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# Submanifolds with parallel Gaussian mean curvature vector in Euclidean spaces

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**Abstract.** In the present paper, we prove a rigidity theorem for complete submanifolds with parallel Gaussian mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$ under an integral curvature pinching condition, which is a unified generalization of some rigidity results for self-shrinkers and the $\lambda$-hypersurfaces in Euclidean spaces.

## 1. Introduction

Let $X : M^n \to \mathbb{R}^{n+p}$ be an $n$-dimensional smooth immersed submanifold in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. Define the Gaussian mean curvature vector $\xi$ of $M$ by

$$\xi = H + \frac{X^N}{2}, \tag{1}$$

where $H$ is the mean curvature vector of $M$ and $(\ )^N$ denotes the normal part of a vector field on $\mathbb{R}^{n+p}$. We call $\xi$ the Gaussian mean curvature vector since it is related to the mean curvature vector $\tilde{H}$ of $M$ when it is considered as a submanifold of the Gaussian space $(\mathbb{R}^{n+p}, e^{-\frac{|x|^2}{2n}}\delta)$ by $\xi = e^{-\frac{|X|^2}{2n}}\tilde{H}$, where $\delta$ denotes the Euclidean metric on $\mathbb{R}^{n+p}$. $M$ is called a submanifold with parallel Gaussian mean curvature vector if $\xi$ is parallel in the normal bundle. Submanifolds of this type were first investigated by Li and Chang [20].

Let $X : M \to \mathbb{R}^{n+p}$ be a submanifold with parallel Gaussian mean curvature vector. Obviously, when $\xi = 0$, $M$ is a self-shrinker of the mean curvature flow, which plays a very important role in the study of the mean curvature flow [10,15,16,27]. The pinching problems of self-shrinkers have been studied extensively. For example, Le and Sesum [17] proved that any smooth self-shrinker with polynomial volume growth and satisfying $|A|^2 < \frac{1}{2}$ is a hyperplane. Here $A$ denote the second fundamental form of an immersion. Cao and Li [1] generalized this

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result to arbitrary codimension and proved that any smooth complete self-shrinker with polynomial volume growth and $|A|^2 \leq \frac{1}{2}$ is one of generalized cylinders. On the other hand, Ding and Xin [11] showed that any immersed self-shrinker satisfying $\left(\int_M |A|^n \mathrm{d}\mu\right)^{1/n} < C$ for certain positive constant sufficiently small is a linear space. For more curvature pinching theorems for self-shrinkers, see [1,2,4,5,11,12,17,19,21] and references therein.

If $p = 1$, (1) is reduced to

$$H_0 + \frac{\langle X, N\rangle}{2} = \lambda, \tag{2}$$

where $H_0$ is the mean curvature function, $N$ is the inward pointing unit normal and $\lambda$ is a constant. A hypersurface satisfying (2) is called a $\lambda$-hypersurface, which was introduced by Cheng–Wei [7] and McGonagle–Ross [23]. The geometric properties of $\lambda$-hypersurfaces are recently investigated by Cheng, Guang, Ogata, Wang, Wei, Xu, Zhao [3,7,13,26], etc. As generalizations of self-shrinkers of the mean curvature flow, Cheng and Wei [7] classified complete $\lambda$-hypersurfaces with polynomial area growth and $H - \lambda \geq 0$. They also defined an $\mathcal{F}$-functional and studied $\mathcal{F}$-stability of $\lambda$-hypersurfaces. Cheng, Ogata and Wei [3] proved some gap and rigidity theorems for complete $\lambda$-hypersurfaces. Wang, Xu and Zhao [26] investigate the integral curvature pinching theorems for $\lambda$-hypersurfaces. See [6,13,24], etc. for more results on the rigidity of $\lambda$-hypersurfaces.

In this paper, we study the integral curvature pinching theorems for submanifolds with parallel Gaussian mean curvature vector. We firstly prove the following $L^n$-pinching theorem of the second fundamental form.

**Theorem 1.** *Let $X : M^n \to \mathbb{R}^{n+p} (n \geq 3)$ be a complete submanifold with parallel Gaussian mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$. If*

$$\left(\int_M |A|^n \mathrm{d}\mu\right)^{1/n} < K(n, |\xi|),$$

*where $K(n, |\xi|)$ is an explicit positive expression of $n$ and $|\xi|$, then $M$ is isometric to $\mathbb{R}^n$.*

*Remark 1.* If $\xi = 0$, then $M$ is a self-shrinker. Hence our theorem is a generalization of the $L^n$-pinching theorem proved by Ding and Xin [11] to submanifold with parallel Gaussian mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$.

Let $\mathring{A}$ denote the tracefree second fundamental form, which is defined by $\mathring{A} = A - \frac{H}{n} g$ with $g$ denoting the induced metric on $M$. We prove an $L^n$-pinching theorem of the tracefree second fundamental form for submanifolds with parallel Gaussian mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$ provided that the mean curvature vector is suitably bounded.

**Theorem 2.** . *Let $X : M^n \to \mathbb{R}^{n+p} (n \geq 3)$ be a complete submanifold with parallel Gaussian mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$. Suppose the mean curvature vector satisfies $\sup_M |H| < \sqrt{\frac{n}{2} + |\xi|^2} - |\xi|$. If*

$$\left(\int_M |\mathring{A}|^n \mathrm{d}\mu\right)^{1/n} < D(n, |\xi|, \sup_M |H|),$$

*where* $D(n, |\xi|, \sup_M |H|)$ *is an explicit positive expression of* $n$, $|\xi|$ *and* $\sup_M |H|$, *then* $M$ *is isometric to* $\mathbb{R}^n$.

When $\xi = 0$, we have the following corollary, which is obtained by [2].

**Corollary 1.** *Let* $X : M^n \to \mathbb{R}^{n+p} (n \geq 3)$ *be a complete self-shrinker in the Euclidean space* $\mathbb{R}^{n+p}$. *Suppose the mean curvature vector satisfies* $\sup_M |H| < \sqrt{\frac{n}{2}}$. *If*

$$\left( \int_M |\mathring{A}|^n \mathrm{d}\mu \right)^{1/n} < D(n, \sup_M |H|),$$

*where* $D(n, \sup_M |H|)$ *is an explicit positive expression of* $n$ *and* $\sup_M |H|$, *then* $M$ *is isometric to* $\mathbb{R}^n$.

For the case $n = 2$, we obtain the following results.

**Theorem 3.** *Let* $X : M^2 \to \mathbb{R}^{2+p}$ *be a complete surface with parallel Gaussian mean curvature vector in the Euclidean space* $\mathbb{R}^{2+p}$. *If*

$$\left( \int_M |A|^4 \mathrm{d}\mu \right)^{1/2} < K(|\xi|),$$

*where* $K(|\xi|)$ *is an explicit positive expression of* $|\xi|$, *then* $M$ *is isometric to* $\mathbb{R}^2$.

**Theorem 4.** *Let* $X : M^2 \to \mathbb{R}^{2+p}$ *be a complete surface with parallel Gaussian mean curvature vector in the Euclidean space* $\mathbb{R}^{2+p}$. *Suppose the mean curvature vector satisfies* $\sup_M |H| \leq \sqrt{|\xi|^2 + 1} - |\xi|$. *If*

$$\left( \int_M |\mathring{A}|^4 \mathrm{d}\mu \right)^{1/2} < D(|\xi|, \sup_M |H|),$$

*where* $D(|\xi|, \sup_M |H|)$ *is an explicit positive expression of* $|\xi|$ *and* $\sup_M |H|$, *then* $M$ *is isometric to* $\mathbb{R}^2$.

Our global pinching theorems for submanifolds with parallel Gaussian mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$ are originally motivated by the global pinching theorems for submanifolds with parallel mean curvature vector in space forms, see [30,31], etc. Please refer to [22,25,28–33] for more rigidity theorems for submanifolds with parallel mean curvature vector. For another generalization of self-shrinkers please see [8,9], etc.

The rest of our paper is organized as follows. Some notations and several lemmas are prepared in Sect. 2. In Sect. 3, we prove Theorems 1 and 2. Theorems 3 and 4 will be proved in Sect. 4.

## 2. Preliminaries

Let $X : M^n \to \mathbb{R}^{n+p}$ be an $n$-dimensional immersed submanifold. Denote by $g$ the induced metric on $M$. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, \ldots \leq n+p, \quad 1 \leq i, j, \ldots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Choose a local field of orthonormal frame field $\{e_A\}$ in $\mathbb{R}^{n+p}$ such that, restricted to $M$, the $e_i$'s are tangent to $M^n$. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of $\mathbb{R}^{n+p}$, respectively. Restricting these forms to $M$, we have

$$\begin{aligned}
\omega_{\alpha i} &= \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji},\\
A &= \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha = \sum_{ij} h_{ij}\omega_i \otimes \omega_j,\\
H &= \sum_{\alpha,i} h^\alpha_{ii} e_\alpha = \sum_\alpha H^\alpha e_\alpha,\\
R_{ijkl} &= \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right),\\
R_{\alpha\beta kl} &= \sum_i \left(h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}\right),
\end{aligned}$$

where $A$, $H$, $R_{ijkl}$, $R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the Riemannian curvature tensor, the normal curvature tensor of $M$, respectively. The tracefree second fundamental form is defined by $\mathring{A} = A - \frac{1}{n} g \otimes H$.

Denoting the first and second covariant derivatives of $h^\alpha_{ij}$ by $h^\alpha_{ijk}$ and $h^\alpha_{ijkl}$ respectively, we have

$$\begin{aligned}
\sum_k h^\alpha_{ijk}\omega_k &= dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{kj}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha},\\
\sum_l h^\alpha_{ijkl}\omega_l &= dh^\alpha_{ijk} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha},
\end{aligned}$$

Then we have

$$\begin{aligned}
h^\alpha_{ijk} &= h^\alpha_{ikj},\\
h^\alpha_{ijkl} - h^\alpha_{ijlk} &= \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{mj}R_{mikl} - \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}.
\end{aligned}$$

The Laplacian of the second fundamental form is given by

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk} = \sum_k h^\alpha_{kkij} + \sum_k \left(\sum_m h^\alpha_{km}R_{mijk} + \sum_m h^\alpha_{mi}R_{mkjk} - \sum_\beta h^\beta_{ki}R_{\alpha\beta jk}\right). \quad (3)$$

For an Euclidean submanifold $M$, an elliptic operator $\mathcal{L}$ is given by

$$\mathcal{L} = \Delta - \frac{1}{2}\langle X, \nabla(\cdot)\rangle = e^{\frac{|X|^2}{4}} \operatorname{div}\left(e^{-\frac{|X|^2}{4}} \nabla(\cdot)\right),$$

where $\Delta$, div and $\nabla$ denote the Laplacian, divergence and the gradient operator on $M$, respectively. The $\mathcal{L}$ operator was introduced by Colding and Minicozzi [10] when they investigated self-shrinkers. They showed that $\mathcal{L}$ is self-adjoint respect to the measure $e^{-\frac{|X|^2}{4}}\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the volume form on $M$. We denote $\rho = e^{-\frac{|X|^2}{4}}$ and $\mathrm{d}\mu$ might be omitted in the integrations for notational simplicity.

In order to prove our results, we give the following lemma first.

**Lemma 1.** *Let $M^n$ be a submanifold with parallel Gaussian mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$. Then we have*

$$\begin{aligned} \mathcal{L}|A|^2 = {} & 2|\nabla A|^2 + |A|^2 + 2\sum_{i,j,k,\alpha,\beta} \xi^\beta h^\beta_{jk} h^\alpha_{ij} h^\alpha_{ik} - 2\sum_{\alpha,\beta}\left(\sum_{i,j} h^\alpha_{ij} h^\beta_{ij}\right)^2 \\ & - 2\sum_{i,j,\alpha,\beta}\left(\sum_p (h^\alpha_{ip}h^\beta_{pj} - h^\alpha_{jp}h^\beta_{pi})\right)^2, \end{aligned} \tag{4}$$

$$\mathcal{L}|H|^2 = 2|\nabla H|^2 + |H|^2 + 2\sum_{\alpha,\beta,i,j} H^\alpha \xi^\beta h^\alpha_{ij} h^\beta_{ij} - 2\sum_{i,j}\left(\sum_\alpha H^\alpha h^\alpha_{ij}\right)^2, \tag{5}$$

*where $\xi^\alpha = H^\alpha + \frac{1}{2}\langle X, e_\alpha\rangle$ and $H^\alpha = \sum_i h^\alpha_{ii}$.*

*Proof.* Since the Gaussian mean curvature vector is parallel in the normal bundle, we have

$$\nabla_{e_i}\left(H^\alpha + \frac{\langle X, e_\alpha\rangle}{2}\right) = 0.$$

Then we obtain

$$\nabla_i H^\alpha = \frac{1}{2}\sum_k h^\alpha_{ik}\langle X, e_k\rangle,$$

and

$$\nabla_j\nabla_i H^\alpha = \frac{1}{2}h^\alpha_{ij} + \frac{1}{2}\sum_{\beta,k}\langle X, e_\beta\rangle h^\beta_{jk} h^\alpha_{ik} + \frac{1}{2}\sum_k h^\alpha_{kij}\langle X, e_k\rangle. \tag{6}$$

Combining (3) and (6), we get

$$\begin{aligned}\sum_{i,j,\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} &= \sum_{i,j,\alpha} h^\alpha_{ij}\nabla_j\nabla_i H^\alpha \\ &\quad + \sum_{i,j,k,\alpha} h^\alpha_{ij}\left(\sum_m h^\alpha_{km}R_{mijk} + \sum_m h^\alpha_{mi}R_{mkjk} - \sum_\beta h^\beta_{ki}R_{\alpha\beta jk}\right) \\ &= \frac{1}{2}|A|^2 + \frac{1}{2}\sum_{i,j,k,\alpha,\beta}\langle X, e_\beta\rangle h^\beta_{jk}h^\alpha_{ij}h^\alpha_{ik} + \frac{1}{4}\langle X, \nabla|A|^2\rangle \\ &\quad + \sum_{i,j,k,\alpha,\beta} H^\beta h^\beta_{jk}h^\alpha_{ij}h^\alpha_{ik} - \sum_{\alpha,\beta}\left(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\right)^2 \\ &\quad - \sum_{i,j,\alpha,\beta}\left(\sum_p (h^\alpha_{ip}h^\beta_{pj} - h^\alpha_{jp}h^\beta_{pi})\right)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}|A|^2 &= \Delta|A|^2 - \frac{1}{2}\langle X, \nabla|A|^2\rangle \\ &= 2\sum_{i,j,\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} + 2|\nabla A|^2 - \frac{1}{2}\langle X, \nabla|A|^2\rangle \\ &= 2|\nabla A|^2 + |A|^2 + 2\sum_{i,j,k,\alpha,\beta}\xi^\beta h^\beta_{jk}h^\alpha_{ij}h^\alpha_{ik} \\ &\quad - 2\sum_{\alpha,\beta}\left(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\right)^2 - 2\sum_{i,j,\alpha,\beta}\left(\sum_p (h^\alpha_{ip}h^\beta_{pj} - h^\alpha_{jp}h^\beta_{pi})\right)^2,\end{aligned}$$

where $\xi^\alpha = H^\alpha + \frac{1}{2}\langle X, e_\alpha\rangle$.

From (6) one has

$$\Delta|H|^2 = 2|\nabla H|^2 + |H|^2 + \sum_{\alpha,\beta,i,j} H^\alpha h^\alpha_{ij}\langle X, e_\beta\rangle h^\beta_{ij} + \frac{1}{2}\langle X, \nabla|H|^2\rangle,$$

where $H^\alpha = \sum_i h^\alpha_{ii}$.

Then it follows that

$$\begin{aligned}\mathcal{L}|H|^2 &= \Delta|H|^2 - \frac{1}{2}\langle X, \nabla|H|^2\rangle \\ &= 2|\nabla H|^2 + |H|^2 + \sum_{\alpha,\beta,i,j} H^\alpha h^\alpha_{ij}\langle X, e_\beta\rangle h^\beta_{ij} \\ &= 2|\nabla H|^2 + |H|^2 + 2\sum_{\alpha,\beta,i,j} H^\alpha\xi^\beta h^\alpha_{ij}h^\beta_{ij} - 2\sum_{i,j}\left(\sum_\alpha H^\alpha h^\alpha_{ij}\right)^2.\end{aligned}$$

□

The following two lemmas will be used in the proof of our theorems.

**Lemma 2.** ([14,30]) *Let $M^n (n \geq 3)$ be a complete submanifold in the Euclidean space $\mathbb{R}^{n+p}$. Let f be a nonnegative $C^1$ function with compact support. Then we have*

$$\|f\|^2_{\frac{2n}{n-2}} \leq D^2(n)\left[\frac{4(n-1)^2(1+t)}{(n-2)^2}\|\nabla f\|_2^2 + \left(1+\frac{1}{t}\right)\frac{1}{n^2}\||H|f\|_2^2\right],$$

*where $D(n) = 2^n(1+n)^{\frac{n+1}{n}}(n-1)^{-1}\sigma_n^{-\frac{1}{n}}$, and $\sigma_n$ denotes the volume of the unit ball in $\mathbb{R}^n$.*

**Lemma 3.** ([25])

*Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers satisfying $\sum_i a_i = \sum_i b_i = 0$, $\sum_i a_i^2 = a$ and $\sum_i b_i^2 = b$. Then*

$$\left|\sum_i a_i b_i^2\right| \leq \frac{n-2}{\sqrt{n(n-1)}}\sqrt{a}b,$$

*and the equality holds if and only if either $ab = 0$, or at least $n-1$ pairs of numbers of $(a_i, b_i)$'s are the same.*

## 3. Gap Theorems for $n \geq 3$

In this section, we assume that the Gaussian mean curvature vector of $M$ is parallel in the normal bundle.

In general, when the codimension $p \geq 2$, we know from [18] that

$$\sum_{\alpha,\beta}\left(\sum_{i,j} h^\alpha_{ij}h^\beta_{ij}\right)^2 + \sum_{i,j,\alpha,\beta}\left(\sum_p (h^\alpha_{ip}h^\beta_{pj} - h^\alpha_{jp}h^\beta_{pi})\right)^2 \leq \frac{3}{2}|A|^4.$$

Combining (4) and the above inequality, we have

$$\begin{aligned}\mathcal{L}|A|^2 &\geq 2|\nabla A|^2 + |A|^2 + 2\sum_{i,j,k,\alpha,\beta}\xi^\beta h^\beta_{jk}h^\alpha_{ij}h^\alpha_{ik} - 3|A|^4 \\ &\geq 2|\nabla A|^2 + |A|^2 - 2|\xi||A|^3 - 3|A|^4.\end{aligned} \tag{7}$$

Firstly, we give the proof of Theorem 1.

*Proof.* If $\xi = 0$, then $M$ is a self-shrinker of the mean curvature flow, and Theorem 1 follows by [11]. Now we assume $\xi \neq 0$. We aim to show that this is impossible for submanifolds satisfying an suitable integral inequality.

It follows from (7) and the inequality $|\nabla A|^2 \geq |\nabla|A||^2$ at all points where $A \neq 0$, which is an easy consequence of the Schwartz inequality, that the inequality

$$\mathcal{L}|A|^2 \geq 2|\nabla|A||^2 + |A|^2 - 2|\xi||A|^3 - 3|A|^4 \tag{8}$$

holds on $M$ in the sense of distribution. For a fixed point $x_0 \in M$ and every $r > 0$, define a smooth cut-off function $\phi_r$ by

$$\phi_r(x) = \begin{cases} 1, & x \in B_r(x_0), \\ \phi_r(x) \in [0,1] \text{ and } |\nabla \phi_r| \le \frac{2}{r}, & x \in B_{2r}(x_0) \setminus B_r(x_0), \\ 0, & x \in M \setminus B_{2r}(x_0), \end{cases}$$

where $B_r(x_0)$ is the geodesic ball in $M$ with radius $r$ centered at $x_0 \in M$. Multiplying both side of (8) by $|A|^{n-2}\phi_r^2$ and integrating by parts with respect to the measure $\rho d\mu$ on $M$ yield

$$\begin{aligned} 0 &\ge 2\int_M |\nabla |A||^2 |A|^{n-2}\phi_r^2\rho + \int_M |A|^n\phi_r^2\rho - 2\int_M |\xi||A|^{n+1}\phi_r^2\rho \\ &\quad - 3\int_M |A|^{n+2}\phi_r^2\rho - \int_M |A|^{n-2}\phi_r^2\mathcal{L}|A|^2\rho \\ &= 2(n-1)\int_M |\nabla |A||^2 |A|^{n-2}\phi_r^2\rho + \int_M |A|^n\phi_r^2\rho - 2\int_M |\xi||A|^{n+1}\phi_r^2\rho \\ &\quad - 3\int_M |A|^{n+2}\phi_r^2\rho + 4\int_M \phi_r |A|^{n-1}\langle \nabla |A|, \nabla \phi_r\rangle\rho \\ &\ge 2(n-1)\int_M |\nabla |A||^2 |A|^{n-2}\phi_r^2\rho + \int_M |A|^n\phi_r^2\rho \\ &\quad - 2|\xi|\left(\frac{\tau}{2}\int_M |A|^n\phi_r^2\rho + \frac{1}{2\tau}\int_M |A|^{n+2}\phi_r^2\rho\right) \\ &\quad - 3\int_M |A|^{n+2}\phi_r^2\rho + 4\int_M \phi_r |A|^{n-1}\langle \nabla |A|, \nabla \phi_r\rangle\rho \\ &\ge \left(2(n-1) - \frac{(4-\sigma)\varrho}{2}\right)\int_M |\nabla |A||^2 |A|^{n-2}\phi_r^2\rho \\ &\quad + (1-|\xi|\tau)\int_M |A|^n\phi_r^2\rho - \left(3 + \frac{|\xi|}{\tau}\right)\int_M |A|^{n+2}\phi_r^2\rho \\ &\quad + \sigma\int_M \phi_r |A|^{n-1}\langle \nabla |A|, \nabla \phi_r\rangle\rho - \frac{4-\sigma}{2\varrho}\int_M |A|^n|\nabla \phi_r|^2\rho, \end{aligned} \tag{9}$$

where $\tau, \varrho \in \mathbb{R}^+$ and $\sigma \in (0,4)$. For the last inequality of (9), we have used

$$\begin{aligned} &4\int_M \phi_r |A|^{n-1}\langle \nabla |A|, \nabla \phi_r\rangle\rho \\ &= \sigma\int_M \phi_r |A|^{n-1}\langle \nabla |A|, \nabla \phi_r\rangle\rho + (4-\sigma)\int_M \phi_r |A|^{n-1}\langle \nabla |A|, \nabla \phi_r\rangle\rho \\ &\ge \sigma\int_M \phi_r |A|^{n-1}\langle \nabla |A|, \nabla \phi_r\rangle\rho - \frac{(4-\sigma)\varrho}{2}\int_M |\nabla |A||^2 |A|^{n-2}\phi_r^2\rho \\ &\quad - \frac{4-\sigma}{2\varrho}\int_M |A|^n|\nabla \phi_r|^2\rho. \end{aligned}$$

By a direct computation, we have

$$|\nabla(|A|^{\frac{n}{2}}\phi_r)|^2 = |A|^n|\nabla \phi_r|^2 + n\phi_r|A|^{n-1}\langle \nabla \phi_r, \nabla |A|\rangle + \frac{n^2}{4}|A|^{n-2}|\nabla |A||^2\phi_r^2. \tag{10}$$

Pick $\sigma, \varrho > 0$ such that $2(n-1) - \frac{(4-\sigma)\varrho}{2} = \frac{n\sigma}{4}$. Then we get from (9) that

$$
\begin{aligned}
0 \geq & \frac{n\sigma}{4}\left(\frac{4}{n^2}\int_M |\nabla(|A|^{\frac{n}{2}}\phi_r)|^2\rho - \frac{4}{n^2}\int_M |A|^n|\nabla\phi_r|^2\rho\right. \\
& \left. -\frac{4}{n}\int_M \phi_r|A|^{n-1}\langle\nabla\phi_r, \nabla|A|\rangle\rho\right) \\
& + (1-|\xi|\tau)\int_M |A|^n\phi_r^2\rho - \left(3+\frac{|\xi|}{\tau}\right)\int_M |A|^{n+2}\phi_r^2\rho \\
& + \sigma\int_M \phi_r|A|^{n-1}\langle\nabla|A|, \nabla\phi_r\rangle\rho - \frac{4-\sigma}{2\varrho}\int_M |A|^n|\nabla\phi_r|^2\rho \\
= & \frac{\sigma}{n}\int_M |\nabla(|A|^{\frac{n}{2}}\phi_r)|^2\rho + (1-|\xi|\tau)\int_M |A|^n\phi_r^2\rho \\
& - \left(3+\frac{|\xi|}{\tau}\right)\int_M |A|^{n+2}\phi_r^2\rho - \left(\frac{\sigma}{n}+\frac{4-\sigma}{2\varrho}\right)\int_M |A|^n|\nabla\phi_r|^2\rho.
\end{aligned}
\tag{11}
$$

Set $f = |A|^{\frac{n}{2}}\rho^{\frac{1}{2}}\phi_r$. Integrating by parts, we obtain

$$
\begin{aligned}
\int_M |\nabla f|^2 = & \int_M |\nabla(|A|^{\frac{n}{2}}\phi_r)|^2\rho + \frac{1}{2}\int_M \nabla(|A|^n\phi_r^2)\nabla\rho \\
& + \int_M |A|^n\phi_r^2|\nabla\rho^{\frac{1}{2}}|^2 \\
= & \int_M |\nabla(|A|^{\frac{n}{2}}\phi_r)|^2\rho - \frac{1}{2}\int_M |A|^n\phi_r^2\Delta\rho \\
& + \frac{1}{16}\int_M |A|^n\phi_r^2|X^T|^2\rho.
\end{aligned}
\tag{12}
$$

Since

$$
\begin{aligned}
\Delta|X|^2 &= 2|\nabla X|^2 + 2\langle X, \Delta X\rangle \\
&= 2n + 2\langle X^N, H\rangle \\
&= 2n + 4\langle\xi, H\rangle - 4|H|^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
\Delta\rho &= -\frac{\rho}{4}\Delta|X|^2 + \frac{\rho}{16}|\nabla|X|^2|^2 \\
&= -\frac{n}{2}\rho - \rho\langle\xi, H\rangle + \rho|H|^2 + \frac{\rho}{4}|X^T|^2.
\end{aligned}
\tag{13}
$$

Substituting (13) into (12) yields

$$\begin{aligned}\int_M |\nabla f|^2 = &\int_M |\nabla(|A|^{\frac{n}{2}}\phi_r)|^2\rho + \frac{n}{4}\int_M |A|^n\phi_r^2\rho \\ &+ \frac{1}{2}\int_M |A|^n\phi_r^2\langle\xi, H\rangle\rho - \frac{1}{2}\int_M |A|^n\phi_r^2|H|^2\rho \\ &- \frac{1}{16}\int_M |A|^n\phi_r^2|X^T|^2\rho \\ \le &\int_M |\nabla(|A|^{\frac{n}{2}}\phi_r)|^2\rho + \frac{n}{4}\int_M |A|^n\phi_r^2\rho \\ &+ \frac{1}{2}\int_M |A|^n\phi_r^2\langle\xi, H\rangle\rho - \frac{1}{2}\int_M |A|^n\phi_r^2|H|^2\rho.\end{aligned} \tag{14}$$

Combining the Sobolev inequality in Lemma 2, (11) and (14), we have

$$\begin{aligned}0 \ge &\frac{\sigma}{n}\int_M |\nabla f|^2 + \frac{\sigma}{2n}\int_M |A|^n\phi_r^2|H|^2\rho - \frac{\sigma}{2n}\int_M |A|^n\phi_r^2\langle\xi, H\rangle\rho \\ &+ \left(1 - |\xi|\tau - \frac{\sigma}{4}\right)\int_M |A|^n\phi_r^2\rho - \left(3 + \frac{|\xi|}{\tau}\right)\int_M |A|^{n+2}\phi_r^2\rho \\ &- \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^n|\nabla\phi_r|^2\rho \\ \ge &\frac{(n-2)^2\sigma}{4n(n-1)^2D^2(n)(1+t)}|||A|^n\phi_r^2\rho||_{\frac{n}{n-2}} \\ &+ \left(\frac{\sigma}{2n} - \frac{(n-2)^2\sigma}{4n^3(n-1)^2t}\right)\int_M |A|^n\phi_r^2|H|^2\rho - \frac{\sigma}{2n}\int_M |A|^n\phi_r^2\langle\xi, H\rangle\rho \\ &+ \left(1 - |\xi|\tau - \frac{\sigma}{4}\right)\int_M |A|^n\phi_r^2\rho - \left(3 + \frac{|\xi|}{\tau}\right)\int_M |A|^{n+2}\phi_r^2\rho \\ &- \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^n|\nabla\phi_r|^2\rho.\end{aligned} \tag{15}$$

Choose $t = \frac{(n-2)^2}{2n^2(n-1)^2} \in \mathbb{R}^+$ such that $\frac{\sigma}{2n} = \frac{(n-2)^2\sigma}{4n^3(n-1)^2t}$. Then (15) becomes

$$\begin{aligned}0 \ge &\frac{n(n-2)^2\sigma}{2D^2(n)[2n^2(n-1)^2 + (n-2)^2]}|||A|^n\phi_r^2\rho||_{\frac{n}{n-2}} \\ &- \frac{\sigma}{2n}\int_M |A|^n\phi_r^2\langle\xi, H\rangle\rho + \left(1 - |\xi|\tau - \frac{\sigma}{4}\right)\int_M |A|^n\phi_r^2\rho \\ &- \left(3 + \frac{|\xi|}{\tau}\right)\int_M |A|^{n+2}\phi_r^2\rho \\ &- \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^n|\nabla\phi_r|^2\rho.\end{aligned} \tag{16}$$

On the other hand, for any $\theta > 0$, we have

$$\begin{aligned} -\frac{\sigma}{2n}\int_M |A|^n\phi_r^2\langle\xi, H\rangle\rho &\geq -\frac{\sigma}{2n}\int_M |A|^n\phi_r^2|\xi||H|\rho \\ &\geq -\frac{\sigma|\xi|}{2n}\int_M |A|^n\phi_r^2\left(\frac{1}{2\theta}+\frac{\theta}{2}|H|^2\right)\rho \\ &= -\frac{\sigma|\xi|}{4n\theta}\int_M |A|^n\phi_r^2\rho - \frac{\sigma\theta|\xi|}{4n}\int_M |A|^n\phi_r^2|H|^2\rho \\ &\geq -\frac{\sigma|\xi|}{4n\theta}\int_M |A|^n\phi_r^2\rho - \frac{\sigma\theta|\xi|}{4}\int_M |A|^{n+2}\phi_r^2\rho. \end{aligned} \tag{17}$$

Combing (16) and (17), we get

$$\begin{aligned} 0 \geq &\frac{n(n-2)^2\sigma}{2D^2(n)[2n^2(n-1)^2+(n-2)^2]}|||A|^n\phi_r^2\rho||_{\frac{n}{n-2}} \\ &+\left(1-|\xi|\tau-\frac{\sigma}{4}-\frac{\sigma|\xi|}{4n\theta}\right)\int_M |A|^n\phi_r^2\rho \\ &-\left(3+\frac{|\xi|}{\tau}+\frac{\sigma\theta|\xi|}{4}\right)\int_M |A|^{n+2}\phi_r^2\rho \\ &-\left(\frac{\sigma}{n}+\frac{4-\sigma}{2\varrho}\right)\int_M |A|^n|\nabla\phi_r|^2\rho. \end{aligned}$$

Now we let $\tau$ satisfy $\tau < \frac{1}{|\xi|}$. Then

$$0 < \frac{4n\theta(1-|\xi|\tau)}{n\theta+|\xi|} < \frac{4n\theta}{n\theta+|\xi|} < \frac{4n\theta}{n\theta} = 4.$$

We choose $\sigma = \frac{4n\theta(1-|\xi|\tau)}{n\theta+|\xi|} \in (0, 4)$ such that

$$|\xi|\tau + \frac{\sigma}{4} + \frac{\sigma|\xi|}{4n\theta} = 1.$$

Hence

$$\begin{aligned} 0 \geq &\frac{n(n-2)^2}{2D^2(n)[2n^2(n-1)^2+(n-2)^2]}\cdot\frac{4n\theta(1-|\xi|\tau)}{n\theta+|\xi|}|||A|^n\phi_r^2\rho||_{\frac{n}{n-2}} \\ &-\left[\frac{n|\xi|\theta^2(1-|\xi|\tau)}{n\theta+|\xi|}+3+\frac{\xi}{\tau}\right]\int_M |A|^{n+2}\phi_r^2\rho \\ &-\left(\frac{\sigma}{n}+\frac{4-\sigma}{2\varrho}\right)\int_M |A|^n|\nabla\phi_r|^2\rho. \end{aligned} \tag{18}$$

By the Hölder inequality, we have

$$\int_M |A|^{n+2}\phi_r^2\rho \leq |||A|^n\phi_r^2\rho||_{\frac{n}{n-2}} \cdot |||A|^2||_{\frac{n}{2}}.$$

Hence it follows from (18) that

$$0 \geq \left[\frac{4\theta(1-|\xi|\tau)}{\kappa(n)(n\theta+|\xi|)} - \left(\frac{n|\xi|\theta^2(1-|\xi|\tau)}{n\theta+|\xi|} + 3 + \frac{|\xi|}{\tau}\right)||A||_n^2\right] |||A|^n\phi_r^2\rho||_{\frac{n}{n-2}} - \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^n|\nabla\phi_r|^2\rho, \tag{19}$$

where

$$\kappa(n) = \frac{2D^2(n)[2n^2(n-1)^2+(n-2)^2]}{n^2(n-2)^2}.$$

Set

$$K(n,|\xi|,\tau,\theta) = \sqrt{\frac{4\theta\tau(1-|\xi|\tau)}{[n\tau|\xi|\theta^2(1-|\xi|\tau)+(3\tau+|\xi|)(n\theta+|\xi|)]\kappa(n)}}.$$

By direct computations, one has

$$\frac{\partial}{\partial\tau}K^2(n,|\xi|,\tau,\theta) = \frac{4\theta|\xi|(n\theta+|\xi|)(-3\tau^2-2|\xi|\tau+1)}{\kappa(n)[n\tau|\xi|\theta^2(1-|\xi|\tau)+(3\tau+|\xi|)(n\theta+|\xi|)]^2},$$
$$\frac{\partial}{\partial\theta}K^2(n,|\xi|,\tau,\theta) = \frac{4\tau|\xi|(1-|\xi|\tau)[-n\tau\theta^2(1-|\xi|\tau)+(3\tau+|\xi|)]}{\kappa(n)[n\tau|\xi|\theta^2(1-|\xi|\tau)+(3\tau+|\xi|)(n\theta+|\xi|)]^2}.$$

It is easy to see that, when restricted in $(0, \frac{1}{|\xi|}) \times (0, \infty)$, the system

$$\begin{cases} -3\tau^2 - 2|\xi|\tau + 1 = 0 \\ -n\tau\theta^2(1-|\xi|\tau) + (3\tau+|\xi|) = 0 \end{cases}$$

has only one solution

$$\tau = \tau_0 := \frac{\sqrt{|\xi|^2+3}-|\xi|}{3},\ \theta = \theta_0 := \frac{3}{\sqrt{n}(\sqrt{|\xi|^2+3}-|\xi|)}.$$

By the monotonicity of $K(n,|\xi|,\tau,\theta)$ as a function of $\tau$ and $\theta$ in $(0, \frac{1}{|\xi|}) \times (0, \infty)$, we see that $K(n,|\xi|,\tau,\theta)$ achieves its maximum

$$K(n,|\xi|) = K_{\max}(n,|\xi|,\tau,\theta) = \sqrt{\frac{4(\sqrt{|\xi|^2+3}-|\xi|)}{[3(n|\xi|+2\sqrt{n}|\xi|+n\sqrt{|\xi|^2+3})]\kappa(n)}}$$

when $\tau = \tau_0,\ \theta = \theta_0$.

Since we have picked $\sigma$ and $\varrho$ such that $2(n-1) - \frac{(4-\sigma)\varrho}{2} = \frac{n\sigma}{4}$ and $\sigma \in (0,4)$, one has

$$0 < \frac{\sigma}{n} + \frac{4-\sigma}{2\varrho} = \frac{\sigma}{n} + \frac{(4-\sigma)^2}{8(n-1)-n\sigma} < \frac{4}{n} + \frac{4}{n-2}.$$

Since $|\nabla\phi_r| \le \frac{2}{r}$ and $\int_M |A|^n \mathrm{d}\mu < \infty$, we have

$$\lim_{r\to\infty}\int_M |A|^n|\nabla\phi_r|^2\rho = 0.$$

Since $||A||_n < K(n, |\xi|)$, (19) imples

$$0 \ge \left(K(n,|\xi|)^2 - ||A||_n^2\right)\lim_{r\to\infty}|||A|^n\phi_r^2\rho||_{\frac{n}{n-2}} \ge 0.$$

Hence $|||A|^n e^{-\frac{|X|^2}{4}}||_{\frac{n}{n-2}} = 0$, which implies that $|A| = 0$. Hence $M$ is a linear subspace of $\mathbb{R}^{n+p}$. This implies $\xi = 0$, which is a contradiction. This completes the proof of Theorem 1. □

Combining (4) and (5), we have

$$\begin{aligned}
\mathcal{L}|\mathring{A}|^2 &= \mathcal{L}|A|^2 - \frac{1}{n}\mathcal{L}|H|^2\\
&= 2|\nabla\mathring{A}|^2 + |\mathring{A}|^2 - 2\sum_{\alpha,\beta}\left(\sum_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta}\right)^2\\
&\quad - 2\sum_{i,j,\alpha,\beta}\left(\sum_p(h_{ip}^{\alpha}h_{pj}^{\beta} - h_{jp}^{\alpha}h_{pi}^{\beta})\right)^2 + \frac{2}{n}\sum_{ij}\left(\sum_{\alpha}H^{\alpha}h_{ij}^{\alpha}\right)^2\\
&\quad + 2\sum_{i,j,k,\alpha,\beta}\xi^{\beta}h_{jk}^{\beta}h_{ij}^{\alpha}h_{ik}^{\alpha} - \frac{2}{n}\sum_{\alpha,\beta,i,j}H^{\alpha}\xi^{\beta}h_{ij}^{\alpha}h_{ij}^{\beta}.
\end{aligned} \tag{20}$$

At the point where the mean curvature vector is zero, we have

$$\begin{aligned}
&-2\sum_{\alpha,\beta}\left(\sum_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta}\right)^2 - 2\sum_{i,j,\alpha,\beta}\left(\sum_p(h_{ip}^{\alpha}h_{pj}^{\beta} - h_{jp}^{\alpha}h_{pi}^{\beta})\right)^2 + \frac{2}{n}\sum_{i,j}\left(\sum_{\alpha}H^{\alpha}h_{ij}^{\alpha}\right)^2\\
&= -2\sum_{\alpha,\beta}N(A^{\alpha}A^{\beta} - A^{\beta}A^{\alpha}) - 2\sum_{\alpha,\beta}[\mathrm{tr}(A^{\alpha}A^{\beta})]^2\\
&\geqslant -3|A|^4,
\end{aligned} \tag{21}$$

where $A^{\alpha} = (h_{ij}^{\alpha})_{n\times n}$ and we have used Theorem 1 in [18] to get the inequality.

For a fixed $\alpha$, choose a local orthonormal frame field $\{e_i\}$ such that $h_{ij}^{\alpha} = \lambda_i^{\alpha}\delta_{ij}$. Then by Lemma 3 we have the following

$$\begin{aligned}
2\sum_{i,j,k,\beta}\xi^{\beta}h_{jk}^{\beta}h_{ij}^{\alpha}h_{ik}^{\alpha} &= 2\sum_{i,\beta}(\lambda_i^{\alpha})^2h_{ii}^{\beta}\xi^{\beta}\\
&\ge -\frac{2(n-2)}{\sqrt{n(n-1)}}\sum_{\beta}\left(\sum_i(\lambda_i^{\alpha})^2\left(\sum_i(h_{ii}^{\beta})^2\right)^{\frac{1}{2}}|\xi^{\beta}|\right)\\
&= -\frac{2(n-2)}{\sqrt{n(n-1)}}\sum_{i,j}(h_{ij}^{\alpha})^2\sum_{\beta}\left(\sum_{i,j}(h_{ij}^{\beta})^2\right)^{\frac{1}{2}}|\xi^{\beta}|.
\end{aligned}$$

Hence we get that for any $\eta > 0$

$$
\begin{aligned}
2\sum_{i,j,k,\alpha,\beta}\xi^\beta h^\beta_{jk}h^\alpha_{ij}h^\alpha_{ik} &\geq -\frac{2(n-2)}{\sqrt{n(n-1)}}\sum_{i,j,\alpha}\left(h^\alpha_{ij}\right)^2\sum_\beta\left(\frac{1}{2\eta}\sum_{i,j}\left(h^\beta_{ij}\right)^2+\frac{\eta}{2}|\xi^\beta|^2\right)\\
&= -\frac{n-2}{\sqrt{n(n-1)}}\left(\frac{1}{\eta}|A|^4+\eta|\xi|^2|A|^2\right).
\end{aligned}
\tag{22}
$$

At the point where the mean curvature vector is nonzero, we choose $e_{n+1}=\frac{H}{|H|}$. The second fundamental form can be written as $A=\sum_\alpha h^\alpha e_\alpha$, where $h^\alpha$, $n+1\leq \alpha\leq n+p$, are symmetric 2-tensors. By the choice of $e_{n+1}$, we see that $\mathrm{tr}h^{n+1}=|H|$ and $\mathrm{tr}h^\alpha=0$ for $\alpha\geq n+2$. The traceless second fundamental form may be rewritten as $\mathring{A}=\sum_\alpha \mathring{h}^\alpha e_\alpha$, where $\mathring{h}^{n+1}=h^{n+1}-\frac{|H|}{n}\mathrm{Id}$ and $\mathring{h}^\alpha=h^\alpha$ for $\alpha\geq n+2$. We set $A_H=h^{n+1}e_{n+1}$, $A_I=\sum_{\alpha\geq n+2}h^\alpha e_\alpha$, $\mathring{A}_H=\mathring{h}^{n+1}e_{n+1}$ and $\mathring{A}_I=\sum_{\alpha\geq n+2}\mathring{h}^\alpha e_\alpha$. Then we have

$$
\begin{aligned}
|A_I|^2 &= \sum_{\alpha\geq n+2}|h^\alpha|^2=|A|^2-|A_H|^2,\\
|\mathring{A}_I|^2 &= \sum_{\alpha\geq n+2}|\mathring{h}^\alpha|^2=|\mathring{A}|^2-|\mathring{A}_H|^2.
\end{aligned}
$$

Note that $|\mathring{A}_H|^2=|A_H|^2-\frac{|H|^2}{n}$ and $|\mathring{A}_I|^2=|A_I|^2$. Since $e_{n+1}$ is chosen globally, $|A_H|^2$, $|\mathring{A}_H|^2$ and $|A_I|^2$ are defined globally and independent of the choice of $e_i$.

Then we have

$$
\begin{aligned}
\sum_{\alpha,\beta}\left(\sum_{i,j}h^\alpha_{ij}h^\beta_{ij}\right)^2 &= |\mathring{A}_H|^4+\frac{2}{n}|H|^2|\mathring{A}_H|^2+\frac{1}{n^2}|H|^4\\
&\quad +2\sum_{\alpha\neq n+1}\left(\sum_{i,j}\mathring{h}^{n+1}_{ij}\mathring{h}^\alpha_{ij}\right)^2+\sum_{\alpha,\beta\neq n+1}\left(\sum_{i,j}\mathring{h}^\alpha_{ij}\mathring{h}^\beta_{ij}\right)^2,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\sum_{i,j,\alpha,\beta}\left(\sum_p(h^\alpha_{ip}h^\beta_{pj}-h^\alpha_{jp}h^\beta_{pi})\right)^2 &= 2\sum_{\alpha\neq n+1}\sum_{i,j}\left(\sum_p\left(h^{n+1}_{ip}\mathring{h}^\alpha_{pj}-h^{n+1}_{jp}\mathring{h}^\alpha_{pi}\right)\right)^2\\
&\quad +\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\left(\sum_p\left(\mathring{h}^\alpha_{ip}\mathring{h}^\beta_{pj}-\mathring{h}^\alpha_{jp}\mathring{h}^\beta_{pi}\right)\right)^2,
\end{aligned}
\tag{24}
$$

$$
\sum_{i,j}\left(\sum_\alpha H^\alpha h^\alpha_{ij}\right)^2=|H|^2|\mathring{A}_H|^2+\frac{1}{n}|H|^4. \tag{25}
$$

From (23), (24) and (25) we obtain the following

$$
\begin{aligned}
&2\sum_{\alpha,\beta}\left(\sum_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta}\right)^2+2\sum_{i,j,\alpha,\beta}\left(\sum_{p}(h_{ip}^{\alpha}h_{pj}^{\beta}-h_{jp}^{\alpha}h_{pi}^{\beta})\right)^2-\frac{2}{n}\sum_{i,j}\left(\sum_{\alpha}H^{\alpha}h_{ij}^{\alpha}\right)^2\\
&=2|\mathring{A}_H|^4+\frac{2}{n}|H|^2|\mathring{A}_H|^2\\
&\quad+4\sum_{\alpha\neq n+1}\left(\sum_{i,j}\mathring{h}_{ij}^{n+1}\mathring{h}_{ij}^{\alpha}\right)^2+4\sum_{\alpha\neq n+1}\sum_{i,j}\left(\sum_{p}(h_{ip}^{n+1}\mathring{h}_{pj}^{\alpha}-h_{jp}^{n+1}\mathring{h}_{pi}^{\alpha})\right)^2\\
&\quad+2\sum_{\alpha,\beta\neq n+1}\left(\sum_{i,j}\mathring{h}_{ij}^{\alpha}\mathring{h}_{ij}^{\beta}\right)^2+2\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\left(\sum_{p}(\mathring{h}_{ip}^{\alpha}\mathring{h}_{pj}^{\beta}-\mathring{h}_{jp}^{\alpha}\mathring{h}_{pi}^{\beta})\right)^2.
\end{aligned}
\tag{26}
$$

We choose $\{e_i\}$ such that $h_{ij}^{n+1}=\lambda_i\delta_{ij}$. Then $\mathring{h}_{ij}^{n+1}=\mathring{\lambda}_i\delta_{ij}$, where $\mathring{\lambda}_i=\lambda_i-\frac{|H|}{n}$. We first have the following estimate.

$$
\begin{aligned}
4\sum_{\alpha\neq n+1}\left(\sum_{i,j}\mathring{h}_{ij}^{n+1}\mathring{h}_{ij}^{\alpha}\right)^2&=4\sum_{\alpha\neq n+1}\left(\sum_{i}\mathring{\lambda}_i\mathring{h}_{ii}^{\alpha}\right)^2\\
&\leq 4\left(\sum_{i}\mathring{\lambda}_i^2\right)\left(\sum_{\alpha\neq n+1}\sum_{i}\left(\mathring{h}_{ii}^{\alpha}\right)^2\right)\\
&=4|\mathring{A}_H|^2\sum_{\alpha\neq n+1}\sum_{i}\left(\mathring{h}_{ii}^{\alpha}\right)^2.
\end{aligned}
$$

For any fixed $i, j=1,\ldots,n,\ i\neq j$, one has

$$
|\mathring{A}_H|^2=\sum_{k}\mathring{\lambda}_k^2=\mathring{\lambda}_i^2+\mathring{\lambda}_j^2+\sum_{k\neq i,j}\mathring{\lambda}_k^2\geq\mathring{\lambda}_i^2+\mathring{\lambda}_j^2.
$$

Hence

$$
\begin{aligned}
4\sum_{\alpha\neq n+1}\sum_{i,j}\left(\sum_{p}\left(h_{ip}^{n+1}\mathring{h}_{pj}^{\alpha}-h_{jp}^{n+1}\mathring{h}_{pi}^{\alpha}\right)\right)^2&=4\sum_{\alpha\neq n+1}\sum_{i\neq j}(\lambda_i-\lambda_j)^2\left(\mathring{h}_{ij}^{\alpha}\right)^2\\
&=4\sum_{\alpha\neq n+1}\sum_{i\neq j}(\mathring{\lambda}_i-\mathring{\lambda}_j)^2\left(\mathring{h}_{ij}^{\alpha}\right)^2\\
&\leq 8\sum_{\alpha\neq n+1}\sum_{i\neq j}\left(\mathring{\lambda}_i^2+\mathring{\lambda}_j^2\right)\left(\mathring{h}_{ij}^{\alpha}\right)^2\\
&\leq 8|\mathring{A}_H|^2\sum_{\alpha\neq n+1}\sum_{i\neq j}(\mathring{h}_{ij}^{\alpha})^2\\
&=8|\mathring{A}_H|^2\left(|\mathring{A}_I|^2-\sum_{\alpha\neq n+1}\sum_{i}\left(\mathring{h}_{ii}^{\alpha}\right)^2\right).
\end{aligned}
$$

By an inequality in [18], we have

$$2\sum_{\alpha,\beta\neq n+1}\left(\sum_{i,j}\mathring{h}^{\alpha}_{ij}\mathring{h}^{\beta}_{ij}\right)^2+2\sum_{\alpha,\beta\neq n+1}\sum_{i,j}\left(\sum_p\left(\mathring{h}^{\alpha}_{ip}\mathring{h}^{\beta}_{pj}-\mathring{h}^{\alpha}_{jp}\mathring{h}^{\beta}_{pi}\right)\right)^2\leq 3|\mathring{A}_I|^4.$$

Hence, we have the following estimate

$$\begin{aligned}\frac{2}{n}\sum_{ij}\left(\sum_{\alpha}H^{\alpha}h^{\alpha}_{ij}\right)^2-2\sum_{\alpha,\beta}\left(\sum_{i,j}h^{\alpha}_{ij}h^{\beta}_{ij}\right)^2-2\sum_{i,j,\alpha,\beta}\left(\sum_p\left(h^{\alpha}_{ip}h^{\beta}_{pj}-h^{\alpha}_{jp}h^{\beta}_{pi}\right)\right)^2\\ \geq -4|\mathring{A}|^4-\frac{2}{n}|H|^2|\mathring{A}|^2.\end{aligned} \tag{27}$$

We also have

$$\begin{aligned}2\sum_{i,j,k,\alpha,\beta}\xi^{\beta}h^{\beta}_{jk}h^{\alpha}_{ij}h^{\alpha}_{ik}-\frac{2}{n}\sum_{\alpha,\beta,i,j}H^{\alpha}\xi^{\beta}h^{\alpha}_{ij}h^{\beta}_{ij}&=2\sum_{i,j,k,\alpha,\beta}\xi^{\beta}\left(h^{\alpha}_{ik}-\frac{1}{n}H^{\alpha}\delta_{ik}\right)h^{\alpha}_{ij}h^{\beta}_{jk}\\ &=2\sum_{i,j,k,\alpha,\beta}\xi^{\beta}\mathring{h}^{\alpha}_{ik}h^{\alpha}_{ij}h^{\beta}_{jk}\\ &=2\sum_{i,j,k,\alpha,\beta}\xi^{\beta}\mathring{h}^{\alpha}_{ik}\\ &\quad\left(\mathring{h}^{\alpha}_{ij}+\frac{1}{n}H^{\alpha}\delta_{ij}\right)\left(\mathring{h}^{\beta}_{jk}+\frac{1}{n}H^{\beta}\delta_{jk}\right)\\ &=2\sum_{i,j,k,\alpha,\beta}\xi^{\beta}\mathring{h}^{\alpha}_{ij}\mathring{h}^{\alpha}_{ik}\mathring{h}^{\beta}_{jk}+\frac{2}{n}\sum_{i,j,\alpha,\beta}H^{\beta}\xi^{\beta}\left(\mathring{h}^{\alpha}_{ij}\right)^2\\ &\quad+\frac{2}{n}\sum_{i,j,\alpha,\beta}H^{\alpha}\xi^{\beta}\mathring{h}^{\alpha}_{ij}\mathring{h}^{\beta}_{ij},\end{aligned} \tag{28}$$

where $\mathring{h}^{\alpha}_{ij}=h^{\alpha}_{ij}-\frac{1}{n}H^{\alpha}\delta_{ij}$.

As (22), for any $\eta>0$ we have

$$2\sum_{i,j,k,\alpha,\beta}\xi^{\beta}\mathring{h}^{\alpha}_{ij}\mathring{h}^{\alpha}_{ik}\mathring{h}^{\beta}_{jk}\geq-\frac{n-2}{\sqrt{n(n-1)}}\left(\frac{1}{\eta}|\mathring{A}|^4+\eta|\xi|^2|\mathring{A}|^2\right). \tag{29}$$

On the other hand, we have

$$\frac{2}{n}\sum_{i,j,\alpha,\beta}H^{\beta}\xi^{\beta}(\mathring{h}^{\alpha}_{ij})^2=\frac{2}{n}\langle H,\xi\rangle|\mathring{A}|^2\geq-\frac{2}{n}|H||\xi||\mathring{A}|^2, \tag{30}$$

$$\frac{2}{n}\sum_{i,j,\alpha,\beta}H^{\alpha}\xi^{\beta}\mathring{h}^{\alpha}_{ij}\mathring{h}^{\beta}_{ij}=\frac{2}{n}\sum_{i,j}\langle H,\mathring{h}_{ij}\rangle\langle\xi,\mathring{h}_{ij}\rangle\geq-\frac{2}{n}|H||\xi|\sum_{i,j}|\mathring{h}_{ij}|^2=-\frac{2}{n}|H||\xi||\mathring{A}|^2. \tag{31}$$

Combining (20), (21), (22), (26), (27), (28), (29), (30) and (31) together, we get that at any point of $M$, there holds

$$\begin{aligned}\mathcal{L}|\mathring{A}|^2\geq 2|\nabla\mathring{A}|^2+|\mathring{A}|^2-4|\mathring{A}|^4-\frac{2}{n}|H|^2|\mathring{A}|^2\\ -\frac{n-2}{\sqrt{n(n-1)}}\left(\frac{1}{\eta}|\mathring{A}|^4+\eta|\xi|^2|\mathring{A}|^2\right)-\frac{4}{n}|H||\xi||\mathring{A}|^2.\end{aligned} \tag{32}$$

By using (32), we give the proof of Theorem 2 as follows.

*Proof.* If $\xi = 0$, then M is a self-shrinker of the mean curvature flow, and Theorem 2 follows by [2]. Now we assume $\xi \neq 0$. Similar to Theorem 1, we aim to show that this is impossible for submanifolds satisfying an suitable integral inequality.

From (32), the following inequality holds on $M$ in the sense of distribution.

$$\mathcal{L}|\mathring{A}|^2 \geq 2|\nabla|\mathring{A}||^2 + \left(1 - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2\right)|\mathring{A}|^2 - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)|\mathring{A}|^4 - \frac{2}{n}|H|^2|\mathring{A}|^2 - \frac{4}{n}|H||\xi||\mathring{A}|^2. \tag{33}$$

Let $\phi_r$ be a smooth function on $M$ with compact support as in the proof of Theorem 1. Multiplying both side of (33) by $|\mathring{A}|^{n-2}\phi_r^2$ and integrating by parts with respect to the measure $\rho d\mu$ on $M$ yield

$$\begin{aligned}
0 &\geq 2\int_M |\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\phi_r^2\rho + \left(1 - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2\right)\int_M |\mathring{A}|^n\phi_r^2\rho \\
&\quad - \frac{2}{n}\int_M |H|^2|\mathring{A}|^n\phi_r^2\rho - \frac{4}{n}\int_M |H||\xi||\mathring{A}|^n\phi_r^2\rho \\
&\quad - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho - \int_M |\mathring{A}|^{n-2}\phi_r^2\mathcal{L}|\mathring{A}|^2\rho \\
&= 2(n-1)\int_M |\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\phi_r^2\rho + \left(1 - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2\right)\int_M |\mathring{A}|^n\phi_r^2\rho \\
&\quad - \frac{2}{n}\int_M |H|^2|\mathring{A}|^n\phi_r^2\rho - \frac{4}{n}\int_M |H||\xi||\mathring{A}|^n\phi_r^2\rho \\
&\quad - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho + 4\int_M \phi_r|\mathring{A}|^{n-1}\langle\nabla|\mathring{A}|, \nabla\phi_r\rangle\rho \\
&\geq \left(2(n-1) - \frac{(4-\sigma)\varrho}{2}\right)\int_M |\nabla|\mathring{A}||^2|\mathring{A}|^{n-2}\phi_r^2\rho \\
&\quad + \left(1 - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2\right)\int_M |\mathring{A}|^n\phi_r^2\rho - \frac{2}{n}\int_M |H|^2|\mathring{A}|^n\phi_r^2\rho \\
&\quad - \frac{4}{n}\int_M |H||\xi||\mathring{A}|^n\phi_r^2\rho - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho \\
&\quad + \sigma\int_M \phi_r|\mathring{A}|^{n-1}\langle\nabla|\mathring{A}|, \nabla\phi_r\rangle\rho - \frac{4-\sigma}{2\varrho}\int_M |\mathring{A}|^n|\nabla\phi_r|^2\rho.
\end{aligned} \tag{34}$$

Here $\varrho \in \mathbb{R}^+$ and $\sigma \in (0, 4)$.

As in (10), we have

$$|\nabla(|\mathring{A}|^{\frac{n}{2}}\phi_r)|^2 = |\mathring{A}|^n|\nabla\phi_r|^2 + n\phi_r|\mathring{A}|^{n-1}\langle\nabla\phi_r, \nabla|\mathring{A}|\rangle + \frac{n^2}{4}|\mathring{A}|^{n-2}|\nabla|\mathring{A}||^2\phi_r^2. \tag{35}$$

Pick $\sigma, \varrho > 0$ such that $2(n-1) - \frac{(4-\sigma)\varrho}{2} = \frac{n\sigma}{4}$. Combining (34) and (35), we get

$$\begin{aligned} 0 \geq & \frac{\sigma}{n} \int_M |\nabla(|\mathring{A}|^{\frac{n}{2}}\phi_r)|^2\rho + \left(1 - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2\right)\int_M |\mathring{A}|^n\phi_r^2\rho \\ & - \frac{2}{n}\int_M |H|^2|\mathring{A}|^n\phi_r^2\rho - \frac{4}{n}\int_M |H||\xi||\mathring{A}|^n\phi_r^2\rho \\ & - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho - \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |\mathring{A}|^n|\nabla\phi_r|^2\rho. \end{aligned} \tag{36}$$

Set $f = |\mathring{A}|^{\frac{n}{2}}\rho^{\frac{1}{2}}\phi_r$. As in the proof of Theorem 1, we have

$$\begin{aligned} \int_M |\nabla f|^2 \leq & \int_M |\nabla(|\mathring{A}|^{\frac{n}{2}}\phi_r)|^2\rho + \frac{n}{4}\int_M |\mathring{A}|^n\phi_r^2\rho + \frac{1}{2}\int_M |\mathring{A}|^n\phi_r^2\langle\xi, H\rangle\rho \\ & - \frac{1}{2}\int_M |H|^2|\mathring{A}|^n\phi_r^2\rho. \end{aligned} \tag{37}$$

Combining the Sobolev inequality, (36) with (37), we obtain

$$\begin{aligned} 0 \geq & \frac{\sigma}{n}\int_M |\nabla f|^2 + \left(1 - \frac{\sigma}{4} - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2\right)\int_M |\mathring{A}|^n\phi_r^2\rho \\ & + \left(\frac{\sigma}{2n} - \frac{2}{n}\right)\int_M |\mathring{A}|^n\phi_r^2|H|^2\rho - \frac{\sigma}{2n}\int_M |\mathring{A}|^n\phi_r^2\langle\xi, H\rangle\rho \\ & - \frac{4}{n}\int_M |H||\xi||\mathring{A}|^n\phi_r^2\rho - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho \\ & - \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |\mathring{A}|^n|\nabla\phi_r|^2\rho \\ \geq & \frac{\sigma}{n}\int_M |\nabla f|^2 + \left(1 - \frac{\sigma}{4} - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2 - \frac{\sigma}{2n}|H||\xi| - \frac{4}{n}|H||\xi|\right)\int_M |\mathring{A}|^n\phi_r^2\rho \\ & + \left(\frac{\sigma}{2n} - \frac{2}{n}\right)\int_M |\mathring{A}|^n\phi_r^2|H|^2\rho - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho \\ & - \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |\mathring{A}|^n|\nabla\phi_r|^2\rho \\ \geq & \frac{(n-2)^2\sigma}{4n(n-1)^2D^2(n)(1+t)}|||\mathring{A}|^n\phi_r^2\rho||_{\frac{n}{n-2}} \\ & + \left(1 - \frac{\sigma}{4} - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2 - \frac{\sigma}{2n}|H||\xi| - \frac{4}{n}|H||\xi|\right)\int_M |\mathring{A}|^n\phi_r^2\rho \\ & + \left(\frac{\sigma}{2n} - \frac{2}{n} - \frac{(n-2)^2\sigma}{4(n-1)^2n^3t}\right)\int_M |\mathring{A}|^n\phi_r^2|H|^2\rho \\ & - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho \\ & - \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |\mathring{A}|^n|\nabla\phi_r|^2\rho. \end{aligned}$$

Since $|H| \leq \sup_M|H| < \sqrt{|\xi|^2 + \frac{n}{2}} - |\xi|$ by the assumption of Theorem 2 and $\sigma \in (0, 4)$, we have

$$
\begin{aligned}
0 \geq\ & \frac{(n-2)^2\sigma}{4n(n-1)^2D^2(n)(1+t)}|||\mathring{A}|^n\phi_r^2\rho||_{\frac{n}{n-2}} \\
&+ \left(1 - \frac{\sigma}{4} - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2 - \frac{\sigma}{2n}|\xi|\sup_M|H|\right. \\
&\left.-\frac{4}{n}|\xi|\sup_M|H|\right)\int_M |\mathring{A}|^n\phi_r^2\rho \\
&+ \left(\frac{\sigma}{2n} - \frac{2}{n} - \frac{(n-2)^2\sigma}{4(n-1)^2n^3t}\right)\sup_M|H|^2\int_M |\mathring{A}|^n\phi_r^2\rho \\
&- \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho \\
&- \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |\mathring{A}|^n|\nabla\phi_r|^2\rho,
\end{aligned}
\tag{38}
$$

where $\eta, \sigma$ are positive constants such that

$$
2n\left(1 - \frac{\sigma}{4} - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2 - \frac{\sigma}{2n}|\xi|\sup_M|H| - \frac{4}{n}|\xi|\sup_M|H|\right) + (\sigma - 4)\sup_M|H|^2 > 0.
$$

Set

$$
\begin{aligned}
U &= U(n, |\xi|, \sup_M|H|, \sigma, \eta) \\
&= 2n\left(1 - \frac{\sigma}{4} - \frac{n-2}{\sqrt{n(n-1)}}\eta|\xi|^2 - \frac{\sigma}{2n}|\xi|\sup_M|H| - \frac{4}{n}|\xi|\sup_M|H|\right) \\
&\quad + (\sigma - 4)\sup_M|H|^2.
\end{aligned}
$$

We choose

$$
t = \frac{\sigma(n-2)^2\sup_M|H|^2}{2n^2(n-1)^2U},
$$

such that

$$
U - \frac{(n-2)^2\sigma}{2n^2(n-1)^2t}\sup_M|H|^2 = 0.
$$

Hence we get from (38) that

$$
\begin{aligned}
0 \geq\ & \frac{\sigma n(n-2)^2U}{2D^2(n)[\sigma(n-2)^2\sup_M|H|^2 + 2n^2(n-1)^2U]}|||\mathring{A}|^n\phi_r^2\rho||_{\frac{n}{n-2}} \\
&- \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right)\int_M |\mathring{A}|^{n+2}\phi_r^2\rho \\
&- \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right)\int_M |\mathring{A}|^n|\nabla\phi_r|^2\rho.
\end{aligned}
$$

By the Hölder inequality, we have

$$
\begin{aligned}
0 \geq \Bigg\{ & \frac{\sigma n(n-2)^2 U}{2D^2(n)[\sigma(n-2)^2 \sup_M |H|^2 + 2n^2(n-1)^2 U]} \\
& - \left(4 + \frac{n-2}{\sqrt{n(n-1)}\eta}\right) ||\mathring{A}||_n^2 \Bigg\} |||\mathring{A}|^n \phi_r^2 \rho||_{\frac{n}{n-2}} \\
& - \left(\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}\right) \int_M |\mathring{A}|^n |\nabla \phi_r|^2 \rho.
\end{aligned}
\tag{39}
$$

Set

$$
\begin{aligned}
D &= D(n, |\xi|, \sup_M |H|, \sigma, \eta) \\
&= \sqrt{\frac{\sigma \eta n(n-2)^2 \sqrt{n(n-1)} U}{2D^2(n)(4\eta\sqrt{n(n-1)} + n - 2)[\sigma(n-2)^2 \sup_M |H|^2 + 2n^2(n-1)^2 U]}}.
\end{aligned}
$$

We take

$$
\begin{aligned}
\sigma = \sigma_0 &:= \frac{n[\varsigma + (n-2)^2|\xi|^2 - (n-2)|\xi|\sqrt{\varsigma + (n-2)^2|\xi|^2}]}{\sqrt{2}(n-2)\sup_M |H| \sqrt{\tilde{\sigma}} + 2n(n-1)\tilde{\sigma}}, \\
\eta = \eta_0 &:= \frac{\sqrt{\varsigma + (n-2)^2|\xi|^2} - (n-2)|\xi|}{4\sqrt{n(n-1)}|\xi|},
\end{aligned}
$$

where $\tilde{\sigma} = \frac{n}{2} + |\xi| \sup_M |H| - \sup_M |H|^2 > 0$, $\varsigma = 4(n-1)(n - 4|\xi| \sup_M |H| - 2\sup_M |H|^2) > 0$. One has

$$
\begin{aligned}
0 &< \frac{n[\varsigma + (n-2)^2|\xi|^2 - (n-2)|\xi|\sqrt{\varsigma + (n-2)^2|\xi|^2}]}{\sqrt{2}(n-2)\sup_M |H| \sqrt{\tilde{\sigma}} + 2n(n-1)\tilde{\sigma}} \\
&< \frac{n\varsigma}{2n(n-1)\tilde{\sigma}} \\
&= \frac{4(n-1)(n - 4|\xi|\sup_M |H| - 2\sup_M |H|^2)}{2(n-1)(\frac{n}{2} + |\xi|\sup_M |H| - \sup_M |H|^2)} \\
&\leq 4 \times \frac{n - 4|\xi|\sup_M |H| - 2\sup_M |H|^2}{n + 2|\xi|\sup_M |H| - 2\sup_M |H|^2} \\
&\leq 4.
\end{aligned}
$$

Here the second inequality is strict since $\xi \neq 0$. Hence $\sigma \in (0, 4)$. We also have $U = U(n, |\xi|, \sup_M |H|, \sigma_0, \eta_0) > 0$. As in the proof of Theorem 1, $D$ achieves its maximum $D(n, |\xi|, \sup_M |H|)$ with

$$
\begin{aligned}
D(n, |\xi|, \sup_M |H|) &= \frac{\sqrt{n}(n-2)}{4\sqrt{n-1}D(n)} \\
&\times \frac{\sqrt{4(n-1)(n - 4|\xi|\sup_M |H| - 2\sup_M |H|^2) + (n-2)^2|\xi|^2} - (n-2)|\xi|}{(n-2)\sup_M |H| + \sqrt{2}n(n-1)\sqrt{\frac{n}{2} + |\xi|\sup_M |H| - \sup_M |H|^2}}.
\end{aligned}
$$

As in the proof of Theorem 1, $\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}$ has an upper bounded $E(n)$ that depends only on $n$. Since $|\nabla\phi_r| \le \frac{2}{r}$ and $\int_M |\mathring{A}|^n \mathrm{d}\mu < \infty$, we have

$$\lim_{r\to\infty} \int_M |\mathring{A}|^n |\nabla\phi_r|^2 \rho = 0.$$

Since $||\mathring{A}||_n < D(n, |\xi|, \sup_M |H|)$, then we get from (39)

$$0 \ge \left( D(n, |\xi|, \sup_M |H|)^2 - ||\mathring{A}||_n^2 \right) \lim_{r\to\infty} |||\mathring{A}|^n \phi_r^2 \rho||_{\frac{n}{n-2}} \ge 0.$$

Hence $|||\mathring{A}|^n e^{-\frac{|X|^2}{4}}||_{\frac{n}{n-2}} = 0$, which implies that $\mathring{A} = 0$. Therefore, $M$ is totally umbilical, i.e., $M$ is $\mathbb{S}^n(\sqrt{|\xi|^2 + 2n} - |\xi|)$ or $\mathbb{R}^n$. Since we have assumed that $\sup_M |H| < \sqrt{\frac{n}{2} + |\xi|^2} - |\xi|$, the first case is excluded. So, $M$ is $\mathbb{R}^n$ and $\xi = 0$, which is a contradiction to the assumption. This completes the proof of Theorem 2. □

## 4. Gap Theorems for $n = 2$

We need another Sobolev type inequality in dimension 2, which was proved by Xu and Gu in [31].

$$\tilde{c}^{-1} \left( \int_M f^4 \mathrm{d}\mu \right)^{\frac{1}{2}} \le \frac{1}{t} \int_M |\nabla f|^2 \mathrm{d}\mu + t \int_M f^2 \mathrm{d}\mu + \frac{1}{2} \int_M |H| f^2 \mathrm{d}\mu, \forall f \in C_c^\infty(M) \tag{40}$$

for all $t \in \mathbb{R}^+$, where $\tilde{c} = \frac{12\sqrt{3\pi}}{\pi}$. Now we give the proof of Theorem 3.

*Proof.* Set $f = |A|\rho^{\frac{1}{2}}\phi_r$. As in the proof of Theorem 1, for any $\tau, \varrho \in \mathbb{R}^+, \sigma \in (0, 4)$, we have

$$\begin{aligned} 0 \ge &\frac{\sigma}{2} \int_M |\nabla f|^2 + \frac{\sigma}{4} \int_M |A|^2 \phi_r^2 |H|^2 \rho - \frac{\sigma}{4} \int_M |A|^2 \phi_r^2 \langle \xi, H \rangle \rho \\ &+ \left(1 - |\xi|\tau - \frac{\sigma}{4}\right) \int_M |A|^2 \phi_r^2 \rho - \left(3 + \frac{|\xi|}{\tau}\right) \int_M |A|^4 \phi_r^2 \rho \\ &- \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right) \int_M |A|^2 |\nabla\phi_r|^2 \rho. \end{aligned} \tag{41}$$

Combining the Sobolev inequality (40) and (41), we obtain

$$\begin{aligned} 0 \ge &\frac{\sigma}{2} \left[ \frac{t}{\tilde{c}} \left( \int_M f^4 \right)^{\frac{1}{2}} - t^2 \int_M |A|^2 \phi_r^2 \rho - \frac{t}{2} \int_M |H||A|^2 \phi_r^2 \rho \right] \\ &+ \frac{\sigma}{4} \int_M |A|^2 \phi_r^2 |H|^2 \rho - \frac{\sigma}{4} \int_M |A|^2 \phi_r^2 \langle \xi, H \rangle \rho \\ &+ \left(1 - |\xi|\tau - \frac{\sigma}{4}\right) \int_M |A|^2 \phi_r^2 \rho - \left(3 + \frac{|\xi|}{\tau}\right) \int_M |A|^4 \phi_r^2 \rho \\ &- \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right) \int_M |A|^2 |\nabla\phi_r|^2 \rho. \end{aligned} \tag{42}$$

By using the Cauchy inequality, for any $\theta > 0$, we get from (42)

$$\begin{aligned}
0 \geq\ & \frac{\sigma t}{2\tilde{c}}\left(\int_M f^4\right)^{\frac{1}{2}} - \frac{\sigma t}{4}\int_M \left(\frac{\theta}{2}|H|^2 + \frac{1}{2\theta}\right)|A|^2\phi_r^2\rho \\
& + \frac{\sigma}{4}\int_M |A|^2\phi_r^2|H|^2\rho - \frac{\sigma}{4}\int_M |A|^2\phi_r^2\langle\xi, H\rangle\rho \\
& + \left(1 - |\xi|\tau - \frac{\sigma}{4} - \frac{\sigma t^2}{2}\right)\int_M |A|^2\phi_r^2\rho - \left(3 + \frac{|\xi|}{\tau}\right)\int_M |A|^4\phi_r^2\rho \\
& - \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^2|\nabla\phi_r|^2\rho \\
\geq\ & \frac{\sigma t}{2\tilde{c}}\left(\int_M f^4\right)^{\frac{1}{2}} + \left(\frac{\sigma}{4} - \frac{\theta\sigma t}{8}\right)\int_M |A|^2\phi_r^2|H|^2\rho \\
& + \left(1 - |\xi|\tau - \frac{\sigma}{4} - \frac{\sigma t}{8\theta} - \frac{\sigma t^2}{2}\right)\int_M |A|^2\phi_r^2\rho \\
& - \frac{\sigma}{4}\int_M |A|^2\phi_r^2\langle\xi, H\rangle\rho - \left(3 + \frac{|\xi|}{\tau}\right)\int_M |A|^4\phi_r^2\rho \\
& - \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^2|\nabla\phi_r|^2\rho.
\end{aligned}$$

We choose $t = \frac{2}{\theta}$, such that $\frac{\sigma}{4} = \frac{\theta\sigma t}{8}$. Hence

$$\begin{aligned}
0 \geq\ & \frac{\sigma}{\theta\tilde{c}}\left(\int_M f^4\right)^{\frac{1}{2}} + \left(1 - |\xi|\tau - \frac{\sigma}{4} - \frac{9\sigma}{4\theta^2}\right)\int_M |A|^2\phi_r^2\rho \\
& - \frac{\sigma}{4}\int_M |A|^2\phi_r^2\langle\xi, H\rangle\rho - \left(3 + \frac{|\xi|}{\tau}\right)\int_M |A|^4\phi_r^2\rho \\
& - \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^2|\nabla\phi_r|^2\rho.
\end{aligned} \tag{43}$$

On the other hand, for any $\omega > 0$, we have

$$\begin{aligned}
-\frac{\sigma}{4}\int_M |A|^2\phi_r^2\langle\xi, H\rangle\rho &\geq -\frac{\sigma}{4}\int_M |A|^2\phi_r^2|\xi||H|\rho \\
&\geq -\frac{\sigma|\xi|}{4}\int_M |A|^2\phi_r^2\left(\frac{1}{2\omega} + \frac{\omega}{2}|H|^2\right)\rho \\
&\geq -\frac{\sigma|\xi|}{8\omega}\int_M |A|^2\phi_r^2\rho - \frac{\sigma|\xi|\omega}{4}\int_M |A|^4\phi_r^2\rho.
\end{aligned} \tag{44}$$

Combining (43) and (44), we get

$$\begin{aligned}
0 \geq\ & \frac{\sigma}{\theta\tilde{c}}\left(\int_M f^4\right)^{\frac{1}{2}} + \left(1 - |\xi|\tau - \frac{\sigma}{4} - \frac{9\sigma}{4\theta^2} - \frac{\sigma|\xi|}{8\omega}\right)\int_M |A|^2\phi_r^2\rho \\
& - \left(3 + \frac{|\xi|}{\tau} + \frac{\sigma|\xi|\omega}{4}\right)\int_M |A|^4\phi_r^2\rho \\
& - \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^2|\nabla\phi_r|^2\rho.
\end{aligned}$$

Let $\tau$ satisfy $1 - |\xi\tau| > 0$. We take

$$\sigma = \frac{8\omega\theta^2(1-|\xi|\tau)}{2\omega\theta^2 + 18\omega + |\xi|\theta^2},$$

which satisfies

$$0 < \sigma < \frac{8\omega\theta^2}{2\omega\theta^2 + 18\omega + |\xi|\theta^2} < \frac{8\omega\theta^2}{2\omega\theta^2} = 4$$

since $\omega > 0$, such that

$$1 - |\xi|\tau - \frac{\sigma}{4} - \frac{9\sigma}{4\theta^2} - \frac{\sigma|\xi|}{8\omega} = 0.$$

Hence we obtain

$$\begin{aligned}
0 \geq\; & \frac{8\omega\theta(1-|\xi|\tau)}{\tilde{c}(2\omega\theta^2 + 18\omega + |\xi|\theta^2)}\left(\int_M f^4\right)^{\frac{1}{2}} \\
& - \left(3 + \frac{|\xi|}{\tau} + \frac{2\omega^2\theta^2|\xi|(1-|\xi|\tau)}{2\omega\theta^2 + 18\omega + |\xi|\theta^2}\right)\int_M |A|^4\phi_r^2\rho \\
& - \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^2|\nabla\phi_r|^2\rho \\
\geq\; & \left[\frac{8\omega\theta(1-|\xi|\tau)}{\tilde{c}(2\omega\theta^2 + 18\omega + |\xi|\theta^2)}\right. \\
& \left. - \left(3 + \frac{|\xi|}{\tau} + \frac{2\omega^2\theta^2|\xi|(1-|\xi|\tau)}{2\omega\theta^2 + 18\omega + |\xi|\theta^2}\right)\left(\int_M |A|^4\right)^{\frac{1}{2}}\right]\left(\int_M f^4\right)^{\frac{1}{2}} \\
& - \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right)\int_M |A|^2|\nabla\phi_r|^2\rho.
\end{aligned} \tag{45}$$

Set

$$K(|\xi|, \tau, \omega, \theta) = \frac{8\omega\theta(1-|\xi|\tau)}{\tilde{c}[(2\omega\theta^2 + 18\omega + |\xi|\theta^2)\left(3 + \frac{|\xi|}{\tau}\right) + 2\omega^2\theta^2|\xi|(1-|\xi|\tau)]}.$$

Similar as in the proof of Theorem 1, $K(|\xi|, \tau, \omega, \theta)$ achieves its maximum

$$K(|\xi|) = \frac{2}{3\tilde{c}(\sqrt{|\xi|^2+3} + |\xi|)}\sqrt{\frac{1}{(\sqrt{|\xi|^2+3} + |\xi|)(\sqrt{|\xi|^2+3} + (1+\sqrt{2})|\xi|)}},$$

when

$$\tau = \frac{\sqrt{|\xi|^2+3} - |\xi|}{3}, \omega = \frac{3}{\sqrt{2}(\sqrt{|\xi|^2+3} - |\xi|)} = \frac{1}{\sqrt{2}\tau},$$

$$\theta = \frac{3\sqrt{3}}{\sqrt{3 + \sqrt{2}|\xi|(\sqrt{|\xi|^2+3} - |\xi|)}}.$$

As in the proof of Theorem 1, $\frac{\sigma}{n}+\frac{4-\sigma}{2\varrho}$ is bounded from above. Since $|\nabla\phi_r|\le\frac{2}{r}$ and $\int_M|A|^2\mathrm{d}\mu<\infty$, we have

$$\lim_{r\to\infty}\int_M|A|^2|\nabla\phi_r|^2\rho=0.$$

Since $\left(\int_M|A|^4\mathrm{d}\mu\right)^{1/2}<K(|\xi|)$, we get from (45) that

$$0\ge\left[K(|\xi|)-\left(\int_M|A|^4\mathrm{d}\mu\right)^{\frac{1}{2}}\right]\lim_{r\to\infty}\left(\int_M|A|^4\phi_r^4\rho^2\right)^{\frac{1}{2}}\ge 0.$$

Hence $\int_M|A|^4\rho^2=0$, which implies that $|A|=0$. Hence $M$ is a linear subspace of $\mathbb{R}^{2+p}$. This completes the proof of Theorem 3. □

Using a similar argument, we give the proof of Theorem 4.

*Proof.* Set $f=|\mathring{A}|\rho^{\frac{1}{2}}\phi_r$. For $n=2$, we have

$$\begin{aligned}0\ge&\frac{\sigma}{2}\int_M|\nabla f|^2\mathrm{d}\mu+\left(\frac{\sigma}{4}-1\right)\int_M|\mathring{A}|^2\phi_r^2|H|^2\rho\\&+\left(1-\frac{\sigma}{4}-\frac{\sigma}{4}|\xi||H|-2|\xi||H|\right)\int_M|\mathring{A}|^2\phi_r^2\rho\\&-4\int_M|\mathring{A}|^4\phi_r^2\rho-\left(\frac{\sigma}{2}+\frac{4-\sigma}{2\varrho}\right)\int_M|\mathring{A}|^2|\nabla\phi_r|^2\rho.\end{aligned}\tag{46}$$

Combining the Sobolev inequality (40) and (46), we obtain

$$\begin{aligned}0\ge&\frac{t\sigma}{2\tilde{c}}\left(\int_M f^4\right)^{\frac{1}{2}}+\left(\frac{\sigma}{4}-1\right)\int_M|\mathring{A}|^2\phi_r^2|H|^2\rho\\&+\left(1-\frac{\sigma}{4}-\frac{t^2\sigma}{2}-\frac{t\sigma}{4}|H|-\frac{\sigma}{4}|\xi||H|-2|\xi||H|\right)\int_M|\mathring{A}|^2\phi_r^2\rho\\&-4\int_M|\mathring{A}|^4\phi_r^2\rho-\left(\frac{\sigma}{2}+\frac{4-\sigma}{2\varrho}\right)\int_M|\mathring{A}|^2|\nabla\phi_r|^2\rho.\end{aligned}\tag{47}$$

Since $|H|\le\sup_M|H|<\sqrt{|\xi|^2+1}-|\xi|$ by the assumption of Theorem 4 and $\sigma\in(0,4)$, (47) becomes

$$\begin{aligned}0\ge&\frac{t\sigma}{2\tilde{c}}\left(\int_M f^4\right)^{\frac{1}{2}}+\left(\frac{\sigma}{4}-1\right)\sup_M|H|^2\int_M|\mathring{A}|^2\phi_r^2\rho\\&+\left(1-\frac{\sigma}{4}-\frac{t^2\sigma}{2}-\frac{t\sigma}{4}\sup_M|H|-\frac{\sigma}{4}|\xi|\sup_M|H|-2|\xi|\sup_M|H|\right)\int_M|\mathring{A}|^2\phi_r^2\rho\\&-4\int_M|\mathring{A}|^4\phi_r^2\rho-\left(\frac{\sigma}{2}+\frac{4-\sigma}{2\varrho}\right)\int_M|\mathring{A}|^2|\nabla\phi_r|^2\rho.\end{aligned}\tag{48}$$

We take

$$\sigma=\frac{4-8|\xi|\sup_M|H|-4\sup_M|H|^2}{1+2t^2+t\sup_M|H|+|\xi|\sup_M|H|-\sup_M|H|^2},$$

which satisfies

$$0 < \sigma < \frac{4(1 - 2|\xi|\sup_M|H| - \sup_M|H|^2)}{1 + |\xi|\sup_M|H| - \sup_M|H|^2} \leq 4$$

since $t > 0$, such that

$$1-\frac{\sigma}{4}-\frac{t^2\sigma}{2}-\frac{t\sigma}{4}\sup_M|H|-\frac{\sigma}{4}|\xi|\sup_M|H|-2|\xi|\sup_M|H| = \left(1-\frac{\sigma}{4}\right)\sup_M|H|^2.$$

Hence we obtain from (48)

$$\begin{aligned} 0 \geq\ & \frac{2t(1 - 2|\xi|\sup_M|H| - \sup_M|H|^2)}{\tilde{c}(1 + 2t^2 + t\sup_M|H| + |\xi|\sup_M|H| - \sup_M|H|^2)}\left(\int_M f^4\right)^{\frac{1}{2}} \\ & - 4\int_M |\mathring{A}|^4\phi_r^2\rho - \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right)\int_M |\mathring{A}|^2|\nabla\phi_r|^2\rho \\ \geq & \left[\frac{2t(1 - 2|\xi|\sup_M|H| - \sup_M|H|^2)}{\tilde{c}(1 + 2t^2 + t\sup_M|H| + |\xi|\sup_M|H| - \sup_M|H|^2)}\right. \\ & \left. -4\left(\int_M |\mathring{A}|^4\right)^{\frac{1}{2}}\right]\left(\int_M f^4\right)^{\frac{1}{2}} - \left(\frac{\sigma}{2} + \frac{4-\sigma}{2\varrho}\right)\int_M |\mathring{A}|^2|\nabla\phi_r|^2\rho. \end{aligned} \tag{49}$$

Set

$$D(|\xi|, \sup_M|H|, t) = \frac{t(1 - 2|\xi|\sup_M|H| - \sup_M|H|^2)}{2\tilde{c}(1 + 2t^2 + t\sup_M|H| + |\xi|\sup_M|H| - \sup_M|H|^2)}.$$

We choose

$$t = \sqrt{\frac{1 + |\xi|\sup_M|H| - \sup_M|H|^2}{2}},$$

such that $D(|\xi|, \sup_M|H|, t)$ achieves its maximum $D(|\xi|, \sup_M|H|)$ with

$$D(|\xi|, \sup_M|H|) = \frac{1 - 2|\xi|\sup_M|H| - \sup_M|H|^2}{2\tilde{c}(2\sqrt{2}\sqrt{1 + |\xi|\sup_M|H| - \sup_M|H|^2} + \sup_M|H|)}.$$

As in the proof of Theorem 1, $\frac{\sigma}{n} + \frac{4-\sigma}{2\varrho}$ is bounded from above. Since $|\nabla\phi_r| \leq \frac{2}{r}$ and $\int_M |\mathring{A}|^2 \mathrm{d}\mu < \infty$, we have

$$\lim_{r\to\infty}\int_M |\mathring{A}|^2|\nabla\phi_r|^2\rho = 0.$$

Since $\left(\int_M |\mathring{A}|^4 \mathrm{d}\mu\right)^{1/2} < D(|\xi|, \sup_M|H|)$, then we get from (49) that

$$0 \geq \left[D(|\xi|, \sup_M|H|) - \left(\int_M |\mathring{A}|^4 \mathrm{d}\mu\right)^{\frac{1}{2}}\right]\lim_{r\to\infty}\left(\int_M |\mathring{A}|^4\phi_r^4\rho^2\right)^{\frac{1}{2}} \geq 0.$$

Hence $\int_M |\mathring{A}|^4\rho^2 = 0$, which implies that $\mathring{A} = 0$. Therefore, $M$ is totally umbilical, i.e., $M$ is $\mathbb{S}^2(\sqrt{|\xi|^2 + 4} - |\xi|)$ or $\mathbb{R}^2$. Since we have assumed that $\sup_M|H| < \sqrt{1 + |\xi|^2} - |\xi|$, the first case is excluded. This completes the proof of Theorem 4. □

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