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On the space of *f*-minimal surfaces with bounded *f*-index in weighted smooth metric spaces

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Abstract. In this note we prove a compactness theorem for the space of connected closed embedded f-minimal surfaces, of bounded f-index, in a simply connected smooth metric measure space $(M^3, g, e^{-f}d\mu)$. This result is similar to that proved by Li and Wei (J Geom Anal 25:421–435, 2015). Li and Wei assumed $Ric_f \geq k > 0$, where Ric_f is the Bakry-Émery Ricci curvature, and that the embedded f-minimal surfaces have fixed genus. Here we suppose $R_f^P + \frac{1}{2}|\overline{\nabla} f|^2 > 0$, where R_f^P is the Perelman scalar curvature, and uniform bound on the f-index of the embedded f-minimal surfaces.

1. Introduction

The study of sequences of minimal surfaces in 3-dimensional Riemannian manifolds is a classic topic in differential geometry. That kind of compactness is important, for instance, because it leads us to use the Degree Theory in order to study the existence of certain kind of minimal surfaces. The seminal work of Allard [1] deals with weak convergence of sequences of minimal surfaces. However, one of the most celebrated compactness result for closed minimal surfaces in 3-dimensional Riemannian manifolds is the Choi-Schoen's Compactness Theorem [4] for closed minimal surfaces embedded in a 3-dimensional Riemannian manifold with positive Ricci curvature and bounded genus. Later, B. White generalized the result of Choi and Schoen to stationary points of arbitrary elliptic functionals defined on the space of embeddings of a compact surface in a 3-dimensional Riemannian manifold, minimal surfaces being stationary points of that area functional. White's result guarantees that compactness holds for such surfaces if we also assume a bound on the area.

As we mentioned above, in 1985, Choi and Schoen [4] proved the following result.

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Theorem 1. *Let (M^3, g) be a closed (i.e. compact with empty boundary) Riemannian manifold with positive Ricci curvature. Then the space of closed embedded minimal surfaces of fixed topological type in M is compact in the C^k topology for any $k \geq 2$.*

Motived by this result, in 2015, Li and Wei proved, [9], a generalization of the Choi-Schoen's Theorem for the class of the f-minimal surfaces.

Theorem 2. *Let $(M^3, g, e^{-f}d\mu)$ be a closed smooth metric measure space with $Ric_f \geq \kappa > 0$. Then the space of closed embedded f-minimal surfaces of fixed topological type in $(M^3, g, e^{-f}d\mu)$ is compact in the smooth topology.*

Li and Wei used the standard argument in Choi–Schoen's paper. Initially they needed a first eigenvalue estimate of the f-Laplacian for f-minimal surfaces in a manifold with positive Bakry-Émery Ricci curvature $Ric_f \geq k > 0$ (proved by Ma and Du in [10]), then they get an a priori upper bound on the weighted area of the f-minimal surface in terms of the topology. Later they get an upper bound for the total curvature of an f-minimal surface and using a singular compactness result and a contradiction argument they get the smooth compactness Theorem 2.

This paper will prove a similar compactness theorem. Instead $Ric_f \geq k > 0$, we suppose that the smooth metric measure space $(M^3, g, e^{-f}d\mu)$ contains no immersed f-stable f-minimal surface, that M is simply connected and $R_f^P + \frac{1}{2}|\overline{\nabla} f|^2 > 0$, where R_f^P is the Perelman scalar curvature and $\overline{\nabla}$ is the Levi-Civita connection of g. Note that $Ric_f \geq k > 0$ implies those two first assumptions but does not imply the latter. In fact, R_f^P is not the trace of Ric_f. The key idea is to explore the following property: that a surface $\Sigma \subset M$ is an f-minimal surface in $(M^3, g, e^{-f}d\mu)$ if and only if Σ is a minimal surface in $(M^3, \widetilde{g})$, where $\widetilde{g} = e^{-f}g$.

Using results of [1,3,12], we get the following theorem.

Theorem 3. *Let $(M^3, g, e^{-f}d\mu)$ be a simply connected and closed smooth metric measure space for which $R_f^P + \frac{1}{2}|\overline{\nabla} f|^2 > 0$ where R_f^P is the Perelman scalar curvature and $\overline{\nabla}$ is the Levi-Civita connection of g. Assume that $(M^3, g, e^{-f}d\mu)$ contains no closed f-stable f-minimal surface. Then the space of connected closed embedded f-minimal surfaces of bounded f-index, in $(M^3, g, e^{-f}d\mu)$, is compact in the C^k topology for all $k \geq 2$.*

Corollary 1. *Let $(M^3, g, e^{-f}d\mu)$ be a simply connected and closed smooth metric measure space for which the Perelman scalar curvature is positive. Assume that $(M^3, g, e^{-f}d\mu)$ contains no closed f-stable f-minimal surface. Then the space of connected closed embedded f-minimal surfaces of bounded f-index, in $(M^3, g, e^{-f}d\mu)$, is compact in the C^k topology for all $k \geq 2$.*

It is worth pointing out the following facts. Let $(M^{n+1}, g, e^{-f}d\mu)$ be a smooth metric measure space with $Ric_f > 0$, then M contains no compact immersed f-stable f-minimal hypersurface (see [8]). If $(M^3, g, e^{-f}d\mu)$ is a complete smooth metric measure space such that $Ric_f \geq k > 0$, then M has finite weighted volume and finite fundamental group (see [11,13]); in this case, since $|\pi_1(M)| < \infty$ after passing to the universal covering, we may assume that M is simply connected. Moreover, if $Ric_f \geq k > 0$; for immersed f-minimal surfaces, uniform bound on their genus imply uniform bound on their f-index (see [3,6,9]).

2. Preliminaries

In this section, we will speak a little bit about smooth metric measure spaces, the reader can refer to [2,7–9,11,13] and references there in for more information about such spaces. The intention here is only to introduce the definitions of some curvatures and objects that will be used.

By a smooth metric measure space we mean a triple $(M^{n+1}, g, e^{-f} d\mu)$, where (M^{n+1}, g) is a $(n+1)$-dimensional smooth Riemannian manifold, f is a smooth function on M and $d\mu$ is the Riemannian volume measure induced from g. Here, the word closed will mean a space that is compact with empty boundary. For the sake of exposition, we assume here that the hypersurfaces are all two-sided and the ambient manifold is oriented.

The Bakry-Émery Ricci curvature of $(M^{n+1}, g, e^{-f} d\mu)$ is defined by

$$Ric_f := Ric + \overline{\nabla}^2 f, \tag{2.1}$$

where Ric, $\overline{\nabla}^2$ and $\overline{\nabla}$ are the Ricci curvature, the Hessian operator and the Levi-Civita connection of g, respectively.

The Perelman's scalar curvature of $(M^{n+1}, g, e^{-f} d\mu)$ is defined by

$$R_f^P := R + 2\overline{\Delta} f - \left|\overline{\nabla} f\right|^2, \tag{2.2}$$

where R is the scalar curvature of (M^{n+1}, g). Note that R_f^P is not the trace of Ric_f. In fact, $tr(Ric_f) = R + \overline{\Delta} f$.

Suppose that Σ is a smooth embedded hypersurface in M, the f-mean curvature of Σ is defined by

$$H_f := H - \langle \overline{\nabla} f, \nu \rangle, \tag{2.3}$$

where ν is the unit normal vector field on Σ and H is the mean curvature of Σ, in (M^{n+1}, g), with respect to ν. Σ is called an f-minimal hypersurface if its f-mean curvature vanishes everywhere. This definition is motivated by the first variation formula of the weighted volume of Σ. The weighted volume of Σ is defined by

$$\mathrm{vol}_f(\Sigma) := \int_\Sigma e^{-f} dv, \tag{2.4}$$

where dv is the volume form of Σ with respect to the metric induced from the ambient space (M^{n+1}, g). When $n = 2$, we use area_f instead of vol_f. For any normal variation Σ_s of Σ with compactly supported variation vector field $X = \varphi\nu$ (for some smooth compactly supported function φ on Σ), the first variation formula of vol_f is given by

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{vol}_f(\Sigma_s) = \int_\Sigma \varphi H_f e^{-f} dv, \tag{2.5}$$

and the second variation is given by

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \mathrm{vol}_f\,(\Sigma_s) = -\int_\Sigma \varphi L_f \varphi\, e^{-f}\, dv, \tag{2.6}$$

where $L_f := \Delta_f + |A|^2 + Ric_f(\nu,\nu)$, $\Delta_f := \Delta + \nabla f \cdot \nabla$, A is the second fundamental form of Σ, Δ is the Laplacian operator on Σ, and ∇ is the Levi-Civita connection of (Σ, g) induced from $\overline{\nabla}$. The operator Δ_f is called f-Laplacian. More generally, there is the concept of constant weighted mean curvature, see [7].

The f-index of Σ, denoted by f-index(Σ), is defined to be the dimension of the subspace (of the space of smooth compactly supported functions φ on Σ) such that the following quadratic form is strictly negative definite

$$Q^\Sigma(\varphi,\varphi) := -\int_\Sigma \varphi L_f \varphi\, e^{-f}\, dv, \tag{2.7}$$

when f-index$(\Sigma) = 0$, we say that Σ is f-stable.

Let $\Sigma \subset (M^{n+1}, g, e^{-f}d\mu)$ be an embedded f-minimal surface. Define a conformal metric $\widetilde{g} = e^{-\frac{2}{n}f} g$ on M, denoting by $\mathrm{vol}_{\widetilde{g}}(\Sigma)$, $H_{\widetilde{g}}(\Sigma)$, $\mathrm{index}_{\widetilde{g}}(\Sigma)$ and $\widetilde{R}$ the volume, the mean curvature, the index of $\Sigma \subset (M^{n+1}, \widetilde{g})$, and the scalar curvature of $(M^{n+1}, \widetilde{g})$ respectively; then $\mathrm{vol}_{\widetilde{g}}(\Sigma) = \mathrm{vol}_f(\Sigma)$, $H_{\widetilde{g}}(\Sigma) = e^{\frac{f}{n}} H_f(\Sigma)$, f-index$(\Sigma) = \mathrm{index}_{\widetilde{g}}(\Sigma)$, and $\widetilde{R} = e^{\frac{2f}{n}}(R_f^P + \frac{1}{n}|\overline{\nabla} f|^2)$.

3. Proof of the result

Let $\Sigma_j \subset (M^3, g, e^{-f}d\mu)$ be a sequence of connected closed embedded f-minimal surfaces such that f-index$(\Sigma_j) \leq I$. Define a conformal metric $\widetilde{g} = e^{-f} g$ on M, so $\mathrm{area}_{\widetilde{g}}(\Sigma_j) = \mathrm{area}_f(\Sigma_j)$, $H_{\widetilde{g}}(\Sigma_j) = e^{\frac{f}{2}} H_f(\Sigma_j)$, f-index$(\Sigma_j) = \mathrm{index}_{\widetilde{g}}(\Sigma_j)$ and $\widetilde{R} = e^f(R_f^P + \frac{1}{2}|\overline{\nabla} f|^2)$. Since $H_{\widetilde{g}}(\Sigma_j) = 0$, then Σ_j is also a sequence of connected, closed, embedded minimal surfaces in $(M^3, \widetilde{g})$. As $R_f^P + \frac{1}{2}|\overline{\nabla} f|^2 > 0$, we have $\widetilde{R} > 0$. From [3, Theorem 1.3]; there is $A_0 = A_0(M, \widetilde{g}, I) < \infty$ and $r_0 = r_0(M, \widetilde{g}, I)$ so that $\mathrm{area}_{\widetilde{g}}(\Sigma_j) \leq A_0$ and $\mathrm{genus}(\Sigma_j) \leq r_0$. Therefore, $\mathrm{area}_{\widetilde{g}}(\Sigma_j) \leq A_0$ and $\mathrm{index}_{\widetilde{g}}(\Sigma_j) \leq I$. From [12, Theorem 2.3] we obtain a closed connected and embedded minimal surface $\Sigma \subset (M, \widetilde{g})$ where $\Sigma_j \overset{(M,\widetilde{g})}{\longrightarrow} \Sigma$ in the varifold sense and $\mathrm{index}_{\widetilde{g}}(\Sigma) \leq I$. Moreover, assuming that $\Sigma_j \neq \Sigma$ eventually, then the convergence is smooth and graphical for all $x \in M \backslash \mathcal{B}$, where $\mathcal{B} \subset M$ is a finite set of points with $|\mathcal{B}| \leq I$.

We have to show that the convergence holds across the points of $\mathcal{B}$, i.e., the convergence is smooth everywhere. By Allard's regularity theorem (see [1]), this follows from showing that the convergence is of multiplicity one.

We know that if the multiplicity of the convergence of a sequence of embedded orientable minimal surfaces in a simply connected 3-manifold is not one, then the limit minimal surface will be stable (see [5]). Therefore, if $\Sigma_j \to \Sigma$ in $(M, \widetilde{g})$ with multiplicity greater than one, then the limit minimal surface $\Sigma \subset (M, \widetilde{g})$ will be stable, therefore $\Sigma \subset (M^3, g, e^{-f}d\mu)$ will be f-stable, this contradicts the hypothesis.

References

[1] Allard, W.K.: On the first variation of a varifold. Ann. Math. **95**, 417–491 (1972)
[2] Barbosa, E., Sharp, B., Wei, Y.: Smooth compactness of f-minimal hypersurfaces with bounded f-index. Proc. Am. Math. Soc. **145**(11), 4945–4961 (2017)
[3] Chodosh, O., Ketover, D., Maximo, D.: Minimal hypersurfaces with bounded index. Invent. Math. **209**(3), 617–664 (2017)
[4] Choi, H.I., Schoen, R.: The space of minimal embeddings of a surface into a three-dimensional manifold of positive ricci curvature. Invent. Math. **81**(3), 387–394 (1985)
[5] Colding, T.H., Minicozzi II W.P.: A Course in Minimal Surfaces. American Mathematical Society, Providence, RI (2011)
[6] Ejiri, N., Micallef, M.: Comparison between second variation of area and second variationof energy of a minimal surface. Adv. Calc. Var. **1**(3), 223–239 (2008)
[7] Espinar, J.M.: Gradient schrödinger operators, manifolds with density and applications. J. Math. Anal. Appl. **455**(2), 1505–1528 (2017)
[8] Fan, E.M.: Topology of three-manifolds with positive p-scalar curvature. Proc. Am. Math. Soc. **136**(9), 3255–3261 (2008)
[9] Li, H., Wei, Y.: f-minimal surface and manifold with positive m-bakry-émery ricci curvature. J. Geom. Anal. **25**(1), 421–435 (2015)
[10] Ma, L., Du, S.-H.: Extension of reilly formula with applications to eigenvalue estimates for drifting laplacians. Comptes Rendus Math. **348**(21–22), 1203–1206 (2010)
[11] Morgan, F.: Manifolds with density. Not. AMS **52**, 853–858 (2005)
[12] Sharp, B.: Compactness of minimal hypersurfaces with bounded index. J. Differ. Geom. **106**(2), 317–339 (2017)
[13] Wei, G., Wylie, W.: Comparison geometry for the bakry-emery ricci tensor. J. Differ. Geom. **83**(2), 377–405 (2009)