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$L^p - L^q$ estimates for the solution of the Dunkl wave equation

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Abstract. In this paper, our main aim is to derive $L^p - L^q$ estimates of the solution $u_k(x,t)$ (t fixed) of the Cauchy problem for the homogeneous linear wave equation associated to the Dunkl Laplacian Δ_k,

$$\Delta_k u_k(x,t) = \partial_t^2 u_k(x,t), \quad \partial_t u_k(x,0) = f(x), \quad u_k(x,0) = g(x).$$

We extend to the Dunkl setting the estimates given by Strichartz (Trans Am Math Soc 148:461–471, 1970) for the ordinary wave equation.

1. Introduction and background

In his seminal paper [5], Dunkl constructed a family of differential-difference operators associated to a finite reflection group on a Euclidean space, which are known as Dunkl operators in the literature. He introduced in [6] an integral transform associated with the eigenfunctions of the Dunkl operators for a root system which generalises the classical Fourier transform and is now called the Dunkl transform. By means of this transform several important results of the classical Fourier analysis have been generalized to Dunkl analysis, giving us new perspectives on familiar topics from Harmonic Analysis and Partial Differential Equations.

In this paper, we are interested in the $L^p - L^q$ estimates of the solutions of wave equations associated to Dunkl Laplace operator, in particular we generalize the estimates for ordinary wave equation due to Strichartz [17]. The techniques used involve interpolation of an appropriate analytic family of operators in a way similar to that used in the classical case. To illustrate our method and collect the results we need, we begin by recalling some preliminary definitions and background materials for the Dunkl analysis. References are [4–6,11,12,18].

Let $G \subset \mathrm{O}(\mathbb{R}^n)$ be a finite reflection group associated to a reduced root system R and $k : R \to [0, +\infty)$ be a G–invariant function (called multiplicity function). Let

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R^+ be a positive root subsystem. The Dunkl operators D_ξ^k on $\mathbb{R}^n$ are the following k-deformations of directional derivatives ∂_ξ by difference operators:

$$D_\xi^k f(x) = \partial_\xi f(x) + \sum_{\upsilon \in R^+} k(\upsilon) \langle \upsilon, \xi \rangle \frac{f(x) - f(\sigma_\upsilon . x)}{\langle \upsilon, x \rangle}, \tag{1.1}$$

where here σ_υ is the reflection with respect to the hyperplane orthogonal to υ and $\langle ., . \rangle$ is the usual Euclidean inner product, we denote $| . |$ its induced norm. If $(e_j)_j$ is the canonical basis of $\mathbb{R}^n$ we simply write D_j^k instead of $D_{e_j}^k$. In analogy to the ordinary Laplacian we define the Dunkl Laplace operator by

$$\Delta_k = \sum_{j=1}^n (D_j^k)^2.$$

The Dunkl operators are antisymmetric with respect to the measure $w_k(x)\, dx$ with density

$$w_k(x) = \prod_{\upsilon \in R^+} | \langle \upsilon, x \rangle |^{2k(\upsilon)}.$$

The operators ∂_ξ and D_ξ^k are intertwined by a Laplace–type operator

$$V_k f(x) = \int_{\mathbb{R}^n} f(y)\, d\nu_x(y)$$

associated to a family of compactly supported probability measures $\{ \nu_x \mid x \in \mathbb{R}^n \}$. Specifically, ν_x is supported in the convex hull $\mathrm{co}(G.x)$.

For every $y \in \mathbb{C}^n$, the simultaneous eigenfunction problem

$$D_\xi^k f = \langle y, \xi \rangle f, \qquad \forall\, \xi \in \mathbb{R}^n,$$

has a unique solution $f(x) = E_k(x, y)$ such that $E_k(0, y) = 1$, called the Dunkl kernel and is given by

$$E_k(x, y) = V_k(e^{\langle ., y \rangle})(x) = \int_{\mathbb{R}^n} e^{\langle z, y \rangle}\, d\nu_x(z) \qquad \forall\, x \in \mathbb{R}^n.$$

When $k = 0$ the Dunkl kernel $E_k(x, y)$ reduces to the exponential $e^{\langle x, y \rangle}$.

The Dunkl transform is defined on $L^1(\mathbb{R}^n, w_k(x)dx)$ by

$$\mathcal{F}_k f(\xi) = c_k^{-1} \int_{\mathbb{R}^n} f(x)\, E_k(x, -i\, \xi)\, w_k(x)\, dx,$$

where

$$c_k = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}}\, w_k(x)\, dx.$$

If $k = 0$ then Dunkl transform coincides with the usual Fourier transform. In the sequel, we denote by $\| . \|_{p,k}$ the norm of $L^p(\mathbb{R}^n, w_k(x)dx)$, $1 \le p < \infty$. Below we summarize some of the useful properties of the Dunkl transform.

(i) For $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space) we have $\mathcal{F}_k(D_\xi^k f)(x) = i\langle \xi, x\rangle \mathcal{F}_k(f)(x)$. In particular

$$\mathcal{F}_k(\Delta_k f)(x) = -|x|^2 \mathcal{F}_k(f)(x).$$

(ii) The Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

(iii) (*Plancherel Theorem*) The Dunkl transform extends to an isometric automorphism of $L^2(\mathbb{R}^n, w_k(x)dx)$.

(iv) (*Inversion formula*) For every $f \in \mathcal{S}(\mathbb{R}^n)$, and more generally for every $f \in L^1(\mathbb{R}^n, w_k(x)dx)$ such that $\mathcal{F}_k f \in L^1(\mathbb{R}^n, w_k(x)dx)$, we have

$$f(x) = \mathcal{F}_k^2 f(-x) \qquad \forall\, x \in \mathbb{R}^n.$$

(v) The Hausdorff-Young inequality: for $1 < p \leq 2$

$$\|\mathcal{F}_k(f)\|_{p',k} \leq c\|f\|_{p,k} \tag{1.2}$$

where c is a positive constant and p' is the conjugate exponent of p. This follows from the Riesz-Thorin Interpolation Theorem for the special cases $p = 1$ and $p = 2$. We refer to [2], where more details on $L^p - L^q$ norm for the Dunkl transform can be found.

(vi) If f is a radial function in $L^1(\mathbb{R}^n, w_k(x)dx)$ such that $f(x) = \widetilde{f}(|x|)$, then $\mathcal{F}_k(f)$ is also radial and

$$\mathcal{F}_k(f)(x) = \frac{1}{|x|^{\gamma_k+n/2-1}} \int_0^\infty \widetilde{f}(s) J_{\gamma_k+n/2-1}(s|x|)\, s^{\gamma_k+n/2} ds; \quad x \in \mathbb{R}^n. \tag{1.3}$$

where

$$\gamma_k = \sum_{\upsilon \in R^+} k(\upsilon) \tag{1.4}$$

and J_ν is the Bessel function,

$$J_\nu(z) = \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}.$$

Let $x \in \mathbb{R}^n$, the Dunkl translation operator $f \to \tau_x(f)$ is defined on $L_k^2(\mathbb{R}^n, w_k(x)dx)$ by

$$\mathcal{F}_k(\tau_x(f))(y) = \mathcal{F}_k f(y)\, E_k(x, iy), \quad y \in \mathbb{R}^n.$$

In particular, when $f \in S(\mathbb{R}^n)$ we have

$$\tau_x(f)(y) = \int_{\mathbb{R}^n} \mathcal{F}_k(f)(\xi) E_k(x, i\xi) E_k(y, -i\xi) w_k(\xi) d\xi$$

and since Dunkl kernel satisfies $|E_k(x, iy)| \leq 1$ (see, e.g. [11, Corollary 5.4]), then we note the following

$$\|\tau_x(f)\|_{\infty,k} \leq \|\mathcal{F}_k(f)\|_{1,k}. \tag{1.5}$$

In the case when $f(x) = \widetilde{f}(|x|)$ is a continuous radial function that belongs to $L^2(\mathbb{R}^n, w_k(x)dx)$, the Dunkl translation is represented by the following integral,

$$\tau_x(f)(y) = \int_{\mathbb{R}^n} \widetilde{f}\left(\sqrt{|y|^2 + |x|^2 + 2\langle y, \eta\rangle}\right) d\nu_x(\eta), \tag{1.6}$$

This formula has been first obtained by M. Rösler [13] when f is a smooth radial function and extended to the mentioned case of functions by F. Dai and H. Wang, [3, Lemma 3.4].

The Dunkl translation operators can be extended to all radial functions f in $L^p(\mathbb{R}^n, w_k(x)dx)$, $1 \le p \le \infty$ and the following holds

$$\|\tau_x(f)\|_{p,k} \le \|f\|_{p,k}. \tag{1.7}$$

It should be mentioned here that (1.7) was justified in [18, Theorem 3.7] for $1 \le p \le 2$ and recently in [7, Theorem 3.7] for $p \ge 2$.

We define the Dunkl convolution product for suitable functions f and g by

$$f *_k g(x) = \int_{\mathbb{R}^N} \tau_x(f)(-y)g(y)w_k(y)dy, \quad x \in \mathbb{R}^n.$$

We note that it is commutative and satisfies the following property,

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \tag{1.8}$$

Moreover, the operator $f \to f *_k g$ is bounded on $L^p(\mathbb{R}^n, w_k(x)dx)$ provide g is a bounded radial function in $L^1(\mathbb{R}^n, w_k(x)dx)$. In particular we have the the following Young's inequality ([18, Theorem 4.1]),

$$\|f *_k g\|_{p,k} \le \|g\|_{1,k}\|f\|_{p,k} . \tag{1.9}$$

We now come to the main subject. We consider the following Cauchy problem for the Dunkl wave equation

$$\Delta_k u_k(x,t) = \partial_t^2 u_k(x,t), \partial_t u_k(x,0) = f(x), u_k(x,0) = g(x); (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

where the functions f and g belong to $S(\mathbb{R}^n)$. The solution is given in term of Dunkl transform by

$$u_k(x,t) = \mathcal{F}_k^{-1}\left(\frac{\sin t|\xi|}{|\xi|}\mathcal{F}_k f(\xi) + \cos(t|\xi|)\mathcal{F}_k(g)(\xi)\right)(x). \tag{1.10}$$

The study of the Dunkl wave equation was initiated by Ben Saïd and Ørsted [14, 15] where they computed the solution u_k and established validity of Huygens' Principles. Also in this context Mejjaoli [9,10] studied the mixed-norm Strichartz type estimates for u_k. The main contribution of our work is the following theorem.

Theorem 1.1. *For $t \neq 0$ there exists $C(t) > 0$ such that*

$$\|u(.,t)\|_{q,k} \leq C(t)\left(\|f\|_{p,k} + \left\|\sum_{j=1}^{n}|D_j^k g|\right\|_{p,k}\right), \tag{1.11}$$

provided that

$$2\,\frac{n+2\gamma_k+1}{n+2\gamma_k+3} \leq p \leq 2; \quad \frac{n+2\gamma_k}{q} = \frac{2\gamma_k+n-1}{2} - \frac{1}{p'} \tag{1.12}$$

and

$$2\frac{n+2\gamma_k}{n+2\gamma_k+2} \leq p \leq 2\frac{n+2\gamma_k+1}{n+2\gamma_k+3}; \quad \frac{1}{q} = \frac{n+2\gamma_k-1}{2} - \frac{n+2\gamma_k}{p'}, \tag{1.13}$$

where γ_k is given by (1.4) and p' is the conjugate exponent of p.

The Proof is based on complex interpolation method much like the proof given in [17]. For the reader's convenience we recall the Stein's Interpolation Theorem [16].

Let (M,μ) and (N,ν) be σ-finite measure spaces and

$$S = \{z \in \mathbb{C};\ a \leq Re(z) \leq b\}, \quad a < b.$$

We suppose that we are given a linear operator T_z, for each $z \in S$, on the space of simple functions in $L^1(M,\mu)$ into the space of measurable functions on N. If f is a simple function in $L^1(M,\mu)$ and g a simple function in $L^1(N,\nu)$, we assume furthermore that $gT_z(f) \in L^1(N,\nu)$. The family of operators $\{T_z\}$ is called admissible if the mapping

$$F : z \to \int_N g\, T_z(f) d\nu$$

is holomorphic in the interior of S and continuous on S, and there exists a constant $c < \pi(b-a)$ such that

$$\sup_{z\in S} e^{-c|Im(z)|} \log|F(z)| < \infty. \tag{1.14}$$

Theorem 1.2. (Stein [16]) *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and $\{T_z\}$, $z \in S$, be an admissible family of linear operators such that*

$$\|T_{a+iy}(f)\|_{q_0} \leq M_0(y)\|f\|_{p_0} \quad \textit{and} \quad \|T_{b+iy}(f)\|_{q_1} \leq M_1(y)\|f\|_{p_1}$$

for each real number y and each simple function $f \in L^1(M,\mu)$. If, in addition, the constants $M_j(y)$, $j = 0, 1$, satisfy

$$\sup_{y\in\mathbb{R}} e^{-c|y|} \log(M_j(y)) < \infty$$

for some $c < \pi(b-a)$, then for all $t \in [0,1]$ there exists a constant M_t such that

$$\|T_{\theta_t}(f)\|_{q_t} \leq M_t\|f\|_{p_t}$$

for all simple functions f provided

$$\theta_t = (1-t)a + tb, \quad \frac{1}{p_t} = \frac{(1-t)}{p_0} + \frac{t}{p_1} \quad \textit{and} \quad \frac{1}{q_t} = \frac{(1-t)}{q_0} + \frac{t}{q_1}$$

2. Interpolation of analytic family of operators

The idea now is to consider the following family of operators $f \to S_z(f)$ given on $L^2(\mathbb{R}^n, w_k(x)dx)$ by

$$S_z(f)(x) = \mathcal{F}_k^{-1}\Big(|\xi|^{\gamma_k+n/2-z} J_{\gamma_k+n/2-z}(|\xi|)\mathcal{F}_k(f)(\xi)\Big)(x)$$

where z can be taken to be complex. Hence in view of (1.10) and by the fact that

$$J_{1/2}(t) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin t}{\sqrt{t}} \quad \text{and} \quad J_{-1/2}(t) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\cos t}{\sqrt{t}}$$

one can write

$$\begin{aligned} u_k(x,t) = &\left(\frac{\pi}{2}\right)^{1/2} t\delta(t) S_{\gamma_k+(n+1)/2}\delta(t^{-1}) f(x) \\ &+ \left(\frac{\pi}{2}\right)^{1/2} \delta(t) S_{\gamma_k+(n-1)/2}\delta(t^{-1}) g(x) \end{aligned} \tag{2.1}$$

where $\delta(t)$ is the dilation operator $\delta(t)f(x) = f(tx)$.

The operator S_z turns out to be the analytic continuation in the parameters z of the convolution operator

$$T_z(f) = \Phi_z *_k f, \quad 0 \leq \text{Re}(z) < 1$$

where

$$\Phi_z(x) = \begin{cases} \frac{2^z}{\Gamma(1-z)}(1-|x|^2)^{-z}, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

This is a consequence of the following proposition.

Proposition 2.1. *The Dunkl transform of* Φ_z, $\;0 \leq \text{Re}(z) < 1$, *is given by*

$$\mathcal{F}_k(\Phi_z)(\xi) = |\xi|^{z-\gamma_k-n/2} J_{\gamma_k+n/2-z}(|\xi|). \tag{2.2}$$

Proof. Since Φ_z is a radial function that belongs to $L^1(\mathbb{R}^n, w_k(x)dx)$ then from (1.3) we have

$$\mathcal{F}_k(\Phi_z)(\xi) = \frac{2^z}{\Gamma(1-z)|\xi|^{\gamma_k+n/2-1}} \int_0^1 (1-s^2)^{-z} J_{\gamma_k+n/2-1}(s|\xi|)\, s^{\gamma_k+n/2} ds.$$

We obtain (2.2) by applying the well known relationship between Bessel functions (Sonine's first finite integral)

$$J_{\mu+\nu+1}(t) = \frac{t^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^1 J_\mu(st) s^{\mu+1}(1-s^2)^\nu ds,$$

for $Re(\mu) > -1$, $Re(\nu) > -1$ and $t > 0$ (see for example 12.11 of [19]). □

We now apply Stein's Interpolation Theorem to the family $\{S_z\}$. The following theorem is the main result of this section.

Theorem 2.2. *Suppose that* $0 \leq \alpha \leq \gamma_k + (n+1)/2$. *Then with a constant* C

$$\|S_\alpha(f)\|_{q,k} \leq C\,\|f\|_{p,k}, \qquad \text{for all} \quad f \in S(\mathbb{R}^n)$$

in each of the following cases :

(a) *For* $1 < p \leq 2 \leq q < \infty$ *and* $1/p - 1/q \leq (2\gamma_k + n + 1 - 2\alpha)/2(n + 2\gamma_k)$.
(b) *For* $p = (n + 1 + 2\gamma_k)/(n + 1 + 2\gamma_k - \alpha)$ *and* $q = p'$
(c) *For* $1/2 \leq \alpha \leq (n+1)/2 + \gamma_k$, *in the following cases:*
 (i) $(n + 1 + 2\gamma_k)/(n + 1 + 2\gamma_k - \alpha) \leq p \leq 2$ *and* $(n + 2\gamma_k)/q = \alpha - 1/p'$.
 (ii) $(n + 2\gamma_k)/(n + 2\gamma_k - \alpha + 1/2) \leq p \leq (n + 1 + 2\gamma_k)/(n + 1 + 2\gamma_k - \alpha)$ *and* $1/q = \alpha - (n + 2\gamma_k)/p'$.

Remark 2.3. We see that condition c)-(ii) can be reduced to c)-(i) by duality.

The argument proceeds similarly to the proof of Theorem 1 of [17]. First, we state the following generalization of the Hardy-Littlewood multiplier theorem

Theorem 2.4. *Let* $0 < t < 2\gamma_k + n$ *and* m *be a measurable function such that some constant* $c > 0$

$$|m(\xi)| \leq \frac{c}{|\xi|^t}. \tag{2.3}$$

Then the operator $T_m = \mathcal{F}_k^{-1}(m\mathcal{F}_k)$ *is bounded from* $L^p(\mathbb{R}^n, w_k(x)dx)$ *to* $L^q(\mathbb{R}^n, w_k(x)dx)$, *provided*

$$1 < p \leq 2 \leq q < \infty\,, \quad \frac{1}{p} - \frac{1}{q} = \frac{t}{2\gamma_k + n}.$$

In the case $k = 0$ this result is contained in Theorem 1.11 of [8]. In the same way we obtain Theorem 2.4 directly from the following lemmas.

Lemma 2.5. *Let* $\varphi \geq 0$ *be a measurable function such that for some constant* $c > 0$

$$\int_{\varphi(x)\geq s} w_k(x)dx \leq \frac{c}{s}, \quad \forall\, s > 0. \tag{2.4}$$

Then for all $1 < p \leq 2$ *there exists a constant* $C_p > 0$ *such that*

$$\int_{\mathbb{R}^n} \left|\frac{\mathcal{F}_k(f)(\xi)}{\varphi(\xi)}\right|^p \varphi(\xi)^2 w_k(\xi)d\xi \leq C_p\|f\|_{p,k}; \quad f \in L^p(\mathbb{R}^n, w_k(x)dx). \tag{2.5}$$

Proof. Put $d\mu_k(\xi) = \varphi(\xi)^2 w_k(\xi)d\xi$ and T the operator $f \to T(f) = \mathcal{F}_k(f)/\varphi$. We have

$$\begin{aligned}
\mu_k\Big\{\xi;\ |T(f)(\xi)| \geq s\Big\} &\leq \mu_k\Big\{\xi;\ \varphi(\xi) \leq \|f\|_{1,k}/s\Big\} \\
&= \int_{\varphi(\xi)\leq \|f\|_{1,k}/s} \varphi(\xi)^2 w_k(\xi)d\xi \\
&= 2\int_{0\,\leq\, t\,\leq\, \varphi(\xi)\,\leq\, \|f\|_{1,k}/s} w_k(\xi)t\,dt\,d\xi \\
&\leq 2\int_0^{\|f\|_{1,k}/s} t\left\{\int_{\varphi(\xi)\,\geq\, t} w_k(\xi)d\xi\right\}dt \\
&\leq \frac{2c\|f\|_{1,k}}{s}.
\end{aligned}$$

This means that the operator T is bounded from $L^1(\mathbb{R}^n, w_k(x)dx)$ into weak space $L^{1,\infty}(\mathbb{R}^n, \mu_k(x)dx)$. On the other hand from Plancherel Theorem

$$\mu_k\{\xi;\ |T(f)(\xi)| \geq s\} \leq \frac{\|f\|_{2,k}^2}{s^2}$$

Then Lemma 2.5 follows from Marcinkiewicz interpolation Theorem. □

Lemma 2.6. *If φ satisfies (2.4) and $1 < p < r < p' < \infty$, then we have*

$$\left(\int_{\mathbb{R}^n} \left|\mathcal{F}_k(f)(\xi)(\varphi(\xi))^{(1/r-1/p')}\right|^r w_k(\xi)d\xi\right)^{1/r} \\ \leq C_p\|f\|_{p,k}; \quad f \in L^p(\mathbb{R}^n, w_k(x)dx).$$

Proof. Put $a = (p'-p)/(p'-r)$, and a' it's conjugate, we have $p/a + p'/a' = r$, $(1-r/p')a = 2-p$ and $(r-p/a)a' = p'$. Then Using Hölder's inequality, (2.5) and the Hausdorff-Young inequality (1.2),

$$\left(\int_{\mathbb{R}^n} |\mathcal{F}_k(f)(\xi)|^r|\varphi(\xi)|^{(1-r/p')}w_k(\xi)d\xi\right)^{1/r} \\ \leq \left(\int_{\mathbb{R}^n} |\mathcal{F}_k(f)(\xi)|^p|\varphi(\xi)|^{2-p}w_k(\xi)d\xi\right)^{1/ra} \\ \times \left(\int_{\mathbb{R}^n} |\mathcal{F}_k(f)(\xi)|^{p'}w_k(\xi)d\xi\right)^{1/ra'} \\ \leq C_p\|f\|_{p,k},$$

which is the desired statement. □

Lemma 2.7. *Let m be a measurable function and $1 < b < \infty$, such that*

$$\int_{|m(x)|\geq s} w_k(x)dx \leq \frac{c}{s^b}, \quad \forall\, s > 0.$$

Then the operator $T_m = \mathcal{F}_k^{-1}(m\mathcal{F}_k)$ is bounded from $L^p(\mathbb{R}^n, w_k(x)dx)$ to $L^q(\mathbb{R}^n, w_k(x)dx)$, provided

$$1 < p \leq 2 \leq q < \infty \quad \textit{and} \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{b}\,.$$

Proof. We may assume first that $p \leq q'$ and we let $\varphi = |m|^b$. Since φ satisfies (2.4), then using Lemma 2.6 with $r = q'$ and the fact that $1/p - 1/q = 1/q' - 1/p' = 1/b$, we obtain, for $f \in L^p(\mathbb{R}^n, w_k(x)dx)$,

$$\|m\mathcal{F}_k(f)\|_{q',k} \leq C_p\|f\|_{p,k}.$$

Therefore Hausdorff-Young inequality implies

$$\|T_m(f)\|_{q,k} \leq \|m\mathcal{F}_k(f)\|_{q',k} \leq C_p\|f\|_{p,k}.$$

When $q' < p = (p')'$, we can apply the similar argument to the adjoint operator $T_m^* = T_{\overline{m}}$, since $1 < q' \le 2 \le p' < \infty$ and $1/q' - 1/p' = 1/b$. Hence by duality it follows that

$$\|T_m(f)\|_{q,k} \le C_p \|f\|_{p,k}.$$

This conclude s the proof of Lemma 2.7. □

Remark 2.8. From Lemma 2.7 we obtain the statement of Theorem 2.4, since

$$\int_{|m(x)|\ge s} w_k(x)dx \le \int_{|x|\le s^{-1/t}} w_k(x)dx \le \frac{c}{s^{(2\gamma_k+n)/t}}.$$

The second fact we shall also require in proving Theorem 2.2 is the following

Lemma 2.9. *Let $\phi \in L^1(\mathbb{R}^n, w_k(x)dx)$ be a radial function. If $\phi \in L^{r,\infty}(\mathbb{R}^n, w_k(x)dx)$ for some $1 < r < \infty$ then the Dunkl-convolution operator with ϕ is of weak type (1,r).*

Proof. Let us recall that ϕ is in $L^{r,\infty}(\mathbb{R}^n, w_k(x)dx)$ if there exists a constant $c > 0$ such that

$$\alpha_\phi(t) = \int_{|\phi(x)|>t} w_k(x)dx \le \frac{c}{t^r}.$$

Let $\lambda > 0$, we decompose $\phi = \phi_1 + \phi_2$ where

$$\phi_1 = \begin{cases} \phi, \text{ if } |\phi| > \lambda, \\ 0 \quad \text{if } |\phi| \le \lambda, \end{cases}, \quad \text{and} \quad \phi_2 = \phi - \phi_1.$$

So, we have

$$\alpha_{\phi_1}(t) = \begin{cases} \alpha_\phi(t), \text{ if } t > \lambda, \\ \alpha_\phi(\lambda) \text{ if } t \le \lambda, \end{cases}$$

and

$$\int_{\mathbb{R}^n} |\phi_1(x)| w_k(x)dx = \int_0^\infty \alpha_{\phi_1}(t)dt = \lambda\alpha_\phi(\lambda) + \int_\lambda^\infty \alpha_\phi(t)dt \le c\lambda^{1-r}.$$

Then, using (1.7) we obtain for $f \in L^1(\mathbb{R}^n, w_k(x)dx)$,

$$\|f *_k \phi_1\|_{1,k} \le \|\phi_1\|_{1,k}\|f\|_{1,k} \le c_0\lambda^{1-r}\|f\|_{1,k} \tag{2.6}$$

and

$$\|f *_k \phi_2\|_\infty \le \|\phi_2\|_\infty\|f\|_{1,k} \le \lambda\|f\|_{1,k} \tag{2.7}$$

Now let $s > 0$ and $\lambda = s/(2\|f\|_{1,k})$. In view of (2.7)

$$\int_{\{|f *_k \phi_2(x)|>s/2\}} w_k(x)dx = 0$$

Thus by Chebyshev inequality and (2.6),

$$\int_{\{|f*_k\phi(x)|>s\}} w_k(x)dx \leq \int_{\{|f*_k\phi_1(x)|>s/2\}} w_k(x)dx + \int_{\{|f*_k\phi_2(x)|>s/2\}} w_k(x)dx$$
$$\leq 2\frac{\|f *_k \phi_1\|_{1,k}}{s} \leq c\left(\frac{\|f\|_{1,k}}{s}\right)^r,$$

which is the desired estimate. □

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Concerning Bessel function we have first to note the following facts.

$$J_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\ \Gamma(\nu+1/2)} \int_{-1}^{1} (1-u^2)^{\nu-1/2} e^{itu} du, \quad Re(\nu) > -1/2,\ t>0 \tag{2.8}$$

and

$$|t^{-(\eta+i\zeta)} J_{\eta+i\zeta}(t)| \leq c_\eta e^{c|\zeta|}(1+t)^{-\eta-\frac{1}{2}}, \quad \eta,\ \zeta \in \mathbb{R} \text{ and } t>0. \tag{2.9}$$

This behavior of Bessel function was mentioned in [17]. In view of (2.9) we have

$$||\xi|^{-(\gamma_k+n/2-\alpha)} J_{\gamma_k+n/2-\alpha}(|\xi|)| \leq \frac{c}{|\xi|^d}; \qquad \xi \neq 0$$

for all $d \leq \gamma_k + (n+1)/2 - \alpha$. We thus obtain (a) by applying Theorem 2.4 to the operator S_α.

To prove (b), We may apply Stein's Interpolation Theorem to the analytic family of the operators S_z, for $0 \leq Re(z) \leq \gamma_k + (n+1)/2$. Indeed, let f and g be simple functions and

$$F(z) = \int_{\mathbb{R}^n} S_z(f)(x)g(x)w_k(x)dx, \quad 0 \leq Re(z) \leq \gamma_k + (n+1)/2.$$

By applying the Cauchy–Schwarz Inequality and Plancherel Theorem the integral converges absolutely. Moreover F can be written as

$$F(z) = \int_{\mathbb{R}^n} |\xi|^{-(\gamma_k+n/2-z)} \mathcal{J}_{\gamma_k+n/2-z}(|\xi|)\mathcal{F}_k(f)(\xi)\mathcal{F}_k(\overline{g})(\xi)w_k(\xi)d\xi.$$

and so in view of (2.8) and (2.9) we see that F is analytic in $\{z \in \mathbb{C},\ 0 < Re(z) < \gamma_k + (n+1)/2\}$ and continuous in $\{z \in \mathbb{C},\ 0 \leq Re(z) \leq \gamma_k + (n+1)/2\}$. (2.9) also implies (1.14). Let us now consider the two boundary lines $Re(z) = 0$ and $Re(z) = \gamma_k + (n+1)/2$. Using (1.8) and the fact that

$$|\Gamma(1-iy)|^{-1} = \left(\frac{\pi \sinh y}{y}\right)^{1/2} \leq ce^{|y|/2} \tag{2.10}$$

we estimate $S_{iy}(f)$ by

$$\|S_{iy}(f)\|_{\infty,k} \leq c\, e^{c\,|y|}\|f\|_{1,k}.$$

However, using (2.9) and Plancherel Theorem we have the estimate

$$\|S_{\gamma_k+(n+1)/2+iy}(f)\|_{2,k} \leq c\, e^{c\,|y|}\|f\|_{2,k}.$$

Therefore the application of Stein's interpolation theorem yields (b). To establish estimate (c) we proceed as follows: when $\alpha > 1/2$ then we obtain (c) from (a) and (b) by the Riesz-Thorin interpolation theorem to the couples (L^{p_1}, L^{q_1}) and (L^{p_2}, L^{q_2}) in the two cases:

$$\begin{cases} p_1 = 2, \quad q_1 = (n+2\gamma_k)/(\alpha - 1/2), \\ p_2 = (n+2\gamma_k+1)/(n+2\gamma_k+1) - \alpha, \quad q_2 = (n+2\gamma_k+1)/\alpha \end{cases}$$

and

$$\begin{cases} p_1 = (n+2\gamma_k)/(n+2\gamma_k+1/2-\alpha), \quad q_1 = 2 \\ p_2 = (n+2\gamma_k+1)/(n+2\gamma_k+1) - \alpha, \quad q_2 = (n+2\gamma_k+1)/\alpha. \end{cases}$$

When $\alpha = 1/2$, one can see that $\varphi_{1/2} \in L^{2,\infty}(\mathbb{R}^n, w_k(x)dx)$ and by Lemma 2.9 the operator $S_{1/2}$ is of weak type $(1, 2)$. Thus, according to the estimates of (b) we obtain (c) by Marcinkiewicz interpolation theorem and duality argument. The proof of Theorem 2.2 is complete. □

2.1. Estimates of Dunkl wave equation

In this section we are going to apply Theorem 2.2 to the Dunkl wave equation. Our goal is to prove Theorem 1.1.

We will need to consider the Riesz transforms for Dunkl transform R_j, $j = 1 \ldots n$ which defined on $L^2(\mathbb{R}^n, w_k(x)dx)$ by

$$\mathcal{F}_k(R_j(f))(\xi) = -i\frac{\xi_j}{|\xi|}\mathcal{F}_k(f)(\xi).$$

We have the following result.

Theorem 2.10. ([1]) *The Riesz transforms R_j, $1 \leq j \leq n$ are bounded operators on $L^p(\mathbb{R}^n, w_k(x)dx)$ for $1 < p < \infty$.*

The second main auxiliary result which will be useful to prove our theorem is the following

Lemma 2.11. *Let ψ be a radial smooth function on $\mathbb{R}^n$ such that $\psi(\xi) = 0$ if $|\xi| \leq 1$ and $\psi(\xi) = 1$ if $|\xi| \geq 2$. Then the Dunkl multiplier defined by $\mathcal{A}_\psi(f) = \mathcal{F}^{-1}\left(\frac{\psi(\xi)}{|\xi|}\mathcal{F}_k(f)\right)$ is a bounded operator from $L^p(\mathbb{R}^n, w_k(x)dx)$ to $L^\infty(\mathbb{R}^n, w_k(x)dx)$ for all $p > n + 2\gamma_k$.*

Proof. Let ρ be a C^∞ function such that $supp(\rho) \subset \{1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{j=-\infty}^{\infty} \rho(2^{-j}\xi) = 1, \qquad \xi \neq 0.$$

Decompose,

$$\frac{\psi(\xi)}{|\xi|}\mathcal{F}_k(f)(\xi) = \sum_{j=0}^{\infty}\rho(2^{-j}\xi)\frac{\psi(\xi)}{|\xi|}\mathcal{F}_k(f)(\xi)$$
$$= \left(\rho(\xi)+\rho\left(\frac{\xi}{2}\right)\right)\frac{\psi(\xi)}{|\xi|}\mathcal{F}_k(f)(\xi)+\sum_{j=2}^{\infty}\frac{2^{-j}\rho(2^{-j}\xi)}{|2^{-j}\xi|}\mathcal{F}_k(f)(\xi)$$
$$= \psi_1(\xi)\mathcal{F}_k(f)(\xi) + \sum_{j=2}^{\infty}2^{-j}\psi_2(2^{-j}\xi)\mathcal{F}_k(f)(\xi),$$

we get

$$\mathcal{A}_\psi(f) = \mathcal{F}_k^{-1}(\psi_1) *_k f + \sum_{j=2}^{\infty}2^{(n+2\gamma_k-1)j}\mathcal{F}_k^{-1}(\psi_2)(2^j.) *_k f.$$

Using Hölder's inequality and (1.7) it follows that, for $p > n + 2\gamma_k$

$$\|\mathcal{A}_\psi(f)\|_\infty \leq \|f\|_{p,k}\left\{\|\mathcal{F}_k^{-1}(\psi_1)\|_{p',k} + \sum_{j=2}^{\infty}2^{j(-1+(n+2\gamma_k)/p)}\|\mathcal{F}_k^{-1}(\psi_2)\|_{p',k}\right\}$$
$$\leq C\,\|f\|_{p,k},$$

which is the desired result. □

We will also need the following lemma

Lemma 2.12. *Let $y \in \mathbb{R}$ and Ψ_j be the function given by $\Psi_j(x) = x_j\Phi_{iy}(x)$, $x \in \mathbb{R}^n$. Then we can find a constant $c > 0$ that does not depend on y and such that*

$$\|\tau_z(\Psi_j)\|_{\infty,k} \leq c\, e^{c|y|},$$

for all $z \in \mathbb{R}^n$.

Proof. Let $\varepsilon > 0$. Define

$$h_\varepsilon(x) = \begin{cases} e^{-\varepsilon/(1-|x|^2)}, & \text{if } |x| < 1,\\ 0, & \text{if } |x| \geq 1,\end{cases}$$

It follows that $h_\varepsilon\ \phi_{iy}$ and $h_\varepsilon\ \phi_{-1+iy}$ are C^∞-functions supported in the unit ball and

$$\frac{\partial}{\partial x_j}\Big(\Phi_{-1+iy}(x)\ h_\varepsilon(x)\Big) = -\Psi_j(x)h_\varepsilon(x) - \frac{\Psi_j(x)}{(1-iy)}\left(\frac{\varepsilon}{1-|x|^2}\ h_\varepsilon(x)\right).$$

Using the dominated convergence theorem we have the following

$$\|\Psi_j h_\varepsilon - \Psi_j\|_{2,k} \to 0, \qquad \text{as} \quad \varepsilon \to 0$$

and

$$\left\| \left(\frac{\varepsilon}{1-|.|^2} h_\varepsilon \right) \Psi_j \right\|_{2,k} \to 0, \qquad \text{as} \quad \varepsilon \to 0$$

which from the boundedness of the Dunkl translation opertaor τ_z on $L^2(\mathbb{R}^n, w_k(x) dx)$ yield that

$$\left\| \tau_z \left(\frac{\partial}{\partial x_j} \left(\Phi_{-1+iy}\, h_\varepsilon \right) \right) + \tau_z(\Psi_j) \right\|_{2,k} \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{2.11}$$

However, since $h_\varepsilon\ \Phi_{-1+iy}$ is a C^∞-radial function we have that

$$\tau_z \left(\frac{\partial}{\partial x_j} \left(\Phi_{-1+iy}\, h_\varepsilon \right) \right) = \tau_z \left(D_j^k \left(\Phi_{-1+iy}\, h_\varepsilon \right) \right) = D_j^k \tau_z \left(\Phi_{-1+iy}\, h_\varepsilon \right).$$

We next compute $D_j^k \tau_z \left(\Phi_{-1+iy}\, h_\varepsilon \right)$ and its limit when $\varepsilon \to 0$. Putting

$$A_z(x,\eta) = \sqrt{|x|^2 + |z|^2 - 2\langle x, \eta \rangle} = \sqrt{|x-\eta|^2 + |z|^2 - |\eta|^2},$$

for $x \in \mathbb{R}^n$ and $\eta \in conv(G.z)$, and using the formula (1.6) and (1.1) we have that

$$\begin{aligned} & D_j^k \tau_z \left(\Phi_{-1+iy}\, h_\varepsilon \right)(x) \\ & = - \int_{\mathbb{R}^n} (x_j - \eta_j) \widetilde{\Phi_{iy}}(A_z(x,\eta)) \widetilde{h_\varepsilon}(A_z(x,\eta)) d\nu_z(\eta) \\ & \quad - \int_{\mathbb{R}^n} \left(\frac{(x_j - \eta_j) \widetilde{\Phi_{iy}}(A_z(x,\eta))}{(1-iy)} \right) \left(\frac{\varepsilon}{1 - A_z(x,\eta)^2}\, \widetilde{h_\varepsilon}(A_z(x,\eta)) \right) d\nu_z(\eta) \\ & \quad + \sum_{\upsilon \in R^+} \frac{k_\upsilon \upsilon_j}{\langle x, \upsilon \rangle} \int_{\mathbb{R}^n} \left(\widetilde{\Phi_{-1+iy}}(A_z(x,\eta)) \widetilde{h_\varepsilon}(A_z(x,\eta)) \right. \\ & \quad \left. - \widetilde{\Phi_{-1+iy}}(A_z(\sigma_\upsilon.x,\eta)) \widetilde{h_\varepsilon}(A_z(\sigma_\upsilon.x,\eta)) \right) d\nu_z(\eta). \end{aligned}$$

Therefore, from (2.11)and dominated convergence Theorem we obtain for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned} \tau_z(\Psi_j)(x) = & \int_{\mathbb{R}^n} (x_j - \eta_j) \widetilde{\Phi_{iy}}(A_z(x,\eta)) d\mu_z(\eta) \\ & - \sum_{\upsilon \in R^+} \frac{k_\upsilon \upsilon_j}{\langle x, \upsilon \rangle} \int_{\mathbb{R}^n} \left(\widetilde{\Phi_{-1+iy}}(A_z(x,\eta)) - \widetilde{\Phi_{-1+iy}}(A_z(\sigma_\upsilon.x,\eta)) \right) d\nu_z(\eta). \end{aligned} \tag{2.12}$$

Note that in the integrands, $|(x_j - \eta_j)| \leq A_z(x,\eta) \leq 1$ and by using (2.10)

$$|(x_j - \eta_j) \widetilde{\Phi_{iy}}(A_z(x,\eta))| \leq c\, e^{c|y|}.$$

Also, if we write

$$\frac{\Phi\widetilde{_{-1+iy}}(A_z(x,\eta)) - \Phi\widetilde{_{-1+iy}}(A_z(\sigma_\upsilon.x,\eta))}{\langle x,\upsilon\rangle}$$
$$= -\sum_{j=1}^{n}\int_0^1 (x_j - t\langle x,\upsilon\rangle\upsilon_j - \eta_j)\upsilon_j\widetilde{\Phi_{iy}}(A_z(x - t\langle x,\upsilon\rangle\upsilon,\eta))dt$$

then we have that

$$\left|\frac{\Phi\widetilde{_{-1+iy}}(A_z(x,\eta)) - \Phi\widetilde{_{-1+iy}}(A_z(\sigma_\upsilon.x,\eta))}{\langle x,\upsilon\rangle}\right| \le c\, e^{c|y|}.$$

Thus in view of (2.12), we conclude the proof of Lemma 2.12. □

Proof of Theorem 1.1. Applying Theorem 2.2-(c), yields

$$\|S_{\gamma_k+(n-1)/2}(f)\|_{q,k} \le c\|f\|_{p,k}; \qquad f \in S(\mathbb{R}^n)$$

for all couple (p,q) satisfying (1.12) or (1.13). So in view of (2.1) it will be enough to prove

$$\|S_{\gamma_k+(n+1)/2}(f)\|_{q,k} \le c\left\|\sum_{j=1}^{n}|D_j^k f|\right\|_{p,k}; \qquad f \in S(\mathbb{R}^n). \tag{2.13}$$

We quote the following

$$\cos(\xi)\mathcal{F}_k(f)(\xi) = \sum_{j=1}^{n}\frac{\xi_j^2}{|\xi|^2}\cos(\xi)\mathcal{F}_k(f)(\xi) = \sum_{j=1}^{n}\frac{\cos(\xi)}{|\xi|}\mathcal{F}_k\Big(R_j(D_j^k f)\Big)(\xi).$$

Hence from Theorem 2.10 one can reduce (2.13) to show that

$$\left\|\mathcal{F}_k^{-1}\left(\frac{\cos(\xi)}{|\xi|}\mathcal{F}_k(f)(\xi)\right)\right\|_{q,k} \le c\|f\|_{p,k}. \tag{2.14}$$

Let ψ be a radial smooth function on $\mathbb{R}^n$ such that $\psi(\xi) = 0$ if $|\xi| \le 1$ and $\psi(\xi) = 1$ if $|\xi| \ge 2$. Then Theorem 2.4 implies

$$\left\|\mathcal{F}_k^{-1}\left((1-\psi(\xi))\frac{\cos(\xi)}{|\xi|}\mathcal{F}_k(f)(\xi)\right)\right\|_{p,k} \le c\|f\|_{p,k}, \tag{2.15}$$

provided, $1 < p \le 2 \le q < \infty$ and $1/p - 1/q \ge 1/(2\gamma_k + n)$. Here clearly conditions (1.12) and (1.13) are also satisfied. Thus we are reduced to showing that

$$\left\|\mathcal{F}_k^{-1}\left(\psi(\xi)\frac{\cos(\xi)}{|\xi|}\mathcal{F}_k(f)(\xi)\right)\right\|_{p,k} \le c\|f\|_{p,k}.$$

For this purpose we define an analytic family of linear operators U_z^j by

$$U_z^j(f) = \mathcal{F}_k^{-1}\Big(\psi(\xi)\xi_j|\xi|^{-n/2-\gamma_k+z-1}J_{n/2+\gamma_k-z-1}(|\xi|)\mathcal{F}_k(f)(\xi)\Big)$$

for $0 \leq Re(z) \leq \gamma_k + (n+1)/2$. We now apply Stein's Interpolation Theorem to the family U_z^j and proceed as in the proof of Theorem 2.2. First on the boundary $Re(z) = \gamma_k + (n+1)/2$ we have

$$\|U_z^j(f)\|_{2,k} \leq c\, e^{c|y|}\, \|f\|_{2,k}$$

which is a simple consequence of (2.9) and Plancherel Theorem.

For $z = iy$, in view of (2.9), the function $\xi \to \psi(\xi)\xi_j|\xi|^{-n/2-\gamma_k+iy-1} J_{n/2+\gamma_k-iy-1}(|\xi|)$ belongs to $L^2(\mathbb{R}^n, w_k(x)dx)$. Then one can write U_{iy}^j as the convolution operator

$$U_{iy}^j(f)(x) = \mathcal{F}_k^{-1}\Big(\psi(\xi)\xi_j|\xi|^{-n/2-\gamma_k+z-1} J_{n/2+\gamma_k-z-1}(|\xi|)\Big) *_k f(x),$$
$$f \in L^2(\mathbb{R}^n, w_k(x)dx)$$

and to obtain a desired $L^1 - L^\infty$ estimate for U_{iy}^j as in Theorem 1.2 it suffices to estimate

$$\Big\|\tau_x \mathcal{F}_k^{-1}\Big(\psi(\xi)\xi_j|\xi|^{-n/2-\gamma_k+z-1} J_{n/2+\gamma_k-z-1}(|\xi|)\Big)\Big\|_{\infty,k}.$$

We begin by recalling the two classical identities for Bessel function

$$\frac{d}{dt}(t^{-\nu} J_\nu(t)) = -t^{-\nu} J_{\nu+1}(t) \tag{2.16}$$

and

$$J_{\nu+1}(t) = 2\nu J_\nu(t)/t - J_{\nu-1}(t). \tag{2.17}$$

Let us observe first that by (2.2) and (2.16)

$$\mathcal{F}_k(x_j\Phi_{iy})(\xi) = i D_j^k \mathcal{F}_k(\Phi_{iy})(\xi) = -i\xi_j|\xi|^{-\gamma_k-n/2+iy-1} J_{\gamma_k+n/2-iy+1}(|\xi|), \tag{2.18}$$

and by (2.17)

$$\mathcal{F}_k(x_j\Phi_{iy})(\xi) = i\xi_j|\xi|^{-\gamma_k-n/2+iy-1} J_{\gamma_k+n/2-iy-1}(|\xi|)$$
$$-i(n+2\gamma_k-iy)\xi_j|\xi|^{-\gamma_k-n/2+iy-2} J_{\gamma_k+n/2-iy}(|\xi|).$$

Hence

$$\mathcal{F}_k^{-1}\Big(\psi(\xi)\xi_j|\xi|^{-\gamma_k-n/2+iy-1} J_{\gamma_k+n/2-iy-1}(|\xi|)\Big) = -i\mathcal{F}_k^{-1}\Big(\psi(\xi)\mathcal{F}_k(x_j\Phi_{iy})(\xi)\Big)$$
$$+(n+2\gamma_k-iy)\mathcal{F}_k^{-1}\Big(\psi(\xi)\xi_j|\xi|^{-\gamma_k-n/2+iy-2} J_{\gamma_k+n/2-iy}(|\xi|)\Big).$$

Now, write

$$\mathcal{F}_k^{-1}\Big(\psi(\xi)\mathcal{F}_k(x_j\Phi_{iy})(\xi)\Big) = \mathcal{F}_k^{-1}\Big((\psi(\xi)-1)\mathcal{F}_k(x_j\Phi_{iy})(\xi)\Big) + x_j\Phi_{iy}(x).$$

The function $\xi \to (\psi(\xi)-1)\mathcal{F}_k(x_j\Phi_{iy})(\xi)$ is a C^∞ with compact support, so by using (1.5), (2.9) and (2.18) it follows that

$$\Big\|\tau_x\Big\{\mathcal{F}_k^{-1}((\psi(\xi)-1)\mathcal{F}_k(x_j\Phi_{iy})(\xi))\Big\}\Big\|_{\infty,k} \leq c\, e^{c|y|}.$$

Thus, in view Lemma 2.12

$$\left\| \tau_x \mathcal{F}_k^{-1}\Big(\psi(\xi)\mathcal{F}_k(x_j\Phi_{iy})(\xi)\Big)\right\|_{\infty,k} \le c\; e^{c|y|}. \tag{2.19}$$

On the other hand, one can write

$$\begin{aligned}\mathcal{F}_k^{-1}\Big(\psi(\xi)\xi_j|\xi|^{-\gamma_k-n/2+iy-2}J_{\gamma_k+n/2-iy}(|\xi|)\Big) &= \mathcal{F}_k^{-1}\left(\frac{\xi_j}{|\xi|}\left(\frac{\psi(\xi)}{|\xi|}\mathcal{F}_k(\Phi_{iy})(\xi)\right)\right)\\ &= i\mathcal{A}_\psi(R_j(\Phi_{iy}))\end{aligned}$$

Now for $p > n + 2\gamma_k$ the radial function ϕ_{iy} belongs to $L^p(\mathbb{R}^n, w_k(x)dx)$, thus we can apply Theorem 2.10, Lemma 2.11 and (1.9) to obtain

$$\begin{aligned}\left\|\tau_x\mathcal{F}_k^{-1}\Big(\psi(\xi)\xi_j|\xi|^{-\gamma_k-n/2+iy-2}J_{\gamma_k+n/2-iy}(|\xi|)\Big)\right\|_{\infty,k} &= \|\mathcal{A}_\psi(R_j(\tau_x(\Phi_{iy})))\|_{\infty,k}\\ &\le c\|\Phi_{iy})\|_{p,k} \le c\; e^{c|y|}.\end{aligned}$$

This together with (2.19) yield

$$\left\|\tau_x\mathcal{F}_k^{-1}\Big(\psi(\xi)\xi_j|\xi|^{-n/2-\gamma_k+z-1}J_{n/2+\gamma_k-z-1}(|\xi|)\Big)\right\|_{\infty,k} \le c\; e^{c|y|}$$

and therefore

$$\|U_{iy}^j(f)\|_{\infty,k} \le c\; e^{c|y|}\; \|f\|_{1,k}.$$

The Stein interpolation theorem now implies the following

$$\|U_\alpha^j(f)\|_{p',k} \le c\; \|f\|_{p,k}$$

for all $0 \le \alpha \le \gamma_k + (n+1)/2$ and $p = (n+1+2\gamma_k)/(n+1+2\gamma_k-\alpha)$. In particular for $\alpha = \gamma_k + (n-1)/2$ and $p = 2(n+2\gamma_k+1)/(n+2\gamma_k+3)$we have that

$$\left\|\mathcal{F}_k^{-1}\left(\psi(\xi)\xi_j\frac{\cos(|\xi|)}{|\xi|^2}\mathcal{F}_k(f)(\xi)\right)\right\|_{p',k} \le c\; \|f\|_{p,k} \tag{2.20}$$

Using the fact that

$$\begin{aligned}\mathcal{F}_k^{-1}\left(\psi(\xi)\frac{\cos(|\xi|)}{|\xi|}\mathcal{F}_k(f)(\xi)\right) &= \sum_{j=1}^n \mathcal{F}_k^{-1}\left(\psi(\xi)\frac{\xi_j^2}{|\xi|^2}\frac{\cos(|\xi|)}{|\xi|}\mathcal{F}_k(f)(\xi)\right)\\ &= \sum_{j=1}^n \mathcal{F}_k^{-1}\left(\psi(\xi)\xi_j\frac{\cos(|\xi|)}{|\xi|^2}\mathcal{F}_k(R_j(f))(\xi)\right)\end{aligned}$$

it follows from (2.20) and Theorem 2.10

$$\left\|\mathcal{F}_k^{-1}\left(\psi(\xi)\frac{\cos(|\xi|)}{|\xi|}\mathcal{F}_k(f)(\xi)\right)\right\|_{p',k} \le c\; \|f\|_{p,k}$$

On the other hand by Theorem 2.4 we have

$$\left\|\mathcal{F}_k^{-1}\left(\psi(\xi)\frac{\cos(|\xi|)}{|\xi|}\mathcal{F}_k(f)(\xi)\right)\right\|_{q,k} \le c\; \|f\|_{p,k}$$

for $1 < p \leq 2 \leq q < \infty$, $1/p - 1/q = 1/(n+2\gamma_k)$. Therefore with the use of the Riesz-Thorin interpolation theorem for the couples (L^{p_1}, L^{q_1}) and (L^{p_2}, L^{q_2}) for

$$\begin{cases} p_1 = 2, \quad q_1 = 2(n+2\gamma_k)/(n+2\gamma_k-2), \\ p_2 = 2(n+2\gamma_k+1)/(n+2\gamma_k+3), \quad q_2 = 2(n+2\gamma_k+1)/(n+2\gamma_k-1) \end{cases}$$

and

$$\begin{cases} p_1 = 2(n+2\gamma_k)/(n+2\gamma_k++2), \quad q_1 = 2 \\ p_2 = 2(n+2\gamma_k+1)/(n+2\gamma_k+3), \quad q_2 = 2(n+2\gamma_k+1)/(n+2\gamma_k-1) \end{cases}$$

we obtain

$$\left\| \mathcal{F}_k^{-1}\left(\psi(\xi)\frac{\cos(|\xi|)}{|\xi|}\mathcal{F}_k(f)(\xi)\right)\right\|_{q,k} \leq c \, \|f\|_{p,k}$$

for all p and q satisfying (1.12) or (1.13). This combined with the estimate (2.15) yields (2.14) and finishes the proof of Theorem 1.1. □

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