_p^{\sigma+1}(0)}\right] E_p^{1+\sigma}(t), \\ E_p(0) = M. \end{cases}$$

Then for $\sigma = \sqrt{\frac{\alpha}{2}} - 1 > 0$, we arrive at

$$E_p(t) \geq \left(\frac{1}{M^\sigma} - \frac{\sigma \int_D \left[u_0^2 + \frac{2}{p}|\nabla_H u_0|^p\right] dx}{M^{\sigma+1}} t\right)^{-\frac{1}{\sigma}}.$$

Then the blow-up time T^* satisfies

$$0 < T^* \leq \frac{M}{\sigma \int_D \left[u_0^2 + \frac{2}{p}|\nabla_H u_0|^p\right] dx}.$$

This completes the proof. □

3.2. Global solution for the pseudo-parabolic equation

We now show that positive solutions, when they exist for some nonlinearities, can be controlled.

Theorem 3.2. *Let $\mathbb{G}$ be a stratified group with N_1 being the dimension of the first stratum. Let $D \subset \mathbb{G}$ be an admissible domain. Let $2 \leq p < \infty$.*

Assume that function f satisfies

$$\alpha F(u) \geq uf(u) + \beta u^p + \alpha\gamma, \quad u > 0, \tag{3.10}$$

where

$$F(u) = \int_0^u f(s)ds,$$

for some

$$\beta \geq \frac{(p-\alpha)}{2} \quad \text{and} \quad \alpha \leq 0, \ \gamma \geq 0. \tag{3.11}$$

Let $u_0 \in L^\infty(D) \cap \mathring{S}^{1,p}(D)$ *satisfy*

$$\mathcal{F}_0 := -\frac{1}{p}\int_D |\nabla_H u_0(x)|^p dx + \int_D (F(u_0(x)) - \gamma)\, dx > 0. \tag{3.12}$$

If u *is a positive local solution of problem (1.2), then it is global and satisfies the following estimate:*

$$\int_D \left[u^2 + \frac{2}{p}|\nabla_H u|^p\right] dx \leq \exp\left(-(p-\alpha)t\right) \int_D \left[u_0^2 + \frac{2}{p}|\nabla_H u_0|^p\right] dx.$$

Proof of Theorem 3.2. Define

$$\mathcal{E}(t) := \int_D \left[u^2 + \frac{2}{p}|\nabla_H u|^p\right] dx.$$

Now we estimate $\mathcal{E}'(t)$ by using assumption (3.10), that gives

$$\begin{aligned}
\mathcal{E}'(t) &= 2\int_D u u_t dx + \frac{2}{p}\int_D (|\nabla_H u|^p)_t dx \\
&= 2\int_D \left[u\mathcal{L}_p u + u\nabla_H \cdot \left(|\nabla_H u|^{p-2}\nabla_H u_t\right) + uf(u)\right] dx \\
&\quad + \frac{2}{p}\int_D (|\nabla_H u|^p)_t dx \\
&= -2\int_D \left[|\nabla_H u|^p + |\nabla_H u|^{p-2}\nabla_H u \cdot \nabla_H u_t\right] dx + 2\int_D uf(u)dx \\
&\quad + \frac{2}{p}\int_D (|\nabla_H u|^p)_t dx \\
&\leq 2\alpha\left[-\frac{1}{p}\int_D |\nabla_H u|^p dx + \int_D (F(u)-\gamma)dx\right] \\
&\quad - \frac{2(p-\alpha)}{p}\int_D |\nabla_H u|^p dx - 2\beta\int_D u^p dx \\
&\leq 2\alpha\left[-\frac{1}{p}\int_D |\nabla_H u|^p dx + \int_D (F(u)-\gamma)dx\right] \\
&\quad - (p-\alpha)\left[E_p(t) - \int_D u^2 dx\right] dx - 2\beta\int_D u^2 dx, \\
&= 2\alpha\mathcal{F}(t) - (p-\alpha)\mathcal{E}(t) + [p-\alpha-2\beta]\int_D u^2 dx,
\end{aligned}$$

with

$$\mathcal{F}(t) := -\frac{1}{p}\int_D |\nabla_H u(x,t)|^p dx + \int_D (F(u(x,t)) - \gamma)dx$$

$$= \mathcal{F}_0 + \int_0^t \int_D u_\tau^2 + |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 dx d\tau.$$

Since $\beta \geq \frac{p-\alpha}{2}$ we arrive at

$$\mathcal{E}'(t) + (p-\alpha)\mathcal{E}(t) \leq 2\alpha \left[\mathcal{F}_0 + \int_0^t \int_D u_\tau^2 + |\nabla_H u|^{p-2}|\nabla_H u_\tau|^2 dx d\tau\right] \leq 0.$$

This implies,

$$\mathcal{E}(t) \leq \exp\left(-(p-\alpha)t\right)\mathcal{E}(0),$$

finishing the proof. □

Declarations

Conflict of interest The authors have no conflict of interest.

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