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G_2-Grassmannians and derived equivalences

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Abstract. We prove the derived equivalence of a pair of non-compact Calabi–Yau 7-folds, which are the total spaces of certain rank 2 bundles on G_2-Grassmannians. The proof follows that of the derived equivalence of Calabi–Yau 3-folds in G_2-Grassmannians by Kuznetsov (Derived equivalence of Ito–Miura–Okawa–Ueda Calabi–Yau 3-folds. arXiv:1611.08386) closely.

1. Introduction

The simply-connected simple algebraic group G of type G_2 has three homogeneous spaces $\mathbf{G} := G/P_1$, $\mathbf{Q} := G/P_2$, and $\mathbf{F} := G/B$ associated with the crossed Dynkin diagrams , , and respectively. The Picard group of $\mathbf{F}$ can be identified with the weight lattice of G, which in turn can be identified with $\mathbb{Z}^2$ as $(a, b) := a\omega_1 + b\omega_2$, where ω_1 and ω_2 are the fundamental weights associated with the long root and the short root respectively. We write the line bundle associated with the weight (k, l) as $\mathcal{O}_{\mathbf{F}}(k, l)$.

Let

$$R := \bigoplus_{k,l=0}^{\infty} H^0\left(\mathcal{O}_{\mathbf{F}}(k, l)\right) \cong \bigoplus_{k,l=0}^{\infty} \left(V_{(k,l)}^G\right)^{\vee} \tag{1.1}$$

be the Cox ring of $\mathbf{F}$, where $\left(V_{(k,l)}^G\right)^{\vee}$ is the dual of the irreducible representation of G with the highest weight (k, l).

The $\mathbb{Z}^2$-grading of R defines a $(\mathbb{G}_m)^2$-action on Spec R, which induces an action of $\mathbb{G}_m$ embedded in $(\mathbb{G}_m)^2$ by the anti-diagonal map $\alpha \mapsto (\alpha, \alpha^{-1})$. We write the geometric invariant theory quotients as

$$\mathbf{V}_+ := \operatorname{Proj} R_+, \quad \mathbf{V}_- := \operatorname{Proj} R_-, \quad \mathbf{V}_0 := \operatorname{Spec} R_0, \tag{1.2}$$

where

$$R_n = \bigoplus_{i \in \mathbb{Z}} R_{i,n-i}, \quad R_+ := \bigoplus_{n=0}^{\infty} R_n, \quad R_- := \bigoplus_{n=0}^{\infty} R_{-n}. \tag{1.3}$$

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$\mathbf{V}_+$ and $\mathbf{V}_-$ are the total spaces of the dual of the equivariant vector bundles of rank 2 on $\mathbf{G}$ and $\mathbf{Q}$ associated with irreducible representations of P_1 and P_2 with the highest weight $(1, 1)$. The computation in [11] shows that the first Chern classes of these rank 2 bundles are minus the first Chern classes of $\mathbf{G}$ and $\mathbf{Q}$ respectively, so that $\mathbf{V}_+$ and $\mathbf{V}_-$ are non-compact Calabi–Yau manifolds. The structure morphisms $\phi_+ : \mathbf{V}_+ \to \mathbf{V}_0$ and $\phi_- : \mathbf{V}_- \to \mathbf{V}_0$ are crepant resolutions which contract the zero-sections. Together with the total space $\mathbf{V}$ of the line bundle $\mathcal{O}_{\mathbf{F}}(1, 1)$ on $\mathbf{F}$, they fit into the commutative diagrams (2.16) and (3.1).

The same construction for the simply-connected simple algebraic group Sp(2) of type C_2, which is accidentally isomorphic to the simply-connected simple algebraic group Spin(5) of type B_2, gives the 5-fold flop discussed in [21], where it is attributed to Abuaf.

The main result in this paper is the following:

Theorem 1.1. $\mathbf{V}_+$ *and* $\mathbf{V}_-$ *are derived-equivalent.*

Theorem 1.1 provides an evidence for the conjecture [4, Conjecture 4.4] [15, Conjecture 1.2] that birationally equivalent smooth projective varieties are K-equivalent if and only if they are D-equivalent.

The proof of Theorem 1.1 closely follows [17], where the derived equivalence of Calabi–Yau complete intersections in $\mathbf{G}$ and $\mathbf{Q}$ defined by sections of the equivariant vector bundles dual to $\mathbf{V}_+$ and $\mathbf{V}_-$. The existence of a derived equivalence between these Calabi–Yau 3-folds in turn follows from Theorem 1.1 using matrix factorizations.

Notations and conventions. We work over a field $\mathbf{k}$ throughout this paper. All pull-back and push-forward are derived. The complexes underlying $\mathrm{Ext}^\bullet(-, -)$ and $H^\bullet(-)$, considered as objects in the derived category of vector spaces, will be denoted by $\mathbf{hom}(-, -)$ and $\mathbf{h}(-)$.

2. The blow-up diagram

As described e.g. in [18, Section 6.4], the G_2-Grassmannian $\mathbf{G}$ is the zero locus $s_\lambda^{-1}(0)$ of the section s_λ of the equivariant vector bundle $\mathcal{Q}^\vee(1)$ of rank 5 on $\mathrm{Gr}(2, V)$, obtained as the tensor product of the dual $\mathcal{Q}^\vee$ of the universal quotient bundle $\mathcal{Q}$ and the hyperplane bundle $\mathcal{O}(1)$. Here $V := V_{(0,1)}^G$ is the 7-dimensional fundamental representation of G_2, and s_λ corresponds to the G_2-invariant 3-form on V under the isomorphism $H^0(\mathrm{Gr}(2, V), \mathcal{Q}^\vee(1)) \cong \bigwedge^3 V^\vee$. We write the G_2-equivariant vector bundle associated with the irreducible representation of P_1 with the highest weight (a, b) as $\mathcal{E}_{(a,b)}$. The restriction $\mathscr{U} := \mathcal{S}|_{\mathbf{G}}$ of the universal subbundle $\mathcal{S}$ of rank 2 on $\mathrm{Gr}(2, V)$ is isomorphic to $\mathcal{E}_{(-1,1)}$.

The G_2-flag variety $\mathbf{F}$ is isomorphic to the total space of the $\mathbb{P}^1$-bundle $\varpi_+ : \mathbb{P}(\mathscr{U}) \to \mathbf{G}$ associated with $\mathscr{U}$ (or any other equivariant vector bundle of rank 2, since all of them are related by a twist by a line bundle). We write the relative hyperplane class of ϖ_+ as h, so that

$$(\varpi_+)_* (\mathcal{O}_{\mathbf{F}}(h)) \cong \mathscr{U}^\vee. \tag{2.1}$$

The pull-back to $\mathbf{F}$ of the ample generator H of $\operatorname{Pic}(\mathbf{G}) \cong \mathbb{Z}$ will be denoted by H again by abuse of notation.

The other G_2-Grassmannian $\mathbf{Q}$ is a quadric hypersurface in $\mathbb{P}(V)$. We write the equivariant vector bundle on $\mathbf{Q}$ associated with the irreducible representation of P_2 with highest weight (a, b) as $\mathcal{F}_{(a,b)}$. The flag variety $\mathbf{F}$ has a structure of a $\mathbb{P}^1$-bundle $\varpi_- : \mathbf{F} \to \mathbf{Q}$, whose relative hyperplane class is given by H. We define a vector bundle $\mathscr{K}$ on $\mathbf{Q}$ by

$$\mathscr{K} := ((\varpi_-)_* (\mathcal{O}_{\mathbf{F}}(H)))^{\vee}, \tag{2.2}$$

so that $\mathbf{F} \cong \mathbb{P}_{\mathbf{G}}(\mathscr{K})$. One can show that $\mathscr{K}$ is isomorphic to $\mathcal{F}_{(1,-3)}$. We write the hyperplane class of $\mathbf{Q}$ as h by abuse of notation, since it pulls back to h on $\mathbf{F}$.

Let $\mathbf{V}$ be the total space of the line bundle $\mathcal{O}_{\mathbf{F}}(-h - H)$ on $\mathbf{F}$. The structure morphism will be denoted by $\pi : \mathbf{V} \to \mathbf{F}$. The Cox ring of $\mathbf{V}$ is the $\mathbb{N}^2$-graded ring

$$S = \bigoplus_{k,l=0}^{\infty} S_{k,l} \tag{2.3}$$

given by

$$\begin{aligned} S_{k,l} &:= H^0 (\mathcal{O}_{\mathbf{V}}(k, l)) && (2.4)\\ &\cong H^0 (\pi_* (\mathcal{O}_{\mathbf{V}}(k, l))) && (2.5)\\ &\cong H^0 (\pi_* \mathcal{O}_{\mathbf{V}} \otimes \mathcal{O}_{\mathbf{F}}(k, l)) && (2.6)\\ &\cong H^0 \left(\left(\bigoplus_{m=0}^{\infty} \mathcal{O}_{\mathbf{F}}(m, m) \right) \otimes \mathcal{O}_{\mathbf{F}}(k, l) \right) && (2.7)\\ &\cong \bigoplus_{m=0}^{\infty} H^0 (\mathcal{O}_{\mathbf{F}}(k + m, l + m)) && (2.8)\\ &\cong \bigoplus_{m=0}^{\infty} \left(V^G_{(k+m,l+m)} \right)^{\vee}, && (2.9) \end{aligned}$$

whose multiple Proj recovers $\mathbf{V}$. Similarly, the Cox ring of the total space $\mathbf{W}_+$ of the bundle $\mathcal{E}^{\vee}_{(1,1)} \cong \mathscr{U}(-H)$ is given by $\bigoplus_{k=0}^{\infty} H^0 \left(\mathcal{O}_{\mathbf{W}_+}(kH) \right)$ where

$$\begin{aligned} H^0 \left(\mathcal{O}_{\mathbf{W}_+}(kH) \right) &\cong H^0 \left(\pi_* \left(\mathcal{O}_{\mathbf{W}_+}(kH) \right) \right) && (2.10)\\ &\cong H^0 \left(\pi_* \mathcal{O}_{\mathbf{W}_+} \otimes \mathcal{O}_{\mathbf{G}}(kH) \right) && (2.11)\\ &\cong \bigoplus_{m=0}^{\infty} H^0 \left(\left(\operatorname{Sym}^m \mathcal{E}_{(1,1)} \right) \otimes \mathcal{O}_{\mathbf{G}}(kH) \right) && (2.12)\\ &\cong \bigoplus_{m=0}^{\infty} H^0 \left(\mathcal{E}_{(m,m)} \otimes \mathcal{E}_{(k,0)} \right) && (2.13)\\ &\cong \bigoplus_{m=0}^{\infty} H^0 \left(\mathcal{E}_{(m+k,m)} \right). && (2.14) \end{aligned}$$

This is isomorphic to R_+, so that $\mathbf{W}_+$ is isomorphic to $\mathbf{V}_+$, and the affinization morphism

$$\mathbf{V} \to \operatorname{Spec} H^0(\mathcal{O}_{\mathbf{V}}) \cong \mathbf{V}_0 \tag{2.15}$$

is the composition of the natural projection $\varphi_+ \colon \mathbf{V} \to \mathbf{V}_+$ and the affinization morphism $\phi_+ \colon \mathbf{V}_+ \to \mathbf{V}_0$. Since $\mathbf{V}_+$ is the total space of $\mathcal{E}_{(1,1)}^\vee$, the ideal sheaf of the zero-section is the image of the natural morphism from $\pi_+^* \mathcal{E}_{(1,1)}$ to $\mathcal{O}_{\mathbf{V}_+}$, and the morphism φ_+ is the blow-up along it. Similarly, the affinization morphism (2.15) also factors into the blow-up $\varphi_- \colon \mathbf{V} \to \mathbf{V}_-$ and the affinization morphism $\phi_- \colon \mathbf{V}_- \to \mathbf{V}_0$, and one obtains the following commutative diagram:

$$\begin{array}{ccccc} & & \mathbf{V} & & \\ & \swarrow^{\varphi_+} & & \searrow^{\varphi_-} & \\ \mathbf{V}_+ & & & & \mathbf{V}_- \\ & \searrow_{\phi_+} & & \swarrow_{\phi_-} & \\ & & \mathbf{V}_0 & & \end{array} \tag{2.16}$$

3. Some extension groups

The zero-sections and the natural projections fit into the following diagram:

$$\begin{array}{ccccc} & & \mathbf{F} & & \\ & \swarrow^{\varpi_+} & \downarrow \iota & \searrow^{\varpi_-} & \\ \mathbf{G} & & \mathbf{V} & & \mathbf{Q} \\ \downarrow & \swarrow_{\phi_+} & & \searrow_{\phi_-} & \downarrow \\ \mathbf{V}+ & & & & \mathbf{V}_- \end{array} \tag{3.1}$$

We write $\mathscr{U}_{\mathbf{F}} := \varpi_+^* \mathscr{U}$, $\mathscr{S}_{\mathbf{F}} := \varpi_-^* \mathscr{S}$, and $\mathscr{U}_{\mathbf{V}} := \pi^* \mathscr{U}_{\mathbf{F}}$. By abuse of notation, we use the same symbol for an object of $D^b(\mathbf{F})$ and its image in $D^b(\mathbf{V})$ by the push-forward ι_*. Since $\mathbf{V}$ is the total space of $\mathcal{O}_{\mathbf{V}}(-h-H)$, one has a locally free resolution

$$0 \to \mathcal{O}_{\mathbf{V}}(h+H) \to \mathcal{O}_{\mathbf{V}} \to \mathcal{O}_{\mathbf{F}} \to 0 \tag{3.2}$$

of $\mathcal{O}_{\mathbf{F}}$ as an $\mathcal{O}_{\mathbf{V}}$-module.

By tensoring $\mathcal{O}_{\mathbf{F}}(-h)$ to [17, Equation (5)], one obtains an exact sequence

$$0 \to \mathcal{O}_{\mathbf{F}}(H-2h) \to \mathscr{U}_{\mathbf{F}}^\vee(-h) \to \mathcal{O}_{\mathbf{F}} \to 0. \tag{3.3}$$

Lemma 3.1 and Proposition 3.2 below are taken from [17]:

Lemma 3.1. ([17, Lemma 1])

(i) Line bundles $\mathcal{O}_{\mathbf{F}}(th - H)$ and $\mathcal{O}_{\mathbf{F}}(tH - h)$ are acyclic for all $t \in \mathbb{Z}$.
(ii) Line bundles $\mathcal{O}_{\mathbf{F}}(-2H)$ and $\mathcal{O}_{\mathbf{F}}(2h - 2H)$ are acyclic and

$$H^\bullet(\mathcal{O}_{\mathbf{F}}(3h - 2H)) \cong \mathbf{k}[-1].$$

(iii) Vector bundles $\mathscr{U}_{\mathbf{F}}(-2H)$, $\mathscr{U}_{\mathbf{F}}(-H)$, $\mathscr{U}_{\mathbf{F}}(h - H)$ and $\mathscr{U}_{\mathbf{F}} \otimes \mathscr{U}_{\mathbf{F}}(-H)$ are acyclic, and

$$H^\bullet(\mathscr{U}_{\mathbf{F}}(h)) \cong \mathbf{k}, \quad H^\bullet(\mathscr{U}_{\mathbf{F}} \otimes \mathscr{U}_{\mathbf{F}}(h)) \cong \mathbf{k}[-1].$$

Proposition 3.2. *([17, Proposition 3 and Lemma 4]) One has an exact sequence*

$$0 \to \mathscr{U}_{\mathbf{F}} \to \mathscr{S}_{\mathbf{F}} \to \mathscr{U}_{\mathbf{F}}^\vee(-h) \to 0. \tag{3.4}$$

Lemma 3.1 immediately implies the following:

Lemma 3.3. $\mathcal{O}_{\mathbf{F}}(-H)$ is right orthogonal to both $\mathscr{U}_{\mathbf{F}}^\vee(-h)$ and $\mathcal{O}_{\mathbf{F}}(-h)$.

Proof. We have

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}(-h), \mathcal{O}_{\mathbf{F}}(-H)) \cong \mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\{\mathcal{O}_{\mathbf{V}}(H) \to \mathcal{O}_{\mathbf{V}}(-h)\}, \mathcal{O}_{\mathbf{F}}(-H)) \tag{3.5}$$

$$\cong \mathbf{h}(\{\mathcal{O}_{\mathbf{F}}(h - H) \to \mathcal{O}_{\mathbf{F}}(-2H)\}) \tag{3.6}$$

and

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\mathscr{U}_{\mathbf{F}}^\vee(-h), \mathcal{O}_{\mathbf{F}}(-H)\right) \cong \mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\left\{\mathscr{U}_{\mathbf{V}}^\vee(H) \to \mathscr{U}_{\mathbf{V}}^\vee(-h)\right\}, \mathcal{O}_{\mathbf{F}}(-H)\right) \tag{3.7}$$

$$\cong \mathbf{h}(\{\mathscr{U}_{\mathbf{F}}(h - H) \to \mathscr{U}_{\mathbf{F}}(-2H)\}), \tag{3.8}$$

both of which vanish by Lemma 3.1. □

Lemma 3.4. One has

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\mathscr{U}_{\mathbf{F}}^\vee(-h), \mathscr{U}_{\mathbf{F}}\right) \cong \mathbf{k}[-1]. \tag{3.9}$$

Proof. One has

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\mathscr{U}_{\mathbf{F}}^\vee(-h), \mathscr{U}_{\mathbf{F}}\right) \cong \mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\left\{\mathscr{U}_{\mathbf{V}}^\vee(H) \to \mathscr{U}_{\mathbf{V}}^\vee(-h)\right\}, \mathscr{U}_{\mathbf{F}}\right) \tag{3.10}$$

$$\cong \mathbf{h}(\{\mathscr{U}_{\mathbf{F}} \otimes \mathscr{U}_{\mathbf{F}}(h) \to \mathscr{U}_{\mathbf{F}} \otimes \mathscr{U}_{\mathbf{F}}(-H)\}). \tag{3.11}$$

Lemma 3.1 shows that the first term gives $\mathbf{k}[-1]$ and the second term vanishes. □

Lemma 3.5. One has

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\mathscr{U}_{\mathbf{F}}^\vee(-h), \mathcal{O}_{\mathbf{F}}\right) \cong \mathbf{k}. \tag{3.12}$$

Proof. One has

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\mathcal{U}_{\mathbf{F}}^{\vee}(-h), \mathcal{O}_{\mathbf{F}}\right) \cong \mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\left\{\mathcal{U}_{\mathbf{V}}^{\vee}(H) \to \mathcal{U}_{\mathbf{V}}^{\vee}(-h)\right\}, \mathcal{O}_{\mathbf{F}}\right) \tag{3.13}$$

$$\cong \mathbf{h}\left(\{\mathcal{U}_{\mathbf{F}}(h) \to \mathcal{U}_{\mathbf{F}}(-H)\}\right). \tag{3.14}$$

Lemma 3.1 shows that the first term gives $\mathbf{k}$ and the second term vanishes. □

Lemma 3.6. One has

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\mathcal{O}_{\mathbf{F}}(H-2h), \mathcal{O}_{\mathbf{F}}(h)\right) \cong 0. \tag{3.15}$$

Proof. One has

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\mathcal{O}_{\mathbf{F}}(H-2h), \mathcal{O}_{\mathbf{F}}(h)\right) \cong \mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}\left(\{\mathcal{O}_{\mathbf{V}}(2H-h) \to \mathcal{O}_{\mathbf{V}}(H-2h)\}, \mathcal{O}_{\mathbf{F}}(h)\right) \tag{3.16}$$

$$\cong \mathbf{h}\left(\{\mathcal{O}_{\mathbf{V}}(3h-H) \to \mathcal{O}_{\mathbf{V}}(2h-2H)\}\right), \tag{3.17}$$

which vanishes by Lemma 3.1. □

4. Derived equivalence by mutation

Recall from [17] that

$$D^b(\mathbf{G}) = \left\langle \mathcal{O}_{\mathbf{G}}(-H), \mathcal{U}, \mathcal{O}_{\mathbf{G}}, \mathcal{U}^{\vee}, \mathcal{O}_{\mathbf{G}}(H), \mathcal{U}^{\vee}(H) \right\rangle \tag{4.1}$$

and

$$D^b(\mathbf{Q}) = \left\langle \mathcal{O}_{\mathbf{Q}}(-3h), \mathcal{O}_{\mathbf{Q}}(-2h), \mathcal{O}_{\mathbf{Q}}(-h), \mathcal{S}, \mathcal{O}_{\mathbf{Q}}, \mathcal{O}_{\mathbf{Q}}(h) \right\rangle. \tag{4.2}$$

It follows from [19] that

$$D^b(\mathbf{V}) = \left\langle \iota_* \varpi_+^* D^b(\mathbf{G}), \Phi_+(D^b(\mathbf{V}_+)) \right\rangle \tag{4.3}$$

and

$$D^b(\mathbf{V}) = \left\langle \iota_* \varpi_-^* D^b(\mathbf{Q}), \Phi_-(D^b(\mathbf{V}_-)) \right\rangle, \tag{4.4}$$

where

$$\Phi_+ := \phi_+^*(-) \otimes \mathcal{O}_{\mathbf{V}}(h) \colon D^b(\mathbf{V}_+) \to D^b(\mathbf{V}) \tag{4.5}$$

and

$$\Phi_- := \phi_-^*(-) \otimes \mathcal{O}_{\mathbf{V}}(H) \colon D^b(\mathbf{V}_-) \to D^b(\mathbf{V}). \tag{4.6}$$

(4.1) and (4.3) gives

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-H), \mathcal{U}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathcal{U}_{\mathbf{F}}^{\vee}, \mathcal{O}_{\mathbf{F}}(H), \mathcal{U}_{\mathbf{F}}^{\vee}(H), \Phi_+(D^b(\mathbf{V}_+)) \right\rangle. \tag{4.7}$$

By mutating $\Phi_+(D^b(\mathbf{V}_+))$ two steps to the left, one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-H), \mathscr{U}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}, \Phi_1(D^b(\mathbf{V}_+)), \mathcal{O}_{\mathbf{F}}(H), \mathscr{U}_{\mathbf{F}}^{\vee}(H) \right\rangle \tag{4.8}$$

where

$$\Phi_1 := \mathbf{L}_{\langle \mathcal{O}_{\mathbf{F}}(H), \mathscr{U}_{\mathbf{F}}^{\vee}(H) \rangle} \circ \Phi_+. \tag{4.9}$$

Recall from [3, Proposition 3.6] that the effect of the left mutation of a semiorthogonal summand from the far right to the far left is given by the action of the Serre functor;

$$\langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{\ell-1}, \mathcal{A}_\ell \rangle \rightsquigarrow \langle S(\mathcal{A}_\ell), \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{\ell-1} \rangle . \tag{4.10}$$

By mutating the last two terms to the far left, one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-h), \mathscr{U}_{\mathbf{F}}^{\vee}(-h), \mathcal{O}_{\mathbf{F}}(-H), \mathscr{U}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}, \Phi_1(D^b(\mathbf{V}_+)) \right\rangle, \tag{4.11}$$

since $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-h-H)$. Lemma 3.3 allows one to move $\mathcal{O}_{\mathbf{F}}(-H)$ to the far left without affecting other objects:

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-H), \mathcal{O}_{\mathbf{F}}(-h), \mathscr{U}_{\mathbf{F}}^{\vee}(-h), \mathscr{U}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}, \Phi_1(D^b(\mathbf{V}_+)) \right\rangle. \tag{4.12}$$

By mutating $\mathscr{U}_{\mathbf{F}}$ one step to the left and using Proposition 3.2 and Lemma 3.4, one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-H), \mathcal{O}_{\mathbf{F}}(-h), \mathscr{S}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}(-h), \mathcal{O}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}, \Phi_1(D^b(\mathbf{V}_+)) \right\rangle. \tag{4.13}$$

By mutating $\mathcal{O}_{\mathbf{F}}(-H)$ to the far right, one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-h), \mathscr{S}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}(-h), \mathcal{O}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}, \Phi_1(D^b(\mathbf{V}_+)), \mathcal{O}_{\mathbf{F}}(h) \right\rangle. \tag{4.14}$$

By mutating $\Phi_1(D^b(\mathbf{V}_+))$ to the right, one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-h), \mathscr{S}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}(-h), \mathcal{O}_{\mathbf{F}}, \mathscr{U}_{\mathbf{F}}^{\vee}, \mathcal{O}_{\mathbf{F}}(h), \Phi_2(D^b(\mathbf{V}_+)) \right\rangle \tag{4.15}$$

where

$$\Phi_2 := \mathbf{R}_{\mathcal{O}_{\mathbf{F}}(h)} \circ \Phi_1. \tag{4.16}$$

By mutating $\mathscr{U}_{\mathbf{F}}^{\vee}(-h)$ one step to the right and using Lemma 3.5 and (3.3), one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-h), \mathscr{S}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(H-2h), \mathscr{U}_{\mathbf{F}}^{\vee}, \mathcal{O}_{\mathbf{F}}(h), \Phi_2(D^b(\mathbf{V}_+)) \right\rangle. \tag{4.17}$$

Similarly, by mutating $\mathscr{U}_{\mathbf{F}}^{\vee}$ one step to the right, one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-h), \mathscr{S}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(H-2h), \mathcal{O}_{\mathbf{F}}(h), \mathcal{O}_{\mathbf{F}}(H-h), \Phi_2(D^b(\mathbf{V}_+)) \right\rangle. \tag{4.18}$$

Lemma 3.6 allows one to exchange $\mathcal{O}_{\mathbf{F}}(H-2h)$ and $\mathcal{O}_{\mathbf{F}}(h)$ to obtain

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-h), \mathcal{S}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \mathcal{O}_{\mathbf{F}}(H-2h), \mathcal{O}_{\mathbf{F}}(H-h), \Phi_2(D^b(\mathbf{V}_+)) \right\rangle. \tag{4.19}$$

By mutating $\Phi_2(D^b(\mathbf{V}_+))$ two steps to the left, one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-h), \mathcal{S}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \Phi_3(D^b(\mathbf{V}_+)), \mathcal{O}_{\mathbf{F}}(H-2h), \mathcal{O}_{\mathbf{F}}(H-h) \right\rangle \tag{4.20}$$

where

$$\Phi_3 := \mathbf{L}_{\langle \mathcal{O}_{\mathbf{F}}(H-2h), \mathcal{O}_{\mathbf{F}}(H-h) \rangle} \circ \Phi_2. \tag{4.21}$$

By mutating the last two terms to the far left, one obtains

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-3h), \mathcal{O}_{\mathbf{F}}(-2h), \mathcal{O}_{\mathbf{F}}(-h), \mathcal{S}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \Phi_3(D^b(\mathbf{V}_+)) \right\rangle. \tag{4.22}$$

By comparing (4.22) with

$$D^b(\mathbf{V}) = \left\langle \mathcal{O}_{\mathbf{F}}(-3h), \mathcal{O}_{\mathbf{F}}(-2h), \mathcal{O}_{\mathbf{F}}(-h), \mathcal{S}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \Phi_-(D^b(\mathbf{V}_-)) \right\rangle \tag{4.23}$$

obtained by combining (4.2) and (4.4), one obtains a derived equivalence

$$\Phi := \Phi_-^! \circ \Phi_3 \colon D^b(\mathbf{V}_+) \xrightarrow{\sim} D^b(\mathbf{V}_-), \tag{4.24}$$

where

$$\Phi_-^!(-) := (\phi_-)_* \left((-) \otimes \mathcal{O}_{\mathbf{V}}(-H) \right) : D^b(\mathbf{V}) \to D^b(\mathbf{V}_-) \tag{4.25}$$

is the left adjoint functor of Φ_-. Note that the left mutation along an exceptional object $\mathcal{E} \in D^b(\mathbf{V})$ is an integral functor $\Phi_{\mathcal{K}}(-) := (p_2)_* \left(p_1^*(-) \otimes \mathcal{K} \right)$ along the diagram

$$\begin{array}{ccccc} & & \mathbf{V} \times_{\mathbf{V}_0} \mathbf{V} & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathbf{V} & & & & \mathbf{V} \end{array} \tag{4.26}$$

whose kernel $\mathcal{K}$ is the cone over the evaluation morphism $\mathrm{ev} \colon \mathcal{E}^\vee \boxtimes \mathcal{E} \to \Delta_{\mathbf{V}}$. The functors $\Phi_+ \colon D^b(\mathbf{V}_+) \to D^b(\mathbf{V})$ and $\Phi_-^! \colon D^b(\mathbf{V}) \to D^b(\mathbf{V}_-)$ are clearly an integral functor, so that the functor (4.24) is also an integral functor, whose kernel is an object of $D^b(\mathbf{V}_+ \times_{\mathbf{V}_0} \mathbf{V}_-)$ obtained by convolution.

5. Matrix factorizations

Let s_+ be a general section of the equivariant vector bundle $\mathcal{E}_{(1,1)}$ on $\mathbf{G}$. The zero X_+ of s_+ is a smooth projective Calabi–Yau 3-fold. Since $\mathbf{V}_+$ is the total space of the dual bundle $\mathcal{E}^\vee_{(1,1)}$ on $\mathbf{G}$, the space of regular functions on $\mathbf{V}_+$ which are linear along the fiber can naturally be identified with the space of sections of $\mathcal{E}_{(1,1)}$. We write the regular function on $\mathbf{V}_+$ associated with $s_+ \in H^0\left(\mathcal{E}_{(1,1)}\right)$ as $\varsigma_+ \in H^0\left(\mathcal{O}_{\mathbf{V}_+}\right)$. The zero D_+ of ς_+ is the union of a line sub-bundle of $\mathbf{V}_+$ and the inverse image of X_+ by the structure morphism $\pi_+ : \mathbf{V}_+ \to \mathbf{G}$. The singular locus of D_+ is given by X_+.

Let ς_- be a regular function on $\mathbf{V}_-$ corresponding to ς_+ under the isomorphism $H^0\left(\mathcal{O}_{\mathbf{V}_+}\right) \cong H^0\left(\mathcal{O}_{\mathbf{V}_0}\right) \cong H^0\left(\mathcal{O}_{\mathbf{V}_-}\right)$ given by the diagram in (2.16), and X_- be the zero of the corresponding section $s_- \in H^0\left(\mathcal{F}_{(1,1)}\right)$, which is a smooth projective Calabi–Yau 3-fold in $\mathbf{Q}$.

The push-forward of the kernel of Φ on $\mathbf{V}_+ \times_{V_0} \mathbf{V}_-$ to $\mathbf{V}_+ \times_{\mathbb{A}^1} \mathbf{V}_-$ gives a kernel of Φ on $\mathbf{V}_+ \times_{\mathbb{A}^1} \mathbf{V}_-$. By taking the base-change along the inclusion $0 \to \mathbb{A}^1$ of the origin and applying [18, Proposition 2.44], one obtains an equivalence $\Phi_0 : D^b(D_+) \cong D^b(D_-)$ of the bounded derived categories of coherent sheaves. By using either of the characterization of perfect complexes as *homologically finite* objects (i.e., objects whose total Ext-groups with any other object are finite-dimensional) or *compact* objects (i.e., objects such that the covariant functors represented by them commute with direct sums), one deduces that Φ_D preserves perfect complexes, so that it induces an equivalence $\Phi_0^{\mathrm{sing}} : D^b_{\mathrm{sing}}(D_+) \cong D^b_{\mathrm{sing}}(D_-)$ of singularity categories (see [16, Section 7] and [5, Theorem 1.1]).

Recall that $\mathbf{V}_+$, $\mathbf{V}_-$ and $\mathbf{V}_0$ are geometric invariant theory quotient of Spec R by the anti-diagonal $\mathbb{G}_m$-action. The residual diagonal $\mathbb{G}_m$-action on both $\mathbf{V}_+$ and $\mathbf{V}_-$ are dilation action on the fiber. The equivalences Φ, Φ_0 and Φ_0^{sing} are equivariant with respect to this $\mathbb{G}_m$-action, and induces an equivalence of $\mathbb{G}_m$-equivariant categories [8, Theorem 1.1], which will be denoted by the same symbol by abuse of notation. Now [14, Theorem 3.6] gives equivalences

$$D^b_{\mathrm{sing}}([D_+/\mathbb{G}_m]) \cong D^b(X_+) \tag{5.1}$$

and

$$D^b_{\mathrm{sing}}([D_-/\mathbb{G}_m]) \cong D^b(X_-) \tag{5.2}$$

between $\mathbb{G}_m$-equivariant singularity categories and derived categories of coherent sheaves (see also [22] where the case of line bundles is discussed independently and around the same time as [14]). By composing these derived equivalences with Φ_0^{sing}, one obtains a derived equivalence between X_+ and X_-. It is an interesting problem to compare this equivalence with the one obtained in [17]. Another interesting problem is to prove the derived equivalence using variation of geometric invariant theory quotient along the lines of [1,2,7,9,20], and use it to produce autoequivalences of the derived category [6,10].

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