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# Supersymmetry and Hodge theory on Sasakian and Vaisman manifolds

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**Abstract.** Sasakian manifolds are odd-dimensional counterpart to Kähler manifolds. They can be defined as contact manifolds equipped with an invariant Kähler structure on their symplectic cone. The quotient of this cone by the homothety action is a complex manifold called Vaisman. We study harmonic forms and Hodge decomposition on Vaisman and Sasakian manifolds. We construct a Lie superalgebra associated to a Sasakian manifold in the same way as the Kähler supersymmetry algebra is associated to a Kähler manifold. We use this construction to produce a self-contained, coordinate-free proof of the results by Tachibana, Kashiwada and Sato on the decomposition of harmonic forms and cohomology of Sasakian and Vaisman manifolds. In the last section, we compute the supersymmetry algebra of Sasakian manifolds explicitly.

**Key words.** Lie superalgebra · Kähler manifold · Sasakian manifold · Vaisman manifold · Reeb field · Hodge theory · Transversally Kähler foliation · Basic forms

## 1. Introduction

Sasakian manifolds are odd-dimensional counterparts to Kähler manifolds. They can be defined as contact manifolds equipped with an invariant Kähler structure on their symplectic cone. Taking a quotient of this cone by the homothety action one

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obtains a complex manifold called Vaisman. Here we develop a version of Hodge theory on Sasakian and Vaisman manifolds.

The modern approach to Hodge theory is inspired by the supersymmetry. Given a manifold with a geometric structure (such as Kähler, hyperkähler, HKT,  $G_2$ -manifold and so on), one takes a bunch of natural operators on the de Rham algebra (such as the de Rham differential, the Lefschetz  $\mathfrak{sl}(2)$ -triple etc.) and proves that these operators generate a finite-dimensional Lie superalgebra. In most cases, the Laplacian is central in this superalgebra. This gives an interesting geometric action on the cohomology.

In this paper we write the natural superalgebra for a Sasakian manifold. Hodge theory for Sasakian manifolds is well developed, but most proofs are written in classical differential geometry style, mixing coordinate computations with invariant ones. We wanted to give a more conceptual proof.

In the end we arrived at a very simple approach to the Hodge theory on Sasakian and Vaisman manifolds (Theorem 6.6, Theorem 8.2). The Sasakian supersymmetry algebra is not used in this development, but it is interesting in itself, and the relations are quite surprising.

When the Sasakian manifold is also Einstein, the superalgebra structure seems to be simpler. Using this approach, J. Schmude ([28]) obtained a closed formula for the de Rham Laplacian in terms of the transversal Laplacian operator.

### *1.1. Supersymmetry in Kähler and non-Kähler geometry*

The connection between de Rham calculus on manifolds with special geometry (such as Kähler and hyperkähler) and their supersymmetry appeared as early as in 1997 ([14]). It is well known that extra supersymmetries of the  $\sigma$ -model force the target space to acquire extra geometric structures: the  $N = 1$  supersymmetry implies Kähler structure on the target space, the  $N = 2$  supersymmetry makes it hyperkähler, and so on. In [14], this supersymmetry was interpreted in terms of the de Rham calculus on the target space.

In [24], the connection between the supersymmetry and rational homotopy theory is further expounded, with the constructions of rational homotopy theory (such as Sullivan's minimal models) interpreted in terms of quantum mechanics.

For Sasakian manifolds, this approach was pioneered by [34], who used the transversal Kähler relations to obtain results about rational homotopy of Sasakian manifolds. This work was applied to homotopy formality of Sasakian manifolds ([4]) and in applications to the geometry of Sasakian nilmanifolds ([8]).

Traditionally, the Kähler identities were obtained using the Levi-Civita connection. Alternatively, one may show that a Kähler manifold can be approximated up to second order by a flat space. Using supersymmetry to prove the Kähler identities (Sect. 2) has many advantages over either of these approaches. In [38], supersymmetry was used to develop the Hodge theory for the HKT manifolds (hyperkähler manifolds with torsion), where the structure tensors are neither preserved by the Levi-Civita connection nor admit a second order approximation. The supersymmetry approach was later used in supersymmetric  $\sigma$ -models associated to HKT manifolds ([15, 16, 32]).

In [40], the supersymmetry approach was used to obtain the Hodge decomposition on nearly Kähler manifolds, where the structure tensors are also non-parallel.

In another direction, one may use the supersymmetry to obtain the superalgebra action on manifolds with parallel differential forms ([39]). The Hodge-theoretic results obtained in this direction were used in [20] to study the geometry of complete  $G_2$ - and  $\text{Spin}(7)$ -manifolds with  $d(\text{linear})$  structure form.

Summing up, finding a superalgebra associated with a given geometric structure seems to be worthwhile, for mathematics as well as for mathematical physics.

### 1.2. Structure of this paper

- In Sect. 2 we relate the Lie superalgebra approach to de Rham calculus and obtain the Kähler identities which lie at the foundation of Hodge theory.
- Section 3 introduces the Sasakian manifolds and explains the basic notions of Sasakian geometry.
- In Sect. 4 we explain how one employs de Rham calculus to obtain the Leray-Serre spectral sequence; this approach was used by A. Hattori in 1960. We need a version of the Leray-Serre spectral sequence which can be applied to smooth foliations with fibers which are not necessarily closed or compact. We compute the differentials of this spectral sequence for Sasakian manifolds explicitly, for later use.
- In Sect. 5 we deal with transversally Kähler structures on smooth foliations. We give simple proofs of the standard results on transversal (basic) cohomology, due to El Kacimi-Alaoui ([12]), much simplified because we need them only for Vaisman and Sasakian manifolds.
- In Sect. 6, we prove the standard results on cohomology for Sasakian manifolds, in Sect. 7 we introduce the Vaisman manifolds, and in Sect. 8 we prove the standard results on cohomology for Vaisman manifolds. The cohomology calculations for Sasakian and Vaisman manifolds are very similar and rely on the same homological algebra argument. Throughout Sect. 8, we use the 1-dimensional transversally Sasakian foliation, generated by the Lee field, and apply the superalgebra computations done in Sect. 6 on Sasakian manifolds. However, the results on Vaisman manifolds cannot be deduced from the results on Sasakian manifolds, because the Lee foliation might have non-closed leaves.
- We finish the paper with an explicit calculation of the Sasakian superalgebra in Sect. 9. This section depends only on Sects. 2–4.

## 2. Lie superalgebras acting on the de Rham algebra

### 2.1. Lie superalgebras and superderivations

In the following, all vector spaces and algebras are considered over  $\mathbb{R}$ . Let  $A$  be a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space,

$$A = A^{\text{even}} \oplus A^{\text{odd}}.$$

We say that  $a \in A$  is **pure** if  $a$  belongs to  $A^{\text{even}}$  or  $A^{\text{odd}}$ . For a pure element  $a \in A$ , we write  $\tilde{a} = 0$  if  $a \in A^{\text{even}}$ , and  $\tilde{a} = 1$  if  $a \in A^{\text{odd}}$ . Consider a bilinear operator

$$\{\cdot, \cdot\} : A \times A \longrightarrow A,$$

called **supercommutator**. Assume that  $\{\cdot, \cdot\}$  is **graded anti-commutative**, that is, satisfies

$$\{a, b\} = -(-1)^{\tilde{a}\tilde{b}}\{b, a\}$$

for pure  $a, b \in A$ . Assume, moreover, that  $\{\cdot, \cdot\}$  is compatible with the grading: the commutator  $\{a, b\}$  is even when both  $a, b$  are even or odd, and odd if one of these elements is odd and another is even. We say that  $(A, \{\cdot, \cdot\})$  is a **Lie superalgebra** if the following identity (called **the graded Jacobi identity**, or **super Jacobi identity**) holds, for all pure elements  $a, b, c \in A$ :

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{\tilde{a}\tilde{b}}\{b, \{a, c\}\}. \quad (2.1)$$

Up to a sign, this is the usual Jacobi identity.

Every reasonable property of Lie algebras has a natural analogue for Lie superalgebras, using the following rule of thumb: every time one would exchange two elements  $a$  and  $b$ , one adds a multiplier  $(-1)^{\tilde{a}\tilde{b}}$ .

*Example 2.1.* Let  $V = V^{\text{even}} \oplus V^{\text{odd}}$  be a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space, and  $\text{End}(V)$  its space of endomorphisms, equipped with the induced grading. We define a supercommutator in  $\text{End}(V)$  by the formula:

$$\{a, b\} = ab - (-1)^{\tilde{a}\tilde{b}}ba$$

It is easy to check that  $(\text{End}(V), \{\cdot, \cdot\})$  is a Lie superalgebra.

*Remark 2.2.* Given a  $\mathbb{Z}$ -graded vector space  $A$ , one defines  $A^{\text{even}}$  as the direct sum of even components, and  $A^{\text{odd}}$  as the direct sum of odd components. Then a  $\mathbb{Z}$ -graded Lie superalgebra is given by a supercommutator on  $A$  satisfying  $\{A^p, A^q\} \subset A^{p+q}$  and satisfying the graded Jacobi identity. In the sequel, all Lie superalgebras we consider are of this type. An endomorphism  $u \in \text{End}(A)$  is called **even** if  $u(A^{\text{odd}}) \subset A^{\text{odd}}$  and  $u(A^{\text{even}}) \subset A^{\text{even}}$ , and **odd** if  $u(A^{\text{even}}) \subset A^{\text{odd}}$  and  $u(A^{\text{odd}}) \subset A^{\text{even}}$ . An endomorphism which is either odd or even is called **pure**.

**Definition 2.3.** A graded algebra  $A$  is called **graded commutative** if  $\{a, b\} = 0$  for all  $a, b \in A$ .

The Grassmann algebra and de Rham algebra are clearly graded commutative.

**Definition 2.4.** Let  $\mathfrak{g}$  be a graded commutative algebra. A map  $\delta : \mathfrak{g} \longrightarrow \mathfrak{g}$  is called an **even derivation** if it is even and satisfies  $\delta(xy) = \delta(x)y + x\delta(y)$ . It is called an **odd derivation** if it is odd and satisfies  $\delta(xy) = \delta(x)y + (-1)^{\tilde{x}}x\delta(y)$ . It is called **graded derivation**, or **superderivation**, if it shifts the grading by  $i$  and satisfies  $\delta(ab) = \delta(a)b + (-1)^{ij}a\delta(b)$ , for each  $a \in A^j$ .

*Remark 2.5.* The supercommutator of two superderivations is again a derivation. Therefore, the derivations form a Lie superalgebra.

## 2.2. Differential operators on graded commutative algebras

We need some algebraic results, which are almost trivial, and well-known for commutative algebras. We extend these statements to graded commutative algebras; the proofs are the same as in the commutative setting (see [9]).

**Definition 2.6.** Let  $A^*$  be a graded commutative algebra. **The algebra of differential operators**  $\text{Diff}(A^*)$  is an associative subalgebra of  $\text{End}(A^*)$  generated by graded derivations and  $A^*$ -linear self-maps. Let  $\text{Diff}^0(A^*)$  be the space of  $A^*$ -linear self-maps,  $\text{Diff}^0(A^*) = A^*$ ,  $\text{Diff}^1(A^*) \supset \text{Diff}^0(A^*)$  the subspace generated by  $\text{Diff}^0(A^*)$  and all graded derivations, and  $\text{Diff}^i(A^*) := \text{Diff}^{i-1}(A^*) \cdot \text{Diff}^1(A^*)$ . This gives a multiplicative filtration  $\text{Diff}^0(A^*) \subset \text{Diff}^1(A^*) \subset \text{Diff}^2(A^*) \subset \dots$ . The elements of  $\text{Diff}^i(A^*)$  are called **differential operators of order  $i$**  on  $A^*$ .

**Claim 2.7.** Let  $D \in \text{Diff}^i(A^*)$ ,  $D' \in \text{Diff}^j(A^*)$  be differential operators on a graded commutative algebra. Then  $\{D, D'\} \in \text{Diff}^{i+j-1}(A^*)$ .

*Proof.* Since the commutator of two derivations is a derivation, one has  $\{\text{Diff}^1(A^*), \text{Diff}^1(A^*)\} \subset \text{Diff}^1(A^*)$ . Then we use induction on  $i$  and the standard commutator identities in the associative algebra.  $\square$

We shall apply this claim to geometric operations on the de Rham algebra, obtaining differential operators of first order. Note that the “differential operator” in the usual sense is a different notion. For example, the interior product operator  $i_v$  of contraction with a vector field  $v$  on a manifold  $M$  is an odd derivation of the de Rham algebra, hence it is a first order differential operator; however,  $i_v$  is  $C^\infty(M)$ -linear, and thus it is not a differential operator in the usual sense.

**Claim 2.8.** Let  $D$  be a differential operator of first order on  $A^*$ . Then  $D(x) = D(1)x + \delta(x)$ , where  $\delta$  is a derivation.

*Proof.* This is seen by defining  $\delta := D - D(1)\cdot$ , then observing that  $\delta(1) = 0$ , which implies that  $\delta \circ a - a \circ \delta = \delta(a)$  (we identify  $a \in A$  with  $a \in \text{End}(A)$ ,  $a(b) = ab$ ). We then have

$$\delta(ab) = (\delta \circ a)(b) = (a \circ \delta)(b) + \delta(a)(b) = a\delta(b) + \delta(a)b,$$

proving that  $\delta$  is a derivation.  $\square$

We use this formalism to compare first order differential operators on  $A^*$  as follows.

**Claim 2.9.** Let  $\delta$  be a derivation on  $A^*$ . Then  $\delta$  is uniquely determined by the values it takes on any set of multiplicative generators of  $A^*$ .  $\square$

**Claim 2.10.** Let  $D$  be a first order differential operator on  $A^*$ . Then  $D$  is uniquely determined by  $D(1)$  and the values it takes on any set of multiplicative generators of  $A^*$ .

*Proof.* By Claim 2.8,  $D - D(1)$  is a derivation, hence Claim 2.9 implies Claim 2.10.  $\square$

**Corollary 2.11.** *Let  $d_1, d_2 : \Lambda^*(M) \rightarrow \Lambda^{*+1}(M)$  be first order differential operators on the de Rham algebra of a manifold  $M$ , satisfying  $d_1^2 = d_2^2 = 0$ , and  $V \subset \Lambda^1(M)$  a subspace such that the space  $d_1(C^\infty(M)) + V$  generates  $\Lambda^1(M)$  as a  $C^\infty(M)$ -module. Suppose that  $d_1|_{C^\infty(M)} = d_2|_{C^\infty(M)}$  and  $d_1|_V = d_2|_V$ . Then  $d_1 = d_2$ .*

*Proof.* Clearly,  $d_1 = d_2$  on  $C^\infty(M) \oplus d_1(C^\infty(M))$ . Then  $d_1 = d_2$  on the set of multiplicative generators  $C^\infty(M) + d_1(C^\infty(M)) + V$ , and Claim 2.10 implies Corollary 2.11.  $\square$

The following claim is used many times in the sequel.

**Claim 2.12.** Let  $d$  be an odd element in a Lie superalgebra  $\mathfrak{h}$ , satisfying  $\{d, d\} = 0$ . Then  $\{d, \{d, u\}\} = 0$  for any  $u \in \mathfrak{h}$ .

*Proof.* By the super Jacobi identity,

$$\{d, \{d, u\}\} = -\{d, \{d, u\}\} + \{\{d, d\}, u\} = -\{d, \{d, u\}\}. \tag{2.2}$$

$\square$

This claim is a special case of the following:

**Claim 2.13.** Let  $d$  be an odd element in a Lie superalgebra  $\mathfrak{h}$ . Then  $2\{d, \{d, u\}\} = \{\{d, d\}, u\}$  for any  $u \in \mathfrak{h}$ .

*Proof.* Follows from (2.2).  $\square$

### 2.3. Supersymmetry on Kähler manifolds

One of the purposes of this paper is to obtain a natural superalgebra acting on the de Rham algebra of a Sasakian manifold. This is modeled on the superalgebra of a Kähler manifold, generated by the de Rham differential, Lefschetz triple, and other geometric operators. To make the analogy more clear, we recall the main results on the supersymmetry algebra of Kähler manifolds. We follow [38, Section 1.3].

Let  $(M, I, g, \omega)$  be a Kähler manifold. Consider  $\Lambda^*(M)$  as a graded vector space. The differentials  $d, d^c := -IdI = IdI^{-1}$  can be interpreted as odd elements in  $\text{End}(\Lambda^*(M))$ , and the Hodge operators  $L, \Lambda, H$  as even elements. As usual, we denote the supercommutator as  $\{\cdot, \cdot\}$ . In terms of the associative algebra,  $\{a, b\} = ab + ba$  when  $a, b$  are odd, and  $\{a, b\} = ab - ba$  if at least one of them is even. Let  $d^* := \{\Lambda, d^c\}$ ,  $(d^c)^* := -\{\Lambda, d\}$ . The usual Kodaira relations can be stated as follows

$$\begin{aligned} \{L, d^*\} &= -d^c, & \{L, (d^c)^*\} &= d, & \{d, (d^c)^*\} &= \{d^*, d^c\} = 0, \\ \{d, d^c\} &= \{d^*, (d^c)^*\} = 0, & \{d, d^*\} &= \{d^c, (d^c)^*\} = \Delta, \end{aligned} \tag{2.3}$$

where  $\Delta$  is the Laplace operator, commuting with  $L, \Lambda, H$ , and  $d, d^c$ .

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Here, as elsewhere, “differential operators on the de Rham algebra” are understood in the algebraic sense as above.

**Definition 2.14.** Let  $(M, I, g, \omega)$  be a Kähler manifold. Consider the Lie superalgebra  $\mathfrak{a} \subset \text{End}(\Lambda^*(M))$  generated by the following operators:

1.  $d, d^*, \Delta$ , constructed out of the Riemannian metric.
2.  $L(\alpha) := \omega \wedge \alpha$ .
3.  $\Lambda(\alpha) := *L*\alpha$ . It is easily seen that  $\Lambda = L^*$ .
4. The Weil operator  $W|_{\Lambda^{p,q}(M)} = \sqrt{-1} (p - q)$

This Lie superalgebra is called **the algebra of supersymmetry of the Kähler manifold**.

Using Theorem 2.15 below, it is easy to see that  $\mathfrak{a}$  is in fact independent from  $M$ .

This Lie superalgebra was studied from the physicists' point of view in [14].

**Theorem 2.15.** *Let  $M$  be a Kähler manifold, and  $\mathfrak{a}$  its supersymmetry algebra acting on  $\Lambda^*(M)$ . Then  $\mathfrak{a}$  has dimension  $(5|4)$  (that is, its odd part is 4-dimensional, and its even part is 5-dimensional). The odd part is generated by  $d, d^c = IdI^{-1}, d^*, (d^c)^*$ , the even part is generated by the Lefschetz triple  $L, \Lambda, H = [L, \Lambda]$ , the Weil operator  $W$  and the Laplacian  $\Delta = \{d, d^*\}$ . Moreover, the Laplacian  $\Delta$  is central in  $\mathfrak{a}$ , hence  $\mathfrak{a}$  also acts on the cohomology of  $M$ . The following are the only non-zero commutator relations in  $\mathfrak{a}$ :*

1.  $\mathfrak{sl}(2)$ -relations in  $\langle L, \Lambda, H = [L, \Lambda] \rangle$ :

$$[H, L] = 2L, \quad [H, \Lambda] = -2\Lambda, \quad [L, \Lambda] = H.$$

*For any operator  $D$  of grading  $k$ , one has  $[H, D] = kD$ .*

2. *The Weil operator acts as a complex structure on the odd part of  $\mathfrak{a}$ :*

$$[W, d] = d^c, \quad [W, d^c] = -d, \quad [W, d^*] = -(d^c)^*, \quad [W, (d^c)^*] = d^*.$$

3. *The Kähler-Kodaira relations between the differentials and the Lefschetz operators are:*

$$[\Lambda, d] = (d^c)^*, \quad [L, d^*] = -d^c, \quad [\Lambda, d^c] = -d^*, \quad [L, (d^c)^*] = d. \quad (2.4)$$

4. *Almost all odd elements supercommute, with the only exception*

$$\Delta = \{d, d^*\} = \{d^c, (d^c)^*\},$$

*and  $\Delta$  is central. In other words, the odd elements of  $\mathfrak{a}$  generate the odd Heisenberg superalgebra, see Claim 2.16.*

*Proof.* These relations are standard in algebraic geometry (see e.g. [18]), but probably the easiest way to prove them is using the results about Lie superalgebras collected in Sect. 2.2.  $\square$

**Proof of the Lefschetz  $\mathfrak{sl}(2)$ -relations:** These relations would follow if we prove that  $H := [L, \Lambda]$  acts on  $p$ -forms by multiplication by  $p - n$ , where  $n = \dim_{\mathbb{C}} M$ . Since  $L, \Lambda, H$  are  $C^\infty(M)$ -linear, it would suffice to prove these relations on a Hermitian vector space.

Let  $V$  be a real vector space equipped with a scalar product, and fix an orthonormal basis  $\{v_1, \dots, v_m\}$ . Denote by  $e_{v_i} : \Lambda^k V \rightarrow \Lambda^{k+1} V$  the operator of multiplication,  $e_{v_i}(\eta) = v_i \wedge \eta$ . Let  $i_{v_i} : \Lambda^k V \rightarrow \Lambda^{k-1} V$  be the operator of contraction with  $v_i$ . The following claim is clear.

**Claim 2.16.** The operators  $e_{v_i}, i_{v_i}, \text{Id}$  form a basis of the **odd Heisenberg Lie superalgebra**, with the only non-trivial supercommutator given by the formula  $\{e_{v_i}, i_{v_j}\} = \delta_{ij} \text{Id}$ . □

Let now  $V$  be an even-dimensional real vector space equipped with a scalar product, and  $\{x_1, \dots, x_n, y_1, \dots, y_n\}$  an orthonormal basis. Consider the complex structure operator  $I$  such that  $I(x_i) = y_i, I(y_i) = -x_i$ . The fundamental symplectic form is given by  $\sum_i x_i \wedge y_i$ , hence

$$L = \sum_i e_{x_i} e_{y_i}, \quad \Lambda = \sum_i i_{x_i} i_{y_i}.$$

Clearly, for any odd elements  $a, b, c, d$  such that  $\{a, b\} = \{a, d\} = \{b, c\} = \{c, d\} = 0$ , one has  $\{ab, cd\} = -\{a, c\}bd + ca\{b, d\}$ . Then

$$\begin{aligned} [L, \Lambda] &= \left[ \sum_i e_{x_i} e_{y_i}, \sum_i i_{x_i} i_{y_i} \right] = \sum_{i=1}^n e_{y_i} i_{y_i} - \sum_{i=1}^n i_{x_i} e_{x_i} \\ &= \sum_{i=1}^n e_{y_i} i_{y_i} + \sum_{i=1}^n (e_{x_i} i_{x_i} - 1) \end{aligned}$$

This term, applied to a monomial  $\alpha$  of degree  $d$ , would give  $(d - n)\alpha$ . This proves the Lefschetz  $\mathfrak{sl}(2)$ -relations.

**Proof of the relations between the Weil operator  $W$  and the odd part of  $\alpha$ .** Clearly, it is enough to prove  $[W, d] = d^c$ , the remaining relations follow by duality or by complex conjugation. Writing the Hodge components of  $d = d^{1,0} + d^{0,1}$ , with  $d^{1,0} = \frac{d + \sqrt{-1}d^c}{2}$  and  $d^{0,1} = \frac{d - \sqrt{-1}d^c}{2}$ , we obtain  $[W, d] = \sqrt{-1}d^{1,0} - \sqrt{-1}d^{0,1} = d^c$ . □

**Proof of the Kähler-Kodaira relations between the Lefschetz  $\mathfrak{sl}(2)$ -operators and the odd part of  $\alpha$ .** As before, it is enough to prove  $[L, d^*] = -d^c$ , the remaining Kähler-Kodaira relations follow by duality or by complex conjugation. The operator  $L$  is  $C^\infty(M)$ -linear, hence it is a differential operator of order 0. The operator  $d^*$  can be written in a frame  $\{v_i\}$  of  $TM$  as

$$d^*(\eta) = \sum_i i_{v_i} \nabla_{v_i} \eta, \tag{2.5}$$

where  $\nabla$  is the Levi-Civita connection of the metric  $g$ . Since  $\nabla_{v_i}$  and  $i_{v_i}$  are both derivations of the de Rham algebra, their product is an order 2 differential operator



(in the algebraic sense as given by Definition 2.6). From Claim 2.7 it follows that  $[L, d^*]$  is a first order operator.

We shall prove  $[L, d^*] = -d^c$  by applying Corollary 2.11. First, let us show that  $[L, [L, d^*]] = 0$ . Clearly,  $[\Lambda, d^*] = ([L, d])^* = 0$ , and  $[H, d^*] = -d^*$ , hence  $d^*$  is the lowest weight vector in a weight 1 representation of  $\mathfrak{sl}(2)$ . This gives  $[L, [L, d^*]] = 0$ . Now, the super Jacobi identity gives

$$\{\{L, d^*\}, \{L, d^*\}\} = \{L, \{d^*, \{L, d^*\}\} + \{d^*, \{L, \{L, d^*\}\}\}.$$

The first term in the RHS vanishes by Claim 2.12, and the second term vanishes because  $[L, [L, d^*]] = 0$ . Then  $([L, d^*])^2 = 0$ . Clearly,  $d^c(C^\infty(M))$  generates  $\Lambda^1(M)$  over  $C^\infty(M)$ . To deduce  $[L, d^*] = -d^c$  from Corollary 2.11, it remains to show that  $[L, d^*]|_{C^\infty(M)} = -d^c|_{C^\infty(M)}$ . This is clear for the following reason. For any function  $f \in C^\infty(M)$ , one has  $[L, d^*]f = -d^*(f\omega)$ . Writing  $d^* = \sum_i i_{v_i} \nabla_{v_i}$  as in (2.5), and using  $\nabla\omega = 0$ , we obtain

$$d^*(f\omega) = \sum_i i_{v_i} \nabla_{v_i}(f\omega) = \sum_i \text{Lie}_{v_i}(f)i_{v_i}(\omega) = \sum_i -\text{Lie}_{v_i}(f)I(v_i)^\flat = -d^c f.$$

This finishes the proof of  $[L, d^*] = -d^c$ .  $\square$

**Proof of the commutator relations between the odd part of  $\mathfrak{a}$ .** We have already shown that  $d^c = [W, d]$ . Then  $\{d, d^c\} = \{d, \{W, d\}\} = 0$  by Claim 2.12. Similarly,  $d^* = -[\Lambda, d^c]$ , giving  $\{d^c, d^*\} = 0$ . The relation  $\{(d^c)^*, d\} = 0$  is obtained by duality. Finally,  $\{d, d^*\} = \{d^c, (d^c)^*\}$  is obtained by applying  $[\Lambda, \cdot]$  to  $\{d, d^c\} = 0$ . Using the Kähler-Kodaira relations (2.4), we obtain

$$0 = \{\Lambda, \{d, d^c\}\} = \{\{\Lambda, d\}, d^c\} + \{d, \{\Lambda, d^c\}\} = \{(d^c)^*, d^c\} - \{d, d^*\},$$

giving  $\{d, d^*\} = \{d^c, (d^c)^*\}$ .

We proved all the relations in the Kähler supersymmetry algebra  $\mathfrak{a}$ , finishing the proof of Theorem 2.15.  $\square$

Further in this paper, we shall develop similar relations for the superalgebra associated with a Sasakian manifold.

### 3. Sasakian manifolds: definition and the basic notions

In this section we provide the necessary background on Sasakian manifolds. For details, please see [5, 6]. The most convenient definition for our context is the one which relates Sasakian and Kähler geometries in terms of Riemannian cones.

**Definition 3.1.** A **Sasakian manifold** is a Riemannian manifold  $(S, g)$  with a Kähler structure on its Riemannian cone  $C(S) := (S \times \mathbb{R}^{>0}, t^2g + dt^2)$ , such that the homothety map  $h_\lambda : C(S) \rightarrow C(S)$  mapping  $(m, t)$  to  $(m, \lambda t)$  is holomorphic.

*Remark 3.2.* (i) A Sasakian manifold is clearly contact, because its cone is symplectic and  $h_\lambda$  acts by symplectic homotheties.

(ii) Let  $S$  be a Sasakian manifold,  $\omega$  the Kähler form on  $C(S)$ , and  $\xi = t \frac{d}{dt}$  the homothety vector field along the generators of the cone (it is also called **Euler**

**field**). The contact form is  $\eta = i_\xi \omega|_{t=1}$ . Then  $\text{Lie}_{I\xi} t = \langle dt, I\xi \rangle = 0$  (where  $\text{Lie}_v$  denotes the Lie derivative in the direction of  $v$ ), and hence  $I\xi$  is tangent to  $S \subset C(S)$ . Then  $\eta(I\xi) = 1$  and  $i_{I\xi} d\eta = 0$ , and thus **the Reeb vector field of a Sasakian manifold is  $\vec{r} = I\xi$** .

(iii) The function  $t^2$  is a Kähler potential on  $C(S)$ . Moreover, the form  $dd^c \log t$  vanishes on  $\langle \xi, I(\xi) \rangle$  and the rest of its eigenvalues are positive.

**Proposition 3.3.** *The Reeb field has constant length and acts on a Sasakian manifold by contact isometries, i.e. the flow of  $\vec{r}$  contains only isometries which preserve the contact subbundle. Moreover, its action lifts to a holomorphic action on its cone.*

*Remark 3.4.* On a  $2n + 1$ -dimensional Sasakian manifold, the contact form satisfies  $\eta \wedge (d\eta)^n \neq 0$ . Since  $i_{\vec{r}} d\eta = 0$ , we see that  $d\eta$  is a volume form on the contact distribution  $\vec{r}^\perp$ .

**Definition 3.5.** A Sasakian (resp. contact) manifold is called **regular** if its Reeb field generates a free action of  $S^1$ ; it is called **quasi-regular** if all orbits of the Reeb field are closed, and it is called **irregular** otherwise.

*Example 3.6.* Let  $S$  be a regular Sasakian manifold, and  $\vec{r}$  its Reeb field. Then the space of  $\vec{r}$ -orbits  $X$  is Kähler. Moreover,  $X$  is equipped with a positive holomorphic Hermitian line bundle  $L$  such that  $S$  is the space of unit vectors in  $L$ . Conversely: if  $X$  is a compact projective manifold, together with an ample line bundle  $L \rightarrow X$ , then the space of unit vectors in  $L$  is a regular Sasakian manifold.

## 4. Hattori differentials on Sasakian manifolds

### 4.1. Hattori spectral sequence and associated differentials

Let  $\pi : M \rightarrow B$  be a smooth fibration, and  $F_k \subset \Lambda^*(M)$  be the ideal generated by  $\pi^* \Lambda^k(B)$ . The Hattori spectral sequence ([19]) is the spectral sequence associated with this filtration.

The  $E_1^{p,q}$ -term of this sequence is  $\Lambda^p(B) \otimes_{\mathbb{R}} R^q \pi_* \mathbb{R}_M$ , where  $R^* \pi_* \mathbb{R}_M$  denotes the local system of cohomology of the fibres, and the  $E_2^{p,q}$ -term is  $H^p(B, R^q \pi_* \mathbb{R}_M)$ .

The same can be done when  $M$  is a manifold equipped with an integrable distribution  $F \subset TM$ , giving a spectral sequence converging to  $H^*(M)$ . This is done as follows.

**Definition 4.1.** Let  $M$  be a manifold, and  $F \subset TM$  an integrable distribution. A  $k$ -form  $\alpha \in \Lambda^*(M)$  is called **basic** if for any vector field  $v \in F$ , one has  $\text{Lie}_v \alpha = 0$  and  $i_v \alpha = 0$ .

If  $F$  is the tangent bundle of the fibres of a fibration  $\pi : M \rightarrow B$ , then the space of basic forms is  $\pi^* \Lambda^k(B)$ . We are going to produce a spectral sequence which gives the standard Hattori spectral sequence when  $F$  is tangent to the leaves of a fibration.

Define the **Hattori filtration associated with  $F$**  as  $F_n \subset F_{n-1} \subset \dots$  by putting  $F_k \subset \Lambda^*(M)$ , where  $F_k$  is the ideal generated by basic  $k$ -forms.

When  $M$  is Riemannian and the metric is compatible with the foliation, this filtered bundle is decomposed into the direct sum of subquotients. This gives a decomposition of the de Rham differential,  $d = d_0 + d_1 + d_2 + \dots + d_{r+1}$ , where  $r = \text{rk } F$  with each successive piece associated with the differential in  $E_k^{p,q}$ , as follows.

Using the metric, we split the cotangent bundle into orthogonal complements as  $\Lambda^1(M) = \Lambda_{\text{hor}}^1(M) \oplus \Lambda_{\text{vert}}^1(M)$ , where  $\Lambda_{\text{hor}}^1(M)$  is generated by basic 1-forms, and  $\Lambda_{\text{vert}}^1(M) = F^*$  is its orthogonal complement. Denote by  $\Lambda_{\text{hor}}^p(M)$ ,  $\Lambda_{\text{vert}}^q(M)$  the exterior powers of these bundles.

This gives the following splitting of the de Rham algebra of  $M$ :

$$\Lambda^m(M) = \bigoplus_p \Lambda_{\text{hor}}^p(M) \otimes \Lambda_{\text{vert}}^{m-p}(M) \quad (4.1)$$

with  $F_p/F_{p-1} = \bigoplus_q \Lambda_{\text{hor}}^p(M) \otimes \Lambda_{\text{vert}}^q(M)$  where  $F_*$  denotes the Hattori filtration.

Consider the associated decomposition of the de Rham differential,  $d = d_0 + d_1 + d_2 + \dots + d_{r+1}$ , where  $r = \text{rk } F$ , and

$$d_i : \Lambda_{\text{hor}}^p(M) \otimes \Lambda_{\text{vert}}^q(M) \longrightarrow \Lambda_{\text{hor}}^{p+i}(M) \otimes \Lambda_{\text{vert}}^{q+1-i}(M)$$

The terms  $d_i$  vanish for  $i > r + 1$  because  $F$  is  $r$ -dimensional, hence for  $i > r + 1$  either  $\Lambda_{\text{vert}}^{q+1-i}(M)$  or  $\Lambda_{\text{vert}}^q(M)$  is 0.

These differentials are related to the differentials in the Hattori spectral sequence in the following way: to find  $E_1^{p,q}$ , one takes the cohomology of  $d_0$ . Then one restricts  $d_1$  to  $E_1^{p,q}$ , and its cohomology gives  $E_2^{p,q}$ , and so on. In this sense, the Hattori differentials are indeed differentials in the Hattori spectral sequence.

*Remark 4.2.* Each of the differentials  $d_i$  is a derivation, because the decomposition  $\Lambda^*(M) = \bigoplus_{p,q} \Lambda_{\text{hor}}^p(M) \otimes \Lambda_{\text{vert}}^q(M)$  is multiplicative.

The spectral sequence which we call ‘‘Hattori spectral sequence’’ was invented independently on several instances after Hattori. In a book [7, Section 1.6] by J.-L. Brylinski it was described as ‘‘Cartan spectral sequence’’, without reference. About the same time, it was described in Vlad Sergiescu’s Ph. D. thesis and in his subsequent papers ([13,30,31]) under the name ‘‘Leray-Serre type spectral sequence’’. This work was quite influential, with a number of publications citing Sergiescu’s papers and his thesis (for example, [1,2,10]). In [1], it was shown that all terms on  $E_2$  page of this spectral sequence are finite-dimensional when the foliation admits a transversal Riemannian structure, and in [10] the same result was proven for cohomology with coefficients in a local system.

#### 4.2. Hattori differentials on Sasakian manifolds

Let now  $Q$  be a Sasakian manifold,  $\vec{r}$  its Reeb field, normalized in such a way that  $|\vec{r}| = 1$ , and  $R \subset TQ$  the 1-dimensional foliation generated by the Reeb field. The

corresponding Hattori differentials are written as  $d = d_0 + d_1 + d_2$ , because  $R$  is 1-dimensional. Since  $d^2 = 0$ , one has  $d_0^2 = d_2^2 = 0$  and  $\{d_0, d_2\} = -\{d_1, d_1\}$ . The differentials  $d_0, d_2$  can be described explicitly as follows.

**Claim 4.3.** Let  $e_{\vec{r}} : \Lambda^*(Q) \longrightarrow \Lambda^{*+1}(Q)$  be the operator of multiplication by the form  $\vec{r}^b = \eta$  dual to  $\vec{r}$ . Then  $d_0 = e_{\vec{r}}\text{Lie}_{\vec{r}}$ .

*Proof.* Locally, the sheaf  $\Lambda^1(Q)$  is generated over  $C^\infty(Q)$  by the basic 1-forms and  $\vec{r}^b$ . Clearly, the differential of a basic 1-form  $\alpha$  belongs to  $\Lambda_{\text{bas}}^2(Q)$ , hence  $d_0(\alpha) = 0$ , and  $e_{\vec{r}}\text{Lie}_{\vec{r}}(\alpha) = 0$ . Also,  $d_0(\vec{r}^b) = e_{\vec{r}}\text{Lie}_{\vec{r}}(\vec{r}^b) = 0$ . By Claim 2.9, to prove  $d_0 = e_{\vec{r}}\text{Lie}_{\vec{r}}$  it remains to show that these two operators are equal on  $C^\infty(Q)$ . However, on  $C^\infty(Q)$ , we have  $d_0 = e_{\vec{r}}\text{Lie}_{\vec{r}}$ , because  $d_0 f$  is the orthogonal projection of  $df$  to  $\Lambda_{\text{vert}}^1(Q) = R^*$  generated by  $e_{\vec{r}}$ , for all  $f \in C^\infty(Q)$ .  $\square$

Recall that the space of leaves of  $R$  on a regular Sasakian manifold is equipped with a complex and Kähler structure, Example 3.6. The corresponding Kähler structure can be described very explicitly.

**Claim 4.4.** Let  $Q$  be a regular Sasakian manifold, and  $\vec{r}^b$  its contact form. Then  $\omega_0 := d(\vec{r}^b)$  is basic with respect to  $R$ , and defines a transversally Kähler structure, that is, the Kähler structure on the space of leaves of  $R$  (Definition 5.1).

*Proof.* Let  $X = Q/\vec{r}$  be the space of orbits. This quotient is well defined and smooth, because  $\vec{r}$  is regular. Then  $X = C(Q)/\mathbb{C}^*$ , where the  $\mathbb{C}^*$ -action is generated by  $\xi = t \frac{d}{dt}$  and  $\vec{r} = I(\xi)$ , and hence it is holomorphic. Therefore,  $X$  is a complex manifold (as a quotient of a complex manifold by a holomorphic action of a Lie group). It is Kähler by Remark 3.2 (iii).  $\square$

**Proposition 4.5.** Let  $L_{\omega_0} : \Lambda^*(Q) \longrightarrow \Lambda^{*+2}(Q)$  be the operator of multiplication by the transversally Kähler form  $\omega_0 = d(\vec{r}^b)$ , and  $i_{\vec{r}}$  the contraction with the Reeb field. Then  $d_2 = L_{\omega_0}i_{\vec{r}}$ .

*Proof.* Clearly, the Hattori differentials

$$d_i : \Lambda_{\text{hor}}^p(Q) \times \Lambda_{\text{vert}}^q(Q) \longrightarrow \Lambda_{\text{hor}}^{p+i}(Q) \otimes \Lambda_{\text{vert}}^{q+1-i}(Q)$$

vanish on  $\Lambda^0(Q)$  unless  $i = 0, 1$ . Therefore, the differentials  $d_2, d_3, \dots$  are always  $C^\infty(Q)$ -linear. By Claim 2.9, it only remains to show that  $d_2 = L_{\omega_0}i_{\vec{r}}$  on some set of 1-forms generating  $\Lambda^1(Q)$  over  $C^\infty(Q)$ .

Clearly, on  $\Lambda_{\text{hor}}^1(Q)$  the differential  $d_2$  should act as

$$d_2 : \Lambda_{\text{hor}}^1(Q) \longrightarrow \Lambda_{\text{hor}}^3(Q) \otimes \Lambda_{\text{vert}}^{-1}(Q),$$

hence  $d_2 \Big|_{\Lambda_{\text{hor}}^1(Q)} = 0$ . To prove Proposition 4.5 it remains to show that  $d_2(\vec{r}^b) = L_{\omega_0}i_{\vec{r}}(\vec{r}^b)$ . However,  $d_2(\vec{r}^b)$  is the  $\Lambda_{\text{hor}}^2(Q)$ -part of  $d(\vec{r}^b)$ , giving  $d_2(\vec{r}^b) = d(\vec{r}^b) = \omega_0$ , and  $L_{\omega_0}i_{\vec{r}}(\vec{r}^b) = \omega_0$  because  $i_{\vec{r}}(\vec{r}^b) = 1$ .  $\square$

The Hattori differential  $d_1$  is, heuristically speaking, the “transversal component” of the de Rham differential. Indeed,  $d_1(\eta) = d(\eta)$  for any basic form  $\eta$ . Since the leaf space of  $R$  is equipped with a complex structure, it is natural to expect that the Hodge components of  $d_1$  have the same properties as the Hodge components of the de Rham differential on a complex manifold.

**Claim 4.6.** Let  $Q$  be a Sasakian manifold, and  $\Lambda^m(Q) = \bigoplus_p \Lambda_{\text{hor}}^p(Q) \otimes \Lambda_{\text{vert}}^{m-p}(Q)$  the decomposition associated with  $R \subset TQ$  as in (4.1). Using the complex structure on the basic forms, consider the Hodge decomposition  $\Lambda_{\text{hor}}^m(Q) = \bigoplus_p \Lambda_{\text{hor}}^{m-p,p}(Q)$ . Then the differential  $d_1 : \Lambda_{\text{hor}}^p(Q) \otimes \Lambda_{\text{vert}}^q(Q) \rightarrow \Lambda_{\text{hor}}^{p+1}(Q) \otimes \Lambda_{\text{vert}}^q(Q)$  has two Hodge components  $d_1^{0,1}$  and  $d_1^{1,0}$ . Moreover, the differential  $d_1^c := d_1^{0,1} - d_1^{1,0}$  satisfies  $d_1^c = Id_1 I^{-1}$ , where  $I$  acts as  $\sqrt{-1}^{p-q}$  on  $\Lambda_{\text{hor}}^{p,q}(Q) \otimes \Lambda_{\text{vert}}^m(Q)$ .

*Proof.* First, let us prove that  $d_1$  has only two non-zero Hodge components. A priori,  $d_1$  could have several Hodge components,  $d_1 = d_1^{-k,k+1} + d_1^{-k+1,k} + \dots + d_1^{k,-k+1} + d_1^{-k,k+1}$ . This is what happens with the Hodge components of the de Rham differential on an almost complex manifold. All these components are clearly derivations. However,

$$d(\Lambda^0(Q)) \subset \Lambda_{\text{vert}}^1(Q) \oplus \Lambda_{\text{hor}}^{1,0}(Q) \oplus \Lambda_{\text{hor}}^{0,1}(Q).$$

Therefore, only  $d_1^{0,1}$  and  $d_1^{1,0}$  are non-zero on functions, the rest of the Hodge components are  $C^\infty(Q)$ -linear. By Claim 2.9, it would suffice to show that the other differentials vanish on a basis in  $\Lambda^1(Q)$ .

Since the space of leaves of  $R$  is a complex manifold, and  $d = d_1$  on basic forms, we have  $d_1 = d_1^{0,1} + d_1^{1,0}$  on basic forms. Since  $d(\bar{r}^b) = \omega_0$ , we find  $d_1(\bar{r}^b) = 0$ , which gives  $d_1 = d_1^{0,1} + d_1^{1,0}$  on  $\bar{r}^b$ . We proved the decomposition  $d_1 = d_1^{0,1} + d_1^{1,0}$ . The relation  $d_1^{0,1} - d_1^{1,0} = Id_1 I^{-1}$  follows in the usual way, because  $\frac{d_1 + \sqrt{-1} d_1^c}{2}$  has Hodge type  $(1, 0)$ , hence satisfies  $d_1^{1,0} = \frac{d_1 + \sqrt{-1} d_1^c}{2}$ .  $\square$

## 5. Transversally Kähler manifolds

**Definition 5.1.** A manifold  $M$  equipped with an integrable distribution  $F \subset TM$  is called a **foliated manifold**. In the sequel, we shall always assume that  $F$  is orientable. Let  $\omega_0 \in \Lambda^2(M)$  be a closed, basic 2-form on a foliated manifold  $(M, F)$ , vanishing on  $F$  and non-degenerate on  $TM/F$ . Let  $g_0 \in \text{Sym}^2(T^*M)$  be a basic bilinear symmetric form which is positive definite on  $TM/F$ . Since  $g_0, \omega$  are basic, the operator  $I := \omega_0^{-1} \circ g_0 : TM/F \rightarrow TM/F$  is well defined on the leaf space  $L$  of  $F$  (locally the leaf space always exists by Frobenius theorem). Assume that  $I$  defines an integrable complex structure on  $L$  for any open set  $U \subset M$  for which the leaf space is well defined. Then  $(M, F, g, \omega)$  is called **transversally Kähler**. A vector field  $v$  such that  $\text{Lie}_v(F) \subset F$ , and  $\text{Lie}_v I = 0$  is called **transversally holomorphic**; it is called **transversally Killing** if, in addition,  $\text{Lie}_v g_0 = 0$ .

**Definition 5.2.** Let  $F \subset TM$  be an integrable distribution and  $\Lambda_{\text{bas}}^*(M)$  the complex of basic forms. Its cohomology algebra is called the **basic**, or **transversal** cohomology of  $M$ . We denote the basic cohomology by  $H_{\text{bas}}^*(M)$ .

*Remark 5.3.* Note that  $H_{\text{bas}}^*(M)$  can be infinite-dimensional even when  $M$  is compact, [29].

The main result of this section is the following theorem, which is a weaker form of the main theorem from [12]. Our result is less general, but the proof is simple and self-contained.

**Theorem 5.4.** *Let  $(M, F, \omega_0, g_0)$  be a compact, transversally Kähler manifold. Assume, moreover that:*

(\*)  *$M$  is equipped with a Riemannian metric  $g$  such that the restriction of  $g$  to the orthogonal complement  $F^\perp = TM/F$  coincides with  $g_0$ , and  $F$  is generated by a collection of Killing vector fields  $v_1, \dots, v_r$ .*

(\*\*) *There exists a closed differential form  $\Phi$  on  $M$  which vanishes on  $F$  and gives a Riemannian volume form on  $TM/F$ .*

*Then the basic cohomology  $H_{\text{bas}}^*(M)$  of  $M$  is finite-dimensional and admits the Hodge decomposition and the Lefschetz  $\mathfrak{sl}(2)$ -action, as in the Kähler case.*

*Proof.* Consider the differential graded algebra of basic forms  $\Lambda_{\text{bas}}^*(M)$ . This algebra is equipped with an action of the superalgebra of Kähler supersymmetry  $\mathfrak{a}$  as in Theorem 2.15. Indeed, define the Lefschetz  $\mathfrak{sl}(2)$ -action by taking the Lefschetz triple  $L_{\omega_0}, \Lambda_{\omega_0} := *L_{\omega_0}*$  and  $H_{\omega_0} := [L_{\omega_0}, \Lambda_{\omega_0}]$ , and the transversal Weil operator  $W$  acting in the standard way on  $\Lambda_{\text{bas}}^*(M)$  and extended to  $\Lambda^*(M)$  by acting trivially on  $\Lambda_{\text{bas}}^1(M)^\perp$  (or in any other way, it does not matter). Together with the de Rham differential  $d : \Lambda_{\text{bas}}^*(M) \rightarrow \Lambda_{\text{bas}}^*(M)$  these operators generate the Lie superalgebra  $\mathfrak{a} \subset \text{End}(\Lambda_{\text{bas}}^*(M))$ , which is isomorphic to the superalgebra of Kähler supersymmetry  $\mathfrak{a}$  (Theorem 2.15), because  $\Lambda_{\text{bas}}^*(M)$  is locally identified with the algebra of differential forms on the leaf space of  $F$  which is Kähler.

Then Theorem 5.4 would follow if we identify  $\ker(\Delta_{\text{bas}}|_{\Lambda_{\text{bas}}^*(M)})$  with the space  $H_{\text{bas}}^*(M)$ , where  $\Delta_{\text{bas}} \in \mathfrak{a}$  is the transversal Laplace operator,  $\Delta_{\text{bas}} = \{d, d_{\text{bas}}^*\}$ , where  $d_{\text{bas}}^*$  denotes the  $d^*$ -operator on the leaf space.

We reduced Theorem 5.4 to the following result. □

**Proposition 5.5.** *Let  $(M, F, g)$ ,  $\text{rk} F = r$ , be a Riemannian foliated manifold,  $d = d_0 + d_1 + \dots + d_{r+1}$  the Hattori decomposition of the differential, and  $\Delta_{\text{bas}} := \{d, d_{\text{bas}}^*\}$  the basic Laplacian defined on basic forms. Assume that:*

(\*)  *$F$  is generated by a collection of Killing vector fields  $v_1, v_2, \dots$*

(\*\*) *The Riemannian volume form  $\Phi \in \Lambda_{\text{vert}}^r(M)$  satisfies  $d_1(\Phi) = 0$ .*

*Then there exists a natural isomorphism between the basic harmonic forms and  $H_{\text{bas}}^*(M)$ .*

---

The assumption (\*) and (\*\*) hold for Sasakian manifolds (Proposition 3.3 and Remark 3.4) and Vaisman manifolds (Remark 7.7).

We start from the following lemma.

**Lemma 5.6.** *In the assumptions of Proposition 5.5, let  $\alpha, \beta \in \Lambda_{\text{bas}}^*(M)$  be two basic forms. Then  $g(d\alpha, \beta) = g(\alpha, d_{\text{bas}}^*\beta)$ , where  $d_{\text{bas}}^*$  denotes the  $d^*$ -operator on the leaf space of  $F$ .*

*Proof.* This is where we use the assumption (\*\*) of Proposition 5.5. Let  $d_h^*$  be the composition of  $d^*$  with the orthogonal projection to the horizontal part  $\Lambda_{\text{hor}}^*(M)$  (see Sect. 4.1). We only need to show that

$$g(d_h^*\alpha, \beta) = g(d_{\text{bas}}^*\alpha, \beta) \text{ for all } \alpha, \beta \in \Lambda_{\text{bas}}^*(M). \quad (5.1)$$

By Theorem 2.15, one has  $d_{\text{bas}}^* = - *_{\text{bas}} d *_{\text{bas}}$  where  $*_{\text{bas}}$  is the Hodge star operator on the leaf space. Let  $r = \text{rk} F$  and  $\Phi \in \Lambda^r F$  be the Riemannian volume form. Using the assumption (\*\*), we obtain that  $d\Phi \in \bigoplus_{i=0}^{r-1} \Lambda_{\text{hor}}^{r-1-i}(M) \otimes \Lambda_{\text{vert}}^i(M)$ . Then  $*\alpha = \Phi \wedge *_{\text{bas}}(\alpha)$ , which gives

$$*d*\alpha = *d(\Phi \wedge *_{\text{bas}}(\alpha)) = d_{\text{bas}}^*\alpha + *(d\Phi \wedge \alpha).$$

The last term belongs to  $\bigoplus_{i=1}^r \Lambda_{\text{hor}}^{r-i}(M) \otimes \Lambda_{\text{vert}}^i(M)$ , hence it is orthogonal to  $\Lambda_{\text{hor}}^*(M)$ . This proves (5.1).  $\square$

Now we can prove Proposition 5.5. By (5.1), for any basic form  $\alpha$  we have

$$g(\Delta_{\text{bas}}\alpha, \alpha) = (d\alpha, d\alpha) + (d_{\text{bas}}^*\alpha, d_{\text{bas}}^*\alpha), \quad (5.2)$$

hence a basic form belongs to  $\ker \Delta_{\text{bas}}$  if and only if it is closed and orthogonal to all exact basic forms. This gives an embedding

$$\ker \Delta_{\text{bas}} \Big|_{\Lambda_{\text{bas}}^*(M)} \hookrightarrow H_{\text{bas}}^*(M). \quad (5.3)$$

It remains only to show that the map (5.3) is surjective. This is where we use the assumption (\*) of Proposition 5.5.

Consider the Hattori decomposition  $d = d_0 + d_1 + \dots + d_{r+1}$  (see Sect. 4.1). Let  $v_1, \dots, v_r \in F$  be the Killing, transversally Killing vector fields, postulated in (\*), and  $\Delta_s$  the “split Laplacian”,  $\Delta_s := \{d_1, d_1^*\} - \sum_i \text{Lie}_{v_i}^2$ . Clearly,  $\Delta_s$  is an elliptic, second order differential operator. By definition,  $g(e_{v_i} i_{v_i} \alpha, \alpha) = g(i_{v_i} \alpha, i_{v_i} \alpha)$ . Since  $v_i$  are Killing, one has  $\text{Lie}_{v_i} = -\text{Lie}_{v_i}^*$ . Therefore  $\Delta_s$  is self-adjoint and positive, with

$$g(\Delta_s \alpha, \alpha) = g(d_1 \alpha, d_1 \alpha) + g(d_1^* \alpha, d_1^* \alpha) + \sum_i g(\text{Lie}_{v_i} \alpha, \text{Lie}_{v_i} \alpha).$$

We obtain that each  $\alpha \in \ker \Delta_s$  satisfies  $i_{v_i} \alpha = \text{Lie}_{v_i} \alpha = 0$ .

Consider the orthogonal projection map  $\Pi_{\text{hor}} : \Lambda^*(M) \longrightarrow \Lambda_{\text{hor}}^*(M)$ . Since the vector fields  $v_i$  are Killing and preserve  $F$ , the projection  $\Pi_{\text{hor}}$  commutes with  $\text{Lie}_{v_i}$ . It is not hard to see that  $\Pi_{\text{hor}}$  commutes with  $d_1$  and  $d_1^*$ . Indeed,  $d_1$  is the part of  $d$  which maps  $\Lambda_{\text{hor}}^p(M) \otimes \Lambda_{\text{vert}}^q(M)$  to  $\Lambda_{\text{hor}}^{p+1}(M) \otimes \Lambda_{\text{vert}}^q(M)$ , and  $d_1^*$  its adjoint. We obtain that  $\Pi_{\text{hor}}$  commutes with  $\Delta_s$ .

Since  $\Delta_s$  is a positive, self-adjoint, Fredholm operator, its eigenvectors are dense in  $\Lambda^*(M)$ . Since  $[\Pi_{\text{hor}}, \Delta_s] = 0$ , the eigenvectors of  $\Delta_s$  are dense in  $\Lambda_{\text{hor}}^*(M)$ .

Let  $G$  be the closure of the Lie group generated by the action of  $e^{\mathbb{R}v_i}$ . Since each  $e^{\mathbb{R}v_i}$  acts by isometries, the group  $G$  is compact. By construction,  $G$  acts on  $M$  preserving the foliation  $F$ , the metric and the transversal Kähler structure, hence it commutes with the Laplacian. Averaging on  $G$ , we obtain that the eigenvectors of  $\Delta_S$  are dense in the space  $\Lambda_{\text{hor}}^*(M)^G = \Lambda_{\text{bas}}^*(M)$  of all basic forms.

On basic forms,  $d = d_1$ , hence on  $\Lambda_{\text{bas}}^*(M)$  one has  $[d, \Delta_S] = 0$ . Let  $\Lambda_{\text{bas}}^*(M)_\lambda$  be the eigenspace of  $\Delta_S|_{\Lambda_{\text{bas}}^*(M)}$  corresponding to the eigenvalue  $\lambda$ . For any closed  $\alpha \in \Lambda_{\text{bas}}^*(M)$  we have  $\lambda\alpha = (dd^* + d^*d)(\alpha) = dd^*\alpha$ . Therefore, any closed form in  $\Lambda_{\text{bas}}^*(M)_\lambda$  is exact when  $\lambda \neq 0$ . We have shown that the only eigenspace of  $\Delta_{\text{bas}}$  which contributes to the basic cohomology is  $\Lambda_{\text{bas}}^*(M)_0 = \ker \Delta_{\text{bas}}|_{\Lambda_{\text{bas}}^*(M)}$ . This proves (5.3); we finished the proof of Proposition 5.5 and Theorem 5.4.  $\square$

## 6. Basic cohomology and Hodge theory on Sasakian manifolds

### 6.1. Cone of a morphism of complexes and cohomology of Sasakian manifolds

We recall first several notions in homological algebra, see [17]:

**Definition 6.1.** A **complex**  $(C_*, d)$  is a collection of vector spaces and homomorphisms

$$\cdots \xrightarrow{d} C_i \xrightarrow{d} C_{i+1} \xrightarrow{d} \cdots$$

(more generally, a collection of objects in an abelian category) such that  $d^2 = 0$ . A **morphism** of complexes is a collection of maps  $C_i \rightarrow C'_i$  from the vector spaces of a complex  $(C_*, d)$  to the vector spaces of  $(C'_*, d)$ , commuting with the differential. The **cohomology groups** of a complex  $(C_*, d)$  are the groups

$$H^i(C_*, d) := \frac{\ker d|_{C_i}}{\text{imd}|_{C_{i-1}}}.$$

Clearly, any morphism induces a homomorphism in cohomology. An **exact sequence of complexes** is a sequence

$$0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$$

of morphisms of complexes such that the corresponding sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$$

are exact for all  $i$ .

The following claim is very basic.



**Claim 6.2.** Let  $0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$  be an exact sequence of complexes. Then there is a natural long exact sequence of cohomology

$$\cdots \rightarrow H^{i-1}(C_*, d_C) \rightarrow H^i(A_*, d_A) \rightarrow H^i(B_*, d_B) \rightarrow H^i(C_*, d_C) \rightarrow \cdots$$

□

**Definition 6.3.** Let  $(C_*, d_C) \xrightarrow{\varphi} (C'_*, d_{C'})$  be a morphism of complexes. Consider the complex  $C(\varphi)_*$ , with  $C(\varphi)_i = C_{i+1} \oplus C'_i$  and differential

$$d := d_C + \varphi - d_{C'} : C_i \oplus C'_{i-1} \rightarrow C_{i+1} \oplus C'_i.$$

or, explicitly:

$$d(c_i, c'_{i-1}) = (d_C(c_i), \varphi(c_i) - d_{C'}(c'_{i-1})).$$

The complex  $(C(\varphi)_*, d)$  is called **the cone of  $\varphi$** .

Denote by  $C_*[1]$  the complex  $C_*$  shifted by one, with  $C_i[1] = C_{i+1}$ . The exact sequence of complexes

$$0 \rightarrow C'_* \rightarrow C(\varphi)_* \rightarrow C_*[1] \rightarrow 0$$

gives the long exact sequence

$$\begin{aligned} \cdots \rightarrow H^i(C) &\xrightarrow{\varphi} H^i(C') \rightarrow H^i(C(\varphi)) \rightarrow \\ &\rightarrow H^{i+1}(C) \xrightarrow{\varphi} H^{i+1}(C') \rightarrow \cdots \end{aligned} \quad (6.1)$$

**Proposition 6.4.** Let  $Q$  be a Sasakian manifold, and  $\vec{r}$  the Reeb field. Denote by  $\Lambda_{\vec{r}}^*(Q)$  the differential graded algebra of  $\text{Lie}_{\vec{r}}$ -invariant forms, and let  $\Lambda_{\text{bas}}^*(Q) \subset \Lambda_{\vec{r}}^*(Q)$  be the algebra of basic forms. Denote by  $\omega_0 \in \Lambda_{\text{bas}}^*(Q)$  the transversal Kähler form (Claim 4.4), and let  $L_{\omega_0} : \Lambda_{\text{bas}}^*(Q) \rightarrow \Lambda_{\text{bas}}^{*+2}(Q)$  be the multiplication map. Then the complex  $\Lambda_{\vec{r}}^*(Q)$  is naturally identified with  $C(L_{\omega_0})[-1]$ , where  $C(L_{\omega_0})$  is the cone of the morphism  $L_{\omega_0} : \Lambda_{\text{bas}}^*(Q) \rightarrow \Lambda_{\text{bas}}^{*+2}(Q)$ .

*Proof.* Consider the Hattori decomposition of the differential in  $\Lambda^*(Q)$ ,  $d = d_0 + d_1 + d_2$  (Sect. 4.2). By Claim 4.3,  $d_0$  vanishes on  $\Lambda_{\vec{r}}^*(Q)$ ; Proposition 4.5 implies that  $d_2 = L_{\omega_0} i_{\vec{r}}$ . Clearly,  $\Lambda_{\vec{r}}^*(Q) = \Lambda_{\text{bas}}^*(Q) \oplus \vec{r}^{\flat} \wedge \Lambda_{\text{bas}}^*(Q)$ . The operator  $d_2 \Lambda_{\omega_0} i_{\vec{r}}$  acts trivially on  $\Lambda_{\text{bas}}^*(Q)$  and maps  $\vec{r}^{\flat} \wedge \alpha$  to  $L_{\omega_0}(\alpha)$  for any  $\alpha \in \Lambda_{\text{bas}}^*(Q)$ . Therefore, under the natural identification

$$\Lambda_{\vec{r}}^*(Q) = \Lambda_{\text{bas}}^*(Q) \oplus \vec{r}^{\flat} \wedge \Lambda_{\text{bas}}^*(Q) = \Lambda_{\text{bas}}^*(Q) \oplus \Lambda_{\text{bas}}^*(Q)[-1].$$

By Proposition 4.5, the differential  $d_1 + d_2$  in  $\Lambda_{\vec{r}}^*(Q)$  gives the same differential as in  $C(L_{\omega_0})[-1] = \Lambda_{\text{bas}}^*(Q) \oplus \Lambda_{\text{bas}}^*(Q)[-1]$ :

$$d(\alpha \oplus \vec{r}^{\flat} \wedge \beta) = d\alpha + L_{\omega_0}(\beta) \oplus (-\vec{r}^{\flat} \wedge d\beta)$$

for any  $\alpha, \beta \in \Lambda_{\text{bas}}^*(Q)$ . □

6.2. Harmonic forms decomposition on Sasakian manifolds

We give a new proof of the main result on harmonic forms on compact Sasakian manifolds (see [6, Proposition 7.4.13], essentially based on a result by Tachibana, [33]).  $\square$

**Theorem 6.5.** *Let  $Q$  be a  $2n + 1$ -dimensional compact Sasakian manifold,  $R$  the Reeb foliation, and  $H_{bas}^*(Q)$  the corresponding basic cohomology. Consider the Lefschetz  $\mathfrak{sl}(2)$ -triple  $L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0}$  acting on the basic cohomology (Theorem 5.4). Then:*

$$H^i(Q) = \begin{cases} \ker L_{\omega_0} \Big|_{H_{bas}^i(Q)}, & \text{for } i \geq n, \\ \frac{H_{bas}^i(Q)}{\text{im } L_{\omega_0}}, & \text{for } i < n. \end{cases}$$

**Proof. Step 1:** *The cohomology of the algebra  $\Lambda_{\vec{r}}^*(Q)$  of  $\text{Lie}_{\vec{r}}$ -invariant forms is equal to the cohomology of  $\Lambda^*(Q)$ . Indeed, let  $G$  be the closure of the action of  $e^{t\vec{r}}$ . Since  $\vec{r}$  is Killing,  $e^{t\vec{r}}$  acts on  $Q$  by isometries, hence its closure  $G$  is compact. Since  $G$  is connected its action on cohomology is trivial. The averaging map  $\text{Av}_G : \Lambda^*(Q) \rightarrow \Lambda_{\vec{r}}^*(Q)$  induces an isomorphism on cohomology.*

**Step 2:** *Applying (6.1), Proposition 6.4, and taking into account the isomorphism  $H^*(\Lambda_{\vec{r}}^*(Q)) = H^*(Q)$ , we obtain the long exact sequence*

$$\begin{aligned} \dots &\rightarrow H_{bas}^{i-2}(Q) \xrightarrow{L_{\omega_0}} H_{bas}^i(Q) \rightarrow H^i(Q) \rightarrow \\ &\rightarrow H_{bas}^{i-1}(Q) \xrightarrow{L_{\omega_0}} H_{bas}^{i+1}(Q) \rightarrow \dots \end{aligned} \tag{6.2}$$

*Since the Lefschetz triple  $L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0}$  induces an  $\mathfrak{sl}(2)$ -action on  $H_{bas}^i(Q)$  (Theorem 5.4), the map  $H_{bas}^{i-2}(Q) \xrightarrow{L_{\omega_0}} H_{bas}^i(Q)$  is injective for  $i \leq n$  and surjective for  $i > n$ . Therefore, the long exact sequence (6.2) gives the short exact sequences*

$$0 \rightarrow H_{bas}^{i-2}(Q) \xrightarrow{L_{\omega_0}} H_{bas}^i(Q) \rightarrow H^i(Q) \rightarrow 0$$

*for  $i \leq n$  and*

$$0 \rightarrow H^i(Q) \rightarrow H_{bas}^{i-1}(Q) \xrightarrow{L_{\omega_0}} H_{bas}^{i+1}(Q) \rightarrow 0$$

*for  $i > n$ .*

**Theorem 6.6.** *Let  $Q$  be a  $2n + 1$ -dimensional compact Sasakian manifold, and  $\mathcal{H}_{bas}^i(Q)$  the space of basic harmonic forms. Let  $\vec{r}^\flat$  be the contact form, dual to the Reeb field. Denote by  $\mathcal{H}^i$  the space of all  $i$ -forms  $\alpha$  on  $Q$  which satisfy*

- for  $i \leq n$ :**  $\alpha$  is basic harmonic (that is, belongs to the kernel of the basic Laplacian) and satisfies  $\Lambda_{\omega_0}(\alpha) = 0$ ;
- for  $i > n$ :**  $\alpha = \beta \wedge \vec{r}^\flat$  where  $\beta$  is basic harmonic and satisfies  $L_{\omega_0}(\beta) = 0$ .

Then all elements of  $\mathcal{H}^*$  are harmonic, and, moreover, all harmonic forms on  $Q$  belong to  $\mathcal{H}^*$ .

**Proof. Step 1:** We prove that all  $\gamma \in \mathcal{H}^*$  are harmonic. Let  $*_{\text{bas}}$  be the Hodge star operator on basic forms. Then for any  $\gamma \in \Lambda_{\text{hor}}^*(Q)$  one has  $*(\gamma) = *_{\text{bas}}(\gamma) \wedge \bar{r}^\flat$ . This implies that the two classes of forms in Theorem 6.6 are exchanged by  $*$ , and it suffices to prove that all  $\alpha \in \Lambda^i(Q)$  which are basic harmonic and satisfy  $\Lambda_{\omega_0}(\alpha) = 0$  for  $i \leq n$  are harmonic. Such a form  $\alpha$  is closed by Theorem 5.4. Since  $\alpha$  is basic hence Lie $_{\bar{r}}$ -invariant, it satisfies  $d_0^*(\alpha) = 0$  (Claim 4.3). By (5.2), a basic form  $\alpha$  is basic harmonic if and only if  $d_1(\alpha) = d_1^*(\alpha) = 0$ . Finally,  $d_2^* = \Lambda_{\omega_0} e_{\bar{r}}$  (Proposition 4.5), hence  $d_2^*(\alpha) = 0$ . This implies  $d^*\alpha = (d_0^* + d_1^* + d_2^*)(\alpha) = 0$ .

**Step 2:** Now we prove that all harmonic forms on  $Q$  belong to  $\mathcal{H}^*$ . By Theorem 6.5, the dimension of  $H^i(Q)$  is equal to the dimension of  $\mathcal{H}^i$ , hence the embedding  $\mathcal{H}^i \rightarrow H^i(Q)$  constructed in Step 1 is also surjective for all indices  $i$ .  $\square$

## 7. Vaisman manifolds

In this section we present, without proofs, the necessary background for LCK and Vaisman manifolds. We refer to [11] and to recent papers of ours for details, e.g. [27].

### 7.1. LCK manifolds

**Definition 7.1.** Let  $(M, I, g)$  be a Hermitian manifold of complex dimension  $n \geq 2$ , with fundamental form  $\omega$ . It is called **locally conformally Kähler (LCK)** if there exists a closed 1-form  $\theta$  (**the Lee form**) such that  $d\omega = \theta \wedge \omega$ . If  $\theta$  is exact,  $M$  is called **globally conformally Kähler (GCK)**. The vector field  $\theta^\sharp$  metrically equivalent with  $\theta$  is called **the Lee field**.

*Remark 7.2.* The definition is conformally invariant: if  $(M, I, g, \theta)$  is LCK, then  $(M, I, e^f g, \theta + df)$  is LCK too.

*Remark 7.3. (Equivalent definition)*  $(M, I, g)$  is LCK if and only if there exists a cover  $\tilde{M}$  admitting a Kähler metric  $\tilde{g}$  such that the deck group  $\Gamma$  acts with holomorphic homotheties w.r.t.  $\tilde{g}$ .

*Remark 7.4.* (i) On a Kähler cover as above, the pull-back of the Lee form is exact and the pull-back of the LCK metric is GCK.

(ii) Let  $(\tilde{M}, I, \tilde{g})$  be the universal cover of  $M$ . The equivalent definition in Remark 7.3 shows the existence of a homothety character  $\chi : \pi_1(M) \rightarrow \mathbb{R}^{>0}$  defined by  $\chi(\gamma) = \frac{\gamma^* \tilde{g}}{\tilde{g}}$ , for all  $\gamma \in \pi_1(M)$ .

**Definition 7.5.** The rank of  $\text{im}(\chi) \subset \mathbb{R}^{>0}$  is called the **LCK rank** of the LCK manifold  $M$ .

7.2. Vaisman manifolds

**Definition 7.6.** An LCK manifold  $(M, I, g, \theta)$  is called **Vaisman** if the Lee form is parallel w.r.t. the Levi-Civita connection of  $g$ .

*Remark 7.7.* (i) The Lee form of a Vaisman metric is, in particular, co-closed, and hence it is a Gauduchon metric. Therefore, on a compact LCK manifold a Vaisman metric, if it exists, is unique in its conformal class (up to constant multiplier).

(ii) Since  $\theta$  is parallel, it has constant norm. The metric  $g$  can be rescaled such that  $|\theta| = 1$ . The fundamental form  $\omega$  of the Vaisman metric with unit length Lee form satisfies the equality:

$$d(I\theta) = \omega - \theta \wedge (I\theta). \tag{7.1}$$

In particular,  $d(I\theta)$  is positive definite on  $\Sigma^\perp$ , thus defining a volume form on  $\Sigma^\perp$ .

(iii) Let  $(M, I, g, \theta)$  be a Vaisman manifold,  $\theta^\sharp$  its Lee field. Then  $\theta^\sharp$  and  $I\theta^\sharp$  are holomorphic ( $\text{Lie}_{\theta^\sharp} I = 0, \text{Lie}_{I\theta^\sharp} I = 0$ ) and Killing ( $\text{Lie}_{\theta^\sharp} g = 0, \text{Lie}_{I\theta^\sharp} g = 0$ ). Therefore, they generate a foliation  $\Sigma$  of real dimension 2 which is complex, Riemannian and totally geodesic. In particular,  $\Sigma$  is transversally Kähler (see Definition 5.1).

(iv) Moreover,  $\Sigma$  is canonical in the following sense: on a compact LCK manifold, the Lee fields of all Vaisman metrics are proportional, and hence the foliation  $\Sigma$  is the same for all Vaisman metrics. Therefore,  $\Sigma$  is called **the canonical foliation**.

(v) Compact complex submanifolds of Vaisman manifolds are Vaisman ([37]).

Remark 7.7 (iii) admits the following converse which is a powerful criterion for the existence of a Vaisman metric in a conformal class (it was proven in [22] for compact LCK manifolds, but a careful analysis of the proof shows that it is valid for non-compact manifolds too).

**Theorem 7.8.** *Let  $(M, I, g, \theta)$  be an LCK manifold equipped with a holomorphic and conformal  $\mathbb{C}$ -action  $\rho$  without fixed points, which lifts to non-isometric homotheties on a Kähler cover  $\tilde{M}$ . Then  $g$  is conformally equivalent with a Vaisman metric.*

A Vaisman manifold  $M$  can have any LCK rank between 1 and  $b_1(M)$ . However, they always admit deformations to Vaisman structures with LCK rank 1:

**Theorem 7.9.** ([27]) *Let  $(M, I, g, \theta)$  be a compact Vaisman manifold,  $\alpha$  a harmonic 1-form such that the deformed 1-form  $\theta' := \theta + \alpha$  has rational cohomology class. Consider the  $(1,1)$ -form  $\omega' := d_{\theta'} = \theta' \wedge I\theta' - d^c\theta'$  obtained as a deformation of  $\omega = \theta \wedge I\theta - d^c\theta$  (cf. (7.1)). Assume that  $\alpha$  is chosen sufficiently small in such a way that  $\omega'$  is positive definite. Then the LCK metric associated  $\omega'$  is conformally equivalent to a Vaisman metric.*

*Remark 7.10.* It is known that a mapping torus of a compact contact manifold is locally conformally symplectic. Correspondingly, a mapping torus of a compact Sasakian manifold is a Vaisman manifold ([36]). Theorem 7.9 can be used to prove the following converse (the Structure Theorem for compact Vaisman manifolds):

**Theorem 7.11.** ([26, 27]) *Every compact Vaisman manifold is biholomorphic to  $C(S)/\mathbb{Z}$ , where  $S$  is Sasakian,  $\mathbb{Z} = \left\langle (x, t) \mapsto (\varphi(x), qt) \right\rangle$ ,  $q > 1$ ,  $\varphi$  is a Sasakian automorphism of  $S$ , and  $C(S)$  is the Sasakian cone considered as a complex manifold.*

*Remark 7.12.* Since  $\theta$  is parallel, the de Rham splitting theorem implies that a Vaisman manifold is locally the product of  $\mathbb{R}$  with a Riemannian manifold which can be shown to be Sasakian. The above Structure Theorem shows that this local decomposition is canonical.

*Example 7.13.* (i) All linear Hopf manifolds  $(\mathbb{C}^n \setminus \{0\})/\langle A \rangle$ , with  $A \in \text{GL}(n, \mathbb{C})$  semi-simple are Vaisman. From Remark 7.7 (v) it follows that all compact submanifolds of semi-simple linear Hopf manifolds are Vaisman.

(ii) Vaisman compact surfaces are classified by Belgun (see [3], [41]): diagonal Hopf surfaces and elliptic surfaces.

(iii) Non semi-simple Hopf manifolds, Kato manifolds ([21]) and some Oeljeklaus-Toma manifolds are LCK manifolds which cannot have Vaisman metrics.

## 8. Hodge theory on Vaisman manifolds

### 8.1. Basic cohomology of Vaisman manifolds

Let  $M$  be a Vaisman manifold,  $\theta^\sharp$  its Lee field, and  $\vec{r} := I(\theta^\sharp)$ . Using the local decomposition of  $M$  as a product of  $\mathbb{R}$  and a Sasakian manifold (Remark 7.12),  $\vec{r}$  can be identified with the Reeb field on its Sasakian component. The manifold  $M$  is equipped with 2 remarkable foliations,  $\Sigma = \langle \vec{r}, \theta^\sharp \rangle$  and  $\mathcal{L} = \langle \theta^\sharp \rangle$ . Both of these foliations satisfy the assumptions of Proposition 5.5 (see also Remark 7.7 (iii)).

Consider the corresponding algebras of basic forms:

- (i)  $\Lambda_{\text{sas}}^*(M)$  – forms which are basic with respect to  $\mathcal{L}$ ,
- (ii)  $\Lambda_{\text{kah}}^*(M)$  – forms which are basic with respect to  $\Sigma$ .

Denote by  $H_{\text{sas}}^*(M)$ ,  $H_{\text{kah}}^*(M)$  the corresponding basic cohomology algebras.

By Theorem 5.4, the algebra  $H_{\text{kah}}^*(M)$  is equipped with the Lefschetz  $\mathfrak{sl}(2)$ -action and the Hodge decomposition in the same way as for Kähler manifolds.

The cohomology of Vaisman manifolds can be expressed non-ambiguously in terms of  $H_{\text{kah}}^*(M)$  and the Lefschetz  $\mathfrak{sl}(2)$ -action. The following theorem is the Vaisman analogue of Theorem 6.5.

**Theorem 8.1.** *Let  $M$  be a compact Vaisman manifold,  $\dim_{\mathbb{R}} M = 2n$ , and  $\theta$  its Lee form. Then*

$$\begin{aligned}
 H^i(M) &\cong H_{\text{sas}}^i(M) \oplus \theta \wedge H_{\text{sas}}^{i-1}(M), \\
 H_{\text{sas}}^i(M) &= \begin{cases} \ker L_{\omega_0} \Big|_{H_{\text{kah}}^i(M)} & \text{for } i \geq n-1, \\ \frac{H_{\text{kah}}^i(M)}{\text{im } L_{\omega_0}} & \text{for } i < n-1 \end{cases} \quad (8.1)
 \end{aligned}$$

where  $\{L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0}\}$  is the Lefschetz  $\mathfrak{sl}(2)$ -triple acting on  $H^i_{\text{kah}}(M)$ .

**Proof. Step 1:** We start by proving (8.1). Let  $G_{\theta^\sharp}$  be the closure of the 1-parametric group  $e^{t\theta^\sharp}$  in the group  $\text{Iso}(M)$  of isometries of  $M$ . Since  $\text{Iso}(M)$  is compact, the group  $G_{\theta^\sharp}$  is also compact. Clearly, averaging on  $G_{\theta^\sharp}$  does not change the cohomology class of a form. Therefore, the algebra of  $G_{\theta^\sharp}$ -invariant forms has the same cohomology as  $\Lambda^*(M)$ . Consider the decomposition (4.1) associated with the rank 1 foliation  $\mathcal{L}$ :

$$\Lambda^m(M) = \bigoplus_p \Lambda^p_{\text{hor}}(M) \otimes \Lambda^{m-p}_{\text{vert}}(M). \tag{8.2}$$

The bundle  $\Lambda^1_{\text{vert}}(M) = \mathcal{L}^*$  is 1-dimensional and generated by  $\theta$ . Therefore, (8.2) gives

$$\Lambda^m(M) = \Lambda^m_{\text{hor}}(M) \oplus \theta \wedge \Lambda^{m-1}_{\text{hor}}(M).$$

Denote by  $\Lambda^m(M)^{G_{\theta^\sharp}}$  the  $G_{\theta^\sharp}$ -invariant part of  $\Lambda^m(M)$ . Since  $\theta$  is  $G_{\theta^\sharp}$ -invariant, and the  $G_{\theta^\sharp}$ -invariant part of  $\Lambda^m_{\text{hor}}(M)$  is identified with  $\Lambda^m_{\text{sas}}(M)$ , this gives

$$\Lambda^m(M)^{G_{\theta^\sharp}} = \Lambda^m_{\text{sas}}(M) \oplus \theta \wedge \Lambda^{m-1}_{\text{sas}}(M).$$

Taking cohomology, we obtain (8.1).

**Step 2:** Now we shall prove  $H^i_{\text{sas}}(M) = \ker L_{\omega_0} \Big|_{H^i_{\text{kah}}(M)}$  for  $i \geq n - 1$  and  $H^i_{\text{sas}}(M) = \frac{H^i_{\text{kah}}(M)}{\text{im} L_{\omega_0}}$  for  $i < n - 1$ .

Let  $\vec{r} \subset TM$  be the Reeb field defined as above, and  $G_{\vec{r}}$  the closure of the 1-parametric group  $e^{t\vec{r}}$ . Then (as in Step 1 and in the proof of Theorem 6.5), the  $G_{\vec{r}}$ -invariant part of  $\Lambda^*_{\text{sas}}(M)$  is written as

$$\Lambda^i_{\text{sas}}(M)^{G_{\vec{r}}} = \Lambda^i_{\text{kah}}(M) \oplus \vec{r}^\flat \wedge \Lambda^{i-1}_{\text{kah}}(M) \tag{8.3}$$

with the differential acting (see Proposition 6.4) as

$$d(\alpha \oplus \vec{r}^\flat \wedge \beta) = d\alpha + L_{\omega_0}(\beta) \oplus (-\vec{r}^\flat \wedge d\beta) \tag{8.4}$$

for any  $\alpha, \beta \in \Lambda^*_{\text{kah}}(M)$ .

Using Cartan’s formula, we notice that  $G_{\vec{r}}$  acts trivially on the cohomology of the complex  $(\Lambda^*_{\text{sas}}(M), d)$ , hence the cohomology of  $(\Lambda^*_{\text{sas}}(M)^{G_{\vec{r}}}, d)$  is identified with  $H^*_{\text{sas}}(M)$ . From (8.3) and (8.4), we obtain that  $\Lambda^i_{\text{sas}}(M)^{G_{\vec{r}}}$  is the cone of the morphism of complexes  $L_{\omega_0} : \Lambda^*_{\text{kah}}(M)[-1] \rightarrow \Lambda^*_{\text{kah}}(M)[1]$ . From the long exact sequence (8.4) we obtain a long exact sequence identical to (6.2):

$$\begin{aligned} \dots &\rightarrow H^{i-2}_{\text{kah}}(M) \xrightarrow{L_{\omega_0}} H^i_{\text{kah}}(M) \rightarrow H^i_{\text{sas}}(M) \rightarrow \\ &\rightarrow H^{i-1}_{\text{kah}}(M) \xrightarrow{L_{\omega_0}} H^{i+1}_{\text{kah}}(M) \rightarrow \dots \end{aligned} \tag{8.5}$$

Since  $L_{\omega_0} : H_{\text{kah}}^{i-2}(M) \rightarrow H_{\text{kah}}^i(M)$  is injective for  $i \leq n-1$  and surjective for  $i > n-1$ , this long exact sequence breaks into short exact sequences of the form

$$0 \rightarrow H_{\text{kah}}^{i-2}(M) \xrightarrow{L_{\omega_0}} H_{\text{kah}}^i(M) \rightarrow H_{\text{sas}}^i(M) \rightarrow 0$$

for  $i \leq n-1$  and

$$0 \rightarrow H^i(M)_{\text{sas}} \rightarrow H_{\text{kah}}^{i-1}(M) \xrightarrow{L_{\omega_0}} H_{\text{kah}}^{i+1}(M) \rightarrow 0$$

for  $i > n$ . We finished the proof of Theorem 8.1.  $\square$

## 8.2. Harmonic forms on Vaisman manifolds

It turns out that (just like it happens in the Sasakian case) the cohomology decomposition obtained in Theorem 8.1 gives a harmonic form decomposition. Together with the Hodge decomposition of the basic cohomology of transversally Kähler structure this allows us to represent certain cohomology classes by forms of a given Hodge type. This theorem was obtained in [23], see also [36] and [35].

Recall that on a compact Vaisman manifold  $(M, I, g, \theta)$  with fundamental form  $\omega$ , the canonical foliation  $\Sigma$  is transversally Kähler (Remark 7.7 (iii)). We denoted  $\vec{r}$  the vector field  $g$ -equivalent with  $I\theta$ ,  $\Lambda_{\text{kah}}^*(M)$  the space of basic forms with respect to  $\Sigma$ , and  $\Delta_{\text{kah}} : \Lambda_{\text{kah}}^*(M) \rightarrow \Lambda_{\text{kah}}^*(M)$  the transversal Laplacian (Proposition 5.5). From Theorem 5.4 we obtain that the space  $\mathcal{H}_{\text{kah}}^i(M)$  of basic harmonic forms is equipped with the Lefschetz  $\mathfrak{sl}(2)$ -action by the operators  $L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0}$ . The main result of this section is:

**Theorem 8.2.** *Let  $(M, I, g, \theta)$  be a compact Vaisman manifold of complex dimension  $n$ , with fundamental form  $\omega$ , and canonical foliation  $\Sigma$ . Denote by  $\mathcal{H}^i$  the space of all basic  $i$ -forms  $\alpha \in \Lambda_{\text{kah}}^*(M)$  which satisfy:*

- for  $i \leq n$ :**  $\alpha$  is basic harmonic (i.e.  $\Delta_{\text{kah}}(\alpha) = 0$ ) and satisfies  $\Lambda_{\omega_0}(\alpha) = 0$ ;
- for  $i > n$ :**  $\alpha = \beta \wedge I\theta$  where  $\beta$  is basic harmonic and satisfies  $L_{\omega_0}(\beta) = 0$ .

*Then all elements of  $\mathcal{H}^* \oplus \theta \wedge \mathcal{H}^*$  are harmonic and, moreover, all harmonic forms on  $M$  belong to  $\mathcal{H}^* \oplus \theta \wedge \mathcal{H}^*$ .*

**Proof. Step 1:** This statement is similar to Theorem 6.6, and the proof is essentially the same. We start by proving that all  $\gamma \in \mathcal{H}^* \oplus \theta \wedge \mathcal{H}^*$  are harmonic. Notice that a product of a parallel form  $\rho$  and a harmonic form is again harmonic (see e.g. [39, Proposition 2.7]). This result is proven in the same way as Theorem 2.15, by considering the Lie superalgebra generated by  $d, d^*$ , and multiplication by  $\rho$ . Since  $\theta$  is parallel, it suffices only to show that all elements in  $\mathcal{H}^*$  are harmonic:

$$\mathcal{H}^* \subset \ker \Delta. \tag{8.6}$$

Consider the foliation  $\mathcal{L} = \langle \theta^\sharp \rangle$ . Since  $M$  is locally a product of a Sasakian manifold and a line (Remark 7.12), all  $\mathcal{L}$ -basic, transversally harmonic forms are harmonic. Therefore, (8.6) would follow if we prove

$$\mathcal{H}^* \subset \ker \Delta_{\text{sas}}. \tag{8.7}$$

where  $\Delta_{\text{sas}}$  is the transversal Laplacian of the foliation  $\mathcal{L}$ .

It would suffice to prove (8.7) locally in  $M$ . However, locally  $\mathcal{L}$  admits a leaf space  $Q$ , which is Sasakian, and we can regard elements from  $\mathcal{H}^*$  as forms on  $Q$  and  $\Delta_{\text{sas}}$  as the usual Laplacian on  $\Lambda^*(Q)$ . Then (8.7) follows from Theorem 6.6, step 1.

**Step 2:** We have shown that all elements of  $\mathcal{H}^* \oplus \theta \wedge \mathcal{H}^*$  are harmonic; this gives a natural linear map

$$\Psi : \mathcal{H}^* \oplus \theta \wedge \mathcal{H}^* \longrightarrow H^*(M).$$

To show that all harmonic form are obtained this way, it would suffice to prove that  $\Psi$  is surjective. This follows from Theorem 8.1 by dimension count.

By Theorem 8.1, the cohomology of  $M$  is isomorphic to  $H_{\text{sas}}^i(M) \oplus \theta \wedge H_{\text{sas}}^{i-1}(M)$ . All forms in  $\mathcal{H}^*$  are closed and belong to  $\Lambda_{\text{sas}}^*(M)$ , which gives a map  $\Psi_{\text{sas}} : \mathcal{H}^* \longrightarrow H_{\text{sas}}^*(M)$ . To prove that  $\Psi$  is surjective, it remains to show that  $\Psi_{\text{sas}}$  is surjective.

However, the dimension of  $H_{\text{sas}}^i(M)$  is equal to  $\dim \ker L_{\omega_0} \Big|_{H_{\text{kah}}^i(M)}$  for  $i \geq n - 1$  and to  $\dim \frac{H_{\text{kah}}^i(M)}{\text{im} L_{\omega_0}}$  for  $i < n - 1$ . The space  $\mathcal{H}^i$  has the same dimension by the transversal Hodge decomposition (Theorem 5.4). This finishes the proof of Theorem 8.2. □

### 9. Supersymmetry on Sasakian manifolds

Let  $Q$  be a Sasakian manifold. In this section we describe the Lie superalgebra  $\mathfrak{q} \subset \text{End}(\Lambda^*(Q))$  reminiscent of the supersymmetry algebra of a Kähler manifold (Theorem 2.15). Unlike the Kähler supersymmetry algebra, the algebra  $\mathfrak{q}$  is infinitely-dimensional; however, it has a simple and compact description, independent of the choice of  $Q$ .

When this project was started, we expected to use Theorem 9.2 to give a more conceptual proof of the classical results on Hodge decomposition of the cohomology of Sasakian and Vaisman manifolds (Theorem 6.6, Theorem 8.2). However, the Lie superalgebra  $\mathfrak{q}$  which we obtained in the end does not contain the de Rham Laplacian operator (the “de Rham Laplacian” is the usual Laplacian operator defined on differential forms on a Riemannian manifold). The de Rham operator on a Sasakian manifold is decomposed onto its Hattori components as  $d_0 + d_1 + d_2$  (Sect. 4.2). The operators  $d_0, d_1$  are elements of  $\mathfrak{q}$ , but  $d_2$  is in fact an element of its universal enveloping algebra  $U_{\mathfrak{q}}$ .

Any attempt to add  $d_2$  or the de Rham Laplacian to  $\mathfrak{q}$  lead to a large subalgebra of  $U_{\mathfrak{q}}$  which is complicated and very difficult to control.

The current version of the proof of Theorem 8.2 is independent from Theorem 9.2.

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The statement of Theorem 6.6 is global, however, the proof of Theorem 6.6, step 1 is local, and this statement is essentially identical to (8.7).



*Remark 9.1.* Some of the relations we derive can be found in P.A.-Nagy's doctoral thesis, in a more general setting, for a compact Riemannian manifold endowed with a unitary vector field (see [25]).

We first recall the notations. Let  $Q$  be a Sasakian manifold,  $\vec{r}$  its Reeb field, and  $R \subset TM$  the corresponding rank 1 distribution. Let

$$\Lambda^*(Q) := \bigoplus \Lambda_{\text{hor}}^{p,q}(Q) \times \Lambda_{\text{vert}}^m(Q)$$

be the corresponding decomposition (Hodge and Hattori) of the de Rham algebra, and  $d = d_0 + d_1 + d_2$  the Hattori differentials (see Sect. 4.2). Denote by  $W$  the Weil operator acting as multiplication by  $\sqrt{-1}(p - q)$  on  $\bigoplus \Lambda_{\text{hor}}^{p,q}(Q) \times \Lambda_{\text{vert}}^m(Q)$ . Let  $d_1, d_1^*, d_1^c$  and  $(d_1^c)^*$  be the differentials defined in Claim 4.6. Let  $L_{\omega_0}, \Lambda_{\omega_0} := L_{\omega_0}^*, H_{\omega_0} := [L_{\omega_0}, \Lambda_{\omega_0}]$  be the Lefschetz  $\mathfrak{sl}(2)$ -triple associated with the transversally Kähler form  $\omega_0$ . For an operator  $A \subset \text{End}(Q)$ , commuting with  $\text{Lie}_{\vec{r}}$ , we denote by  $A(k)$  the composition  $A \circ (\text{Lie}_{\vec{r}})^k$ . Let  $i_{\vec{r}}$  be the contraction with  $\vec{r}$  and  $e_{\vec{r}}$  the dual operator (above denoted  $e_{\vec{r}^\flat}$ ).

Denote by  $\mathfrak{q}$  the Lie superalgebra generated by all operators

$$L_{\omega_0}(i), \Lambda_{\omega_0}(i), H_{\omega_0}(i), d_1(i), e_{\vec{r}}(i), i_{\vec{r}}(i), \text{Id}(i), W(i) \text{ for all } i \in \mathbb{Z}^{\geq 0}.$$

Since the vector field  $\vec{r}$  acts by Sasakian isometries,  $\mathfrak{q}$  commutes with  $\text{Lie}_{\vec{r}}$ ; in other words,  $\text{Lie}_{\vec{r}}$  is central in  $\mathfrak{q}$ . We consider  $\mathfrak{q}$  as an  $\mathbb{R}[t]$ -module, with  $t$  mapping  $A(i)$  to  $A(i + 1)$ . Then the Lie superalgebra  $\mathfrak{q}$  is a free  $\mathbb{R}[t]$ -module of rank  $(6|6)$  (with 6 even and 6 odd generators) over  $\mathbb{R}[t]$ . Its even generators are

$$L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0}, W, \Delta_1 := \{d_1, d_1^*\}, \text{Id},$$

and the odd generators are

$$d_1, d_1^*, d_1^c, (d_1^c)^*, e_{\vec{r}}, i_{\vec{r}}.$$

Notice that  $d_0 = e_{\vec{r}}(1)$  and  $d_0^* = i_{\vec{r}}(1)$  (Claim 4.3).

The main result of this section is:

**Theorem 9.2.** *The only non-zero commutator relations in  $\mathfrak{q}$  can be written as follows.*

(i) *The Lefschetz's  $\mathfrak{sl}(2)$ -action: the even elements  $L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0}$  satisfy the usual  $\mathfrak{sl}(2)$ -relations (Theorem 2.15) and commute with*

$$W, \Delta_1, \Delta_0, d_0, d_0^*.$$

*The operator  $H_{\omega_0}$  acts as multiplication by  $p - n$  on  $\Lambda_{\text{hor}}^p(Q) \times \Lambda_{\text{vert}}^m(Q)$ , where  $\dim_{\mathbb{R}} Q = 2n + 1$ .*

(ii) *The Weil operator satisfies*

$$[W, d] = d_1^c, [W, d_1^c] = -d_1, [W, d_1^*] = -(d_1^c)^*, [W, (d_1^c)^*] = d_1.$$

*Also,  $W$  commutes with the rest of the generators of  $\mathfrak{q}$ .*

(iii) The differentials  $d_1, d_1^*, d_1^c, (d_1^c)^*$  have non-zero square:

$$\{d_1, d_1\} = \{d_1^c, d_1^c\} = -L_{\omega_0}(1), \quad \{d_1^*, d_1^*\} = \{(d_1^c)^*, (d_1^c)^*\} = \Lambda_{\omega_0}(1)$$

Moreover,  $\{d_1, d_1^c\} = \{d_1^*, (d_1^c)^*\} = 0$ .

(iv) The usual Kodaira relations still hold:

$$\begin{aligned} [\Lambda_{\omega_0}, d_1] &= (d_1^c)^*, & [L_{\omega_0}, d_1^*] &= -d_1^c, \\ [\Lambda_{\omega_0}, d_1^c] &= -d_1^*, & [L_{\omega_0}, (d_1^c)^*] &= d_1. \end{aligned} \tag{9.1}$$

(v) Unlike it happens in the Kähler case, the differentials  $d_1^*, d_1^c$ , etc. do not (super-)commute:

$$\{d_1^*, d_1^c\} = \{d_1, (d_1^c)^*\} = -\frac{1}{2}H_{\omega_0}(1). \tag{9.2}$$

(vi) The only non-zero commutator between  $e_{\bar{r}}, i_{\bar{r}}, \text{Id}$  is  $\{e_{\bar{r}}, i_{\bar{r}}\} = \text{Id}$ . These elements commute with the rest of  $\mathfrak{q}$ .

(vii) The Laplacian  $\Delta_1 := \{d_1, d_1^*\}$  satisfies  $\Delta_1 = \{d_1^*, (d_1^c)^*\}$  and commutes with all even generators in  $\mathfrak{q}$  and with  $e_{\bar{r}}, i_{\bar{r}}$ . Its commutators with the other 4 odd generators are expressed as follows

$$\begin{aligned} \{d_1, \Delta_1\} &= -\frac{1}{2}d_1^c(1), & \{d_1^c, \Delta_1\} &= \frac{1}{2}d_1(1), \\ \{d_1^*, \Delta_1\} &= -\frac{1}{2}(d_1^c)^*(1), & \{(d_1^c)^*, \Delta_1\} &= \frac{1}{2}(d_1)^*(1). \end{aligned} \tag{9.3}$$

*Proof of Theorem 9.2 (i):.* The  $\mathfrak{sl}(2)$ -relations and the expression for  $H_{\omega_0}$  are proven in Sect. 2.3; the proof in the Sasakian case is literally the same. Also, from the definition it is clear that  $L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0}$  commute with  $W, i_{\bar{r}}$  and  $e_{\bar{r}}$ . Since  $d_0 = e_{\bar{r}}(1)$  and  $d_0^* = i_{\bar{r}}(1)$  (Claim 4.3), these operators commute with the  $\mathfrak{sl}(2)$ -action and satisfy  $\{d_0, d_0^*\} = \text{Lie}_{\bar{r}}^2$ . We postpone the commutator relation for  $\Delta_1$  until we proved Theorem 9.2 (iv).  $\square$

*Proof of Theorem 9.2 (ii):.* Same as Theorem 2.15, part 2.  $\square$

*Proof of Theorem 9.2 (iii):.* Start from  $\{d_1, d_1\} = -L_{\omega_0}(1)$ .

Since  $d^2 = 0$ , one has  $d_0^2 = d_2^2 = 0$  and  $\{d_0, d_2\} = -\{d_1, d_1\}$ . However,  $d_0 = e_{\bar{r}}(1)$  and  $d_2 = L_{\omega_0}i_{\bar{r}}$ , hence  $\{d_0, d_2\} = \{e_{\bar{r}}, i_{\bar{r}}\}L_{\omega_0}(1) = L_{\omega_0}(1)$ . The squares of the rest of the differentials  $d_1^c$ , etc., are obtained by duality and complex conjugation.

To show that  $\{d_1, d_1^c\} = 0$ , we use  $d_1^c = [W, d_1]$ . By Claim 2.13,

$$\{d_1, d_1^c\} = \{d_1, \{d_1, W\}\} = \frac{1}{2}\{\{d_1, d_1\}, W\}.$$

Since  $\{d_1, d_1\} = -L_{\omega_0}(1)$  and  $W$  commutes with  $L_{\omega_0}$ , this gives  $\{d_1, d_1^c\} = 0$ . The equation  $\{d_1^*, (d_1^c)^*\} = 0$  is obtained by duality. We finished Theorem 9.2 (iii).  $\square$

*Proof of Theorem 9.2 (iv), the Kähler-Kodaira relations:*. Again, it suffices to prove  $[L_{\omega_0}, d_1^*] = -d_1^c$ , the rest is obtained by duality and complex conjugation. We prove it applying the same argument as used in Theorem 2.15. As an operator on  $\Lambda^*(Q)$ , the commutator  $[L_{\omega_0}, d_1^*]$  has first order, because  $L_{\omega_0}$  is zero order, and  $d_1^*$  is order 2 (Claim 2.7).

The differential operators  $[L_{\omega_0}, d_1^*]$  and  $-d_1^c$  are equal on functions for the same reasons as in Theorem 2.15.

Clearly,  $d_1^c(C^\infty(Q))$  generates  $\Lambda_{\text{hor}}^1(Q)$ . Therefore, to prove  $[L_{\omega_0}, d_1^*] = -d_1^c$  on  $\Lambda_{\text{hor}}^1(Q)$ , we need only to show that

$$([L_{\omega_0}, d_1^*])^2 = (-d_1^c)^2. \quad (9.4)$$

This is implied by the graded Jacobi identity, applied as follows. First, we notice that  $[\Lambda_{\omega_0}, d_1^*] = 0$ , because  $d_1(\omega_0) = 0$ . Therefore, the  $\mathfrak{sl}(2)$ -representation generated by the Lefschetz triple  $\langle L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0} \rangle$  from  $d_1^*$  has weight 1, and  $[L_{\omega_0}, [L_{\omega_0}, d_1^*]] = 0$ . Applying the graded Jacobi identity, and using  $[L_{\omega_0}, [L_{\omega_0}, d_1^*]] = 0$ , we obtain

$$\begin{aligned} \{[L_{\omega_0}, d_1^*], [L_{\omega_0}, d_1^*]\} &= [L_{\omega_0}, \{d_1^*, [L_{\omega_0}, d_1^*]\}] \\ &= \frac{1}{2}[L_{\omega_0}(1), [L_{\omega_0}(1), \{d_1^*, d_1^*\}]] \quad \text{by Claim 2.13} \\ &= -\frac{1}{2}[L_{\omega_0}(1), [L_{\omega_0}(1), \Lambda_{\omega_0}]] \quad \text{by Theorem 9.2 (iii)} \\ &= -L_{\omega_0}(1). \end{aligned}$$

However,  $(-d_1^c)^2 = -L_{\omega_0}(1)$  by Theorem 9.2 (iii), which proves (9.4).

This implies that  $[L_{\omega_0}, d_1^*] = -d_1^c$  on  $\Lambda_{\text{hor}}^1(Q)$ . The space  $\Lambda^1(Q)$  is generated over  $C^\infty(Q)$  by  $\Lambda_{\text{hor}}^1(Q)$  and  $V = \langle \vec{r}^b \rangle$ . Applying Corollary 2.11, we obtain that  $[L_{\omega_0}, d_1^*] = -d_1^c$  if  $[L_{\omega_0}, d_1^*](\vec{r}^b) = -d_1^c(\vec{r}^b)$ . However,  $d(\vec{r}^b) = \omega_0$ , hence  $d_1^c(\vec{r}^b) = 0$ . Then,  $[L_{\omega_0}, d_1^*](\vec{r}^b) = d_1^*(\vec{r}^b \wedge \omega_0)$ . It is not hard to see that  $*(\vec{r}^b \wedge \omega_0) = \frac{1}{(n-1)!}\omega_0^{n-1}$ , where  $\dim_{\mathbb{R}} Q = 2n + 1$ . Therefore,

$$d_1^*(\vec{r}^b \wedge \omega_0) = \pm * d_1 * (\omega_0 \wedge \vec{r}^b) = \pm * d_1 \frac{1}{(n-1)!} \omega_0^{n-1} = 0$$

because  $d\omega_0^{n-1} = 0$ . We proved that  $[L_{\omega_0}, d_1^*](\vec{r}^b) = -d_1^c(\vec{r}^b)$  and finished the proof of the Kähler-Kodaira relations.  $\square$

**Proof of Theorem 9.2 (v), the commutators of  $d_1^*$ ,  $d^c$ :** The commutator  $\{d_1^*, d^c\}$  is obtained from the Kähler-Kodaira relations. Indeed,  $\{d_1^*, d^c\} = \{d_1^*, \{d_1^*, L_{\omega_0}\}\}$ . Then Claim 2.13 gives

$$\{d_1^*, \{d_1^*, L_{\omega_0}\}\} = \frac{1}{2}\{\{d_1^*, d_1^*\}, L_{\omega_0}\} = \frac{1}{2}[\Lambda_{\omega_0}(1), L_{\omega_0}] = -\frac{1}{2}H_{\omega_0}.$$

The relation  $\{d_1, (d_1^c)^*\} = -\frac{1}{2}H_{\omega_0}(1)$  is dual to  $\{d_1^*, d_1^c\} = -\frac{1}{2}H_{\omega_0}(1)$ .

**Proof of Theorem 9.2 (vi), the commutators of  $e_{\bar{r}}$  and  $i_{\bar{r}}$ :** The equation  $\{e_{\bar{r}}, i_{\bar{r}}\} = 1$  is standard. Vanishing of the commutators between  $e_{\bar{r}}, i_{\bar{r}}$  and  $L_{\omega_0}, \Lambda_{\omega_0}, H_{\omega_0}, W$  is standard linear algebra. The only commutators for which we have to prove the vanishing is between  $e_{\bar{r}}, i_{\bar{r}}$  and  $d_1, d_1^*, d_1^c, (d_1^c)^*$ . Using duality and complex conjugation, we reduce the vanishing of these commutators to only two of them:  $\{e_{\bar{r}}, d_1\} = 0$  and  $\{e_{\bar{r}}, d_1^*\} = 0$ . As  $d^2 = 0$  and  $\{d_0, d_1\}$  is the grading 1 part of  $d^2$ , one has  $\{d_0, d_1\} = 0$ . Since  $d_0 = e_{\bar{r}}(1)$ , this also implies  $\{e_{\bar{r}}, d_1\} = 0$ . Twisting with  $I$ , we obtain  $\{e_{\bar{r}}, d_1^c\} = 0$ . Applying the graded Jacobi identity to  $\{e_{\bar{r}}, d_1^*\} = -\{e_{\bar{r}}, \{\Lambda_{\omega_0}, d_1^c\}\}$  (Theorem 9.2 (iv)) and using  $\{e_{\bar{r}}, L_{\omega_0}\} = 0$ , we obtain

$$\{e_{\bar{r}}, d_1^*\} = -\{e_{\bar{r}}, \{\Lambda_{\omega_0}, d_1^c\}\} = \{\{e_{\bar{r}}, \Lambda_{\omega_0}\}, d_1^c\} + \{\Lambda_{\omega_0}, \{e_{\bar{r}}, d_1^c\}\} = 0.$$

This finishes the proof of Theorem 9.2 (vi).

*Proof of Theorem 9.2 (vii).* The equation

$$\{d_1, d_1^*\} = \{(d_1^c), (d_1^c)^*\}$$

follows from  $\{d_1, d_1^c\} = 0$  (Theorem 9.2 (iii)) because

$$0 = \{\Lambda, \{d_1, d_1^c\}\} = \{\{\Lambda, d_1\}, d_1^c\} + \{d_1, \{\Lambda, d_1^c\}\} = \{(d_1^c)^*, d_1^c\} - \{d_1, d_1^*\}.$$

This implies, in particular, that  $[W, \{d_1, d_1^*\}] = 0$ . The commutators between the Lefschetz operators and  $\Delta_1$  follow from the Kähler-Kodaira relations:

$$\{L_{\omega_0}, \{d_1, d_1^*\}\} = \{\{L_{\omega_0}, d_1\}, d_1^*\} + \{d_1, \{L_{\omega_0}, d_1^*\}\} = -\{d_1, d_1^c\} = 0.$$

We proved that  $\Delta_1$  commutes with the even part of  $q$ . By duality and complex conjugation, to prove (9.3) it would suffice to prove only one of these relations, say,  $\{d_1, \Delta_1\} = -\frac{1}{2}d_1^c(1)$ . This equation follows from (9.1), (9.2) (i.e. (iv) and (v) of this theorem) and Claim 2.13:

$$\{d_1, \{d_1, d_1^*\}\} = \frac{1}{2}\{\{d_1, d_1\}, d_1^*\} = -\frac{1}{2}\{L_{\omega_0}(1), d_1^*\} = -\frac{1}{2}(d_1^c)^*.$$

We finished the proof of Theorem 9.2. □

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## References

- [1] Álvarez López, J.A.: A finiteness theorem for the spectral sequence of a Riemannian foliation, *Illinois. J. Math.* **33**, 79–92 (1989)
- [2] Álvarez López, J.A.: Duality in the spectral sequence of Riemannian foliations. *Am. J. Math.* **111**(6), 905–926 (1989)
- [3] Belgun, F.A.: On the metric structure of non-Kähler complex surfaces. *Math. Ann.* **317**, 1–40 (2000)
- [4] Biswas, I., Fernández, M., Muñoz, V.: Tralle, Aleksey On formality of Sasakian manifolds. *J. Topol.* **9**(1), 161–180 (2016)
- [5] Blair, D.E.: *Riemannian Geometry of Contact and Symplectic Manifolds*, 2nd edn. *Progress in Mathematics*, **203**. Birkhäuser Boston, Inc., Boston (2010)
- [6] Boyer, C., Galicki, K.: *Sasakian Geometry*. Oxford Mathematical Monographs, Oxford University Press, Oxford (2008)
- [7] Brylinsky, J.-L.: *Loop Spaces, Characteristic Classes and Geometric Quantization*. Birkhäuser, Oxford (2008)
- [8] Cappelletti-Montano, B., De Nicola, A., Marrero, J.C., Yudin, I.: Sasakian nilmanifolds. *Int. Math. Res. Not. IMRN* **72**, 6648–6660 (2015)
- [9] Coutinho, S.C.: *A Primer of Algebraic D-modules*. London Mathematical Society Study, vol. 33. Cambridge University Press, Cambridge (1995)
- [10] Domínguez, D.: A finiteness theorem for transitive foliations and flat vector bundles *Israel. J. Math.* **107**, 253–287 (1998)
- [11] Dragomir, S., Ornea, L.: *Locally Conformally Kähler Manifolds*, *Progress in Mathematics* **55**, Birkhäuser, (1998)
- [12] El Kacimi-Alaoui, A.: Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications. *Compositio Math.* **73**(1), 57–106 (1990)
- [13] El Kacimi-Alaoui, A., Sergiescu, V., Hector, G.: La cohomologie basique d’un feuilletage riemannien est de dimension finie. *Math. Z.* **188**(4), 593–599 (1985)
- [14] Figueroa-O’Farrill, J.M., Koehl, C., Spence, B.: Supersymmetry and the cohomology of (hyper)Kähler manifolds. *Nuclear Phys. B* **503**(3), 614–626 (1997)
- [15] Fedoruk, S., Ivanov, E., Smilga, A.: Generic HKT geometries in the harmonic superspace approach. *J. Math. Phys.* **59**(8), 083501 (2018)
- [16] Fedoruk, S., Smilga, A.: Bi-HKT and bi-Kähler supersymmetric sigma models. *J. Math. Phys.* **57**(4), 042103 (2016)
- [17] Gelfand, S.I., Manin, Y.I.: *Methods of Homological Algebra*. Springer, Berlin (2003)
- [18] Griffiths, Ph., Harris, J.: *Principles of Algebraic Geometry*. Wiley-Interscience, New York (1978)
- [19] Hattori, A.: Spectral sequence in the de Rham cohomology of fibre bundles. *J. Fac. Sci. Univ. Tokyo Sect.* **1**(8), 289–331 (1960)
- [20] Huang, T.:  $L^2$  harmonic forms on complete special holonomy manifolds. *Ann. Global Anal. Geom.* **56**(1), 17–36 (2019)
- [21] Istrati, N., Otîman, A., Pontecorvo, M.: On a class of Kato manifolds. *Int. Math. Res. Not. IMRN* **7**, 5366–5412 (2021)
- [22] Kamishima, Y., Ornea, L.: Geometric flow on compact locally conformally Kähler manifolds. *Tohoku Math. J.* **57**(2), 201–221 (2005)
- [23] Kashiwada, T., Sato, S.: On harmonic forms in compact locally conformal Kähler manifolds with the parallel Lee form. *Ann. Fac. Sci. Univ. Nat. Zaïre (Kinshasa) Sect. Math. Phys.* **6**(1–2), 17–29 (1980)
- [24] Kim, H., Saberi, I.: Real homotopy theory and supersymmetric quantum mechanics. *J. Math. Phys.* **59**(7), 072102 (2018)

- [25] Nagy, P.-A.: Un principe de séparation des variables pour le spectre du laplacien des formes différentielles et applications, Ph. D. Thesis, Université de Savoie (2001)
- [26] Ornea, L., Verbitsky, M.: Structure theorem for compact Vaisman manifolds. *Math. Res. Lett.* **10**, 799–805 (2003)
- [27] Ornea, L., Verbitsky, M.: LCK rank of locally conformally Kähler manifolds with potential. *J. Geom. Phys.* **107**, 92–98 (2016)
- [28] Schmude, J.: Laplace operators on Sasaki-Einstein manifolds. *JHEP* **1404**, 008 (2014)
- [29] Schwarz, G.W.: On the de Rham cohomology of the leaf space of a foliation. *Topology* **13**, 185–187 (1974)
- [30] Sergiescu, V.: Quelques problèmes concernant les feuilletages et les groupes de difféomorphismes, Université de Lille 1, Thèse de doctorat en Mathématiques pures, (1986), <http://ori.univ-lille1.fr/notice/view/univ-lille1-ori-39175>
- [31] Sergiescu, V.: Sur la suite spectrale d'un feuilletage riemannien Proceedings of the XIXth National Congress of the Mexican Mathematical Society, Vol. 2 (Guadalajara, 1986), pp. 33–39, *Aportaciones Mat. Comun.*, 4, Soc. Mat. Mexicana, México (1987)
- [32] Smilga, A.V.: Supercharges in the hyper-Kähler with torsion supersymmetric sigma models. *J. Math. Phys.* **53**(12), 122105 (2012)
- [33] Tachibana, S.: On harmonic tensors in compact Sasakian spaces. *Tohoku Math. J.* **17**, 271–284 (1965)
- [34] Tievsky, A.: Analogues of Kähler Geometry on Sasakian Manifolds, Ph.D. Thesis, M.I.T., (2008)
- [35] Tsukada, K.: The canonical foliation of a compact generalized Hopf manifold. *Differ. Geom. Appl.* **11**(1), 13–28 (1999)
- [36] Vaisman, I.: Generalized Hopf manifolds. *Geom. Dedicata.* **13**, 231–255 (1982)
- [37] Verbitsky, M.: Vanishing theorems for locally conformal hyperkähler manifolds. *Proc. Steklov Inst.* **246**, 54–79 (2004). [arxiv:math.DG/0302219](https://arxiv.org/abs/math/0302219)
- [38] Verbitsky, M.: Hyperkähler manifolds with torsion, supersymmetry and Hodge theory. *Asian J. Math.* **6**(4), 679–712 (2002). [arxiv:math.AG/0112215](https://arxiv.org/abs/math/0112215)
- [39] Verbitsky, M.: Manifolds with parallel differential forms and Kaehler identities for  $G_2$ -manifolds. *J. Geom. Phys.* **61**(6), 1001–1016 (2011)
- [40] Verbitsky, M.: Hodge theory on nearly Kähler manifolds *Geom. Topol.* **15**(4), 2111–2133 (2011)
- [41] Verbitsky, M., Vuletescu, V., Ornea, L.: Classification of non-Kähler surfaces and locally conformally Kähler geometry. *Russian Math. Surv.* **76**, 261–290 (2021)