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Lagrangian L-stability of Lagrangian translating solitons

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Abstract. In this paper, we prove that any Lagrangian translating soliton is Lagrangian L-stable.

Key words. Lagrangian translating soliton · Lagrangian L-stable

1. Introduction

Recent years, motivated by the problem of existence of special Lagrangian submanifolds, Lagrangian mean curvature flow has attracted much attention. It was proved by Chen and Li [2] and Wang [14] that there is no finitie time Type I singularity for almost calibrated Lagrangian mean curvature flow (see also Neves [9] for more discussions). Therefore, there are many works concentrating on Type II singularities of Lagrangian mean curvature flow, especially, on Lagrangian translating solitons ([5,6,10,12,13], etc.).

An n-dimensional submanifold Σ^n in $\mathbb{R}^{n+k}$ is called a *translating soliton* if there exists a constant vector $\mathbf{T} \in \mathbb{R}^{n+k}$, such that

$$\mathbf{T}^{\perp} = \mathbf{H} \tag{1.1}$$

holds on Σ, where $\mathbf{H}$ is the mean curvature vector of Σ^n in $\mathbb{R}^{n+k}$.

Similar to that of self-shrinkers [4], one can also study the translating solitons from variational viewpoint. Actually, translating solitons can be viewed as critical points of the following functional:

$$F(\Sigma) = \int_{\Sigma} e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu, \tag{1.2}$$

where $\mathbf{x}$ is the position vector in $\mathbb{R}^{n+k}$ and $d\mu$ is the induced area element on Σ. Then it is natural to define stability of translating solitons. Shahriyari [11] proved that any translating graph in $\mathbb{R}^3$ is L-stable.

A translating soliton Σ^n in $\mathbb{C}^n$ is called a *Lagrangian translating soliton* if it is also a Lagrangian submanifold of $\mathbb{C}^n$. In [15], L. Yang proved that any Lagrangian

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translating soliton is Hamiltonian L-stable. In this paper, we prove that it is in fact Lagrangian L-stable:

Theorem 1.1. *Any Lagrangian translating soliton is Lagrangian L-stable.*

Theorem 1.1 relies crucially on that the variation is Lagrangian. There are many examples for Lagrangian translating solitons ([3,7], etc.). By Theorem 1.1, they are all Lagrangian L-stable. One natural question is whether we can find examples which are in fact L-stable (not just only Lagrangian L-stable). In [1], we showed that the Grim Reaper cylinder $\Gamma \times \mathbb{R}^{n-1}$ is L-stable in $\mathbb{R}^{n+1}$, where Γ is the Grim Reaper in the plane. This is actually true for any mean convex translating soliton Σ^n in $\mathbb{R}^{n+1}$. In this paper, we will show that:

Theorem 1.2. *The Lagrangian Grim Reaper cylinder $\Gamma \times \mathbb{R}$ in $\mathbb{C}^2$ is L-stable.*

For the relations between Lagrangian F-stable self-shrinkers and Hamiltonian F-stable self-shrinkers, we refer to [8].

2. Preliminaries

In this section, we will recall some results for the first variation and second variation formulas. Since the proofs can be found in Section 4 of [1] with $f = \langle \mathbf{T}, \mathbf{x} \rangle$, where we dealt with more general cases (see also [15]), we omit the details here.

Recall that the F-functional is defined by

$$F(\Sigma) = \int_\Sigma e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu.$$

The following first variation formula is known (Proposition 4.1 of [1]):

Proposition 2.1. *Let $\Sigma_s^n \subset \mathbb{R}^{n+k}$ be a smooth compactly supported variation of Σ with normal variational vector field $\mathbf{V}$, then*

$$\frac{d}{ds}|_{s=0} F(\Sigma_s) = \int_\Sigma \langle \mathbf{T}^\perp - \mathbf{H}, \mathbf{V} \rangle e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu. \tag{2.1}$$

In particular, Σ is a critical point of F if and only if $\mathbf{T}^\perp = \mathbf{H}$, i.e., Σ is a translating soliton in $\mathbb{R}^{n+k}$.

For the second variation formula, we have (see (4.17) of [1]):

Theorem 2.2. *Suppose that Σ is a critical point of F. If $\Sigma_s^n \subset \mathbb{R}^{n+k}$ be a smooth compactly supported variation of Σ with normal variational vector field $\mathbf{V}$, then the second variation formula is given by*

$$F'' := \frac{d^2}{ds^2}|_{s=0} F(\Sigma_s) = -\int_\Sigma \langle L\mathbf{V}, \mathbf{V} \rangle e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu. \tag{2.2}$$

Here, the stability operator L is defined on a normal vector field $\mathbf{V}$ on M by

$$LV = \left(\Delta V^\alpha + \langle \boldsymbol{T}, \nabla V^\alpha \rangle + g^{ik} g^{jl} h_{ij}^\alpha h_{kl}^\beta V^\beta \right) e_\alpha, \tag{2.3}$$

where $\{e_\alpha\}$ is a local orthonormal frame of the normal bundle $N\Sigma$, g_{ij} is the induced metric on Σ and $\mathbf{V} = V^\alpha e_\alpha$.

We can also write the stability operator (2.3) in the following global form:

$$L\mathbf{V} = \Delta_{\Sigma}^{\perp}\mathbf{V} + \langle \mathbf{T}, \nabla^{\perp}\mathbf{V}\rangle + \tilde{\mathbf{A}}(\mathbf{V}), \tag{2.4}$$

where $\Delta_{\Sigma}^{\perp}\mathbf{V}$ is the *Laplacian on the normal bundle* defined in local coordinate by

$$\Delta_{\Sigma}^{\perp}(V^{\alpha}e_{\alpha}) := (\Delta V^{\alpha})e_{\alpha},$$

$\nabla^{\perp}\mathbf{V}$ is the *covariant differential on the normal bundle* defined in local coordinate by

$$\nabla^{\perp}(V^{\alpha}e_{\alpha}) := \nabla V^{\alpha} \otimes e_{\alpha},$$

and $\tilde{\mathbf{A}}$ is the *Simons' operator* defined in local coordinate by

$$\tilde{\mathbf{A}}(\mathbf{V}) := g^{ik}g^{jl}\langle \mathbf{A}(e_i, e_j), \mathbf{V}\rangle \mathbf{A}(e_k, e_l).$$

Definition 2.1. A translating soliton Σ^n in $\mathbb{R}^{n+k}$ is said to be L**-stable** if for every compactly supported normal variational vector field $\mathbf{V}$, we have

$$F'' = -\int_{\Sigma} \langle L\mathbf{V}, \mathbf{V}\rangle e^{\langle \mathbf{T}, \mathbf{x}\rangle} d\mu \geq 0.$$

Now we turn to Lagrangian translating solitons. Let $\bar{\omega}$ and J be the standard Kähler form and complex structure on $\mathbb{C}^n$, respectively. A submanifold Σ^n is said to be a *Lagrangian* submanifold of $\mathbb{C}^n$, if $\bar{\omega}|_{\Sigma} = 0$, or equivalently, J maps the tangent space of Σ onto its normal space at each point of Σ. For a Lagrangian submanfiold, there is a canonical correspondence between the sections of the normal bundle and the space of 1-forms on Σ:

$$\begin{aligned} \Gamma(N\Sigma) &\longrightarrow \Lambda^1(\Sigma) \\ \mathbf{V} &\longleftrightarrow \theta_{\mathbf{V}} := -i_{\mathbf{V}}\bar{\omega}. \end{aligned}$$

A normal vector field $\mathbf{V}$ is a *Lagrangian variation* if $\theta_{\mathbf{V}}$ is closed; a normal vector field $\mathbf{V}$ is a *Hamiltonian variation* if $\theta_{\mathbf{V}}$ is exact.

Definition 2.2. A Lagrangian translating soliton Σ^n in $\mathbb{C}^n$ is said to be **Lagrangian (resp. Hamiltonian)** L**-stable** if for every compactly supported normal Lagrangian (resp. Hamiltonian) variation $\mathbf{V}$, we have

$$F'' = -\int_{\Sigma} \langle L\mathbf{V}, \mathbf{V}\rangle e^{\langle \mathbf{T}, \mathbf{x}\rangle} d\mu \geq 0.$$

3. Lagrangian L-stability of Lagrangian Translating Solitons

In this section, we will prove Theorem 1.1. First, we would like to rewrite the second variation formula for Lagrangian variations.

Let $(\mathbb{C}^n, \bar{g}, J, \bar{\omega})$ be the complex Euclidean space with standard metric $\bar{g}$, complex structure J and Kähler form $\bar{\omega}$ such that $\bar{g} = \bar{\omega}(\cdot, J\cdot)$. Given any Lagrangian submanifold Σ^n in $\mathbb{C}^n$, we choose a local orthonormal frame $\{e_i\}_{i=1}^n$ of $T\Sigma$, and set $\nu_i = Je_i$. Then $\{\nu_i\}_{i=1}^n$ forms a local orthonormal frame of the normal bundle $N\Sigma$. The frame can be chosen so that at a fixed point $x \in \Sigma$, we have $\nabla_{e_i} e_j = 0$, where ∇ is the induced connection on Σ. The second fundamental form is defined by

$$h_{ijk} = \bar{g}\left(\overline{\nabla}_{e_i} e_j, \nu_k\right),$$

which is symmetric in i, j and k. Th mean curvature vector is given by

$$\mathbf{H} = H_k \nu_k = h_{iik}\nu_k.$$

Let $\{\omega^i\}_{i=1}^n$ be the dual basis of $\{e_i\}_{i=1}^n$. Then for any normal vector field $\mathbf{V} = V_i\nu_i$, we have the correspondence

$$\theta_{\mathbf{V}} := -i_{\mathbf{V}}\bar{\omega} = V_i\omega^i.$$

Since $d\theta_{\mathbf{V}} = \nabla_{e_i} V_j \omega^j \wedge \omega^i$, we see that

Proposition 3.1. *A normal vector field* $\mathbf{V}$ *of a Lagrangian submanfiold* Σ^n *in* $\mathbb{C}^n$ *is a Lagrangian variation if and only if* $\nabla_{e_i} V_j = \nabla_{e_j} V_i$.

Using the above notations, we see that the stability operator (2.3) can be rewritten as

$$L\mathbf{V} = \left(\Delta V_i + \langle \mathbf{T}, \nabla V_i\rangle + h_{kli}h_{klj}V_j\right)\nu_i. \tag{3.1}$$

Therefore, we have

Proposition 3.2. *A Lagrangian translating soliton* Σ^n *in* $\mathbb{C}^n$ *is Lagrangian* L*-stable if and only if for every compactly supported normal Lagrangian variation* $\mathbf{V} = V_i\nu_i$*, we have*

$$F'' = -\int_\Sigma \left(V_i\Delta V_i + V_i\langle \boldsymbol{T}, \nabla V_i\rangle + h_{kli}h_{klj}V_iV_j\right)e^{\langle \mathbf{T},\mathbf{x}\rangle}d\mu \geq 0. \tag{3.2}$$

Now, we can prove the first main result. For the purpose of convenience, we rewrite it here:

Theorem 3.3. *Any Lagrangian translating soliton is Lagrangian* L*-stable.*

Proof. By Proposition 3.2, it suffices to prove that (3.2) holds for every compactly supported Lagrangian variation $\mathbf{V} = V_i\nu_i$. Since $\mathbf{V} = V_i\nu_i$ is Lagrangian, by Proposition 3.1, we see that $\nabla_{e_i} V_j = \nabla_{e_j} V_i$. By Ricci identity, we have

$$\Delta V_i = \nabla_j\nabla_j V_i = \nabla_j\nabla_i V_j = \nabla_i\nabla_j V_j + R_{jijk}V_k = \nabla_i\nabla_j V_j + R_{ik}V_k, \tag{3.3}$$

where R_{ik} is the Ricci curvature of the induced metric on Σ. By Gauss equation, we have that

$$R_{ijkl} = h_{pik}h_{pjl} - h_{pil}h_{pjk},$$

which implies that

$$R_{ik} = g^{jl}R_{ijkl} = H_p h_{pik} - h_{pji}h_{pjk}. \tag{3.4}$$

Putting (3.4) into (3.3) yields

$$\Delta V_i = \nabla_i \nabla_j V_j + R_{ik}V_k = \nabla_i \nabla_j V_j + H_p h_{pik}V_k - h_{pji}h_{pjk}V_k.$$

Therefore, we have

$$F'' = -\int_\Sigma \left(V_i \nabla_i \nabla_j V_j + V_i \langle \mathbf{T}, e_j\rangle \nabla_j V_i + H_p h_{pij} V_i V_j\right) e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu. \tag{3.5}$$

Integrating by part, we can compute the first term on the right hand side of (3.5) as:

$$\begin{aligned}
&-\int_\Sigma V_i \nabla_i \nabla_j V_j e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu \\
&= \int_\Sigma (\nabla_i V_i \nabla_j V_j + V_i \nabla_j V_j \nabla_i \langle \mathbf{T}, \mathbf{x}\rangle)\mathbf{e}^{\langle \mathbf{T},\mathbf{x}\rangle}\mathbf{d}\mu \\
&= \int_\Sigma \left[\left(\sum_{j=1}^n \nabla_j V_j\right)^2 + \left(\sum_{j=1}^n \nabla_j V_j\right)\left(\sum_{i=1}^n \langle \mathbf{T}, e_i\rangle V_i\right)\right] e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu.
\end{aligned} \tag{3.6}$$

On the other hand, from the translating soliton equation (1.1), we can easily see that $H_p = \langle \mathbf{T}, \nu_\mathbf{p}\rangle$. Therefore, using the fact that $\nabla_{e_i} V_j = \nabla_{e_j} V_i$, the second term on the right hand side of (3.5) can be computed as:

$$\begin{aligned}
&-\int_\Sigma V_i \langle \mathbf{T}, e_j\rangle \nabla_j V_i e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu = -\int_\Sigma (\nabla_i V_j) V_i \langle \mathbf{T}, e_j\rangle e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu \\
&= \int_\Sigma \left[(\nabla_i V_i)\langle \mathbf{T}, e_j\rangle V_j + V_i V_j \nabla_i \langle \mathbf{T}, e_j\rangle + V_i V_j \langle \mathbf{T}, e_j\rangle \nabla_i \langle \mathbf{T}, \mathbf{x}\rangle\right] e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu \\
&= \int_\Sigma \left[(\nabla_i V_i)\langle \mathbf{T}, e_j\rangle V_j + V_i V_j \langle \mathbf{T}, h_{pij}\nu_p\rangle + V_i V_j \langle \mathbf{T}, e_j\rangle \langle \mathbf{T}, e_i\rangle\right] e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu \\
&= \int_\Sigma \left[\left(\sum_{j=1}^n \nabla_j V_j\right)\left(\sum_{i=1}^n \langle \mathbf{T}, e_i\rangle V_i\right)\right. \\
&\qquad \left. + H_p h_{pij} V_i V_j + \left(\sum_{i=1}^n \langle \mathbf{T}, e_i\rangle V_i\right)^2\right] e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu.
\end{aligned} \tag{3.7}$$

Here, we used the fact that $\overline{\nabla}_{e_i} e_j = h_{pij}\nu_j$ at a fixed point by the choice of the frame. Putting (3.6) and (3.7) into (3.5) yields

$$F'' = \int_{\Sigma} \left(\sum_{j=1}^{n} \nabla_j V_j + \sum_{i=1}^{n} \langle \mathbf{T}, e_i \rangle V_i \right)^2 e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \geq 0.$$

This finishes the proof of the theorem. □

4. The Lagrangian Grim Reaper Cylinder

In the previous section, we proved that any Lagrangian translating soliton is Lagrangian L-stable. However, it is not clear that whether they are L-stable or not. In this section, as an example, we will show that the Grim Reaper cylinder $\Gamma \times \mathbb{R}$ is in fact L-stable in $\mathbb{C}^2$.

First recall that the Grim Reaper Γ in the plane is defined by

$$\begin{aligned} \gamma : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) &\longrightarrow \mathbb{C} \\ x &\longrightarrow \gamma(x) = (-\log\cos x, x). \end{aligned}$$

Then the Grim Reaper cylinder $\Gamma \times \mathbb{R}$ is defined by

$$\begin{aligned} \Phi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} &\longrightarrow \mathbb{C}^2 \\ (x, y) &\longrightarrow \Phi(x, y) = (-\log\cos x, x, y, 0). \end{aligned}$$

We will see that it is a Lagrangian translating soliton and is L-stable.

Theorem 4.1. *The Grim Reaper cylinder $\Sigma = \Gamma \times \mathbb{R}$ is a Lagrangian translating soliton of $\mathbb{C}^2$ and is L-stable.*

Proof. By the definition of Φ, the tangent space of Σ is spanned by

$$\Phi_x = (\tan x, 1, 0, 0), \quad \Phi_y = (0, 0, 1, 0).$$

The orthonormal basis of the normal space can be taken as

$$e_3 = (\cos x, -\sin x, 0, 0), \quad e_4 = (0, 0, 0, -1).$$

The induced metric can be represented as

$$(g_{ij})_{1 \leq i,j, \leq 2} = \begin{pmatrix} \frac{1}{\cos^2 x} & 0 \\ 0 & 1 \end{pmatrix}, \quad (g^{ij})_{1 \leq i,j, \leq 2} = \begin{pmatrix} \cos^2 x & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

The induced area form is given by

$$d\mu = \sqrt{\det(g_{ij})} dx dy = \frac{1}{\cos x} dx dy. \tag{4.2}$$

Since

$$\Phi_{xx} = \left(\frac{1}{\cos^2 x}, 0, 0, 0\right), \quad \Phi_{xy} = (0, 0, 0, 0), \quad \Phi_{yy} = (0, 0, 0, 0),$$

from $h^{\alpha}_{ij} = \langle \Phi_{ij}, e_{\alpha} \rangle$, we see that the second fundamental form are given by

$$h^3_{xx} = \frac{1}{\cos x},\ h^3_{xy} = h^3_{yy} = h^4_{xx} = h^4_{xy} = h^4_{yy} = 0. \tag{4.3}$$

Therefore,

$$H^3 = g^{ij} h^3_{ij} = g^{xx} h^3_{xx} = \cos x,\ H^4 = g^{ij} h^4_{ij} = 0,$$

and the mean curvature vector is given by

$$\mathbf{H} = H^3 e_3 + H^4 e_4 = \cos x e_3.$$

Now if we take $\mathbf{T} = (1, 0, 0, 0) \in \mathbb{C}^2$, then

$$\mathbf{T}^{\perp} = \langle \mathbf{T}, e_3 \rangle e_3 + \langle \mathbf{T}, e_4 \rangle e_4 = \cos x e_3 = \mathbf{H}.$$

Therefore, Σ is a translating soliton in $\mathbb{C}^2$.

Recall that the standard complex structure in $\mathbb{C}^2$ is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Since

$$J\Phi_x = (1, -\tan x, 0, 0) = \frac{1}{\cos x} e_3,\ J\Phi_y = (0, 0, 0, -1) = e_4,$$

we see that Σ is a Lagrangian translating soliton in $\mathbb{C}^2$.

Next we will show that Σ is L-stable. Since

$$\Delta v + \langle \mathbf{T}, \nabla v \rangle = e^{-\langle \mathbf{T}, \mathbf{x} \rangle} div_{\Sigma} (e^{\langle \mathbf{T}, \mathbf{x} \rangle} \nabla v),$$

for any smooth function v on Σ, we can see easily from Theorem 2.2 that Σ is L-stable if and only if

$$\int_{\Sigma} g^{ik} g^{jl} h^{\alpha}_{ij} h^{\beta}_{kl} V^{\alpha} V^{\beta} e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \leq \int_{\Sigma} \sum_{\alpha} |\nabla V^{\alpha}|^2 e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \tag{4.4}$$

holds for every compactly supported normal variation vector field $\mathbf{V} = V^{\alpha} e_{\alpha}$.

In our case, $\langle \mathbf{T}, \mathbf{x} \rangle = \langle (1, 0, 0, 0), (-\log \cos x, x, y, 0) \rangle = -\log \cos x$ so that

$$e^{\langle \mathbf{T}, \mathbf{x} \rangle} = \frac{1}{\cos x}. \tag{4.5}$$

By (4.1) and (4.3), we have

$$g^{ik} g^{jl} h^{\alpha}_{ij} h^{\beta}_{kl} V^{\alpha} V^{\beta} = g^{xx} g^{xx} h^3_{xx} h^3_{xx} V^3 V^3 = \cos^2 x (V^3)^2.$$

Combining with (4.2) and (4.5), we get that

$$\int_{\Sigma} g^{ik} g^{jl} h^{\alpha}_{ij} h^{\beta}_{kl} V^{\alpha} V^{\beta} e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu = \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(V^3\right)^2 dx dy. \tag{4.6}$$

On the other hand, using (4.1), we compute

$$|\nabla V^\alpha|^2 = g^{ij}\frac{\partial}{\partial x^i}V^\alpha \frac{\partial}{\partial x^j}V^\alpha = \cos^2 x\left(\frac{\partial}{\partial x}V^\alpha\right)^2 + \left(\frac{\partial}{\partial y}V^\alpha\right)^2.$$

Therefore,

$$\int_\Sigma \sum_\alpha |\nabla V^\alpha|^2 e^{\langle \mathbf{T},\mathbf{x}\rangle} d\mu = \int_{-\infty}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(\frac{\partial}{\partial x}V^3\right)^2 + \left(\frac{\partial}{\partial x}V^4\right)^2\right] dxdy + \int_{-\infty}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cos^2 x}\left[\left(\frac{\partial}{\partial y}V^3\right)^2 + \left(\frac{\partial}{\partial y}V^4\right)^2\right] dxdy \tag{4.7}$$

Note that $V^3, V^4 \in C_0^\infty\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\times\mathbb{R}\right)$. In particular, for each fixed y, we have $V^3(\cdot, y) \in C_0^\infty\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$. By Wirtinger inequality, we have for each $y \in \mathbb{R}$ that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(V^3(x,y)\right)^2 dx \le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial}{\partial x}V^3(x,y)\right)^2 dx$$

Integrating with respect to y yields

$$\int_{-\infty}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(V^3(x,y)\right)^2 dxdy \le \int_{-\infty}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial}{\partial x}V^3(x,y)\right)^2 dxdy. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), we see that (4.4) holds for every compactly supported normal variation vector field $\mathbf{V} = V^3 e_3 + V^4 e_4$. This shows that the Lagrangian translating soliton Σ is L-stable. □

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