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Bang-Xian Han

Sharp p-Poincaré inequalities under measure contraction property

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Abstract. We obtain sharp estimate on p-spectral gaps, or equivalently optimal constant in p-Poincaré inequalities, for metric measure spaces satisfying measure contraction property. We also prove the rigidity for the sharp p-spectral gap.

Contents

1. Introduction

Sharp estimates on spectral gap for p-Laplacian, or equivalently, the optimal constant in p-Poincaré inequalities is a classical problem in comparison geometry. It addresses the following basic problem. Given a family $\mathcal{F} := \{(X_\alpha, \mathrm{d}_\alpha, \mathfrak{m}_\alpha) : \alpha \in \mathcal{A}\}$ of metric measure spaces, the corresponding optimal constant $\lambda_{\mathcal{F}}$ in p-Poincaré inequalities is defined by

$$\lambda_{\mathcal{F}} := \inf_{\alpha\in\mathcal{A}} \inf \left\{ \frac{\int_{X_\alpha} |\nabla_{\mathrm{d}_\alpha} f|^p \,\mathrm{d}\mathfrak{m}_\alpha}{\int_{X_\alpha} |f|^p \,\mathrm{d}\mathfrak{m}_\alpha} : f \in \mathrm{Lip} \cap L^p, \int_{X_\alpha} f|f|^{p-2} \,\mathrm{d}\mathfrak{m}_\alpha = 0, f \neq 0 \right\}, \tag{1.1}$$

where the local Lipschitz constant $|\nabla_{\mathrm{d}_\alpha} f| : X_\alpha \mapsto \mathbb{R}$ is defined by

$$|\nabla_{\mathrm{d}_\alpha} f|(x) := \varlimsup_{y\to x} \frac{|f(y) - f(x)|}{\mathrm{d}_\alpha(y, x)}.$$

B.-X. Han (✉): Department of Mathematics, Technion-Israel Institute of Technology, Haifa, Israel. e-mail: hanbangxian@gmail.com

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One of the most studied families of metric measure spaces is Riemannian manifolds with lower Ricci curvature bound $K \in \mathbb{R}$, upper dimension bound $N > 0$ and diameter bound $D > 0$. In this case, $\lambda_{\mathcal{F}}$ is the minimum of all first positive eigenvalues of the p-Laplacian (assuming Neumann boundary conditions if the boundary is not empty). Based on a refined gradient comparison technique and a careful analysis of the underlying model spaces, sharp estimate on the first eigenvalue of the p-Laplacian was finally obtained by Valtorta and Naber [22,26].

Another important family is weighted Riemannian manifolds (called smooth metric measure spaces) satisfying $\mathrm{BE}(K, N)$ curvature-dimension condition à la Bakry–Émery [5,6]. More generally, thanks to the deveploment of optimal transport theory, it was realized that Bakry–Émery's condition in smooth setting can be equivalently characterized by convexity of an entropy functional along L^2-Wasserstein geodesics (c.f. [14] and [27]). In this direction, metric measure spaces satisfying $\mathrm{CD}(K, N)$ condition was introduced by Lott–Villani [20] and Sturm [24,25]. This class of metric measure spaces with synthetic lower Ricci curvature bound and upper dimension bound includes the previous smooth examples, and is closed in the measured Gromov–Hausdorff topology. Recently, using measure decomposition technique on Riemannian manifolds developed by Klartag [19] (and by Cavalletti–Mondino [10] on metric measure spaces), sharp p-Poincaré inequalities under the $\mathrm{BE}(K, N)$ condition and the $\mathrm{CD}(K, N)$ condition have been obtained by E. Calderon in his Ph.D thesis [9].

In addition, Measure Contraction Property $\mathrm{MCP}(K, N)$ was introduced independently by Ohta [23] and Sturm [25] as a weaker variant of $\mathrm{CD}(K, N)$ condition. The family $\mathrm{MCP}(K, N)$ is strictly larger than $\mathrm{CD}(K, N)$. It was discovered by Juillet [18] that the n-th Heisenberg group equipped with the left-invariant measure, which is the simplest sub-Riemannian space, does not satisfy any $\mathrm{CD}(K, N)$ condition but do satisfy $\mathrm{MCP}(0, N)$ for $N \geq 2n + 3$. More recently, interpolation inequalities à la Cordero–Erausquin–McCann–Schmuckenshläger [14] were obtained, under suitable modifications, by Barilari and Rizzi [8] in the ideal sub-Riemannian setting, Badreddine and Rifford [4] for Lipschitz Carnot group, and by Balogh, Kristály and Sipos [7] for the Heisenberg group. As a consequence, more and more examples of spaces verifying MCP but not CD have been found, e.g. the generalized H-type groups and the Grushin plane (for more details, see [8]).

In [17], the author and Milman proved a sharp Poincaré inequality for subsets of (essentially non-branching) $\mathrm{MCP}(K, N)$ metric measure spaces, whose diameter is bounded from above by D. The current paper is a subsequent work of [17]. We will study the general p-poincaré inequality within the class of spaces verifying measure contraction property. Thanks to measure decomposition theorem (c.f. Theorem 3.5 [12]), it suffices to study the corresponding eigenvalue problems on one-dimensional model spaces introduced by Milman [21]. In particular, we identify a family of one-dimensional $\mathrm{MCP}(K, N)$-densities with diameter D, not verifying $\mathrm{CD}(K, N)$, achieving the optimal constant $\lambda^p_{K,N,D}$.

As a basic problem in metric geometry, the rigidity theorem helps us to understand more about the spaces under study. For the family of metric measure spaces satisfying $\mathrm{RCD}(K, N)$ condition with $K > 0$, a space that reaches the equality in (1.1) must have maximal diameter $\pi\sqrt{\frac{N-1}{K}}$. By maximal diameter theorem this space is isomorphic to a spherical suspension (see [11] and references therein for details). For MCP(K, N) spaces, the situation is very different. For $K > 0$, due to lack of monotonicity, we do not know whether a space that reaches the minimal spectrum has maximal diameter. For $K \le 0$, by monotonicity (Proposition 3.6) and one-dimensional rigidity (Theorem 3.13) we can prove the rigidity Theorem 4.2.

2. Prerequisites

Let (X, d) be a complete metric space and $\mathfrak{m}$ be a locally finite Borel measure with full support. Denote by $\mathrm{Geo}(X, \mathrm{d})$ the space of geodesics. We say that a set $\Gamma \subset \mathrm{Geo}(X, \mathrm{d})$ is non-branching if for any $\gamma^1, \gamma^2 \in \Gamma$, it holds:

$$\exists t \in (0,1) \ \text{ s.t. } \ \gamma^1_s = \gamma^2_s,\ \forall s \in [0,t] \Rightarrow \gamma^1_s = \gamma^2_s,\ \forall s \in [0,1].$$

Let (μ_t) be a L^2-Wasserstein geodesic. Denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all probability measures $\Pi \in \mathcal{P}(\mathrm{Geo}(X, \mathrm{d}))$ such that $(e_t)_\sharp \Pi = \mu_t$ (c.f. Theorem 2.10 [1]) where e_t denotes the evaluation map $e_t(\gamma) := \gamma_t$. We say that $(X, \mathrm{d}, \mathfrak{m})$ is essentially non-branching if for any $\mu_0, \mu_1 \ll \mathfrak{m}$, any $\Pi \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

It is clear that if (X, d) is a smooth Riemannian manifold then any subset $\Gamma \subset \mathrm{Geo}(X, \mathrm{d})$ is a set of non-branching geodesics, in particular any smooth Riemannian manifold is essentially non-branching. In addition, many sub-Riemannian spaces are also essentially non-branching, which follows from the existence and uniqueness of the optimal transport map on some ideal sub-Riemannian manifolds (c.f. [15]).

Given $K, N \in \mathbb{R}$, with $N > 1$, we set for $(t, \theta) \in [0,1] \times \mathbb{R}^+$,

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \ge (N-1)\pi^2, \\ \frac{\sin(t\theta\sqrt{K/(N-1)})}{\sin(\theta\sqrt{K/(N-1)})}, & \text{if } 0 < K\theta^2 < (N-1)\pi^2, \\ t, & \text{if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/(N-1)})}{\sinh(\theta\sqrt{-K/(N-1)})}, & \text{if } K\theta^2 < 0. \end{cases}$$

and

$$\tau^{(t)}_{K,N} := t^{\frac{1}{N}} \left(\sigma^{(t)}_{K,N-1}\right)^{1-\frac{1}{N}}.$$

Definition 2.1. (*Measure Contraction Property* MCP(K, N)) We say that an essentially non-branching metric measure space $(X, \mathrm{d}, \mathfrak{m})$ satisfies measure contraction property MCP(K, N) if for any point $o \in X$ and Borel set $A \subset X$ with $0 < \mathfrak{m}(A) < \infty$ (and with $A \subset B(o, \sqrt{(N-1)/K}$ if $K > 0$), there is $\Pi \in \mathrm{OptGeo}(\frac{1}{\mathfrak{m}(A)}\mathfrak{m}|_A, \delta_o)$ such that the following inequality holds for all $t \in [0,1]$

$$\frac{1}{\mathfrak{m}(A)}\mathfrak{m} \ge (e_t)_\sharp\left[\tau^{(1-t)}_{K,N}\left(\mathrm{d}(\gamma_0, \gamma_1)\right)^N \Pi(\mathrm{d}\gamma)\right]. \tag{2.1}$$

Theorem 2.2. (Localization for $\mathrm{MCP}(K, N)$ spaces, Theorem 3.5 [12]) *Let $(X, \mathrm{d}, \mathfrak{m})$ be an essentially non-branching metric measure space satisfying $\mathrm{MCP}(K, N)$ condition for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Then for any 1-Lipschitz function u on X, the non-branching transport set T_u associated with u (roughly speaking, T_u coincides with $\{|\nabla u| = 1\}$ up to $\mathfrak{m}$-measure zero set) admits a disjoint family of unparameterized geodesics $\{X_q\}_{q\in\mathfrak{Q}}$ such that*

$$\mathfrak{m}(\mathsf{T}_u \setminus \cup X_q) = 0,$$

and

$$\mathfrak{m}|_{\mathsf{T}_u} = \int_{\mathfrak{Q}} \mathfrak{m}_q \,\mathrm{d}\mathfrak{q}(q), \quad \mathfrak{q}(\mathfrak{Q}) = 1 \;\; \textit{and} \;\; \mathfrak{m}_q(X_q) = 1 \;\; \mathfrak{q} - \textit{a.e. } q \in \mathfrak{Q}.$$

Furthermore, for $\mathfrak{q}$-a.e. $q \in \mathfrak{Q}$, $\mathfrak{m}_q$ is a Radon measure with $\mathfrak{m}_q \ll \mathcal{H}^1|_{X_q}$ and $(X_q, \mathrm{d}, \mathfrak{m}_q)$ satisfies $\mathrm{MCP}(K, N)$.

3. One dimensional models

3.1. One dimensional MCP densities

Let $h \in L^1(\mathbb{R}^+, \mathcal{L}^1)$ be a non-negative Borel function. It is known (see e.g. Lemma 4.1 [17]) that $(\mathrm{supp}\, h, |\cdot|, h\mathcal{L}^1)$ satisfies $\mathrm{MCP}(K, N)$ condition if and only if h is a $\mathrm{MCP}(K, N)$ density in the following sense

$$h(tx_1 + (1-t)x_0) \geq \sigma^{(1-t)}_{K,N-1}(|x_1 - x_0|)^{N-1} h(x_0) \tag{3.1}$$

for all $x_0, x_1 \in \mathrm{supp}\, h$ and $t \in [0, 1]$.

Definition 3.1. Given $K \in \mathbb{R}$, $N > 1$. Denote by $D_{K,N}$ the Bonnet–Meyers diameter upper-bound:

$$D_{K,N} := \begin{cases} \frac{\pi}{\sqrt{K/(N-1)}} & \text{if } K > 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{3.2}$$

For any $D > 0$, we define $\mathcal{F}_{K,N,D}$ as the collection of $\mathrm{MCP}(K, N)$ densities $h \in L^1(\mathbb{R}^+, \mathcal{L}^1)$ with $\mathrm{supp}\, h = [0, D \wedge D_{K,N}]$.

For $\kappa \in \mathbb{R}$, we define the function $s_\kappa : [0, +\infty) \mapsto \mathbb{R}$ (on $[0, \pi/\sqrt{\kappa})$ if $\kappa > 0$)

$$s_\kappa(\theta) := \begin{cases} (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}\theta), & \text{if } \kappa > 0, \\ \theta, & \text{if } \kappa = 0, \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}\theta), & \text{if } \kappa < 0. \end{cases}$$

It can be seen that (3.1) is equivalent to

$$\left(\frac{s_{K/(N-1)}(b - x_1)}{s_{K/(N-1)}(b - x_0)}\right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1 - a)}{s_{K/(N-1)}(x_0 - a)}\right)^{N-1} \tag{3.3}$$

for all $[x_0, x_1] \subset [a, b] \subset \mathrm{supp}\, h$.

Furthermore, we have the following characterization.

Lemma 3.2. *Given $D \le D_{K,N}$, a density h is in $\mathcal{F}_{K,N,D}$ if and only if*

$$\left(\frac{s_{K/(N-1)}(D-x_1)}{s_{K/(N-1)}(D-x_0)}\right)^{N-1} \le \frac{h(x_1)}{h(x_0)} \le \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(x_0)}\right)^{N-1} \quad \forall\, 0 \le x_0 \le x_1 \le D. \tag{3.4}$$

Furthermore, $h \in \mathcal{F}_{K,N,D}$ if and only if $\ln h$ is $\mathcal{L}^1$-a.e. differentiable and

$$-h(x)\cot_{K,N,D}(D-x) \le h'(x) \le h(x)\cot_{K,N,D}(x), \quad \mathcal{L}^1 - a.e.\ x \in [0,D]$$

where the function $\cot_{K,N,D} : [0,D] \mapsto \mathbb{R}$ is defined by

$$\cot_{K,N,D}(x) := \begin{cases} \sqrt{K(N-1)}\cot\left(\sqrt{\frac{K}{N-1}}x\right), & \text{if } K>0,\\ (N-1)/x, & \text{if } K=0,\\ \sqrt{-K(N-1)}\coth\left(\sqrt{\frac{-K}{N-1}}x\right), & \text{if } K<0. \end{cases}$$

Proof. It can be checked that the function

$$a \mapsto \frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)}$$

is non-decreasing on $[0, x_0]$, and the function

$$b \mapsto \frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)}$$

is non-decreasing on $[x_1, D]$. Thus, (3.4) follows from (3.3).

Furthermore, for any $h \in \mathcal{F}_{K,N,D}$, it can be seen that (3.4) holds if and only if

$$x \mapsto \frac{\left(s_{K/(N-1)}(D-x)\right)^{N-1}}{h(x)} \text{ is non-increasing,} \tag{3.5}$$

and

$$x \mapsto \frac{\left(s_{K/(N-1)}(x)\right)^{N-1}}{h(x)} \text{ is non-decreasing.} \tag{3.6}$$

From (3.4) we can see that $\ln h$ is locally Lipschitz, so $\ln h$ is differentiable almost everywhere. So, by (3.5) and (3.6) we know (3.4) is equivalent to

$$\left(\ln s^{N-1}_{K/(N-1)}(D-\cdot)\right)' \le (\ln h)' = \frac{h'}{h} \le \left(\ln s^{N-1}_{K/(N-1)}\right)' \quad \mathcal{L}^1 - \text{a.e. on } [0,D]$$

which is the thesis. □

Notice that the function

$$[0,D] \ni x \mapsto \frac{s_{K/(N-1)}(D-x)}{s_{K/(N-1)}(x)}$$

is decreasing. By Lemma 3.2 (or (3.5) and (3.6)) we immediately obtain the following rigidity result.

Lemma 3.3. (One dimensional rigidity) *Denote* $h^1_{K,N,D} = \big(s_{K/(N-1)}(x)\big)^{N-1}|_{[0,D]}$ *and* $h^2_{K,N,D} = \big(s_{K/(N-1)}(D-x)\big)^{N-1}|_{[0,D]}$. *Then we have* $h^1_{K,N,D}, h^2_{K,N,D} \in \mathcal{F}_{K,N,D}$. *Furthermore,* $h^1_{K,N,D}$ *is the unique* $\mathcal{F}_{K,N,D}$ *density (up to multiplicative constants) satisfying*

$$h'(x) = h(x)\cot_{K,N,D}(x)$$

and $h^2_{K,N,D}$ *is the unique* $\mathcal{F}_{K,N,D}$ *density satisfying*

$$h'(x) = -h(x)\cot_{K,N,D}(D-x).$$

3.2. One dimensional p-Poincaré inequalities

Definition 3.4. For $p \in (1,\infty)$ and $h \in \mathcal{F}_{K,N,D}$, the p-spectral gap associated with h is defined by

$$\lambda^{p,h} := \inf\left\{\frac{\int |u'|^p h\,\mathrm{d}x}{\int |u|^p h\,\mathrm{d}x} : u \in \mathrm{Lip}\cap L^p, \int u|u|^{p-2}h\,\mathrm{d}x = 0, u \neq 0\right\}. \tag{3.7}$$

Definition 3.5. Let $K \in \mathbb{R}$, $D > 0$ and $N > 1$. The optimal constant $\lambda^p_{K,N,D}$ is defined as the infimum of all p-spectral gaps associated with admissible densities, i.e. $\lambda^p_{K,N,D}$ is given by

$$\lambda^p_{K,N,D} := \inf_{h\in\cup_{D'\leq D}\mathcal{F}_{K,N,D'}} \lambda^{p,h}.$$

Proposition 3.6. *Let* $K \in \mathbb{R}$, $D > 0$ *and* $N > 1$. *The function* $D \mapsto \lambda^p_{K,N,D}$ *is non-increasing, and*

$$\lambda^p_{K,N,D} = \inf_{h\in\cup_{D'\leq D}\mathcal{F}_{K,N,D'}\cap C^\infty} \lambda^{p,h}. \tag{3.8}$$

If $K \leq 0$, *the map* $D \mapsto \lambda^p_{K,N,D}$ *is strictly decreasing, and*

$$\lambda^p_{K,N,D} = \inf_{h\in\mathcal{F}_{K,N,D}\cap C^\infty} \lambda^{p,h}. \tag{3.9}$$

Proof. By Lemma 3.2 we know MCP densities are locally Lipschitz. Thus, using a standard mollifier we can approximate h uniformly by smooth MCP densities. Then by a simple approximation argument (see e.g. Proposition 4.8 [17]) we can prove

$$\lambda^p_{K,N,D} = \inf_{h\in\cup_{D'\leq D}\mathcal{F}_{K,N,D'}\cap C^\infty} \lambda^{p,h}.$$

Let $h \in \mathcal{F}_{K,N,D'}$ be a MCP density for some $D' > 0$, and u be an admissible function in (3.7). Then $\bar{h}(x) := h(\frac{D'}{D}x) \in \mathcal{F}_{K',N,D}$ with $K' = \big(\frac{D'}{D}\big)^2 K$, and $\bar{u}(x) := u(\frac{D'}{D}x)$ is also an admissible function. By computation, we have $\frac{\int |\bar{u}'|^p \bar{h}\,\mathrm{d}x}{\int |\bar{u}|^p \bar{h}\,\mathrm{d}x} = \left(\frac{D'}{D}\right)^p \frac{\int |u'|^p h\,\mathrm{d}x}{\int |u|^p h\,\mathrm{d}x}$. Therefore, if $K \le 0$ and $D' < D$, we have

$$\inf_{h\in\mathcal{F}_{K,N,D}} \lambda^{p,h} \le \inf_{h\in\mathcal{F}_{K',N,D}} \lambda^{p,h} \le \left(\frac{D'}{D}\right)^p \big(\inf_{h\in\mathcal{F}_{K,N,D'}} \lambda^{p,h}\big) < \inf_{h\in\mathcal{F}_{K,N,D'}} \lambda^{p,h}$$

and so

$$\lambda^p_{K,N,D} < \lambda^p_{K,N,D'}.$$

Then we obtain (3.9). □

Remark 3.7. The difference between the cases $K \le 0$ and $K > 0$ was already observed in [13] in the isoperimetric context and in [17] in the 2-Poincaré context. It is known that the monotonicity property (3.9) is **false** when $K > 0$.

In order to study the equation (3.18) in Theorem 3.10, we recall some basic facts about generalized trigonometric functions $\sin_p$ and $\cos_p$.

Definition 3.8. For $p \in (1, +\infty)$, define π_p by

$$\pi_p := \int_{-1}^{1} \frac{\mathrm{d}t}{(1-|t|^p)^{\frac{1}{p}}} = \frac{2\pi}{p\sin(\pi/p)} > 0.$$

The periodic C^1 function $\sin_p : \mathbb{R} \mapsto [-1, 1]$ is defined on $[-\pi_p/2, 3\pi_p/2]$ by:

$$\begin{cases} t = \int_0^{\sin_p(t)} \frac{\mathrm{d}s}{(1-|s|^p)^{\frac{1}{p}}} & \text{if } t \in \left[-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right], \\ \sin_p(t) = \sin_p(\pi_p - t) & \text{if } t \in \left[\frac{\pi_p}{2}, \frac{3\pi_p}{2}\right]. \end{cases} \tag{3.10}$$

It can be seen that $\sin_p(0) = 0$ and $\sin_p$ is strictly increasing on $[-\frac{\pi_p}{2}, \frac{\pi_p}{2}]$. Define $\cos_p(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sin_p(t)$, then we have the following generalized trigonometric identity

$$|\sin_p(t)|^p + |\cos_p(t)|^p = 1.$$

Definition 3.9. Let $h^i_{K,N,D}$, $i = 1, 2$ be MCP(K, N) densities defined in Lemma 3.3. Define $h_{K,N,D}$ by

$$h_{K,N,D}(x) := \begin{cases} h^1_{K,N,D}(x) & \text{if } x \in \left[\frac{D}{2}, D\right] \\ h^2_{K,N,D}(x) & \text{if } x \in \left[0, \frac{D}{2}\right]. \end{cases}$$

Define $T_{K,N,D}$ by

$$T_{K,N,D} := \left(\ln h_{K,N,D}\right)' = \begin{cases} \cot_{K,N,D}(x) & \text{if } x \in \left[\frac{D}{2}, D\right] \\ -\cot_{K,N,D}(D-x) & \text{if } x \in \left[0, \frac{D}{2}\right]. \end{cases}$$

By Lemma 3.2 we know $h_{K,N,D}$ is a MCP(K, N) density. It can be seen that (c.f. Lemma 3.4 [13]) $h_{K,N,D}$ does not satisfy any forms of CD condition.

Theorem 3.10. (One dimensional p-spectral gap) *Let $K \in \mathbb{R}$, $N > 1$, $D > 0$. Denote by $\hat{\lambda}^p_{K,N,D}$ the minimal λ such that the following initial value problem has a solution:*

$$\begin{cases} \varphi' = \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} T_{K,N,D} \cos_p^{p-1}(\varphi) \sin_p(\varphi), \\ \varphi(0) = -\frac{\pi_p}{2}, \quad \varphi\left(\frac{D}{2}\right) = 0, \quad \varphi(D) = \frac{\pi_p}{2}. \end{cases} \tag{3.11}$$

Then $\lambda^{p,h} \geq \hat{\lambda}^p_{K,N,D}$ for any $h \in \mathcal{F}_{K,N,D}$.

Proof. **Step 1.** Firstly we will show the existence of $\hat{\lambda}^p_{K,N,D}$.

By Lemma 3.2 we know $T_{K,N,D} \in C^\infty((0, \frac{D}{2}) \cup (\frac{D}{2}, D))$ and $-\cot_{K,N,D}(D - \cdot) \leq T_{K,N,D} \leq \cot_{K,N,D}$. Denote $T = T_{K,N,D}$, and denote by $u = u^{T,\lambda}$ the (unique) solution of the following equation:

$$\begin{cases} \left(u'|u'|^{p-2}\right)' + Tu'|u'|^{p-2} + \lambda u|u|^{p-2} = 0, \\ u(\frac{D}{2}) = 0. \end{cases} \tag{3.12}$$

Next we will study the equation (3.12) using a version of the so-called Pfüfer transformation. Define the functions $e = e^{T,\lambda}$ and $\varphi = \varphi^{T,\lambda}$ by:

$$\alpha := \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}}, \quad \alpha u = e \sin_p(\varphi), \quad u' = e \cos_p(\varphi).$$

By Lemma 3.11 we know that φ, e solve the following equation:

$$\begin{cases} \varphi' = \alpha + \frac{1}{p-1} T |\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\ \frac{\mathrm{d}}{\mathrm{d}t} \ln e = \frac{e'}{e} = -\frac{1}{p-1} T |\cos_p(\varphi)|^p. \end{cases} \tag{3.13}$$

Consider the following initial valued problem on $(0, \frac{D}{2}) \cup (\frac{D}{2}, D)$.

$$\begin{cases} \varphi' = \alpha + \frac{1}{p-1} T |\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\ \varphi(\frac{D}{2}) = 0. \end{cases} \tag{3.14}$$

By Cauchy's theorem we have the existence, uniqueness and continuous dependence on the parameters. Fix an $\epsilon \in (0, \frac{D}{2})$. We can find $\alpha = \alpha(\epsilon) > 0$, such that $\varphi'(x) > \frac{\pi_p}{D - 2\epsilon} > 0$ for all $x \in (\epsilon, \frac{D}{2})$. So there exists $a_\alpha \in [0, \frac{D}{2})$ such that $\varphi(a_\alpha) = -\frac{\pi_p}{2}$. Similarly, there is $b_\alpha \in (\frac{D}{2}, D]$ such that $\varphi(b_\alpha) = \frac{\pi_p}{2}$. Conversely, assume there is $\alpha > 0$ such that the following problem has a solution φ for some $a_\alpha \in [0, \frac{D}{2})$ and $b_\alpha \in (\frac{D}{2}, D]$:

$$\begin{cases} \varphi' = \alpha + \frac{1}{p-1} T |\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\ \varphi(a_\alpha) = -\frac{\pi_p}{2}, \varphi\left(\frac{D}{2}\right) = 0, \varphi(b_\alpha) = \frac{\pi_p}{2}. \end{cases} \tag{3.15}$$

Then for any $\alpha' > \alpha$, the following problem also has a solution for some $a'_\alpha \in (a_\alpha, \frac{D}{2})$ and $b'_\alpha \in (\frac{D}{2}, b_\alpha)$

$$\begin{cases} \varphi' = \alpha' + \frac{1}{p-1} T |\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\ \varphi(a'_\alpha) = -\frac{\pi_p}{2}, \varphi\left(\frac{D}{2}\right) = 0, \varphi(b'_\alpha) = \frac{\pi_p}{2}. \end{cases} \tag{3.16}$$

Therefore, by connectedness, there is a minimal $\bar{\lambda} \geq 0$ such that for any $\lambda > \bar{\lambda}$, there exist $\varphi = \varphi^{T,\lambda}$, $0 \leq a^\lambda < \frac{D}{2}$ and $\frac{D}{2} < b^\lambda \leq D$ such that

$$\begin{cases} \varphi' = \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1} T \cos_p^{p-1}(\varphi)\sin_p(\varphi), \\ \varphi(a^\lambda) = -\frac{\pi_p}{2}, \varphi\left(\frac{D}{2}\right) = 0, \varphi(b^\lambda) = \frac{\pi_p}{2}. \end{cases} \tag{3.17}$$

By continuous dependence on the parameter λ, we know (3.17) has a solution φ_∞ for $\gamma = \bar{\gamma}$, some $a^{\bar{\lambda}} \in [0, \frac{D}{2})$ and $b^{\bar{\lambda}} \in (\frac{D}{2}, D]$. In particular, $\bar{\lambda} > 0$.

Since $T(x) = -T(D-x)$ on $[0, \frac{D}{2}]$, by symmetry and minimality (or domain monotonicity) of $\bar{\lambda}$, we have $a^{\bar{\lambda}} = 0$ and $b^{\bar{\lambda}} = D$ (otherwise we can find a smaller λ). In particular, there is a minimal $\hat{\lambda}^p_{K,N,D}$ such that the initial value problem (3.11) has a solution $\varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}}$.

Step 2. Given $h \in \mathcal{F}_{K,N,D} \cap C^\infty$, we will show that $\hat{\lambda}^p_{K,N,D} \leq \lambda^{p,h}$.

First of all, by a standard variational argument we can see that $\lambda^{p,h}$ is the smallest positive real number such that there exists a non-zero $u \in W^{1,p}([0,D], h\mathcal{L}^1)$ solving the following equation (in weak sense):

$$\Delta_p^h u = -\lambda u|u|^{p-2} \tag{3.18}$$

with Neumann boundary condition, where $\Delta_p^h u$ is the weighted p-Laplacian on $([0,D], |\cdot|, h\mathcal{L}^1)$:

$$\Delta_p^h u = \Delta_p u + u'|u'|^{p-2}(\log h)' = \left(u'|u'|^{p-2}\right)' + u'|u'|^{p-2}\frac{h'}{h}.$$

By regularity theory we know $u \in C^{1,\alpha} \cap W^{1,p}$ for some $\alpha > 0$, and $u \in C^{2,\alpha}$ if $u' \neq 0$. Conversely, for any u solving the Neumann problem (3.18), we have $\int u|u|^{p-2}h\,\mathrm{d}x = 0$ and $\int |u'|^p h\,\mathrm{d}x = \lambda \int |u|^p h\,\mathrm{d}x$.

Assume by contradiction that $\lambda^{p,h} < \hat{\lambda}^p_{K,N,D}$. From the monotonicity argument in **Step 1**, we can see that there is $\lambda < \hat{\lambda}^p_{K,N,D}$ such that the following equation has a (monotone) solution $\varphi = \varphi^{\frac{h'}{h},\lambda}$:

$$\begin{cases} \varphi' = \left(\frac{\lambda}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1}\frac{h'}{h} \cos_p^{p-1}(\varphi)\sin_p(\varphi), \\ \varphi(0) = -\frac{\pi_p}{2}, \varphi(D) = \frac{\pi_p}{2}, \end{cases} \tag{3.19}$$

Without loss of generality (or by symmetry), we may assume there is $a' \in [\frac{D}{2}, D]$ such that $\varphi^{\frac{h'}{h},\lambda}(a') = 0$. Suppose there is a point $x_0 \in [a', D)$ such that $\varphi^{\frac{h'}{h},\lambda}(x_0) = \varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}}(x_0)$. From Lemma 3.2 we know that $\frac{h'}{h} \leq T_{K,N,D}$. So we know

$$\left(\varphi^{\frac{h'}{h},\lambda}\right)'(x_0) < \left(\varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}}\right)'(x_0).$$

Therefore,

$$\varphi^{\frac{h'}{h},\lambda}(x) < \varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}}(x)$$

for all $x \in (a', D]$, which contradicts to the fact that $\varphi^{\frac{h'}{h},\lambda}(\frac{D}{2}) = \varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}}(\frac{D}{2}) = \frac{\pi_p}{2}$. □

The following formulas has been used in [22,26]. We give a proof for completeness.

Lemma 3.11. *Let* e, φ, T *be functions defined in the proof of Theorem* 3.10*. Then we have*

$$\begin{cases} \varphi' = \alpha + \frac{1}{p-1} T |\cos_p(\varphi)|^{p-2} \cos_p(\varphi) \sin_p(\varphi), \\ \frac{\mathrm{d}}{\mathrm{d}t} \ln e = \frac{e'}{e} = -\frac{1}{p-1} T |\cos_p(\varphi)|^p. \end{cases} \tag{3.20}$$

Proof. Firstly, we have

$$\begin{aligned} \left(u'|u'|^{p-2}\right)' &= \left(e \cos_p(\varphi)|e \cos_p(\varphi)|^{p-2}\right)' \\ &= |e \cos_p(\varphi)|^{p-2}\left(e' \cos_p(\varphi) + e \sin''_p(\varphi)\varphi'\right) \\ &\quad + e \cos_p(\varphi)(p-2) e \cos_p(\varphi)|e \cos_p(\varphi)|^{p-4} \\ &\quad \times \left(e' \cos_p(\varphi) + e \sin''_p(\varphi)\varphi'\right) \\ &= |e \cos_p(\varphi)|^{p-2}(p-1)\left(e' \cos_p(\varphi) + e \sin''_p(\varphi)\varphi'\right). \end{aligned}$$

Combining with (3.12) we obtain

$$\begin{aligned} &|e \cos_p(\varphi)|^{p-2}\left(e' \cos_p(\varphi) + e \sin''_p(\varphi)\varphi'\right) \sin_p(\varphi) \\ &\quad + \frac{1}{p-1} T e \cos_p(\varphi) \sin_p(\varphi)|e \cos_p(\varphi)|^{p-2} + \frac{\lambda}{p-1}\alpha^{1-p} e^{p-1} |\sin_p(\varphi)|^p = 0. \end{aligned}$$

Differentiating the equation $\alpha u = e \sin_p(\varphi)$ and substituting u' by $e \cos_p(\varphi)$, we get

$$\alpha e \cos_p(\varphi) = e' \sin_p(\varphi) + e \cos_p(\varphi)\varphi'.$$

Differentiating the identity $|\sin_p(t)|^p + |\cos_p(t)|^p = 1$ we also have

$$|\sin_p(t)|^{p-2} \sin_p(t) \cos_p(t) + |\cos_p(t)|^{p-2} \cos_p(t) \sin''_p(t) = 0.$$

Therefore,

$$\begin{aligned} &|e \cos_p(\varphi)|^{p-2}\left(e' \cos_p(\varphi) + e \sin''_p(\varphi)\varphi'\right) \sin_p(\varphi) \\ &= |e \cos_p(\varphi)|^{p-2}\left(\alpha e \cos^2_p(\varphi) - e \cos^2_p(\varphi)\varphi' + e \sin''_p(\varphi) \sin_p(\varphi)\varphi'\right) \\ &= \alpha e^{p-1}(\varphi)|\cos_p(\varphi)|^p - e^{p-1}\left(|\cos_p(\varphi)|^p + |\sin_p(\varphi)|^p\right)\varphi' \\ &= \alpha e^{p-1}(\varphi)|\cos_p(\varphi)|^p - e^{p-1}\varphi'. \end{aligned}$$

Combining the results above, we prove the lemma. □

Combining Proposition 3.6 and Theorem 3.10, we get the following corollary immediately.

Corollary 3.12. *We have the following sharp* p*-spectral gap estimates for one dimensional models:*

$$\lambda^p_{K,N,D} = \begin{cases} \hat{\lambda}^p_{K,N,D} & \text{if } K \le 0 \\ \inf_{D' \in (0, \min(D, D_{K,N})]} \hat{\lambda}^p_{K,N,D'} & \text{if } K > 0 \end{cases}$$

Theorem 3.13. (One dimensional rigidity) *Given $K \le 0$, $N > 1$ and $D > 0$. If $\lambda^{p,h} = \hat{\lambda}^p_{K,N,D}$ for some $h \in \mathcal{F}_{K,N,D}$. Then $h = h_{K,N,D}$ up to a multiplicative constant.*

Proof. Assume $\lambda^{p,h} = \hat{\lambda}^p_{K,N,D}$ for some $h \in \mathcal{F}_{K,N,D}$. Then there is $h_n \in \mathcal{F}_{K,N,D} \cap C^\infty$ with $h_n \to h$ uniformly, and a decreasing sequence (λ^{p,h_n}) with $\lambda^{p,h_n} \to \hat{\lambda}^p_{K,N,D}$, such that $\varphi_n = \varphi^{\frac{h_n'}{h_n}, \lambda^{p,h_n}}$ solves the following equation:

$$\begin{cases} \varphi_n' = \left(\frac{\lambda^{p,h_n}}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p-1}\frac{h_n'}{h_n}\cos_p^{p-1}(\varphi_n)\sin_p(\varphi_n), \\ \varphi_n(0) = -\frac{\pi_p}{2}, \varphi_n(D) = \frac{\pi_p}{2}. \end{cases} \tag{3.21}$$

From Lemma 2.1 we know that $\{\varphi_n'\}_n$ and $\{\varphi_n\}_n$ are uniformly bounded. By Arzelà–Ascoli theorem we may assume $\varphi_n \to \varphi_\infty$ uniformly for some Lipschitz function φ_∞.

By minimality of $\hat{\lambda}^p_{K,N,D}$ and symmetry, we can see that $\lim_{n\to\infty} \varphi_n^{-1}(t)$ exists for any $t \in [-\frac{\pi_p}{2}, \frac{\pi_p}{2}]$ and

$$\lim_{n\to\infty} \varphi_n^{-1} = \left(\varphi^{T_{K,N,D}, \hat{\lambda}^p_{K,N,D}}\right)^{-1}.$$

In fact, assume by contradiction that $\lim_{n\to\infty} \varphi_n^{-1}(t) \neq \left(\varphi^{T_{K,N,D}, \hat{\lambda}^p_{K,N,D}}\right)^{-1}(t)$ for some $t \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2})$. By symmetry we may assume there are $N_1 \in \mathbb{N}$ and $\delta > 0$, such that $\delta_n := \left(\varphi^{T_{K,N,D}, \hat{\lambda}^p_{K,N,D}}\right)^{-1}(t) - \varphi_n^{-1}(t) \ge \delta$ for all $n \ge N_1$. Define a MCP(K, N) density $\bar{h}_n$ by

$$\bar{h}_n(x) := \begin{cases} h_n(x) & \text{if } x \in [0, \varphi_n^{-1}(t)], \\ \frac{h_n(\varphi_n^{-1}(t))}{h_{K,N,D}(\varphi_n^{-1}(t)+\delta_n)} h_{K,N,D}(x + \delta_n) & \text{if } x \in [\varphi_n^{-1}(t), D - \delta_n]. \end{cases}$$

Then $\bar{\varphi}_n = \varphi^{\frac{\bar{h}_n'}{\bar{h}_n}, \lambda^{p,h_n}}$ satisfies $(\bar{\varphi}_n)^{-1}(\frac{\pi_p}{2}) < D - \frac{\delta}{2}$ for n large enough, which contradicts to Proposition 3.6 and the minimality of $\hat{\lambda}^p_{K,N,D}$.

In conclusion, $\varphi_\infty = \varphi^{T_{K,N,D}, \hat{\lambda}^p_{K,N,D}}$ and we have $\varphi_n \to \varphi^{T_{K,N,D}, \hat{\lambda}^p_{K,N,D}}$ uniformly.

Then we get

$$\begin{aligned}
\frac{\pi_p}{2} &= \varphi_n(\varphi_n^{-1}(0)) - \varphi_n(0)\\
&= \lim_{n\to\infty}\int_0^{\varphi_n^{-1}(0)\wedge\frac{D}{2}} \Big(\frac{\lambda^{p,h_n}}{p-1}\Big)^{\frac{1}{p}} + \frac{1}{p-1}\frac{h_n'}{h_n}\cos_p^{p-1}(\varphi_n)\sin_p(\varphi_n)\,\mathrm{d}x\\
&\le \lim_{n\to\infty}\int_0^{\varphi_n^{-1}(0)\wedge\frac{D}{2}} \Big(\frac{\lambda^{p,h_n}}{p-1}\Big)^{\frac{1}{p}} + \frac{1}{p-1}T_{K,N,D}\cos_p^{p-1}(\varphi_n)\sin_p(\varphi_n)\,\mathrm{d}x\\
&= \int_0^{\frac{D}{2}} \Big(\frac{\hat{\lambda}^p_{K,N,D}}{p-1}\Big)^{\frac{1}{p}}\\
&\quad + \frac{1}{p-1}T_{K,N,D}\cos_p^{p-1}(\varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}})\sin_p(\varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}})\,\mathrm{d}x\\
&= \frac{\pi_p}{2}.
\end{aligned}$$

Therefore,

$$(\ln h_n)' = \frac{h_n'}{h_n} \to T_{K,N,D} = \frac{h'_{K,N,D}}{h_{K,N,D}}$$

in $L^1([0,\frac{D}{2}],\cos_p^{p-1}(\varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}})\sin_p(\varphi^{T_{K,N,D},\hat{\lambda}^p_{K,N,D}})\mathcal{L}^1)$. By symmetry, we can see that $(\ln h_n)' \to (\ln h_{K,N,D})'$ in $L^1([0,D],\mathcal{L}^1)$. Hence $h = h_{K,N,D}$ up to a multiplicative constant. □

4. *p*-spectral gap

4.1. Sharp *p*-spectral gap estimates

Using standard localization argument (c.f. Theorem 1.1 [17], Theorem 4.4 [11]), we can prove the sharp p-Poincaré inequality with one dimensional results.

Theorem 4.1. (The sharp p-spectral gap under $\mathrm{MCP}(K,N)$) *Let* $(X,\mathrm{d},\mathfrak{m})$ *be an essentially non-branching metric measure space satisfying* $\mathrm{MCP}(K,N)$ *for some* $K\in\mathbb{R}$, $N\in(1,\infty)$ *and* $\mathrm{diam}(X)\le D$. *For any* $p>1$, *define* $\lambda^p_{(X,\mathrm{d},\mathfrak{m})}$ *as the optimal constant in* p*-Poincaré inequality on* $(X,\mathrm{d},\mathfrak{m})$:

$$\lambda^p_{(X,\mathrm{d},\mathfrak{m})} := \inf\left\{\frac{\int|\nabla f|^p\,\mathrm{d}\mathfrak{m}}{\int|f|^p\,\mathrm{d}\mathfrak{m}} : f\in\mathrm{Lip}\cap L^p, \int f|f|^{p-2}\,\mathrm{d}\mathfrak{m}=0,\ f\neq 0\right\}.$$

Then we have the following sharp estimate

$$\lambda^p_{(X,\mathrm{d},\mathfrak{m})} \ge \lambda^p_{K,N,D} = \begin{cases}\hat{\lambda}^p_{K,N,D} & \text{if } K\le 0\\ \inf_{D'\in(0,\min(D,D_{K,N})]}\hat{\lambda}^p_{K,N,D'} & \text{if } K>0.\end{cases}$$

Proof. Let $\bar{f} = f|f|^{p-2}$ be a Lipschitz function with $\int \bar{f} = 0$. Let $\bar{f}^{\pm}$ denote the positive and the negative parts of $\bar{f}$ respectively. Then we have $\int \bar{f}^{+} = -\int \bar{f}^{-}$. Consider the L^1-optimal transport problem from $\mu_0 := \bar{f}^{+}\mathfrak{m}$ to $\mu_1 := -\bar{f}^{-}\mathfrak{m}$. By Theorem 2.2, there exists a family of disjoint unparameterized geodesics $\{X_q\}_{q\in\mathfrak{Q}}$ of length at most D, such that

$$\mathfrak{m}(X \setminus \cup X_q) = 0, \quad \mathfrak{m} = \int_{\mathfrak{Q}} \mathfrak{m}_q \, \mathrm{d}\mathfrak{q}(q)$$

where $\mathfrak{m}_q = h_q \mathcal{H}^1|_{X_q}$ for some $h_q \in \mathcal{F}_{K,N,D'_q}$ with $D'_q \le D$, $\mathfrak{m}_q(X_q) = \mathfrak{m}(X)$ and

$$\int \bar{f} h_q \, \mathrm{d}\mathcal{H}^1|_{X_q} = 0$$

for $\mathfrak{q}$-a.e. $q \in \mathfrak{Q}$.

Denote $f_q = f|_{X_q}$. By definition we obtain

$$\int |f'_q|^p h_q \, \mathrm{d}\mathcal{H}^1|_{X_q} \ge \lambda^{p,h_q} \int |f_q|^p h_q \, \mathrm{d}\mathcal{H}^1|_{X_q} \ge \lambda^p_{K,N,D} \int |f_q|^p h_q \, \mathrm{d}\mathcal{H}^1|_{X_q}.$$

Notice that $|f'_q| \le |\nabla f|$. Thus, we have

$$\begin{aligned}
\lambda^p_{K,N,D} \int |f|^p \, \mathrm{d}\mathfrak{m} &= \lambda^p_{K,N,D} \int_{\mathfrak{Q}} \int_{X_q} |f_q|^p \mathfrak{m}_q \, \mathrm{d}\mathfrak{q}(q) \\
&\le \int_{\mathfrak{Q}} \int_{X_q} |f'_q|^p \mathfrak{m}_q \, \mathrm{d}\mathfrak{q}(q) \\
&= \int |\nabla f|^p \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

Combining with Corollary 3.12 we prove the theorem. □

4.2. Rigidity for p-spectral gap

In this part, we will study the rigidity for p-spectral gap under the measure contraction property. We adopt the notation $|\mathrm{D}f|$ to denote the weak upper gradient of a Sobolev function f. We refer the readers to [2] and [16] for details about Sobolev space theory and calculus on metric measure spaces.

Theorem 4.2. (Rigidity for p-spectral gap) *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be an essentially non-branching metric measure space satisfying* $\mathrm{MCP}(K, N)$ *for some* $K \le 0$, $N \in (1, \infty)$ *and* $\mathrm{diam}(X) \le D$. *Assume there is a non-zero Sobolev function* $f \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ *with* $\int f|f|^{p-2} \, \mathrm{d}\mathfrak{m} = 0$ *such that*

$$\int |\mathrm{D}f|^p \, \mathrm{d}\mathfrak{m} - \hat{\lambda}^p_{K,N,D} \int |f|^p \, \mathrm{d}\mathfrak{m} = 0.$$

Then $\mathrm{diam}(X) = D$ *and there are disjoint unparameterized geodesics* $\{X_q\}_{q\in\mathfrak{Q}}$ *of length* D *such that* $\mathfrak{m}(X \setminus \cup X_q) = 0$. *Moreover,* $\mathfrak{m}$ *has the following representation*

$$\mathfrak{m} = \int_{\mathfrak{Q}} h_q \, \mathrm{d}\mathcal{H}^1|_{X_q} \mathrm{d}\mathfrak{q}(q),$$

where $\frac{h_q'}{h_q} = T^p_{K,N,D}$ *for* $\mathfrak{q}$*-a.e.* $q \in \mathfrak{Q}$.

Proof. Similar to the proof of Theorem 4.1, we can find a measure decomposition associated with $\bar{f} := f|f|^{p-2}$, such that

$$\mathfrak{m}(X \setminus \cup X_q) = 0, \quad \mathfrak{m} = \int_{\mathfrak{Q}} \mathfrak{m}_q \, \mathrm{d}\mathfrak{q}(q)$$

where $\mathfrak{m}_q = h_q \mathcal{H}^1|_{X_q}$ for some $h_q \in \mathcal{F}_{K,N,D_q'}$ with $D_q' \le D$, $\mathfrak{m}_q(X_q) = \mathfrak{m}(X)$ and

$$\int \bar{f} h_q \, \mathrm{d}\mathcal{H}^1|_{X_q} = 0$$

for $\mathfrak{q}$-a.e. $q \in \mathfrak{Q}$.

By Theorem 7.3 [3] we know $f_q := f|_{X_q} \in W^{1,q}(X_q)$ and $|\mathrm{D} f_q| \le |\mathrm{D} f|$. Then from the proof of Theorem 4.1 we can see that $\lambda^{p,h_q} = \hat{\lambda}^p_{K,N,D}$ for $\mathfrak{q}$-a.e. $q \in \mathfrak{Q}$. By Proposition 3.6 we know that the function $D \mapsto \hat{\lambda}^p_{K,N,D}$ is strictly decreasing, so $D_q' = D$ and $\mathrm{diam}(X) = D$. Finally, by Theorem 3.13 we know that $\frac{h_q'}{h_q} = T^p_{K,N,D}$. □

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