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The Morse criticality revisited and some new applications to the Morse–Sard theorem

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Abstract. Given a C^r function $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$. Inspired by a recent result due to Moreira and Ruas (Manuscr Math 129:401–408, 2009), we show that for any $x \in U$, there exists a $\delta(x) > 0$, such that for all $y \in B(x, \delta(x)) \cap U$, it holds

$$|f(x) - f(y)| \leq C_1 |df(y)||x - y| + C_2 \sup_{0 \leq t \leq 1} |D^r f(x + t(y - x)) - D^r f(x)||x - y|^r,$$

where C_1, C_2 depends only on n, m. This inequality can be thought as a generalized Bochnak–Łojasiewicz inequality for smooth functions, it contains a "polynomial term" and a correction term from the finite differentiability. When $df(y) = 0$, the inequality improves the classical Morse criticality theorem, therefore, our approach unifies and simplifies various results on Morse criticality theorems, and leads to some streamlined proofs of the Morse–Sard type theorems. To showcase the wide applicability of our inequality, we provide two novel Morse–Sard type results. Define $\Sigma_f^\nu = \{x \in U \mid \text{rank}(df(x)) \leq \nu\}$. In the first place, if $f \in C^{k+(\alpha)}, k \geq 1, k \in \mathbb{N}, 0 < \alpha \leq 1$, ($C^{k+(\alpha)}$, Moreira's class), then the packing dimension $\dim_{\mathcal{P}} f(\Sigma_f^0) \leq \frac{n}{k+\alpha}$. Secondly, we consider $f \in W^{k+s,p}(U; \mathbf{R}^m), n > m, k \geq 1, sp > n, 0 < s \leq 1$, $W^{k+s,p}$ is the (possibly fractional) Sobolev space. We will show that, for $f \in W^{k+s,p}(U; \mathbf{R}^m)$, $\mathscr{L}^m(f(\Sigma_f^\nu)) = 0$ if $k + 1 \geq \frac{n-\nu}{m-\nu}, \nu = 0, 1, \ldots, m - 1$; for $f \in W^{k+s,p}, 0 < s < 1, \mathscr{L}^m(f(\Sigma_f^0)) = 0$, if $k + s \geq \frac{n}{m}$. To the best of our knowledge, it's the first result on the Morse–Sard theorem for fractional Sobolev spaces.

1. Introduction

The classical Morse–Sard theorem, stating that the set of critical values of a sufficiently smooth mapping has Lebesgue measure zero, plays a fundamental role in differential topology and dynamical systems. Especially, transversality theory makes use of the Morse–Sard theorem in an essential way and the theorem has wide applications in many other fields of mathematics.

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We recall the classical Morse–Sard theorem. Let $f : U \to \mathbf{R}^m$ be a C^k function, where U is an open subset of $\mathbf{R}^n$. A point $x \in U$ is a critical point of f if $\text{rank} Df(x) < \min\{n, m\}$, and $y \in U$ is a critical value of f if there is a critical point $x \in U$ with $f(x) = y$. Denote by $\mathscr{L}^m$ the m-dimensional Lebesgue measure.

The Morse–Sard Theorem. *Let $f : U \to \mathbf{R}^m$, where U is open in $\mathbf{R}^n$. If f is C^k where $k \geq \max\{1, n-m+1\}$, then the set of critical values of f has $\mathscr{L}^m$ -measure zero.*

Actually, if $n \leq m$, $f \in C^1$, the theorem is almost trivial. Therefore, the main concern is on the case $n > m$. Morse [15] first proved the theorem in this form when $m = 1$. That is, he showed that for $f : U \to \mathbf{R}, U \subset \mathbf{R}^n$ open, $n > 1$, $f \in C^k$, $k \geq n$, the set of critical values of f has Lebesgue measure zero in $\mathbf{R}$. Later on, Sard [18] extended the theorem to the case $m > 1$. Combined with an earlier famous construction by Whitney [21], we now know that the differentiability order $k = n - m + 1$ cannot be weakened.

There have appeared some new directions and new ideas since the pioneering works of Whitney, Morse, Sard. Nowadays it might be difficult to list all important contributions on the theorem, however, the reader can find an excellent brief survey on the history and backgrounds on the Morse–Sard theorem in De Pascale [6], or one can consult the insightful monograph by Yomdin and Comte [26], to find out some of the most significant contributions on the topic.

A key tool, underling many proofs of various versions of the Morse–Sard theorem, is the so-called Morse criticality theorem [15]. The theorem tells us the behavior of a C^k function near its critical set.

Theorem 1.1. (Morse criticality Lemma). *Let U be an open set of $\mathbf{R}^n$, $n \geq 1$. If $f : U \to \mathbf{R}$ is of class C^k, define $B := \{x \in U | df(x) = 0\}$. Then there exist sets $B_0, B_1, \ldots$, so that*

- $B = \bigcup_{i=0}^{\infty} B_i$;
- *B_0 is countable;*
- *for $i > 0$, the set B_i is bounded and has no isolated points;*
- *for every $x \in B_i$, it holds*

$$\lim_{\substack{y \to x \\ y \in B_i}} \frac{f(y) - f(x)}{|y - x|^k} = 0.$$

Morse used a clever double induction on k and n to show the lemma. Observe that for $k = 1$ or $h = 1$, the results are just consequences of calculus. In particular, if $n = 1$, x_0 is an non-isolated critical point of f, then $f'(x_0) = f''(x_0) = \cdots = f^{(k)}(x_0) = 0$. However, for $n \geq 2$, at an non-isolated critical point x_0, $D^2 f(x_0) \neq 0$ might happen. To overcome the difficulty, Morse tactically singled out a set $U^* \subset C_f$, which roughly is the critical set of df, ie, on U^*, $d^2 f = 0$, then one can apply the induction hypothesis on $df \in C^{k-1}$. On $C_f \backslash U^*$, the implicit function theorem can be used to reduce the dimension n.

Making an essential use of the equality $\lim_{\substack{y \to x \\ y \in B_i}} \frac{f(y)-f(x)}{|y-x|^k} = 0$, when $k \geq n$, one can deduce easily that $\mathscr{L}^1(f(B_i)) = 0$, hence the Morse–Sard theorem for $m = 1$

follows. Thereafter, analogous arguments combined with some clever applications of the implicit function theorem, led Sard [18] to obtain the corresponding statement in $m > 1$.

This has been an exciting area of research to extend the Morse–Sard theorem to other function spaces. In many of these works, the compelling Morse criticality argument (or Morse's decomposition) has been adapted and again played a vital role, typical examples include, e.g, those in Hölder spaces [17], in $C^{k+(\alpha)}$ class [5], in Sobolev spaces [6], etc.

In this paper we will pursue some further advances in the Morse–Sard type theorem, the main results here will be the natural consequences of a new inequality, which in turn can be viewed as a quantitative Morse criticality theorem.

The ideas presented here are motivated by a result of Moreira–Ruas [14], where some techniques from semi-algebraic geometry were used to obtain the following remarkable proposition.

Theorem 1.2. (Moreira–Ruas). *Given a C^k function $f : U \subset \mathbf{R}^n \to \mathbf{R}, k \geq 1$. Then we have*

$$\lim_{\substack{y \to x \\ y \in crit(f)}} \frac{f(y) - f(x)}{|y - x|^k} = 0 \quad \text{for all } x \in Crit(f)' \cap U. \tag{1.1}$$

where $Crit(f) = \{x \in U | df(x) = 0\}$, $Crit(f)'$ is the set of accumulation points of $Crit(f)$.

This formula replaces the Morse criticality lemma, and it leads Moreira and Ruas to a simple and elegant proof of the classical Morse–Sard theorem. More importantly, the methodology behind is compelling, and more flexible for other situations.

The idea is simple. Moreira and Ruas observed, using a classical inequality of Bochnak and Łojasiewicz from real algebraic geometry, that (1.1) can be derived from Taylor polynomial approximation to functions with finite smoothness.

In fact, this line of research has long been followed by Yomdin in a series of works [23–25], etc. In [23], tools form semi-algebraic geometry allows him to establish a so-called quantitative Morse–Sard theorem in Hölder spaces, that is, the sharp upper bounds of entropy dimension for critical sets are derived. The same strategy has been wonderfully used, by Yomdin and his collaborators, to deal with delicate problems from a wide range of pure and applied mathematics, including dynamical systems, singularity theory, complexity theory and robotics. A comprehensive introduction to this topic is [26].

Nevertheless, we want to emphasize that, it is Moreira and Ruas who first realized that the Morse criticality can be deduced in this way, we also remark that sometimes it is not an easy task to apply Yomdin's result directly. On the other hand, the Morse criticality usually provides elementary and direct treatments to these kinds of problems.

Continuing in this spirit, we will establish a new inequality in this paper, which plays the same role as the Morse criticality does in the classical Morse–Sard theorem. Our idea is similar to that by Moreira–Ruas. However, the quantitative nature of the inequality makes it very useful in various function spaces, and our work provides a unifying framework for Morse criticality theorem in these spaces.

Here is our starting point.

Theorem 1.3. *Let $f : U \to \mathbf{R}^m$, U open in $\mathbf{R}^n$, $f \in C^r, r \geq 1$. Then given any $x \in U$, there exists $\delta = \delta(x) > 0$, for all $y \in B(x, \delta(x)) \cap U$, we have*

$$|f(y) - f(x)| \leq C_1 |df(y)||y - x| + C_2 \sup_{0 \leq t \leq 1} |D^r f(x + t(y - x)) - D^r f(x)||y - x|^r, \tag{1.2}$$

where C_1, C_2 are constants depending only on n, m.

Notice that if f is a polynomial mapping, choosing r larger than the degrees of the components of f, (1.2) becomes the well known Bochnak–Łojasiewicz inequality in real algebraic geometry. While if $df(y) = 0$, (1.2) is similar to the estimates in Morse's criticality. Therefore, it is fair to call (1.2) "the Bochnak–Łojasiewicz-Morse inequality".

To showcase the utility of the above inequality, we will prove the following

Theorem 1.4. (Quantitative Morse–Sard theorem for $C^{k+(\alpha)}$). *Let $f : U \to \mathbf{R}^m$ be of $C^{k+(\alpha)}$, $0 < \alpha \leq 1$, here $C^{k+(\alpha)}$ denotes Moreira classes, see definitions below. Define $\Sigma_f^\nu = \{x \in U \mid rank(df(x)) \leq \nu\}$. Then*

$$\dim_{\mathcal{P}} f\left(\Sigma_f^0\right) \leq \frac{n}{k + \alpha}.$$

here $\dim_{\mathcal{P}} S$ is the packing dimension of a set $S \subset \mathbf{R}^m$ (the definition of packing dimension can be found in Sect. 3).

Moreira [5] defines $C^{k+(\alpha)}$ as follows: $f \in C^{k+(\alpha)}$ if $f \in C^k$, and for any $x \in U$, there exist $\delta_x > 0$, $C_x > 0$ such that $|D^k f(x) - D^k f(y)| \leq C_x |x - y|^\alpha$, for all $y \in U$, $|y - x| \leq \delta_x$. Trivially, $C^{k+(\alpha)}$ includes the usual locally Hölder (or Lipschitz) spaces $C^{k,\alpha}$, where C_x, δ_x can be chosen uniformly on compact subsets. While, it is easy to see that the inclusion is strict. For example, $C^{k+(s)}$ characterizes all $f \in C^k$ and $D^{k+1} f$ exists almost everywhere, which is the content of Stepanov's theorem. Consequently, $C^{k,1} \subsetneqq C^{k+(1)}$.

It is instructive to compare Theorem 1.4 with [5,23]. Yomdin showed, for $f \in C^{k,\alpha}$, that $\dim_e f(\Sigma_f^\nu) \leq \nu + \frac{n-\nu}{k+\alpha}$, $\dim_e$ is the entropy dimension. Moreira showed, for $f \in C^{k+(\alpha)}$, $\mathcal{H}^{\nu + \frac{n-\nu}{k+\alpha}}(f(\Sigma_f^\nu)) = 0$, which implies $\dim_{\mathcal{H}}(f(\Sigma_f^\nu)) \leq \nu + \frac{n-\nu}{k+\alpha}$, where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure, $\dim_{\mathcal{H}}$ the Hausdorff dimension.

As far as we know, Yomdin's original method cannot cover the $C^{k+(\alpha)}$ case, at least in some straightforward way. On the other hand, it is known [13] that, for a set $S \subset \mathbf{R}^m$, $\dim_{\mathcal{H}} S \leq \dim_{\mathcal{P}} S \leq \dim_e S$, and inequalities are strict generically. Therefore, in view of dimension estimates, our result is stronger that Moreira's. On the other hand, we do not get similar estimates for $\nu > 0$.

Lately, there has been some interest in the Morse–Sard theorem in Sobolev spaces, see [1,4,6,10], etc.

Our contribution in this respect is the following

Theorem 1.5. (Morse–Sard theorem for $W^{k+s,p}$). *Let $f : U \to \mathbf{R}^m$, U open in $\mathbf{R}^n$, $n > m$. If $f \in W^{k+s,p}, 0 < s \le 1, sp > n$, then for $\nu = 0, 1, \ldots, m-1$,*

- *for $s = 1, k+1 \ge \frac{n-\nu}{m-\nu}$, we have $\mathscr{L}^m(f(\Sigma_f^\nu)) = 0$;*
- *for $0 < s < 1, k+s \ge \frac{n}{m}$, we have $\mathscr{L}^m(f(\Sigma_f^0)) = 0$, where $\mathscr{L}^m$ denotes the m -dimensional Lebesgue measure.*

In the above, when $0 < s < 1$, $W^{k+s,p}$ is so-called fractional Sobolev space, its precise definition will be given in Sect. 4. For $s = 1$, $W^{k+s,p}$ is the usual Sobolev space, De Pascale [6] showed (see [10] for a short proof) that, for $f \in W^{k+1,p}$, $p > n$, if $k \ge n - m$, then $\mathscr{L}^m(f(\Sigma_f^{m-1})) = 0$. Hence, our result can be thought as a natural extension of De Pascale's work. The key step is again the Bochnak–Łojasiewicz–Morse inequality (1.2). To the best of our knowledge, there has been no previous work on Morse–Sard theorems in fractional Sobolev spaces.

The rest of the paper is organized as follows. In Sect. 2, we will prove the main inequality Theorem 1.3. Sections 3 and 4 are devoted to the proofs of Theorem 1.4, Theorem 1.5, respectively.

2. Bochnak–Łojasiewicz–Morse's inequality

Recall the celebrated Bochnak–Łojasiewicz inequality [3].

Lemma 2.1. (Bochnak–Łojasiewicz). *Let $p : U \subset \mathbf{R}^n \to \mathbf{R}$ be a polynomial. Given any $x \in U$, any $C > 1$, there exists a neighborhood $W \subset U$ of x, such that*

$$|p(x) - p(y)| \le C|x - y||dp(y)|, \text{ for any } y \in W.$$

Note that this inequality is trivial unless x is a critical point for p.

Now we turn to the proof of Theorem 1.3, with the help of Lemma 2.1.

Proof of Theorem 1.3. We first assume $m = 1$.

Let $f : U \to \mathbf{R}$ be of C^r, $r \ge 1$. Given any point $x \in U$. Define

$$p(y) = \sum_{|\alpha|=0}^{r} \frac{D^\alpha f(x)}{\alpha!}(y - x)^\alpha$$

i.e., $p(y)$ is the Taylor expansion of $f(y)$ around x, and $p(x) = f(x)$. By Taylor's formula with integral remainder term, one has, for y in some convex neighborhood of x,

$$\begin{aligned} f(y) - p(y) = \frac{1}{(r-1)!}\int_0^1 &(1-t)^{r-1}(D^r f(x + t(y-x)) \\ &- D^r f(x))dt \cdot (y-x)^r, \end{aligned} \tag{2.1}$$

here we use the notation for $w, v \in \mathbf{R}^n$,

$$D^r g(w) \cdot v^r = \sum_{|\beta|=r} \frac{D^\beta g(w)}{\beta!} v^\beta,$$

here and in what follows the multi-index notation will be used without further mention.

Define

$$R_r(y) = \sup_{0 \le t \le 1} |D^r f(x + t(y - x)) - D^r f(x)|$$

By (2.1), we get

$$|f(y) - p(y)| \le \frac{1}{r!} R_r(y)|x - y|^r. \tag{2.2}$$

Now we define

$$|df(y) - dp(y)| = \max_{1 \le i \le n} \left| \frac{\partial f}{\partial x_i}(y) - \frac{\partial p}{\partial x_i}(y) \right|.$$

Similarly,

$$|df(y) - dp(y)| \le \frac{1}{(r-1)!} R_r(y)|x - y|^{r-1}. \tag{2.3}$$

Now, by Lemma 2.1, for a fixed constant $C > 1$, say $C = 2$, one can choose $\delta = \delta(x) > 0$, so that for all $y \in B(x, \delta) \cap U$,

$$|p(y) - p(x)| \le C|dp(y)||y - x| \tag{2.4}$$

Therefore, by (2.2), (2.4), and (2.3), we get

$$\begin{aligned}
|f(y) - f(x)| &\le |p(y) - p(x)| + \frac{1}{r!} R_r(y)|x - y|^r \\
&\le C|y - x||dp(y)| + \frac{1}{r!} R_r(y)|x - y|^r \\
&\le C|y - x| \left(|df(y)| + \frac{1}{(r-1)!} R_r(y)|x - y|^{r-1} \right) \\
&\quad + \frac{1}{r!} R_r(y)|x - y|^r \\
&= C|y - x||df(y)| + \left(\frac{C}{(r-1)!} + \frac{1}{r!} \right) R_r(y)|x - y|^r \\
&\le C|y - x||df(y)| + (C + 1) R_r(y)|x - y|^r
\end{aligned}$$

Hence the inequality for $m = 1$.

In general, if $f : U \to \mathbf{R}^m$, $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$, we can apply the above $1 - d$ case to $f_i(x)$, $i = 1, 2, \ldots, m$, to obtain

$$|f(y) - f(x)| \le C_1 |df(y)||y - x| + C_2 R_r(y)|x - y|^r,$$

this is the desired Bochnak–Łojasiewicz-Morse inequality. The universal constants C_1, C_2 depend on the peculiar forms of the norms in $\mathbf{R}^n, \mathbf{R}^m$, so one can think C_1, C_2 rely on n, m only. The precise estimates on C_1, C_2 are irrelevant to our discussions below. □

Now we explain why the inequality can be viewed as a replacement of Morse's criticality theorem.

Let $f : U \to \mathbf{R}^m$ be of C^r, $r \geq 1$. Define $B = \{x \in U | df(x) = 0\}$.

Corollary 2.1. (Morse's Criticality). *If* $x \in B$, *there exists* $\delta = \delta(x)$*, for all* $y \in B$, $|y - x| \leq \delta(x)$*, we have*

$$|f(y) - f(x)| \leq C \sup_{0 \leq t \leq 1} |D^r f(x + t(y - x)) - D^r f(x)||y - x|^r, \quad (2.5)$$

where $C = C(n, m) > 0$.

Since $D^r f$ is continuous, Moreira–Ruas's result Theorem 1.2 follows trivially from Corollary 2.1.

Remark. (2.5) trivially holds if x is an isolated critical point. Hence, in the following one does not need to treat isolated critical points separately.

It turns out that Corollary 2.1, or inequality (2.5) will unify some previous results on the refined Morse criticality theorem.

Recall that $f \in C^{k,\alpha}(U; \mathbf{R}^m)$, if $f : U \to \mathbf{R}^m$ belongs to C^k, and for any compact $K \subset U$, there exists $C_k > 0$, so that $|D^k f(x) - D^k f(y)| \leq C_K |x - y|^\alpha$, here $0 < \alpha \leq 1$. The $C^{k,\alpha}$ is the local Hölder (or Lipschitz, if $\alpha = 1$) space. The Morse–Sard theorem in Hölder spaces was initiated by A. Norton [17] in 1986, as the natural spaces to answer a question of Whitney. The final form of the extended Morse–Sard theorem in Hölder spaces was obtained only some years later, by S. M. Bates [2]. The key in these works is a refined version of the Morse criticality theorem first observed by Norton.

Following Morse's original argument, Norton noticed that, for $B = \{x \in U | df(x) = 0\}$, $f \in C^{k,\alpha}$, there exists a decomposition $B = B_0 \cup \cup_{i=1}^{\infty} B_i$, B_0 the set of all isolated points in B; on B_i, there holds

$$|f(x) - f(y)| \leq C_i |x - y|^{k+\alpha}, \text{ for all } x, y \in B_i.$$

Now using Corollary 2.1, one actually has

Corollary 2.2. *(*Morse criticality for $C^{k,\alpha}$*). For any* $x \in B \cap K$*,* K *any compact subset of* U*, there exists* $C_K > 0$*, such that*

$$|f(x) - f(y)| \leq C_K |x - y|^{k+\alpha}, \textit{for all } y \in B \cap K, |y - x| \leq \delta(x). \quad (2.6)$$

where $f \in C^{k,\alpha}, 0 < \alpha \leq 1$.

Regardless of some slight differences between (2.6) and Norton's version of the Morse criticality for $C^{k,\alpha}$, however, (2.6) suffices to recover the Morse–Sard theorem for $C^{k,\alpha}$, originally showed by Bates [2]. Anyway, we will turn back to the issue in Sect. 3.

Now we recall that Moreira's classes $C^{k+(\alpha)}$ are defined as: $f : U \to \mathbf{R}$, $f \in C^{k+(\alpha)}(U)$ if $f \in C^k(U)$, and for any $x \in U$, there exists $C_x, \delta_x > 0$, such that for any $y \in B(x, \delta_x) \cap U$, it holds $|D^k f(x) - D^k f(y)| \leq C_x |x - y|^\alpha$. Naturally, for a mapping $f : U \to \mathbf{R}^m$, $f \in C^{k+(\alpha)}$ means that $f_i \in C^{k+(\alpha)}(U)$, where $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$, $i = 1, 2, \ldots, m$.

The following implication of Corollary 2.1 is now easy.

Corollary 2.3. *(*Morse criticality for $C^{k+(\alpha)}$*). Let $f : U \to \mathbf{R}^m$ be of $C^{k+(\alpha)}$. Define $B = \{x \in U | df(x) = 0\}$. For any $x \in B$, there exist two constants $C_x, \delta_x > 0$, so that if $y \in B \cap B(x, \delta_x)$, we have*

$$|f(x) - f(y)| \le C_x |x - y|^{k+\alpha}. \tag{2.7}$$

Again, estimates like (2.7) has appeared in Moreira's works on the Morse–Sard theorem for $C^{k+(\alpha)}$, and the derivations of such estimates are quite complicated, though basic ideas followed the original idea of Morse.

In Sect. 3 we will use (2.7) as a crucial step to get some new quantitative Morse–Sard type theorem for $C^{k+(\alpha)}$.

As the last application of Bochnak–Łojasiewicz-Morse's inequality (or Corollary 2.1), we state the corresponding Criticality theorem for $W^{k+s,p}(U; \mathbf{R}^m)$, $0 < s \le 1$, $sp > n$. The definition of fractional Sobolev spaces will be postponed to Sect. 4.

Corollary 2.4. *(*Morse criticality for $W^{k+s,p}$*). Let $f : U \to \mathbf{R}^m$ be of $W^{k+s,p}$, $k \ge 1$, $0 < s \le 1$, and $sp > n$. Define $B = \{x \in U | df(x) = 0\}$. Then for any $x \in B$, there exists $\delta_x > 0$, if $y \in B \cap B(x, \delta_x)$, we have, for $0 < s < 1$*

$$|f(x) - f(y)| \le C|x - y|^{k+s-\frac{n}{p}} \left(\int_{B(x,r)} |D^k f(z)| dz + \int_{B(x,r)} \int_{B(x,r)} \frac{|D^k f(u) - D^k f(v)|^p}{|u - v|^{n+sp}} du dv \right)^{\frac{1}{p}}. \tag{2.8}$$

and for $s = 1$,

$$|f(x) - f(y)| \le C|x - y|^{k+1-\frac{n}{p}} \left(\int_{B(x,r)} |D^{k+1} f(z)|^p dz \right)^{\frac{1}{p}}. \tag{2.9}$$

where $r = |x - y|$, C is a constant depending on n, m only.

The inequality (2.8) for $W^{k+s,p}$ $(0 < s < 1)$ is something new while (2.9) has appeared in [6,10], where (2.9) is derived for $x, y \in \widetilde{B} = \{x \in U | D^\alpha f = 0$, for all $0 < |\alpha| \le k\}$. Note that $\widetilde{B}$ is just a small part of B.

Our presentation somehow is more transparent and conceptually simpler. We shall discuss Sobolev spaces in more details in Sect. 4, there Corollary 2.4 will be proved, some new Morse–Sard theorems in these spaces then follow.

It is very interesting to note that Corollaries 2.2, 2.3, 2.4 all follow from the main inequality in an uniform pattern: by replacing the oscillations of higher derivatives with suitable forms, one can get very precise information on oscillations of the functions on sets where all first derivatives vanish. In previous results [5,6,17] similar but weaker criticality theorems are obtained case by case, proofs usually are more difficult, and sometimes the key points needed in the Morse–Sard type theorems are hidden.

We believe that the Bochnak–Łojasiewicz-Morse inequality will be applicable to many other problems. Instead of Bochnak–Łojasiewicz's inequality, if one uses Łojasiewicz's inequality $|p(y) - p(x)| \le C|dp(y)|^\beta$, $\beta > 1$, one can get the corresponding Łojasiewicz-Morse's inequality. However, the inequality makes no difference with (1.2) on points where first derivatives vanish.

3. New Morse–Sard type theorems for $C^{k+(\alpha)}$

In this section, we always assume U is a bounded open subset of $\mathbf{R}^n$. One of the main results in this section is Theorem 1.4. Before proving the theorem, we first define some standard terminology, which can be found in, say [13].

Let $f : U \to \mathbf{R}^m$ be of $C^{k+(\alpha)}$, the class of Moreira that has been explained before, where $k \geq 1, 0 < \alpha \leq 1$. Recall for a bounded set $S \subset \mathbf{R}^n$, we can define $M(\varepsilon, S)$ the minimal number of closed balls of radius ε in $\mathbf{R}^n$ covering S. The entropy dimension (or the Minkowski dimension, upper boxing counting dimension) of S is defined as

$$\dim_e S = \overline{\lim_{\varepsilon\to 0}} \frac{\log M(\varepsilon, S)}{-\log \varepsilon}$$

The packing dimension of S is then defined as:

$$\dim_{\mathcal{P}} S = \inf \left\{ \sup_{1\leq i\leq\infty} \dim_e A_i, S = \bigcup_{i=1}^{\infty} A_i \right\},$$

where the infimum is taken over all A_i's such that $\bigcup_{i=1}^{\infty} A_i = S$. It's well known that $\dim_{\mathcal{H}} S \leq \dim_{\mathcal{P}} S \leq \dim_e S$, where $\dim_{\mathcal{H}} S$ is the Hausdorff dimension of S, and generically these inequalities can be strict.

It turns out that Corollary 2.3 will be the key to our proof.

Proof of Theorem 1.4. Denote $B = \Sigma_f^0 = \{x \in U | df(x) = 0\}$. The Morse criticality Corollary 2.3 tells us, for any $x \in B$, there exist $C_x, \delta_x > 0$, so that if $y \in B \cap B(x, \delta_x)$, we have

$$|f(x) - f(y)| \leq C_x |x - y|^{k+\alpha}. \tag{3.1}$$

The lack of uniformity, i.e., the dependence of C or δ on x, of course will cause some trouble, as usual in analysis. Our idea is to decompose B into a countable union of sets, on each set, the constants are uniform.

By (3.1), the decomposition can be done easily as $B = \bigcup_{N=1}^{\infty} B_N$, where

$$B_N = \left\{ x \in B |\text{ if } y \in B, |y - x| < \frac{1}{N}, \text{ then } |f(y) - f(x)| \leq N|y - x|^{k+\alpha} \right\}$$

We note that each B_N is a closed set, and it actually holds that for any $x, y \in B_N$, if $|y - x| < \frac{1}{N}$, then $|f(y) - f(x)| \leq N|y - x|^{k+\alpha}$.

Now we make an obvious observation: for all $x, y \in B_N$, one has

$$|f(x) - f(y)| \leq C_N |x - y|^{k+\alpha}, \tag{3.2}$$

C_N a constant independent of x, y.

The assertion is straightforward: for $x, y \in B_N$, $|x - y| < \frac{1}{N}$, one takes $C = N$ in (3.2); for $|x - y| \geq \frac{1}{N}$, one has $|f(x) - f(y)| \leq 2M = 2MN^{k+\alpha}(\frac{1}{N})^{k+\alpha} \leq 2MN^{k+\alpha}|x - y|^{k+\alpha}$, here $M = \sup_{x\in U} |f(x)|$. Therefore, in general, one can choose $C_N = \max(N, 2MN^{k+\alpha})$.

In summary, on the compact set B_N, we have the estimate: there exists a $C_N > 0$, such that $|f(x) - f(y)| \leq C_N|x-y|^{k+\alpha}$ for all $x, y \in B_N$. Now employing Whitney's extension theorem [22] (see also [11,19]), there is an $F : \mathbf{R}^n \to \mathbf{R}^m$, $F \in C^{k,\alpha}$, and $F|_{B_N} = f|_{B_N}$, $D^\alpha F(x) = 0$ for $0 < |\alpha| \leq k$, $x \in B_N$. Therefore $f(B_N) = F(B_N)$, $B_N \subset \{x \in \mathbf{R}^n | dF(x) = 0\}$. Using Yomdin's result ([23] or [26, p. 113]) treating F as a $C^{k,\alpha}$ mapping defined on some ball $B_r(0) \supset U$, $\dim_e f(B_N) = \dim_e F(B_N) \leq \frac{n}{k+\alpha}$.

On the other hand, by definition, since $f(B) = \bigcup_{N=1}^\infty f(B_N)$, so $\dim_{\mathcal{P}} f(B) \leq \sup\limits_N \dim_e f(B_N) \leq \frac{n}{k+\alpha}$. □

We now make a few remarks to clarify some issues.

Remark 1. We shall remark that our result $\dim_{\mathcal{P}} f(\Sigma_f^0) \leq \frac{n}{k+\alpha}$ is stronger than both the statement $\dim_{\mathcal{H}} f(\Sigma_f^0) \leq \frac{n}{k+\alpha}$ and Moreira's result $\mathcal{H}^{\frac{n}{k+\alpha}}(f(\Sigma_f^0)) = 0$. In fact, for any $a, b \in \mathbf{R}$, $0 < a < b < 1$, there exists a Cantor-like set $S \subset [0,1]$, such that $\dim_{\mathcal{H}} S = a$, $\dim_{\mathcal{P}} S = b$, see [12]. Hence, if $a < \frac{n}{k+\alpha} < b$, our results shows that S cannot be the critical set for any $f \in C^{k+(\alpha)} : \mathbf{R}^n \to \mathbf{R}$, while it's impossible to rule out the possibility by Hausdorff estimates.

2. The packing dimension estimate $\dim_{\mathcal{P}} f(\Sigma_f^0) \leq \frac{n}{k+\alpha}$ is sharp in the following sense. Yomdin and Comte [26, pp. 113–114] construct a sequence of functions $f_i : U \to \mathbf{R}$, with $\dim_{\mathcal{H}} f_i(\Sigma_{f_i}^0) = \dim_e f_i(\Sigma_{f_i}^0) \underset{i\to\infty}{\to} \frac{n}{k+\alpha}$. Realizing that $C^{k,\alpha} \subset C^{k+(\alpha)}$ and $\dim_{\mathcal{H}} \leq \dim_{\mathcal{P}} \leq \dim_e$, we have $\dim_{\mathcal{P}} f_i(\Sigma_{f_i}^0) \to \frac{n}{k+\alpha}$ as $i \to \infty$, where $f_i \in C^{k+(\alpha)}$. Hence, for any $r < \frac{n}{k+\alpha}$, $\dim_{\mathcal{P}} f(\Sigma_f^0) \leq r$ cannot hold for all $f \in C^{k+(\alpha)}$.

3. It would be very nice to get a better estimate $\dim_e f(\Sigma_f^0) \leq \frac{n}{k+\alpha}$ for $f \in C^{k+(\alpha)}$. Anyway, we don't know how to extend Yomdin's powerful arguments for $C^{k,\alpha}$ to the more local case $C^{k+(\alpha)}$. It might be plausible to deduce such a result by combining Yomdin's arguments with the arguments used above.

The following theorem in the spirit is more similar to the classical Morse–Sard theorem concerning the Lebesgue measure.

Theorem 3.1. *Let $f : U \to \mathbf{R}^m$ be of $C^{k+(\alpha)}$, $k \geq 1$, $0 < \alpha \leq 1$, and $n > m$. If $k + \alpha \geq \frac{n-\nu}{m-\nu}$, $\nu = 0, 1, 2, \ldots, m-1$, then*

$$\mathscr{L}^m(f(\Sigma_f^\nu)) = 0.$$

Proof. It is sufficient to consider the $\nu = 0$ case, the general case can be reduced to $\nu = 0$ by a standard argument, see [2].

Recall that we have decomposed $B = \{x \in U | df(x) = 0\} = \Sigma_f^0$ as $B = \bigcup_{N=1}^\infty B_N$, and on B_N there exists $C_N > 0$ such that for all $x, y \in B_N$, it holds

$$\forall x, y \in B_N, |f(x) - f(y)| \leq C_N|x-y|^{k+\alpha}$$

Now we employ the following lemma, which is explicitly proved in [2] , without stating it as a lemma. According to [8], Ferry has already found the lemma in 1976 [9]. We state a simple version of the lemma, more general version on metric spaces can be found in [8].

Lemma 3.2. (Ferry's lemma). *Let $f : E \to \mathbf{R}^m$, $E \subset \mathbf{R}^n$. Suppose that there are p and M such that*

$$\forall x, y \in E, |f(x) - f(y)| \leq M|x - y|^p.$$

If $p > 1$, then the $\frac{n}{p}$-dimensional Hausdorff measure of $f(E)$ is 0.

Applying Ferry's lemma with $E = B_N$, $p = k + \alpha$, it leads to the conclusion $\mathcal{H}^{\frac{n}{k+\alpha}}(f(B_N)) = 0$. Therefore, $\mathcal{H}^{\frac{n}{k+\alpha}}(f(B)) \leq \sum_{N=1}^{\infty} \mathcal{H}^{\frac{n}{k+\alpha}}(f(B_N)) = 0$.

Suppose $k + \alpha \geq \frac{n}{m}$. Then $\frac{n}{k+\alpha} \leq m$, which implies $\mathcal{H}^m(f(B)) = \mathscr{L}^m(f(B)) = 0$. Hence the statement is proved in $\nu = 0$ case.

As we said before, $\nu > 0$ can be reduced to $\nu = 0$, as done in [2,18]. Here one needs to verify the validity of the inverse function theorem in $C^{k+(\alpha)}$, but this fact is obvious, by following the argument in the appendix of [17].

Remark 3.3. Moreira's theorem [5] states that, if $f : U \to \mathbf{R}^m$ be of $C^{k+(\alpha)}$, then $\mathcal{H}^{\frac{n-\nu}{k+\alpha}+\nu}(f(\Sigma_f^\nu)) = 0$. In particular, it implies that if $k+\alpha \geq \frac{n-\nu}{m-\alpha}$, $\mathscr{L}^m(f(\Sigma_f^\nu)) = 0$. Therefore, our theorem is a consequence of Moreira's theorem. However, we notice that our argument is more elementary and much simpler. The reason of such simplicity is, on one hand, our Morse's criticality Theorem Corollary 2.3 provides a shortcut, and on the other hand, we only proved a weaker statement, in this case Fubini's theorem holds when dealing with $\mathscr{L}^m$.

It's well known that Fubini's theorem does not hold anymore for non-integral Hausdorff measures, Moreira's contribution is to overcome this obstacle concerning the use of Fubini's theorem, by highly technical arguments.

Remark 3.4. Since $C^{k,\alpha} \subset C^{k+(\alpha)}$, Theorem 3.1 implies that, if $f : U \to \mathbf{R}^m$ be of $C^{k,\alpha}$, $k + \alpha \geq \frac{n-\nu}{m-\alpha}$, then $\mathscr{L}^m(f(\Sigma_f^\nu)) = 0$. This result is due to Bates [2]. Hence, the condition $k+\alpha \geq \frac{n-\nu}{m-\alpha}$ is optimal to guarantee $\mathscr{L}^m(f(\Sigma_f^\nu)) = 0$. For if $k+\alpha < \frac{n-\nu}{m-\alpha}$, Bates constructs some $f \in C^{k,\alpha} \subset C^{k+(\alpha)}$, with $\mathscr{L}^m(f(\Sigma_f^\nu)) > 0$.

4. The Morse–Sard theorem in fractional Sobolev spaces

This section is divided into three subsections. The first one is on the Morse criticality in Sobolev spaces. The rest two subsections are on the Morse–Sard theorems in the corresponding spaces. Among these results, the theorems about fractional Sobolev spaces are new.

4.1. *The Morse criticality for $W^{k+s,p}$*

We will prove Corollary 2.4 in the subsection, and these Morse's criticality theorems will be used as key tools in next subsections.

The definitions below are quite standard, they can be found in classical books [7,27]. For fractional Sobolev spaces, a short and very helpful introduction is [16].

Let $f : U \to \mathbf{R}^m$, where U is a bounded open set in $\mathbf{R}^n$. We say $f \in W^{k+1,p}(U)$, where $k \geq 1$, $p \in [1, +\infty)$, if there exist all the distributional derivations of f with order up to $k+1$, and for all multi-index α, $|\alpha| \leq k+1$, we have $D^\alpha f \in L^p(U)$. If f is a mapping from U to $\mathbf{R}^m$, i.e. $f : U \to \mathbf{R}^m$, we say $f \in W^{k+1,p}(U; \mathbf{R}^m)$ (or simply $W^{k+1,p}(U)$, or $W^{k+1,p}$ without confusion), if all components of f belongs to $W^{k+1,p}(U)$, i.e., if $f(x) = (f_1(x), \ldots, f_m(x))$, $f_i \in W^{k+1,p}(U)$, $i = 1, \ldots, m$.

Now we switch from 1 to $s \in (0, 1)$. Fix $s \in (0, 1)$. For any $p \in [1, +\infty)$, we define $W^{s,p}(U)$ as follows:

$$W^{s,p}(U) = \left\{ f \in L^p(U) : \frac{|f(x) - f(y)|}{|x - y|^{\frac{n}{p}+s}} \in L^p(U \times U) \right\};$$

i.e., an intermediate Banach space between $L^p(U)$ and $W^{1,p}(U)$, endowed with the natural norm:

$$\|f\|_{W^{s,p}(U)} = \left(\int_U |f|^p dx + \int_U \int_U \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}},$$

where the term

$$[f]_{W^{s,p}(U)} := \left(\int_U \int_U \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}$$

is the so-called Gagliardo seminorm of f. Then $W^{k+s,p}(U)$ is defined as

$$W^{k+s,p}(U) = \left\{ f \in W^{k,p}(U) : D^\alpha f \in W^{s,p}(U) \text{ for any } \alpha \in \mathbb{N} \text{ with } |\alpha| = k \right\}.$$

and this is a Banach space with respect to the norm

$$\|f\|_{W^{k+s,p}(U)} := \left(\|f\|^p_{W^{k,p}(U)} + \sum_{|\alpha|=k} \left\| D^\alpha f \right\|^p_{W^{s,p}} \right)^{\frac{1}{p}}.$$

Similarly, if f is a mapping, $f \in W^{k+s,p}(U; \mathbf{R}^m)$ if all components of f belong to $W^{k+s,p}(U)$.

We first deal with the more familiar $s = 1$ case. Recall the well-known Morrey's inequality, which says that if $p > n$, then $W^{k+1,p}(U) \subset C^{k,1-\frac{n}{p}}(U)$. That means that in each Lebesgue equivalent class of $W^{k+1,p}$, one can find a continuous representative. Thereafter, when we talk about $f \in W^{k+1,p}$, we always think f is continuous. The Hö lder regularity is from Morrey's lemma [7]: If $f \in W^{k+1,p}(U; \mathbf{R}^m)$, $k \geq 1$, $p > n$, then it holds: for $x, y \in U$,

$$|D^k f(x) - D^k f(y)| \leq C|x - y|^{1-\frac{n}{p}} \left(\int_{B(x,r)} |D^{k+1} f(z)|^p dz \right)^{\frac{1}{p}}, \quad (4.1.1)$$

where $C = C(n, m, p)$ is a universal constant, $r = |x - y|$. Note that in the above we apply Morrey's inequality on $D^k f$, which is on $W^{1,p}$.

Now one can prove the inequality (2.9) in Corollary 2.4. From Corollary 2.1, if $f : U \to \mathbf{R}^m$ be of $W^{k+1,p} \subset C^{k,1-\frac{n}{p}}$ and $x \in B = \{x \in U | df(x) = 0\}$, there exists a $\delta(x) > 0$, for all $y \in B \cap B(x, \delta(x))$, one has

$$\begin{aligned} |f(x) - f(y)| &\le C \sup_{0\le t\le 1} |D^k f(x + t(y - x)) - D^k f(x)||x - y|^k \\ &\le C \sup_{0\le t\le 1} \left\{ (t|x - y|)^{1-\frac{n}{p}} \left(\int_{B(x,t|x-y|)} |D^{k+1} f(z)|^p dz \right)^{\frac{1}{p}} \right\} \\ &\quad |x - y|^k \text{ (by (4.1.1))} \\ &\le C|x - y|^{k+1-\frac{n}{p}} \left(\int_{B(x,|x-y|)} |D^{k+1} f(z)|^p dz \right)^{\frac{1}{p}}, \end{aligned}$$

where $C = C(n, m, p)$ is a constant independent of x. Hence, the inequality (2.9) is derived.

For fractional Sobolev spaces $W^{k+s,p}(U; \mathbf{R}^m)$, $0 < s < 1$, if $sp > n$, an analogous embedding theorem also holds: $W^{k+s,p} \subset C^{k,s-\frac{n}{p}}$.

From [16] formula (8.8) and (8.4), we know if $f \in W^{k+s,p}$, $sp > n$, then for all $x, y \in U$

$$\begin{aligned} |D^k f(x) - D^k f(y)| &\le C[D^k f]_{p,sp} |x - y|^{s-\frac{n}{p}} \\ &\le C \left\| D^k f \right\|_{W^{s,p}(B(x,r))} |x - y|^{s-\frac{n}{p}} \end{aligned} \tag{4.1.2}$$

where C is a constant independent of f and x, $r = |x - y|$, and

$$[g]_{p,sp} := \left(\sup_{\substack{x_0\in U \\ \rho>0}} \rho^{-sp} \int_{B(x_0,\rho)\cap U} |f(x) - \langle f \rangle_{B(x_0,\rho)\cap U}|^p dx \right)^{\frac{1}{p}}.$$

and

$$\langle f \rangle_{B(x_0,\rho)\cap U} := \frac{1}{|B(x_0, \rho) \cap U|} \int_{B(x_0,\rho)\cap U} f(x) dx.$$

Employing Corollary 2.1 again, if $f : U \to \mathbf{R}^m$ be of $W^{k+s,p}$, $p \in [1, +\infty)$, $sp > n$, then for any $x \in B := \{x \in U | df(x) = 0\}$, there exists $\delta(x) > 0$, for all $y \in B \cap B(x, \delta(x))$, we have

$$\begin{aligned} |f(x) - f(y)| &\le C \sup_{0\le t\le 1} |D^k f(x + t(y - x)) - D^k f(x)||x - y|^k \\ &\le C \left\| D^k f \right\|_{W^{s,p}(B(x,r))} |x - y|^{k+s-\frac{n}{p}}, \text{ (by (4.1.2))} \end{aligned}$$

which is (2.8).

Remark. Usually the validity of the embedding $W^{k+s,p}(U) \subset C^{k,s-\frac{n}{p}}$ requires U to be an extension domain of $W^{s,p}$. However, since in the above argument we actually treat f as a mapping defined on balls $B(x, \delta(x))$, we do not need further requirement on U, except U is open.

4.2. The Morse–Sard theorem in $W^{k+1,p}$

We will treat two cases $s = 1$ and $s \in (0, 1)$ separately. As we shall see, a unified proof can be given in both cases. However, in integral Sobolev spaces, one can apply a beautiful trick due to De Pascale [6], to check the so-called N_0-property for Sobolev functions. Hence in this case a much shorter argument can be given.

Theorem 4.2.1. *Let $f : U \to \mathbf{R}^m$, U open in $\mathbf{R}^n, n > m$. Suppose $f \in W^{k+1,p}, k \geq 1, p > n$. Define $\Sigma_f^\nu = \{x \in U| \ \ rankdf(x) \leq \nu\}, \nu = 0, 1, \dots, m-1$. If $k+1 \geq \frac{n-\nu}{m-\nu}$, then $\mathscr{L}^m(f(\Sigma_f^\nu)) = 0$.*

Some remarks are in order.

Remark 4.2.2. As mentioned in [6], the embedding $W^{k+1,p} \subset C^{k,1-\frac{n}{p}}$ itself is not powerful enough to get our conclusion, this is the effect of the existence of another weak derivative summable enough.

Remark 4.2.3. When $\nu = m-1$, our theorem recovers the result by De Pascale. However, we shall note that, using De Pascale or Figalli's methods, one cannot get the corresponding statements for general $\nu \neq m-1$. De Pascale's argument depends on his Sobolev version of Morse's criticality or decomposition theorem, which is too complicated to treat the general cases. On the other hand, Figalli [10] found an alternative simple proof of De Pascale's theorem, based on the Kneser–Glaeser rough composition theorem. The key step in his argument is to use the implicit function theorem for the dimension reduction. Unfortunately, dimension reduction works only for $\nu = m-1$ case. The power of our argument relies on the special form of Morse's criticality Corollary 2.4, which holds on the whole set Σ_f^0.

Remark 4.2.4. More general statements concerning Hausdorff measures of critical values have been obtained quite recently, in [1,4] for example. These authors borrowed heavy tools from geometric measure theory to obtain more precise results on Sard-type properties. Compared with the heavy machinery they used, our methods are much more elementary, though our results are weaker.

Proof of Theorem 4.2.1. It suffices to treat $\nu = 0$ case, i.e., we will show if $k+1 \geq \frac{n}{m}$, then $\mathscr{L}^m(f(\Sigma_f^0)) = 0$. The $0 < \nu \leq m-1$ cases can be reduced to this case by the implicit function theorem, and the reduction is now standard, see Figalli [10, p. 3677].

De Pascale observed that, it is sufficient to show the so-called N_0 property: $E \subset \Sigma_f^0, \mathcal{H}^n(E) = 0 \Rightarrow \mathcal{H}^m(f(E)) = 0$.

For the sake of completeness, we now explain De Pascale's observation. Let $f \in W^{k+1,p}(U; \mathbf{R}^m)$. It is well known [7] that, for any $\varepsilon > 0$, there exist a set U_ε and a mapping $f_\varepsilon \in C^{k+1}(U; \mathbf{R}^m)$, such that $D^\alpha f_\varepsilon|_{U_\varepsilon} = D^\alpha f|_{U_\varepsilon}, |\alpha| \leq 1$, and $\mathscr{L}^n(U \backslash U_\varepsilon) < \varepsilon$.

Now decompose Σ_f^0 as $\Sigma_f^0 = \cup_{N=1}^\infty \left(\Sigma_{f_{1/N}}^0 \cap U_{1/N}\right) \cup E$, where $E = \Sigma_f^0 \backslash \cup_{N=1}^\infty \left(\Sigma_{f_{1/N}}^0 \cap U_{1/N}\right)$

It is clear $\mathscr{L}^n(E) = 0$. By the classical Morse–Sard theorem for C^{k+1}, if $k+1 \geq \frac{n}{m}$,

$$\mathscr{L}^m\big(f\big(\Sigma^0_{f_{1/N}} \cap U_{1/N}\big)\big) = \mathscr{L}^m\big(f_{1/N}\big(\Sigma^0_{f_{1/N}} \cap U_{1/N}\big)\big) = 0.$$

So, if one can show $\mathscr{L}^m(f(E)) = 0$, then

$$\mathscr{L}^m\big(f\big(\Sigma^0_f\big)\big) \leq \sum_{N=1}^{\infty} \mathscr{L}^m(f_{1/N}(\Sigma^0_{f_{1/N}} \cap U_{1/N})) + \mathscr{L}^m(f(E)) = 0$$

Hence, Theorem 4.2.1 is reduced to proving □

Lemma 4.2.5. *If $E \subset \Sigma^0_f$, $\mathscr{L}^m(E) = 0$, and $k+1 \geq \frac{n}{m}$, then $\mathscr{L}^m(f(E)) = 0$.*

Proof of the Lemma. From Corollary 2.4, for any $x \in \Sigma^0_f$, there exists a $\delta(x) > 0$, such that for arbitrary $y \in \Sigma^0_f$, $|y - x| \leq \delta(x)$, one has

$$|f(x) - f(y)| \leq C|x-y|^{k+1-\frac{n}{p}} \left(\int_{B(x,r)} |D^{k+1} f(z)|^p dz\right)^{\frac{1}{p}},$$

where $r = |x - y|$.

Therefore, by Young's inequality

$$\begin{aligned} |f(x) - f(y)|^m &\leq C|x-y|^{m(k+1-\frac{n}{p})} \left(\int_{B(x,r)} |D^{k+1} f(z)|^p dz\right)^{\frac{m}{p}} \\ &\leq C|x-y|^{\frac{pm}{p-m}(k+1-\frac{n}{p})} + C\int_{B(x,r)} |D^{k+1} f(z)|^p dz. \end{aligned}$$

Since $k+1 \geq \frac{n}{m}$, we have $\frac{pm}{p-m}(k+1-\frac{n}{p}) \geq \frac{pm}{p-m}(\frac{n}{m} - \frac{n}{p}) = n$. So, if $|x-y| \leq \min(\delta(x), 1)$, $x, y \in \Sigma^0_f$,

$$|f(x) - f(y)|^m \leq C\int_{B(x,r)} \left(1 + |D^{k+1} f(z)|^p\right) dz. \tag{4.2.1}$$

Applying (4.2.1) on $x \in E$ and noting that $\mathscr{L}^n(E) = 0$, it is quite routine to obtain the conclusion $\mathscr{L}^m(f(E)) = 0$. The left argument can be viewed as an exercise on Vitali's covering theorem, the reader is invited to write down the details. The reader can also follow Figalli's argument [10, p. 3678], by realizing that formula (3) in [10] is identical to (4.2.1) above, with k replace by $k+1$. □

Remark. The condition $k+1 \geq \frac{n-\nu}{m-\nu}$ is sharp for $\mathscr{L}^m(f(\Sigma^\nu_f)) = 0$, for if $k+1 < \frac{n-\nu}{m-\nu}$, by Bate's example [2], there is an $f \in C^{k,1}(U) \subset W^{k+1,p}(U)$ with $\mathscr{L}^m(f(\Sigma^\nu_f)) > 0$.

4.3. The Morse–Sard theorem for $W^{k+s,p}$, $0 < s < 1$

The statement is

Theorem 4.3.1. *Let* $f : U \to \mathbf{R}^m$ *be of* $W^{k+s,p}$, $sp > n > m, s \in (0,1)$. *If* $k + s \geq \frac{n}{m}$, $\nu = 0, 1, \ldots, m-1$, *then* $\mathscr{L}^m(f(\Sigma_f^0)) = 0$.

Proof. We note that when $0 < s < 1$, De Pascale's trick does not work any more. Although there are some results concerning approximation by Hö lder functions in fractional Sobolev spaces (see [20] for example), these results are not very suitable for our purpose.

Instead, we will use Figalli's argument in [10], which in turn is a clever adaptation in Sobolev spaces of an argument by Bates [2].

First, we see how the condition $k + s \geq \frac{n}{m}$ works in our proof. We know from Corollary 2.4, for $x, y \in \Sigma_f^0$, $|x - y| \leq \delta(x)$,

$$|f(x) - f(y)| \leq C|x-y|^{k+s-\frac{n}{p}} \left\| D^k f \right\|_{W^{s,p}(B(x,r))}$$

where $r = |x - y|$.

Therefore, by Young's inequality

$$\begin{aligned} |f(x) - f(y)|^m &\leq C|x-y|^{m(k+s-\frac{n}{p})} \left\| D^k f \right\|^m_{W^{s,p}(B(x,r))} \\ &\leq C|x-y|^{\frac{pm}{p-m}(k+s-\frac{n}{p})} + C \left\| D^k f \right\|^p_{W^{s,p}(B(x,r))}. \end{aligned}$$

If $k + s \geq \frac{n}{m}$, then $\frac{pm}{p-m}(k + s - \frac{n}{p}) \geq \frac{pm}{p-m}(\frac{n}{m} - \frac{n}{p}) = n$, so the above inequality becomes, if $|x - y| \leq \min(1, \delta(x))$, $x, y \in \Sigma_f^0$, then

$$\begin{aligned} |f(x) - f(y)|^m &\leq C \int_{B(x,r)} \left(1 + |D^k f(z)|^p\right) dz \\ &\quad + C \int_{B(x,r)} \int_{B(x,r)} \frac{|D^k f(u) - D^k f(v)|^p}{|u-v|^{n+sp}} du dv \\ &\leq C \int_{B(x,r)} \left(1 + |D^k f(z)|^p + g(z)\right) dz, \end{aligned} \tag{4.3.1}$$

where $g(z) = \int_U \frac{|D^k f(z) - D^k f(w)|^p}{|z-w|^{n+sp}} dw$. Note that $g \in L^1(U)$ because $f \in W^{k+s,p}$.

We now use the idea of Figalli or Bates, to write $\Sigma_f^0 = F_1 \cup F_2$, where

$$F_1 = \left\{\text{density points for } \Sigma_f^0\right\} \cap \left\{\text{Lebesgue points of } |D^k f|^p + g\right\}$$

and $F_2 = \Sigma_f^0 \backslash F_1$.

Obviously, $\mathscr{L}^n(F_2) = 0$. From (4.3.1), $\mathscr{L}^m(f(F_2)) = 0$, as proved by arguments similar to those in [10, p. 3678].

On the other hand, following the arguments on p. 3679 of [10] almost word by word, we know $\mathscr{L}^m(f(F_1)) = 0$. We do not want to repeat Figalli's argument, instead, we only mention some necessary modification in our case: firstly, all appearance of $1+|D^k f(x)|^p$ should be replaced by $1+|D^k f(x)|^p+g(x)$ in our case, secondly, formula (6) in [10] should be replaced by

$$|f(x)-f(y)|^m \le CP^{m(1-k-s+\frac{n}{p})}\int_{B(x,r_x)\cap F_1}\Big(1+|D^k f(z)|^p+g(z)\Big)\,dz,$$

$\forall y \in B(x,r_x)\cap \Sigma_f^0$.

Note that the conditions $k \ge 1, sp > n$ guarantee that $1-k-s+\frac{n}{p} < 0$, so Figalli's argument with above modifications works.

The proof is complete. □

Theorem 1.5 is just the combination of Theorems 4.2.1 and 4.3.1.

Remark 4.3.1. Unlike the case $W^{k+1,p}$, for $W^{k+s,p}, 0 < s < 1$, one cannot reduce rank $\nu > 0$ to rank $\nu = 0$ case by standard arguments (i.e, [10, p. 3677]). The key difference between integral and fractional Sobolev spaces lies on the following : for $f \in W^{k+1,p}(\mathbf{R}^n)$, $x=(x_1,\dots,x_n)$, we have for almost all $x'=(x_1,\dots,x_\nu)\in \mathbf{R}^\nu$, $f_{x'}(x'') := f(x',x'')$ belongs to $W^{k+1,p}(\mathbf{R}^{n-\nu})$, where $x''=(x_{\nu+1},\dots,x_n)$, $x=(x',x'')\in\mathbf{R}^n$. But if $f\in W^{k+s,p}$, $0<s<1$, this kind of slicing property does not hold. Therefore, usual reduction argument does not work. Somehow we believe it is still true that if $k+s \ge \frac{n-\nu}{m-\nu}$, $\nu = 0,1,\dots,m-1$, then $\mathscr{L}^m(f(\Sigma_f^\nu)) = 0$. We don't know how to prove (or disprove) the statement.

Remark 4.3.2. The above proof also works for $s=1$, by simply neglecting the term $g(z)$.

Remark 4.3.3. It is very instructive to take $p \to +\infty$. A well known fact [7] on Sobolev spaces is that $W^{k+1,\infty}(U)$ can be identical to $C^{k,1}(U)$, and since $W^{k+1,\infty}(U) \subset W^{k+1,p}(U)$, hence Theorem 4.2.1 can be viewed as an extension of Bates' work on $C^{k,1}$ version of Morse–Sard's theorem. For $0<s<1$, if we define $W^{k+s,\infty}(U)$ as the space of functions

$$\left\{u : D^k u \in L^\infty(U),\ \frac{|D^k u(x)-D^k u(y)|}{|x-y|^s} \in L^\infty(U\times U)\right\},$$

$W^{k+s,\infty}(U)$ boils down to $C^{k,s}(U)$ [16]. Theorem 4.3.1 can be viewed as a sort of extension of Bates's work on $C^{k,s}$ Morse–Sard's theorem. We note that in general $W^{k+s,\infty}(U) \nsubseteq W^{k+s,p}(U)$, $1<p<\infty$. But it holds $C^{k,s'}(U)\subset W^{k+s'',p}(U)$ for $s' > s''$. We leave the claim as an exercise.

Remark 4.3.4. Following the preceding comment, it is easy to see that $k+s \ge \frac{n}{m}$ is a sharp condition for $\mathscr{L}^m(f(\Sigma_f^0)) = 0$. For if $k+s < \frac{n}{m}$, one chooses some $f \in C^{k,s'} \subset W^{k+s,p}$ such that $\mathscr{L}^m(f(\Sigma_f^0)) \ne 0$, where $s' > s$, $k+s' < \frac{n}{m}$.

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