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Motives of moduli spaces on K3 surfaces and of special cubic fourfolds

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Abstract. For any smooth projective moduli space M of Gieseker stable sheaves on a complex projective K3 surface (or an abelian surface) S, we prove that the Chow motive $\mathfrak{h}(M)$ becomes a direct summand of a motive $\bigoplus \mathfrak{h}(S^{k_i})(n_i)$ with $k_i \leq \dim(M)$. The result implies that finite dimensionality of $\mathfrak{h}(M)$ follows from finite dimensionality of $\mathfrak{h}(S)$. The technique also applies to moduli spaces of twisted sheaves and to moduli spaces of stable objects in $\mathrm{D}^{\mathrm{b}}(S, \alpha)$ for a Brauer class $\alpha \in \mathrm{Br}(S)$. In a similar vein, we investigate the relation between the Chow motives of a K3 surface S and a cubic fourfold X when there exists an isometry $\widetilde{H}(S, \alpha, \mathbb{Z}) \cong \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$. In this case, we prove that there is an isomorphism of transcendental Chow motives $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$.

Introduction

Given a moduli space M of stable sheaves on a K3 surface S, one expects that certain invariants of M are determined by the geometry of S. We will study the relation between the Chow groups and motives of M and S. The analogous question for moduli spaces of stable vector bundles on a curve has been settled by del Baño [15]. He showed that the Chow motive of the moduli space is contained in the full pseudo-abelian tensor subcategory generated by the motive of the curve and the Lefschetz motive.

For surfaces, a natural notion of stability for sheaves is provided by Gieseker stability. More generally, we will consider stability for α-twisted sheaves with $\alpha \in \mathrm{Br}(S)$ a Brauer class. The case of a moduli space of Gieseker stable sheaves corresponds to the trivial Brauer class $\alpha = 1$. The first main result of this paper is the following:

Theorem 0.1. *Let S be a complex projective K3 surface or an abelian surface and $\alpha \in \mathrm{Br}(S)$. Assume that M is one of the following:*

- *a smooth projective moduli space of Gieseker stable α-twisted sheaves or*
- *a smooth projective moduli space of stable objects in $\mathrm{D}^{\mathrm{b}}(S, \alpha)$.*

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Then the Chow motive $\mathfrak{h}(M)$ *of* M *is a direct summand of a motive* $\bigoplus \mathfrak{h}(S^{k_i})(n_i)$ *for some* $1 \leq k_i \leq \dim M$, $n_i \in \mathbb{Z}$.

The theorem extends a result of Arapura [5, Thm. 7.8] to the level of Chow groups and, therefore, allows an application to Chow motives. In fact, our result holds true for any surface with effective anti-canonical bundle, see Remark 2.2 for details. The argument also applies to curves and gives a significantly easier proof of del Baño's result.

As a direct consequence, finite dimensionality of the motive of S implies the same for M:

Corollary 0.2. *Let* S *and* M *be as above. If* $\mathfrak{h}(S)$ *is finite dimensional, then* $\mathfrak{h}(M)$ *is finite dimensional as well.* □

The motive of any abelian variety is known to be finite dimensional [26, Ex. 9.1] and consequently we obtain the following unconditional result:

Corollary 0.3. *Let* S *be an abelian surface and* M *a smooth projective moduli space of stable sheaves on* S*. Then the Chow motive* $\mathfrak{h}(M)$ *is finite dimensional.* □

Although finite dimensionality is expected for all motives of smooth projective varieties, only a few families of K3 surfaces with finite dimensional motives are known. Even fewer examples are known in higher dimension; one example is provided by the Hilbert scheme $S^{[n]}$ of a K3 surface S with finite dimensional motive, see [14, Thn. 6.2.1].

The second half of this paper has a similar flavour; we investigate the relation between K3 surfaces and cubic fourfolds on the level of algebraic cycles. Recall that cubic fourfolds admitting a labelling of discriminant d form a divisor $\mathcal{C}_d \subseteq \mathcal{C}$ inside the moduli space of smooth complex cubic fourfolds (see Sect. 3.1 for a brief review of the relevant notions). For a cubic fourfold X, we denote by $\mathcal{A}_X \subseteq \mathrm{D}^{\mathrm{b}}(X)$ the Kuznetsov component of the derived category [28]. We prove the following result:

Theorem 0.4. *Let* $X \in \mathcal{C}_d$ *be a special cubic fourfold. Assume that there exist a K3 surface* S*, a Brauer class* $\alpha \in \mathrm{Br}(S)$ *and a Hodge isometry* $\widetilde{H}(S, \alpha, \mathbb{Z}) \cong \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$*. Then there is a cycle* $Z \in \mathrm{CH}^3(S \times X)$ *inducing an isomorphism of Chow groups* $\mathrm{CH}_0(S)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom}}$ *and transcendental motives* $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$*. Furthermore,* $\mathfrak{h}(X) \cong \mathbb{1} \oplus \mathfrak{h}(S)(1) \oplus \mathbb{L}^2 \oplus \mathbb{L}^4$ *and, therefore,* $\mathfrak{h}(S)$ *is finite dimensional if and only if* $\mathfrak{h}(X)$ *is finite dimensional.*

A (twisted) K3 surface and a Hodge isometry $\widetilde{H}(S, \alpha, \mathbb{Z}) \cong \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ as above exist if and only if d satisfies a certain numerical condition $(**')$, see for example 3.1.

The two results fit into the following picture. For a variety X we denote by $\mathrm{Mot}(X)$ the full pseudo-abelian tensor subcategory of motives generated by $\mathfrak{h}(X)$ and the Lefschetz motive $\mathbb{L}$. Let now X be a cubic fourfold and F its Fano variety of lines, which is a hyperkähler variety of dimension four. It is known that the motive of F is contained in $\mathrm{Mot}(X)$ (we say that $\mathfrak{h}(F)$ is motivated by $\mathfrak{h}(X)$ following

Arapura [5]). Indeed, Laterveer [31, Thm. 5] proved a formula for Chow motives, which is similar to the result obtained by Galkin–Shinder [19, Thm. 5.1] in the Grothendieck ring of varieties:

$$\mathfrak{h}(F)(2) \oplus \bigoplus_{i=0}^{4} \mathfrak{h}(X)(i) \cong \mathfrak{h}(X^{[2]}).$$

Since the Hilbert scheme $X^{[2]}$ can be described as a blow-up of the symmetric product $X^{(2)}$ along the diagonal, its motive is motivated by $\mathfrak{h}(X)$. In Sect. 2.2 we will argue that $\mathfrak{h}(X)$ is also motivated by $\mathfrak{h}(F)$, see also [11, Thm. 4.5]:

Corollary 0.5. *Let X be a cubic fourfold and F its Fano variety of lines. The full pseudo-abelian tensor categories of motives generated by the Lefschetz motive and $\mathfrak{h}(X)$ and $\mathfrak{h}(F)$ resp., agree:*

$$\mathrm{Mot}(X) = \mathrm{Mot}(F).$$

In particular, $\mathfrak{h}(X)$ is finite dimensional if and only if $\mathfrak{h}(F)$ is finite dimensional.

To compare this result with Theorem 0.1, assume that X is a special cubic fourfold satisfying condition $(**')$, which is equivalent to the Fano variety F being birational to a moduli space M of stable twisted sheaves on a K3 surface S, cf. [23, Prop. 4.1]. In this case, all of the following categories of motives agree:

$$\mathrm{Mot}(S) = \mathrm{Mot}(M) = \mathrm{Mot}(F) = \mathrm{Mot}(X).$$

Indeed, we know that birational hyperkähler varieties have isomorphic Chow motives, see Proposition 1.4. This induces the middle equality. It follows from Theorem 0.4 that $\mathrm{Mot}(S)$ and $\mathrm{Mot}(X)$ coincide. For an arbitrary complex projective K3 surface and a moduli space M as in Theorem 0.1 we have at least an inclusion $\mathrm{Mot}(M) \subseteq \mathrm{Mot}(S)$ which we expect to be an equality as well, see Remark 2.3 for some comments.

Notations and Conventions

We will work over the complex numbers unless otherwise stated. The bounded derived category of coherent sheaves on a smooth projective variety X is denoted by $\mathrm{D}^{\mathrm{b}}(X)$. Throughout, all motives are meant to be Chow motives with rational coefficients, see Sect. 1.

1. Preliminaries

We briefly review the main facts about Chow motives of K3 surfaces and cubic fourfolds. The objects of the category $\mathrm{Mot}_{\mathbb{C}}$ of Chow motives are triples (X, p, m), with X a smooth projective variety over $\mathbb{C}$, $p \in \mathrm{CH}^{\dim(X)}(X \times X)_{\mathbb{Q}}$ a projector (with respect to convolution) and m an integer. Morphisms are defined by

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q \circ \mathrm{CH}^{\dim(X)+n-m}(X \times Y)_{\mathbb{Q}} \circ p.$$

The motive of a smooth projective variety X is defined as $\mathfrak{h}(X) = (X, [\Delta_X], 0)$. We denote the motive of a point by $\mathbb{1}$ and the Lefschetz motive by $\mathbb{L}$. Let S be a projective K3 surface and $\rho(S)$ the Picard number of S. Recall that there is a decomposition (see e.g. [36, Ch. 6.3]):

$$\mathfrak{h}(S) \cong \mathbb{1} \oplus \mathbb{L}^{\oplus \rho(S)} \oplus \mathfrak{t}(S) \oplus \mathbb{L}^2.$$

The only mysterious part is the *transcendental motive* $\mathfrak{t}(S) = (S, \pi_S^{2,\mathrm{tr}}, 0)$. The motive of a cubic fourfold X splits similarly (cf. [11, Sec. 4]):

$$\mathfrak{h}(X) \cong \mathbb{1} \oplus \mathbb{L} \oplus (\mathbb{L}^2)^{\oplus \rho_2} \oplus \mathfrak{t}(X) \oplus \mathbb{L}^3 \oplus \mathbb{L}^4,$$

where $\rho_2 = \dim H^{2,2}(X, \mathbb{Q})$. Again, the only part which remains unclear is the transcendental motive $\mathfrak{t}(X) = (X, \pi_X^{4,\mathrm{tr}}, 0)$. The above decompositions are so called *refined Chow–Künneth decompositions*, see [36, Ch. 6.1]. The Chow and cohomology groups of the transcendental motives are given by:

$$H^*(\mathfrak{t}(S)) = H^2(\mathfrak{t}(S)) = T(S)_{\mathbb{Q}} \quad \text{and} \quad \mathrm{CH}^*(\mathfrak{t}(S)) = \mathrm{CH}^2(\mathfrak{t}(S)) = \mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}},$$
$$H^*(\mathfrak{t}(X)) = H^4(\mathfrak{t}(X)) = T(X)_{\mathbb{Q}} \quad \text{and} \quad \mathrm{CH}^*(\mathfrak{t}(X)) = \mathrm{CH}^3(\mathfrak{t}(X)) = \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}},$$

where $T(S)$ and $T(X)$ are the transcendental lattices.

Remark 1.1. One can also consider the following (coarser) decomposition of the motive of a cubic fourfold X, which will be used in the proof of Theorem 0.4. Let $h \in \mathrm{CH}^1(X)$ be the class of a hyperplane section and pt the class of any closed point. Define the primitive projector $\pi_X^{\mathrm{pr}} = [\Delta_X] - [\mathrm{pt} \times X] - \frac{1}{3}[h^3 \times h] - \frac{1}{3}[h^2 \times h^2] - \frac{1}{3}[h \times h^3] - [X \times \mathrm{pt}]$ and the *primitive motive* $\mathfrak{h}^{\mathrm{pr}}(X) = (X, \pi_X^{\mathrm{pr}}, 0)$. There is a decomposition:

$$\mathfrak{h}(X) \cong \mathbb{1} \oplus \mathbb{L} \oplus \mathbb{L}^2 \oplus \mathfrak{h}^{\mathrm{pr}}(X) \oplus \mathbb{L}^3 \oplus \mathbb{L}^4.$$

Recall the notion of a surjective morphism of motives $f\colon M \to N$. It means that the induced map $\mathrm{CH}^*(M \otimes \mathfrak{h}(Z)) \to \mathrm{CH}^*(N \otimes \mathfrak{h}(Z))$ is surjective for all smooth projective varieties Z, cf. [36, Sec. 5.4]. Equivalently, f admits a right inverse and N becomes a direct summand of M, see [36, Ex. 2.3.(vii), Lem. 5.4.3]. It is well known (cf. [45, Lem. 3.2], [11, Lem. 4.3]) that it suffices to check surjectivity of $\mathrm{CH}^i(M_K) \to \mathrm{CH}^i(N_K)$ for all function fields:

Lemma 1.2. *Let $M = (X, p, m)$, $N = (Y, q, n) \in \mathrm{Mot}_{\mathbb{C}}$ and $f \in \mathrm{Hom}(M, N)$ a morphism of motives. Assume that $(f_K)_*\colon \mathrm{CH}^i(M_K) \to \mathrm{CH}^i(N_K)$ is surjective for all finitely generated field extensions $\mathbb{C} \subseteq K$. Then f is surjective.*

Proof. Let Z be any variety over $\mathbb{C}$. The proof proceeds by induction on the dimension of Z, the case of dimension zero being trivial. Let K be the function field of Z and $\gamma \in \mathrm{CH}^i(N \otimes \mathfrak{h}(Z))$. We write $\gamma|_{N_K}$ for the pullback of γ to N_K. By assumption, there exists $\delta \in \mathrm{CH}^i(M_K)$ such that $(f_K)_*\delta = \gamma|_{N_K}$. Denote by $\bar{\delta}$ the closure of δ in $X \times Z$. Then $\gamma - (f_Z)_*\bar{\delta}$ is supported on $Y \times Z'$ for some closed proper subvariety $Z' \subseteq Z$ and we conclude by induction. □

In Sect. 3 we will also include some comments on the notion of finite dimensionality in the sense of Kimura and O'Sullivan, see e.g. [36, Ch. 4]. The following key result is essentially due to Kimura:

Proposition 1.3. *Let $M \to N$ be a surjective morphism of motives. If M is finite dimensional, then N is finite dimensional. If $M \cong M_1 \oplus M_2$, then M_1 and M_2 are finite dimensional if and only if M is finite dimensional. Moreover, if $X \to Y$ is a dominant morphism of smooth projective varieties and $\mathfrak{h}(X)$ is finite dimensional, then so is $\mathfrak{h}(Y)$.*

Proof. The first assertion is proven in [26, Prop. 6.9] and the statement about direct summands is an immediate consequence. The last assertion follows from the observation that any dominant morphism of varieties gives rise to a surjective morphism of motives, see [36, Ex. 5.4.2]. $\square$

To conclude this section, observe that the Chow motive of a hyperkähler variety is in fact a birational invariant. Indeed, for two birational hyperkähler varieties X and X' one can always find families $\mathcal{X}$ and $\mathcal{X}'$ over a smooth quasi-projective curve C, which are isomorphic away from a point $0 \in C$ with central fibres $X = \mathcal{X}_0$ resp. $X' = \mathcal{X}'_0$ (cf. [22, Thm. 10.12], [39, Prop. 2.1]). This can be used to show that their Chow rings $\mathrm{CH}^*(X)$ and $\mathrm{CH}^*(X')$ are isomorphic [39, Thm. 3.2]. The same proof also shows that their Chow motives are isomorphic, see also [44, Sec. 1.6]:

Proposition 1.4. *Let X and X' be birational hyperkähler varieties. There is an isomorphism of Chow motives*

$$\mathfrak{h}(X) \cong \mathfrak{h}(X').$$

$\square$

Our result therefore also applies to any hyperkähler variety which is birational to a moduli space as in Theorem 0.1.

2. Motives of moduli spaces of stable sheaves

2.1. Moduli spaces of stable sheaves on a K3 surface

This section contains the proof of Theorem 0.1. Let S be a projective K3 surface or an abelian surface. Assume first that M is a smooth projective moduli space of stable sheaves on S. The general case of a moduli space of stable objects in $\mathrm{D}^{\mathrm{b}}(S, \alpha)$ is treated at the end of the proof.

Proof of Theorem 0.1. Let E be a quasi-universal sheaf on $M \times S$ and F its transpose on $S \times M$. We use the following notation for the projections:

$$\begin{tikzcd} & M \times S \times M \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi"] \arrow[dr, "\pi_{23}"] & \\ M \times S & M \times M & S \times M \end{tikzcd}$$

and $\mathcal{E} = \pi_{12}^*(E)$, $\mathcal{F} = \pi_{23}^*(F)$ for the pullbacks. Consider the complex

$$W = \mathcal{E}xt_\pi^\bullet(\mathcal{E}, \mathcal{F})[1] \in \mathrm{D}^{\mathrm{b}}(M \times M)$$

whose cohomology sheaves are the relative extension sheaves $\mathcal{E}xt_\pi^i(\mathcal{E}, \mathcal{F}) = \mathbf{R}^i(\pi_* \circ \mathcal{H}om)(\mathcal{E}, \mathcal{F})$. Note that in our case only $\mathcal{E}xt_\pi^1(\mathcal{E}, \mathcal{F})$ and $\mathcal{E}xt_\pi^2(\mathcal{E}, \mathcal{F})$ are non-zero. A computation of the Chern classes due to Markman [33, Thm. 1] yields

$$c_m([W]) = [\Delta_M] \in \mathrm{CH}^m(M \times M), \tag{1}$$

where m is the dimension of M.

Consider the Chow groups $\mathrm{CH}^*(M \times M)_{\mathbb{Q}}$ as a unital ring with convolution of cycles and unit given by the diagonal. Define the following two-sided ideal generated by correspondences which factor through some power of S:

$$I = \langle \beta \circ \alpha \mid \alpha \in \mathrm{CH}^*(M \times S^k)_{\mathbb{Q}}, \beta \in \mathrm{CH}^*(S^k \times M)_{\mathbb{Q}}, k \geq 1 \rangle \subseteq \mathrm{CH}^*(M \times M)_{\mathbb{Q}}.$$

We will prove that I is closed under intersection products. Let $\alpha \in \mathrm{CH}^*(M \times S^k)_{\mathbb{Q}}$, $\beta \in \mathrm{CH}^*(S^k \times M)_{\mathbb{Q}}$, $\alpha' \in \mathrm{CH}^*(M \times S^{k'})_{\mathbb{Q}}$, $\beta' \in \mathrm{CH}^*(S^{k'} \times M)_{\mathbb{Q}}$ and denote by τ the involution of $M \times M \times M \times M$ interchanging the middle two factors:

$$\begin{aligned}(\beta \circ \alpha) \cdot (\beta' \circ \alpha') &= \left[{}^{\mathrm{t}}\Gamma_{\Delta_{M\times M}}\right]_* (\beta \circ \alpha \times \beta' \circ \alpha') = \left[{}^{\mathrm{t}}\Gamma_{\Delta_{M\times M}}\right]_* \circ \tau_* (\beta \times \beta' \circ \alpha \times \alpha') \\ &= \left[{}^{\mathrm{t}}\Gamma_{\tau\circ\Delta_{M\times M}}\right]_* (\beta \times \beta' \circ \alpha \times \alpha') = \left(\left[{}^{\mathrm{t}}\Gamma_{\Delta_M}\right] \times \left[{}^{\mathrm{t}}\Gamma_{\Delta_M}\right]\right)_* (\beta \times \beta' \circ \alpha \times \alpha') \\ &= \left(\left[{}^{\mathrm{t}}\Gamma_{\Delta_M}\right] \circ \beta \times \beta'\right) \circ \left(\alpha \times \alpha' \circ \left[\Gamma_{\Delta_M}\right]\right).\end{aligned}$$

The last equality follows from Lieberman's Lemma, cf. [36, Prop. 2.1.3]. We obtain a correspondence which factors through $S^{k+k'}$, so it is contained in I. We will conclude by showing that the class of the diagonal is contained in I.

A Grothendieck–Riemann–Roch computation gives:

$$\begin{aligned}\mathrm{ch}([W]) &= -\mathrm{ch}\Big(\pi_![\mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{F})]\Big) = -\pi_*\Big(\mathrm{ch}[\mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{F})] \cdot \pi_2^*\mathrm{td}(S)\Big) \\ &= -\pi_*\Big(\pi_{12}^*\mathrm{ch}(E^\vee) \cdot \pi_{23}^*\mathrm{ch}(F) \cdot \pi_2^*\mathrm{td}(S)\Big),\end{aligned} \tag{2}$$

where $E^\vee = \mathbf{R}\mathcal{H}om(E, \mathcal{O}_{M\times S})$ denotes the derived dual of E and π_2 is the projection to S. Let $\alpha = \oplus\alpha^i = \mathrm{ch}(E^\vee) \cdot \pi_2^*\sqrt{\mathrm{td}(S)}$, $\beta = \oplus\beta^i = \mathrm{ch}(F) \cdot \pi_2^*\sqrt{\mathrm{td}(S)}$ and $n \in \mathbb{N}$. Considering only the codimension n part of (2) we find that the nth Chern character is contained in I:

$$\mathrm{ch}_n([W]) = -\sum_{i+j=n+2} \pi_*\left(\pi_{12}^*\alpha^i \cdot \pi_{23}^*\beta^j\right) \in I.$$

The codimension n part of the Chern character is given as a sum $\dfrac{(-1)^{n-1}}{(n-1)!}c_n + p$, where p is a polynomial in the Chern classes of degree less than n. Note that $c_1 = \mathrm{ch}_1$ is contained in I and, therefore, also $c_2 = \frac{1}{2}c_1^2 - \mathrm{ch}_2 \in I$. It follows iteratively that $c_n \in I$ for all n and therefore $[\Delta_M] \in I$ by (1). Thus, there are cycles $\gamma_i \in \mathrm{CH}^{e_i}(M \times S^{k_i})_{\mathbb{Q}}$, $\delta_i \in \mathrm{CH}^{d_i}(S^{k_i} \times M)_{\mathbb{Q}}$, for some $k_i \in \mathbb{N}$, such that

$$[\Delta_M] = \sum \delta_i \circ \gamma_i \in \mathrm{CH}^m(M \times M)_{\mathbb{Q}}. \tag{3}$$

Let $\delta = \bigoplus \delta_i$ viewed as a morphism of motives $\bigoplus \mathfrak{h}(S^{k_i})(n_i) \to \mathfrak{h}(M)$ with $n_i = d_i - 2k_i$. Equation (3) asserts that $\gamma = \bigoplus \gamma_i$ defines a right inverse for δ, i.e. the following composition is the identity:

$$\mathfrak{h}(M) \xrightarrow{\gamma} \bigoplus \mathfrak{h}(S^{k_i})(n_i) \xrightarrow{\delta} \mathfrak{h}(M).$$

Hence, $\mathfrak{h}(M)$ is a direct summand of $\bigoplus \mathfrak{h}(S^{k_i})(n_i)$.

Moreover, we obtain a bound for the exponents k_i. Consider the filtration I_k of I generated by correspondences which factor through S^l with $l \leq k$. With the above notation we have $\mathrm{ch}_n \in I_1$ for all n and $I_k \cdot I_{k'} \subseteq I_{k+k'}$. Thus $k_i \leq m = \dim M$ for all i.

To conclude the proof, we consider the general case of a smooth projective moduli space M of σ-stable objects in $\mathrm{D^b}(S, \alpha)$ for a Brauer class $\alpha \in \mathrm{Br}(S)$ and stability condition σ. It has been explained in [32, pp. 2–3] that Markman's formula (1) can be obtained analogously in this case. The two crucial ingredients are the vanishing of $\mathcal{E}xt^i(\mathcal{E}_x, \mathcal{F}_x)$ for $i \neq 0, 1, 2$ and $x \in M \times M$ and the fact that $\mathcal{E}xt^2_\pi(\mathcal{E}, \mathcal{F})$ is a line bundle on the diagonal in $M \times M$. Both assertions still hold true for stable α-twisted sheaves and similarly for stable objects in $\mathrm{D^b}(S, \alpha)$. □

Corollary 2.1. *Let S and M be as above. If $\mathfrak{h}(S)$ is finite dimensional, then $\mathfrak{h}(M)$ is finite dimensional as well.* □

Remark 2.2. As mentioned in the introduction, the proof of Theorem 0.1 also applies to moduli spaces of stable vector bundles of coprime rank and degree on a curve, thus recovering del Baño's result [15, Thm. 4.5]. Indeed, it was observed by Beauville [8] that in this case Eq. (1) follows from Porteous formula. The same argument may be used in the case of a non-symplectic surface S with effective anti-canonical bundle and a moduli space of stable sheaves of positive rank, see [34, Thm. 8]. Here, the vanishing of the extension group $\mathrm{Ext}^2(E, F)$ for any two stable sheaves E and F is the key ingredient. For moduli spaces M of stable sheaves on $\mathbb{P}^2$, a description of the Chow ring $\mathrm{CH}^*(M)$ was given by Ellingsrud and Strømme [16, Thm. 1.1(iii)]. They proved that the cycle class map is an isomorphism and, therefore, the Chow motive $\mathfrak{h}(M)$ is a sum of Lefschetz powers.

Remark 2.3. We expect also that $\mathfrak{h}(S)$ is motivated by $\mathfrak{h}(M)$ (see the introduction). This holds for example in the case of a Hilbert scheme. For fine moduli spaces it would follow from a conjecture of Addington [2, Conj.]: A universal sheaf induces a Fourier–Mukai transform $F\colon \mathrm{D^b}(S) \to \mathrm{D^b}(M)$ with right adjoint R. Addington conjectured that the composition of F and R splits as follows:

$$R \circ F \cong \mathrm{id} \oplus \mathrm{id}[-2] \oplus \cdots \oplus \mathrm{id}[-2n+2].$$

If v and w are the Mukai vectors of the Fourier–Mukai kernels, we obtain:

$$[\Delta_S] = \frac{1}{n} v \circ w \in \mathrm{CH}^2(S \times S)_{\mathbb{Q}}.$$

It follows as above that $\mathfrak{h}(S)$ is a direct summand of $\bigoplus \mathfrak{h}(M)(n_i)$ for some $n_i \in \mathbb{Z}$. See for example [4, Thm. A] for some progress on the conjecture in the case of a moduli space of torsion sheaves.

2.2. The Fano variety of lines

We provide a short proof of Corollary 0.5. Let X be a cubic fourfold and F its Fano variety of lines. The Chow groups and motive of F were investigated in detail by Shen and Vial [44]. They studied Fourier transforms inducing a (particularly interesting) decomposition of the Chow ring, similar to the case of an abelian variety. The relation between the Chow groups of F and X given via the universal line (viewed as a correspondence) has been elucidated as well. We refrain from going into the details and recommend op. cit. for further reading.

Proposition 2.4. *Let X be a cubic fourfold and F its Fano variety of lines. Then the transcendental motive $\mathfrak{t}(X)$ is a direct summand of $\mathfrak{h}(F)(-1)$. In particular, the motive of X is contained in* $\mathrm{Mot}(F)$.

Proof. The universal line $L \in \mathrm{CH}^3(F \times X)$ induces a morphism f of motives:

$$\mathfrak{h}(F)(-1) \xrightarrow{L} \mathfrak{h}(X) \xrightarrow{\pi_X^{4,\mathrm{tr}}} \mathfrak{t}(X).$$

Let K be any finitely generated field extension of $\mathbb{C}$. The only non-trivial rational Chow group of $\mathfrak{t}(X_K)$ is $\mathrm{CH}^3(\mathfrak{t}(X_K)) \cong \mathrm{CH}_1(X_K)_{\mathrm{hom},\mathbb{Q}}$. Indeed, choose an embedding of K into the complex numbers and denote by Y the base change of X_K to $\mathbb{C}$, which is a smooth complex cubic fourfold. It is well known that the base change map $\mathrm{CH}^i(\mathfrak{t}(X_K)) \to \mathrm{CH}^i(\mathfrak{t}(Y))$ induced by a field extension is injective up to torsion, see e.g. [9, Lem. 1A.3] and [45, Lem. 1.2]. Now use that $\mathrm{CH}^i(\mathfrak{t}(Y))$ vanishes for $i \neq 3$. The Chow group of one-cycles is universally generated by lines [43, Thm. 1.7] and the assertion thus follows from Lemma 1.2. □

3. Motives of special cubic fourfolds

3.1. Special cubic fourfolds

Recall that cubic fourfolds admitting a labelling of discriminant d form a divisor $\mathcal{C}_d \subseteq \mathcal{C}$ inside the moduli space of smooth complex cubic fourfolds, see [21, Sec. 3.1]. The existence of an associated K3 surface (in a suitable sense) can be characterized solely in terms of d. The following numerical conditions have been introduced over the past years (we use the notation of Addington [3]):

$$\exists a, n \in \mathbb{Z} : a^2 d = 2n^2 + 2n + 2, \qquad (***)$$

$$\exists n \in \mathbb{Z} : d \mid 2n^2 + 2n + 2, \qquad (**)$$

$$\exists k, d_0 \in \mathbb{Z} : d_0 \text{ satisfies } (**) \text{ and } d = k^2 d_0. \qquad (**')$$

There are (strict) inclusions of subsets inside the moduli space $\mathcal{C}$ of cubic fourfolds:

$$\bigcup_{(***)} \mathcal{C}_d \subseteq \bigcup_{(**)} \mathcal{C}_d \subseteq \bigcup_{(**')} \mathcal{C}_d.$$

A cubic fourfold admits a labelling of discriminant d satisfying $(**')$ if and only if there exist a K3 surface S, a Brauer class $\alpha \in \mathrm{Br}(S)$ and a Hodge isometry $\widetilde{H}(S, \alpha, \mathbb{Z}) \cong \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ [23, Thm. 1.3]. In this case, we prove that there is an isomorphism of Chow motives $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$. This generalizes work of Bolognesi and Pedrini [11], and Laterveer [30]. In [11, Thm. 4.13], the authors obtained such an isomorphism in the case when $F(X) \cong S^{[2]}$. Injectivity has been proven in [30, Thm. 3.1] for cubic fourfolds invariant under a certain involution. Both cases are instances of Theorem 0.4, see the comments in Sect. 3.2. We start with a well known fact:

Lemma 3.1. *Let S be a projective K3 surface and X a cubic fourfold. Then $\mathrm{CH}_0(S)_{\mathrm{hom}}$ and $\mathrm{CH}_1(X)_{\mathrm{hom}}$ are divisible and torsion-free.*

Proof. Divisibility of $\mathrm{CH}_0(S)_{\mathrm{hom}}$ is well known and follows easily by constructing a curve through any two given points and using the Jacobian of the normalization. The theorem of Rojtman [40] implies that this group is torsion-free. Let F be the Fano variety of lines in X. It is a hyperkähler variety, so its first Betti number vanishes and it follows as above that $\mathrm{CH}_0(F)_{\mathrm{hom}}$ is divisible and torsion-free. The universal line L induces a surjection

$$\mathrm{CH}_0(F)_{\mathrm{hom}} \xrightarrow{L_*} \mathrm{CH}_1(X)_{\mathrm{hom}},$$

hence the assertion follows from the divisibility of $\mathrm{Ker}(L_*)$ which was proven by Shen and Vial [44, Thm. 20.5, Lem. 20.6]. □

Proof of Theorem 0.4. Since $\mathbb{C}$ is a universal domain, it suffices to prove the isomorphism on Chow groups. By a variant of Manin's identity principle (cf. [20, Lem. 1], [45, Lem. 3.2] or [11, Lem. 4.3]) this implies $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$. The results of Addington–Thomas [1, Thm. 1.1] and Huybrechts [23, Thm. 1.4] imply that there is an exact equivalence $\mathrm{D^b}(S) \simeq \mathcal{A}_X$ (resp. $\mathrm{D^b}(S, \alpha) \simeq \mathcal{A}_X$) if $X \in \mathcal{C}_d$ is genericand we consider this case first. Assume that $\alpha = 1$, i.e. d satisfies $(**)$. Consider the composition Φ of an exact equivalence $\mathrm{D^b}(S) \simeq \mathcal{A}_X$ and the inclusion $\mathcal{A}_X \subseteq \mathrm{D^b}(X)$. By [37, Thm. 2.2], this functor is of Fourier–Mukai type, i.e. there is a complex $\mathcal{E} \in \mathrm{D^b}(S \times X)$, such that for all $\mathcal{G} \in \mathrm{D^b}(S)$:

$$\Phi(\mathcal{G}) \cong p_*(\mathcal{E} \otimes q^*(\mathcal{G})),$$

where p and q are the projections. It follows that the left adjoint to Φ is of Fourier–Mukai type as well, say with kernel $\mathcal{F}$. Let $v = \mathrm{ch}(\mathcal{E}) \cdot \sqrt{\mathrm{td}(S \times X)}$ (resp. w) be the Mukai vector of $\mathcal{E}$ (resp. $\mathcal{F}$). It is an algebraic cycle with $\mathbb{Q}$-coefficients on $S \times X$ which needs not be of pure dimension. Denote by v^i (resp. w^i) its codimension i part. Since Φ is fully faithful, the convolution $w \circ v$ is rationally equivalent to the class of the diagonal $[\Delta_S]$ on $S \times S$. More precisely, the following equality holds in $\mathrm{CH}^2(S \times S)_{\mathbb{Q}}$:

$$[\Delta_S] = w^0 \circ v^6 + w^1 \circ v^5 + w^2 \circ v^4 + w^3 \circ v^3 + w^4 \circ v^2 + w^5 \circ v^1 + w^6 \circ v^0. \quad (4)$$

At the moment, an equivalence $\mathrm{D^b}(S) \cong \mathcal{A}_X$ (resp. $\mathrm{D^b}(S, \alpha) \cong \mathcal{A}_X$) is established only for generic $X \in \mathcal{C}_d$. This gap is expected to be filled soon and would make the last step of the proof superfluous (see the upcoming work of Bayer et al. [7]).

Recall that the homologically trivial part of the Chow groups of S and X are concentrated in codimension two and three, respectively. The induced action of v on Chow groups is compatible with the action on cohomology. Thus, $w^3 \circ v^3$ is the only summand on the right hand side of (4) acting non-trivially on $\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}}$, i.e. the following composition is the identity:

$$\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}} \xrightarrow{v_*^3} \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}} \xrightarrow{w_*^3} \mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}}.$$

This proves injectivity of v_*^3. For the surjectivity consider the following diagram:

$$\begin{tikzcd}
\mathrm{K}(S)_{\mathbb{Q}} \arrow[r, "\sim"] \arrow[d, "v"] & \mathrm{K}(\mathcal{A}_X)_{\mathbb{Q}} \arrow[r] \arrow[dr, "\phi"] & \mathrm{K}(X)_{\mathbb{Q}} \arrow[d, "v"] \\
\mathrm{CH}^*(S)_{\mathbb{Q}} \arrow[rr, "v_*"] & & \mathrm{CH}^*(X)_{\mathbb{Q}} \\
\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}} \arrow[u, hook] \arrow[rr, "v_*^3"] & & \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}. \arrow[u, hook]
\end{tikzcd}$$

Commutativity of the middle diagram follows from the Grothendieck–Riemann–Roch Theorem. It suffices to show that the image of $\phi\colon \mathrm{K}(\mathcal{A}_X)_{\mathbb{Q}} \to \mathrm{CH}^*(X)_{\mathbb{Q}}$ contains $\mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$. Indeed, this would imply that any $\beta \in \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$ lifts to some $\alpha \in \mathrm{CH}^*(S)_{\mathbb{Q}}$ such that $v_*(\alpha) = \beta$. Since the action of v on cohomology is injective, α is homologically trivial, i.e. $\alpha \in \mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}}$.

Recall that $\mathrm{CH}_1(X)$ is generated by lines by a result of Paranjape [38], see also [42, Cor. 4.3]. Let $i : \ell \subseteq X$ be the inclusion of a line and consider the associated second syzygy sheaf $\mathcal{F}_\ell$ of $\mathcal{I}_\ell(1)$ defined by:

$$0 \longrightarrow \mathcal{F}_\ell \longrightarrow H^0(X, \mathcal{I}_\ell(1)) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} \mathcal{I}_\ell(1) \longrightarrow 0.$$

Here, $\mathcal{O}_X(1)$ is the induced polarization of $X \subseteq \mathbb{P}^5$ and ev is the evaluation map which is surjective, cf. [27, Lem. 5.1]. A straightforward computation in op. cit. shows that $\mathcal{F}_\ell$ is contained in $\mathcal{A}_X$. Next, we compute the Mukai vector of $\mathcal{F}_\ell$:

$$v(\mathcal{F}_\ell) = v(\mathcal{O}_X^{\oplus 4}) - v(\mathcal{I}_\ell(1)) = v(\mathcal{O}_X^{\oplus 4}) - v(\mathcal{O}_X(1)) + v(\mathcal{O}_\ell(1)).$$

Using the Grothendieck–Riemann–Roch Theorem one finds:

$$\begin{aligned} v(\mathcal{O}_\ell(1)) &= \mathrm{ch}(\mathcal{O}_\ell) \cdot \mathrm{ch}(\mathcal{O}_X(1)) \cdot \mathrm{td}(X)^{\frac{1}{2}} = i_*(\mathrm{td}(\ell)) \cdot \mathrm{ch}(\mathcal{O}_X(1)) \cdot \mathrm{td}(X)^{-\frac{1}{2}} \\ &= ([\ell] + [\mathrm{pt}]) \cdot \mathrm{ch}(\mathcal{O}_X(1)) \cdot \mathrm{td}(X)^{-\frac{1}{2}}, \end{aligned}$$

where $[\mathrm{pt}] \in \mathrm{CH}_0(X) \cong \mathbb{Z}$ is the class of any closed point (X is rationally connected). The Todd class of X is a polynomial in the class of a hyperplane section $h = c_1(\mathcal{O}_X(1))$, in fact

$$\mathrm{td}(X) = 1 + \frac{3}{2}h + \frac{5}{4}h^2 + \frac{3}{4}h^3 + \frac{1}{3}h^4.$$

Therefore, $v(\mathcal{O}_\ell(1)) = [\ell] + \frac{5}{4}[\mathrm{pt}]$ and

$$\phi([\mathcal{F}_\ell] - [\mathcal{F}_{\ell'}]) = v(\mathcal{O}_\ell(1)) - v(\mathcal{O}_{\ell'}(1)) = [\ell] - [\ell'],$$

for each pair of lines ℓ and ℓ', which proves surjectivity of ϕ since $\mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$ is generated by cycles of this form.

So far, we proved that $Z = v^3$ induces an isomorphism $\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$. As mentioned earlier, a variant of Manin's identity principle gives that Z also induces an isomorphism of motives $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$, which extends to an isomorphism $\mathfrak{h}(S)(1) \cong \mathbb{L} \oplus \mathfrak{h}^{\mathrm{pr}}(X) \oplus \mathbb{L}^3$. Indeed, the Picard rank ρ of S equals $\rho_2 - 1$ with $\rho_2 = \dim H^{2,2}(X, \mathbb{Q})$. Thus, there are cycles $W, W' \in \mathrm{CH}^3(S \times X)_{\mathbb{Q}}$ such that

$$^{\mathrm{t}}W' \circ W = [\Delta_S], \quad W \circ {}^{\mathrm{t}}W' = \frac{1}{3}[h^3 \times h] + \pi_X^{\mathrm{pr}} + \frac{1}{3}[h \times h^3]. \tag{5}$$

This will be useful for the specialization argument below.

Next, assume that d satisfies $(**')$, i.e. $\mathrm{D}^{\mathrm{b}}(S, \alpha) \cong \mathcal{A}_X$. The composition with the inclusion is again of Fourier–Mukai type by [12, Thm. 1.1]) and the formalism of Mukai vectors works in the twisted case as well, see [25, Sec. 1] for details. For $E \in \mathrm{Coh}(S \times X, \alpha^{-1} \boxtimes 1)$ locally free and $n = \mathrm{ord}(\alpha)$ the order of the Brauer class, $E^{\otimes n}$ is naturally an untwisted sheaf and one defines (cf. [24, Sec. 2.1])

$$v(E) = \sqrt[n]{\mathrm{ch}(E^{\otimes n})} \cdot \sqrt{\mathrm{td}(S \times X)}.$$

The nth root can be obtained formally, since $\mathrm{rk}(E) \neq 0$. Using a locally free resolution, this definition extends to all twisted coherent sheaves. Define the cycle Z as above. The proof now works analogously, replacing $\mathrm{D}^{\mathrm{b}}(S)$ by $\mathrm{D}^{\mathrm{b}}(S, \alpha)$ and $\mathrm{K}(S)$ by $\mathrm{K}(S, \alpha)$.

Finally, we prove the assertion for any $X_0 \in \mathcal{C}_d$ via specialization. Let $T \subseteq \mathcal{C}_d$ be a curve passing through the point corresponding to X_0 such that there are families of K3 surfaces (resp. cubic fourfolds) $\mathcal{S}$ and $\mathcal{X}$ over T with an exact equivalence $\mathrm{D}^{\mathrm{b}}(\mathcal{S}_s) \cong \mathcal{A}_{\mathcal{X}_s}$ over a very general point $s \in T$ and $\mathcal{X}_0 \cong X_0$ for a closed point $0 \in T$, see [1, Thm. 1.1]. Write S_0 for the fibre of $\mathcal{S}$ over 0.

By a standard argument (see e.g. [41, Lem. 8]) we may assume that T is the spectrum of a complete discrete valuation ring $R \cong \mathbb{C}[\![t]\!]$ with generic point η and closed point 0. Write $K = \mathbb{C}(\!(t)\!)$ for its fraction field and $\bar{K}$ for an algebraic closure of K.

Let $W, W' \in \mathrm{CH}^3(\mathcal{S}_{\bar{\eta}} \times_{\bar{K}} \mathcal{X}_{\bar{\eta}})$ be as above, such that (5) holds. In fact, all cycles of (5) are defined over a finite extension $\mathbb{C}(\!(t^{\frac{1}{n}})\!)$ of K. Replacing R by $\mathbb{C}[\![t^{\frac{1}{n}}]\!]$, we may assume that the cycles W and W' are defined over K. Recall the specialization map for Chow groups (see [18, Ch. 10.1] for details), which is compatible with intersection product, pullback and proper pushforward. We obtain cycles $W_0, W_0' \in \mathrm{CH}^3(S_0 \times X_0)_{\mathbb{Q}}$ such that equalities of the form (5) hold. Thus, W_0 induces an isomorphism of motives $\mathfrak{h}(S_0)(1) \cong \mathbb{L} \oplus \mathfrak{h}^{\mathrm{pr}}(X_0) \oplus \mathbb{L}^3$. The action on Chow groups restricts to an isomorphism of homologically trivial cycles

$\mathrm{CH}_0(S)_{\mathrm{hom},\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom},\mathbb{Q}}$ induced by $\pi^{4,\mathrm{tr}}_{X_0} \circ W_0 \circ \pi^{2,\mathrm{tr}}_{S_0}$. In fact, $\mathrm{CH}_0(S)_{\mathrm{hom}}$ and $\mathrm{CH}_1(X)_{\mathrm{hom}}$ are both divisible and torsion-free, see Lemma 3.1. Hence, tensoring with $\mathbb{Q}$ is a bijection and we obtain an isomorphism of integral Chow groups. □

Corollary 3.2. *Let $X \in \mathcal{C}_d$ be a special cubic fourfold with d satisfying $(**')$ and S an associated (twisted) K3 surface. Then $\mathfrak{h}(X)$ is finite dimensional if and only if $\mathfrak{h}(S)$ is finite dimensional. Moreover, if $\rho_2 = \dim H^{2,2}(X, \mathbb{Q}) \geq 20$, then $\mathfrak{h}(X)$ is finite dimensional.*

Proof. The above theorem evidently implies $\mathfrak{h}(X) \cong \mathbb{1} \oplus \mathfrak{h}(S)(1) \oplus \mathbb{L}^2 \oplus \mathbb{L}^4$. This proves the first assertion. If $\rho_2 = \dim H^{2,2}(X, \mathbb{Q}) \geq 20$, then the Picard rank of S is at least 19 and, therefore, S admits a Shioda–Inose structure, cf. [35, Cor. 6.4]. The motive of an abelian variety is finite dimensional, see e.g. [36, Ch. 4.6, Thm. 2.7.2]. Thus, $\mathfrak{h}(S)$ is finite dimensional and we conclude using Proposition 1.3. □

3.2. Examples

This section contains a comparison with the work of Bolognesi, Pedrini [11] and some applications of Theorem 0.4. In each example, the relation on the level of motives between the K3 surface and the cubic fourfold becomes visible by a concrete geometric construction.

Example 3.3. (*Cubic fourfolds containing a plane*) Consider the divisor $\mathcal{C}_8 \subseteq \mathcal{C}$. It corresponds exactly to the cubic fourfolds X containing a plane, cf. [46, Sec. 3]. In this case, there is the following standard construction: Let $\widetilde{X}$ be the blow-up of X along a plane P. Projecting X from P onto a disjoint plane in $\mathbb{P}^5$ yields a rational map which can be resolved to give a morphism $q : \widetilde{X} \to \mathbb{P}^2$. The fiber of q over a point $x \in \mathbb{P}^2$ is the residual surface of the intersection $\overline{xP} \cap X$. Generically, it is a smooth quadric surface, i.e. isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and has two different rulings. The discriminant divisor of q is a sextic curve in $\mathbb{P}^2$, which is smooth if and only if X does not contain any other plane meeting P, see e.g. [6, Prop. 4.1]. The fibres over points of the discriminant curve are singular with only one ruling. More precisely, let $F(\widetilde{X}/\mathbb{P}^2)$ be the relative Fano variety of lines with universal line $L \subseteq F(\widetilde{X}/\mathbb{P}^2) \times \widetilde{X}$. The projection $L \to \mathbb{P}^2$ factors through a double cover $S \to \mathbb{P}^2$ branched along a sextic curve, which is smooth for a general choice of X. Thus, S is a K3 surface. The projection $L \to S$ is a $\mathbb{P}^1$-bundle (a Brauer–Severi variety) and induces a Brauer class $\alpha \in \mathrm{Br}(S)$. Kuznetsov showed that there is an exact equivalence $\mathrm{D}^{\mathrm{b}}(S, \alpha) \cong \mathcal{A}_X$, cf. [28, Thm. 4.3].

It is well known that rationality of the cubic fourfold X follows, if q has a rational section. This holds true if there is an additional surface $W \subseteq X$ such that $\deg(W) - \langle P, W \rangle$ is odd. In this case, it was observed in [11, Sec. 8] that the isomorphism $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$ would follow from finite dimensionality of $\mathfrak{h}(S)$. In fact, Theorem 0.4 implies that the isomorphism $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$ holds without any further assumptions.

Example 3.4. (*Cubic fourfolds with an automorphism of order three*) Let X be a cubic fourfold given by an equation of the form

$$f(x_0, x_1, x_2) - g(x_3, x_4, x_5) = 0,$$

where f and g are homogeneous polynomials of degree three. Denote by ζ_3 a primitive third root of unity. Then X is invariant under the automorphism σ of $\mathbb{P}^5$ given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : x_1 : x_2 : \zeta_3 x_3 : \zeta_3 x_4 : \zeta_3 x_5].$$

Thus, there is an induced automorphism σ_F of the Fano variety $F(X)$, which is in fact symplectic, i.e. $\sigma_F|_{H^{2,0}} = \mathrm{id}$, see e.g. [17] for a classification of polarized symplectic automorphisms of $F(X)$. Consider the cubic surfaces $Z_1 = \{f(x_0, x_1, x_2) - s^3 = 0\}$ and $Z_2 = \{g(x_3, x_4, x_5) - t^3 = 0\}$ in $\mathbb{P}^3$ with s resp. t as additional variables. The rational map

$$([x_0 : x_1 : x_2 : s], [x_3 : x_4 : x_5 : t]) \mapsto \left[\frac{x_0}{s} : \frac{x_1}{s} : \frac{x_2}{s} : \frac{x_3}{t} : \frac{x_4}{t} : \frac{x_5}{t}\right]$$

induces a degree three morphism $\widetilde{Z_1 \times Z_2} \to X$ from the blow-up of $Z_1 \times Z_2$ along $E_1 \times E_2$. Here, E_i is the cubic curve in Z_i defined by the vanishing of s resp. t, see e.g. [13, Prop. 1.2].

Note that finite dimensionality of $\mathfrak{h}(X)$ follows from Proposition 1.3 since rational surfaces have finite dimensional motives. Moreover, this morphism can be used to find two disjoint planes P_1 and P_2 contained in X; if $\ell_i \subseteq Z_i$ are lines (recall that Z_i contains 27 of them) the image of the product $\ell_1 \times \ell_2$ is a plane in X and certain choices of lines produce disjoint planes, cf. [13, Rem. 2.4]. There is a birational map from $P_1 \times P_2$ to X sending a pair of points (x, y) to the residual point of the intersection $\overline{xy} \cap X$. The indeterminacy locus $S \subseteq P_1 \times P_2$ parametrizes lines contained in X joining the two planes. It is a complete intersection of divisors of type $(1, 2)$ and $(2, 1)$, i.e. S is a K3 surface, see [19, Ex. 5.9]. Resolving the indeterminacy locus gives an isomorphism $\mathrm{Bl}_S(P_1 \times P_2) \xrightarrow{\sim} \mathrm{Bl}_{P_1 \cup P_2}(X)$ which induces $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$ by comparing homologically trivial cycles. In fact, the cubic fourfold X satisfies condition $(\ast\ast\ast)$, since the Fano variety of X is birational to the Hilbert scheme $S^{[2]}$.

Example 3.5. (*Cubic fourfolds with an involution*) Consider the involution σ on $\mathbb{P}^5$ given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : x_1 : x_2 : x_3 : -x_4 : -x_5].$$

A cubic X invariant under σ is always of the form

$$\{F(x_0, x_1, x_2, x_3) + x_4^2 L_1 + x_5^2 L_2 + x_4 x_5 L_3 = 0\},$$

where F is homogeneous of degree three and the L_i are linear forms in $x_0, \dots, x_3$. Note that the fixed locus of σ in $\mathbb{P}^5$ is the union of $\mathbb{P}^3 = \{x_4 = x_5 = 0\}$ and the line $\ell = \{[0 : 0 : 0 : 0 : x_4 : x_5]\}$. Thus, the fixed locus in X consists of a cubic surface W and the line ℓ.

The fixed locus of the induced symplectic involution on the Fano variety $F(X)$ can be described as follows. It consists of the line ℓ, the 27 lines contained in W and a K3 surface S. The surface S parametrizes lines contained in X joining W and ℓ. It is a double cover of the cubic W branched along the degree 6 curve $L_3^2 - L_1 L_2$. This suggests that S is associated to X: The inclusion $S \subseteq F(X)$ induces an isomorphism $H^{2,0}(F(X)) \cong H^{2,0}(S)$ and an isomorphism of transcendental lattices. Composing with the incidence correspondence, we get $T(S)(-1) \cong T(X)$. It is not directly obvious that this is an isometry. An isomorphism $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$ was nevertheless established by Bolognesi and Pedrini [11, Sec. 5.2] building on work of Laterveer [30, Thm. 3.1].

Example 3.6. (*Cyclic cubic fourfolds*) Let $f(x_0, \ldots, x_4)$ be a homogeneous polynomial of degree three, defining a smooth cubic threefold $Y \subseteq \mathbb{P}^4$. A cyclic cubic fourfold is a triple cover $X \to \mathbb{P}^4$ ramified along Y. It is a smooth cubic hypersurface $X \subseteq \mathbb{P}^5$ with an equation:

$$f(x_0, \ldots, x_4) + x_5^3 = 0$$

and covering automorphism $\sigma\colon X \xrightarrow{\sim} X$ given by:

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4 : \zeta_3 x_5].$$

It was shown in [29, Thm. 3.1] that the motive of a cyclic cubic fourfold X is finite dimensional. If X satisfies condition $(**')$ and S is an associated (twisted) K3 surface, then $\mathfrak{t}(S)(1) \cong \mathfrak{t}(X)$ and $\mathfrak{h}(S)$ is finite dimensional as well. Unfortunately, it is not clear which K3 surfaces can be associated to X as above. Note that the family of cyclic cubic fourfolds contains the Fermat cubic, so in particular it has non-trivial intersection with the divisor $\mathcal{C}_8$ of cubic fourfolds containing a plane. However, there exists an example of a cyclic Pfaffian cubic fourfold containing no plane, see [10, Prop. 5.1].

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