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Gradient estimates for a nonlinear parabolic equation and Liouville theorems

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Abstract. We establish local elliptic and parabolic gradient estimates for positive smooth solutions to a nonlinear parabolic equation on a smooth metric measure space. As applications, we determine various conditions on the equation's coefficients and the growth of solutions that guarantee the nonexistence of nontrivial positive smooth solutions to many special cases of the nonlinear equation. In particular, we apply gradient estimates to discuss some Yamabe-type problems of complete Riemannian manifolds and smooth metric measure spaces.

1. Introduction and main results

In this paper we will study gradient estimates for positive smooth solutions $u(x, t)$ to a parabolic equation

$$\left(\Delta_f - \frac{\partial}{\partial t}\right)u + \mu(x, t)u + p(x, t)u^\alpha + q(x, t)u^\beta = 0 \quad (1.1)$$

on a smooth metric measure space $(M, g, e^{-f} dv_g)$, where $\mu(x, t)$, $p(x, t)$ and $q(x, t)$ are all smooth space-time functions, and $\alpha, \beta \in \mathbb{R}$. As applications, we give Liouville-type theorems for various special cases of Eq. (1.1). In particular, since Eq. (1.1) is related to Yamabe-type problems (see the explanation below), we also apply gradient estimates to study some Yamabe-type problems of complete Riemannian manifolds and smooth metric measure spaces.

A smooth metric measure space is a tuple $(M, g, e^{-f} dv_g)$ of an n -dimensional complete Riemannian manifold (M, g) , and a weighted measure $e^{-f} dv_g$ determined by some $f \in C^\infty(M)$ and the Riemannian volume element dv_g of the metric g . Such spaces arise in many contexts, for example as collapsed measured Gromov–Hausdorff limits [31]. On $(M, g, e^{-f} dv_g)$, the f -Laplacian is defined by

$$\Delta_f = \Delta - \nabla f \cdot \nabla,$$

where Δ is the usual Laplacian, which is self-adjoint with respect to $e^{-f} dv_g$. For any number $m \geq 0$, the m -Bakry–Émery Ricci tensor introduced by Bakry and

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Émery [5] is defined by

$$\text{Ric}_f^m := \text{Ric} + \text{Hess } f - \frac{1}{m} df \otimes df,$$

where Ric is the Ricci tensor of (M, g) , and Hess is the Hessian of metric g . When $m = 0$, it means that f is constant and Ric_f^m returns to the usual Ricci tensor Ric . In [32], the weighted scalar curvature related to Ric_f^m is defined by

$$S_f^m := S + 2\Delta f - \frac{m+1}{m} |\nabla f|^2,$$

where S is the scalar curvature of (M, g) . In general, S_f^m is not the trace of Ric_f^m , except when f is constant. When $m \rightarrow \infty$, we have the Perelman's scalar curvature (see [34])

$$S_f^\infty := S + 2\Delta f - |\nabla f|^2$$

and the $(\infty-)$ Bakry–Émery Ricci tensor

$$\text{Ric}_f := \text{Ric}_f^\infty.$$

It is easy to see that $\text{Ric}_f^m \geq c$ implies $\text{Ric}_f \geq c$, but not vice versa.

On a smooth metric measure space $(M, g, e^{-f} dv_g)$, if

$$\text{Ric}_f = \lambda g$$

for some $\lambda \in \mathbb{R}$, then $(M, g, e^{-f} dv_g)$ is a gradient Ricci soliton, which is a generalization of an Einstein manifold. Gradient Ricci solitons play a fundamental role in the formation of singularities of the Ricci flow, and have been studied by many authors; see [10, 22] and references therein for nice surveys.

There have been many gradient estimates and Liouville-type theorems about special cases of Eq. (1.1). In 1980s, Gidas and Spruck [19] studied the equation

$$\Delta u + p(x)u^\alpha = 0, \quad 1 \leq \alpha < \frac{n+2}{n-2} \quad (1.2)$$

on an n -dimensional manifold. The case $\alpha = 3$ is relevant to Yang-Mills equations (see [7]). The case $\alpha < 0$ is related to a steady state of the thin film (see [20]). Gidas and Spruck [19] proved that any nonnegative solution to Eq. (1.2) is identically zero when the Ricci tensor of manifold is nonnegative. Yang [47] showed that if $\alpha < 0$ and $p(x)$ is positive constant, then Eq. (1.2) does not admit any positive solution on a complete manifold with the nonnegative Ricci tensor. Li [27] proved the Gidas–Spruck's result under some weaker restrictions of $p(x)$ for $1 < \alpha < \frac{n}{n-2}$ ($n \geq 4$). He also proved Li–Yau gradient estimates and Harnack inequalities for the nonlinear parabolic equation

$$\left(\Delta - \frac{\partial}{\partial t} \right) u + p(x, t)u^\alpha = 0, \quad \alpha > 0 \quad (1.3)$$

on a manifold. In biomathematics, Eq. (1.3) could be interpreted as the population dynamics (see [9]). Recently, Zhu [50,51] gave elliptic gradient estimates and Liouville-type theorems for positive ancient solutions to Eq. (1.3).

Apart from the relation to the above equations, the famous and widely studied special example of Eq. (1.1) is related to conformally deformation of the scalar curvature on a manifold. Indeed, for any n -dimensional ($n \geq 3$) complete manifold (M, g) , consider a pointwise conformal metric

$$\tilde{g} = u^{\frac{4}{n-2}} g$$

for some $0 < u \in C^\infty(M)$. Then the scalar curvature \tilde{S} of metric \tilde{g} related to the scalar curvature S of metric g is given by (see [33])

$$\Delta u - \frac{n-2}{4(n-1)} S u + \frac{n-2}{4(n-1)} \tilde{S} u^{\frac{n+2}{n-2}} = 0, \tag{1.4}$$

which is a special form of Eq. (1.1). If M is compact and \tilde{S} is constant, the existence of a positive solution u is the well-known Yamabe problem and it has been solved in the affirmative by the combined efforts of Yamabe [45], Trudinger [39], Aubin [2] and Schoen [36]; see the survey [25] for more details. However, if M is noncompact (\tilde{S} is still constant), Jin [23] gave examples of complete metrics on the noncompact manifold on which there do not exist a positive smooth solution of (1.4). When \tilde{S} is a smooth function, the geometry of manifolds plays a large role in the existence and nonexistence of positive solutions of (1.4) on compact or noncompact manifolds. The interested reader can refer to [6,24,28,33,37,49] and references therein.

Another important reason of studying Eq. (1.1) is that a static form of Eq. (1.1) is related to the weighted Yamabe problem posed by Case [15]. Recall that, for any $m \geq 0$, Case [15] introduced the weighted Yamabe quotient

$$Q(u) := \frac{\left(\int_M |\nabla u|^2 + \frac{m+n-2}{4(m+n-1)} S_f^m u^2\right) \left(\int_M |u|^{\frac{2(m+n-1)}{m+n-2}} e^{\frac{f}{m}}\right)^{\frac{2m}{n}}}{\left(\int_M |u|^{\frac{2(m+n)}{m+n-2}}\right)^{\frac{2m+n-2}{n}}}$$

on a smooth metric measure space $(M, g, e^{-f} dv_g)$, where all integrals are taken with respect to the weighted measure $e^{-f} dv_g$. The weighted Yamabe quotient is conformally invariant in the sense that if

$$\left(M^n, \tilde{g}, e^{-\tilde{f}} dv_{\tilde{g}}\right) = \left(M^n, e^{\frac{2\rho}{m+n-2}} g, e^{\frac{(m+n)\rho}{m+n-2}} e^{-f} dv_g\right)$$

for some $\rho \in C^\infty(M)$, then $\tilde{Q}(u) = Q(e^{\frac{\rho}{2}} u)$ (see [15]). The weighted Yamabe constant is defined by

$$\Lambda[g, e^{-f} dv_g] := \inf \{ Q(u) \mid 0 < u \in C^\infty(M) \},$$

which is a generalization of the Yamabe constant. Indeed, if $f = 0$ and $m = 0$, the weighted Yamabe constant returns to the classical Yamabe constant. In [15]

Case observed that u is a critical point of the weighted Yamabe quotient $\mathcal{Q}(u)$ on a smooth metric measure space $(M, g, e^{-f} dv_g)$ if and only if it satisfies

$$\Delta_f u - \frac{m+n-2}{4(m+n-1)} S_f^m u - c_1 e^{\frac{f}{m}} u^{\frac{m+n}{m+n-2}} + c_2 u^{\frac{m+n+2}{m+n-2}} = 0, \tag{1.5}$$

which is a special elliptic case of (1.1) in some setting. Here,

$$c_1 = \frac{2m(m+n-1)\mathcal{Q}(u)}{n(m+n-2)} \left(\int_M u^{\frac{2(m+n)}{m+n-2}} \right)^{\frac{2m+n-2}{n}} \left(\int_M u^{\frac{2(m+n-1)}{m+n-2}} e^{\frac{f}{m}} \right)^{-\frac{2m+n}{n}},$$

$$c_2 = \frac{(2m+n-2)(m+n)\mathcal{Q}(u)}{n(m+n-2)} \left(\int_M u^{\frac{2(m+n)}{m+n-2}} \right)^{\frac{2m-2}{n}} \left(\int_M u^{\frac{2(m+n-1)}{m+n-2}} e^{\frac{f}{m}} \right)^{-\frac{2m}{n}},$$

where all integrals are taken with respect to $e^{-f} dv_g$. Obviously, c_1 and c_2 have the same sign. When $\Lambda[g, e^{-f} dv_g] = 0$, we have $c_1 = c_2 = 0$ and the critical point of \mathcal{Q} is in fact a minimizer of Λ . Case [15] proved that minimizers always exist on a compact smooth metric measure space provided the weighted Yamabe constant is strictly less than its value on Euclidean space.

In this paper, we will give local elliptic and parabolic gradient estimates for positive solutions to Eq. (1.1) on a smooth metric measure space with the Bakry–Émery Ricci tensor bounded below. As applications, we will determine various conditions on the growth of solutions and coefficients that guarantee the nonexistence of nontrivial positive smooth solutions to many special cases of Eq. (1.1). In particular, we can apply gradient estimates to analyze Yamabe-type problems of equations (1.4) and (1.5) on a complete manifold and a smooth metric measure space, respectively.

In order to state the results, we introduce some notations. On an n -dimensional complete smooth metric measure space $(M, g, e^{-f} dv)$, let ∇ and $|\cdot|$ stand for the Levi-Civita connection and the norm with respect to metric g , respectively. For a fixed point $x_0 \in M$ and $R > 0$, let $r(x)$ (or $d(x, x_0)$) denote a distance function to x from x_0 with respect to g , and $B(x_0, R)$ denote the geodesic ball centered at x_0 of radius R . In the elliptic gradient estimate setting, let $Q_{R,T}$ be

$$Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty), \quad t_0 \in \mathbb{R} \text{ and } T > 0.$$

In the parabolic gradient estimate setting, let $H_{R,T}$ be

$$H_{R,T} := B(x_0, R) \times [0, T], \quad T > 0.$$

For any $\mu \in C^\infty(Q_{R,T})$, denote

$$\mu^+ := \sup_{(x,t) \in Q_{R,T}} \{\mu^+(x, t), 0\} \text{ and } \mu^- := \inf_{(x,t) \in Q_{R,T}} \{\mu^-(x, t), 0\},$$

where $\mu^+(x, t) := \max\{\mu(x, t), 0\}$ and $\mu^-(x, t) := \min\{\mu(x, t), 0\}$. For $\mu \in C^\infty(H_{R,T})$, we similarly define μ^+ and μ^- in $H_{R,T}$ as above. We also introduce the geometric quantities

$$\sigma := \max_{\{x|d(x,x_0)=1\}} \Delta_f r(x) \text{ and } \sigma^+ := \max\{\sigma, 0\},$$

which will appear in our theorems.

We now give one of main theorems, a local elliptic (space-only) gradient estimates for positive smooth solutions to Eq. (1.1) when Ric_f is bounded below.

Theorem 1.1. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space. Assume that $\text{Ric}_f \geq -(n - 1)K$ for some constant $K \geq 0$ in $B(x_0, R)$, where $x_0 \in M$ and $R \geq 2$. Let $0 < u(x, t) \leq D$ for some constant D , be a smooth solution to Eq. (1.1) in $Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0]$. There exists a constant c depending only on n , such that*

$$|\nabla \ln u| \leq c \left(1 + \ln \frac{D}{u} \right) \left[\frac{1}{R} + \sqrt{\frac{\sigma^+}{R}} + \frac{1}{\sqrt{t - t_0 + T}} + \sqrt{K} + \sqrt{\mu^+} + \sup_{Q_{R,T}} |\nabla \mu|^{\frac{1}{3}} \right. \\ \left. + \sqrt{[(\alpha - 1)p]^+ + p^+} \sup_{Q_{R,T}} \{u^{\frac{\alpha-1}{2}}\} + \sup_{Q_{R,T}} |\nabla p|^{\frac{1}{3}} \sup_{Q_{R,T}} \{u^{\frac{\alpha-1}{3}}\} \right. \\ \left. + \sqrt{[(\beta - 1)q]^+ + q^+} \sup_{Q_{R,T}} \{u^{\frac{\beta-1}{2}}\} + \sup_{Q_{R,T}} |\nabla q|^{\frac{1}{3}} \sup_{Q_{R,T}} \{u^{\frac{\beta-1}{3}}\} \right]$$

in $Q_{R/2,T}$ with $t \neq t_0 - T$.

Remark 1.2. If f is constant, the term $\sqrt{\frac{\sigma^+}{R}}$ is unnecessary in the above estimate. If $\mu(x, t)$, $p(x, t)$ and $q(x, t)$ are identically zero, the theorem returns to [42]. Recently, Dung et al. [18] proved similar results when $\mu(x, t)$, $p(x, t)$, $q(x, t)$, α and β are special constants.

Besides, we can give a local parabolic (space-time) gradient estimate for positive smooth solutions to Eq. (1.1) when Ric_f^m is bounded below.

Theorem 1.3. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space. Assume that $\text{Ric}_f^m \geq -(m + n - 1)K$ ($m < \infty$) for some constant $K \geq 0$ in $B(x_0, 2R)$, where $x_0 \in M$ and $R > 0$. Let $u(x, t)$ be a positive smooth solution to Eq. (1.1) in $H_{2R,T} := B(x_0, 2R) \times [0, T]$. Also assume that*

$$|\nabla p| \leq a_1, \quad \Delta_f p \geq b_1 \quad \text{for some constants } a_1 \text{ and } b_1; \\ |\nabla q| \leq a_2, \quad \Delta_f q \geq b_2 \quad \text{for some constants } a_2 \text{ and } b_2; \\ |\nabla \mu| \leq a_3, \quad \Delta_f \mu \geq b_3 \quad \text{for some constants } a_3 \text{ and } b_3$$

in $B(x_0, 2R)$. For any $\lambda > 1$ and $\varepsilon \in (0, 1)$ satisfying $\Psi \geq 0$, there exists a universal positive constant c_1 independent of the geometry of $(M, g, e^{-f} dv)$ such

that

$$\begin{aligned} \frac{|\nabla u|^2}{\lambda u^2} + pu^{\alpha-1} + qu^{\beta-1} + \mu - \frac{u_t}{u} &\leq \frac{(m+n)\lambda}{2t} + \sqrt{\frac{m+n}{2}} \Psi^{\frac{1}{2}} \\ &+ \frac{m+n}{2R^2} \lambda \left[(m+n)c_1(1 + R\sqrt{K}) + 2c_1^2 + \frac{(m+n)c_1^2\lambda^2}{4(\lambda-1)} \right] \\ &+ \frac{m+n}{2} \lambda \left\{ [(\alpha-1)p]^+ \sup_{H_{2R,T}} \{u^{\alpha-1}\} + [(\beta-1)q]^+ \sup_{H_{2R,T}} \{u^{\beta-1}\} \right\} \end{aligned}$$

in $B(x_0, R) \times (0, T]$, where

$$\begin{aligned} \Psi &:= \frac{3}{2} \left[\frac{(m+n)\lambda^2}{4\varepsilon(\lambda-1)^2} \right]^{\frac{1}{3}} \gamma^{\frac{4}{3}} + \frac{(m+n)\lambda^2 \tilde{K}^2}{2(1-\varepsilon)(\lambda-1)^2} \\ &\quad - \lambda \left[\inf_{H_{2R,T}} (u^{\alpha-1}b_1 + u^{\beta-1}b_2) + b_3 \right], \\ \gamma &:= a_1 |\lambda\alpha - 1| \sup_{H_{2R,T}} \{u^{\alpha-1}\} + a_2 |\lambda\beta - 1| \sup_{H_{2R,T}} \{u^{\beta-1}\} + a_3(\lambda - 1) \end{aligned}$$

and

$$\begin{aligned} \tilde{K} &:= (m+n-1)K - \frac{1}{2} [(\alpha-1)(\lambda\alpha-1)p]^- \sup_{H_{2R,T}} \{u^{\alpha-1}\} \\ &\quad - \frac{1}{2} [(\beta-1)(\lambda\beta-1)q]^- \sup_{H_{2R,T}} \{u^{\beta-1}\}. \end{aligned}$$

Remark 1.4. If f is constant, $p(x, t)$ and $q(x, t)$ are identically zero, then the theorem returns to the well-known Li–Yau gradient estimate [29]. More parabolic gradient estimates for special cases of Eq. (1.1) were proved in [11, 13, 27, 30].

Theorem 1.1 describes local elliptic gradient estimates under only Ric_f bounded below, whose assumption on Ric_f is obviously weaker than the assumption on Ric_f^m ($m < \infty$). Theorem 1.3 describes local Li–Yau gradient estimates under the assumption on Ric_f^m ($m < \infty$) rather than Ric_f , because, according to [42], there seems essential obstacles to obtain Li–Yau gradient estimates for Eq. (1.1) when Ric_f is bounded below, even assuming growth assumption on f .

Theorems 1.1 and 1.3 have many applications. On one hand, we apply Theorem 1.1 to get parabolic Liouville-type theorems for special cases of Eq. (1.1). Here, we only provide two typical results. More related results will be discussed in Sect. 4.

Theorem 1.5. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space with $\text{Ric}_f \geq 0$. Assume that there exist two constants $s > 0$ and $\kappa > 0$, such that $\mu(x)$ and $p(x)$ in the following equation*

$$\left(\Delta_f - \frac{\partial}{\partial t} \right) u + \mu(x)u + p(x)u^\alpha = 0, \quad \alpha > 1, \quad p(x) \not\equiv 0, \quad (1.6)$$

satisfy

- (1) $\mu^+|_{B(x_0,R)} = o(R^{-s})$ and $\sup_{B(x_0,R)} |\nabla\mu| = o(R^{-s})$, as $R \rightarrow \infty$;
- (2) $p^+|_{B(x_0,R)} = o[R^{-\kappa(\alpha-1)}]$ and $\sup_{B(x_0,R)} |\nabla p| = o[R^{-\kappa(\alpha-1)}]$, as $R \rightarrow \infty$.

Let $u(x, t)$ be a positive ancient solution to Eq. (1.6) (that is, a solution defined in all space and negative time) such that

$$u(x, t) = o[(r(x) + |t|)^{\tilde{\kappa}}]$$

for some $\tilde{\kappa} \in (0, \kappa)$ near infinity. Then $u(x, t) \equiv c^{\frac{1}{\alpha-1}}$ and $\mu(x) \equiv -cp(x)$ for some constant $c > 0$.

Theorem 1.6. Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space with $\text{Ric}_f \geq 0$. Assume that there exist two constants $s > 0$ and $\kappa > 0$, such that $\mu(x)$ and $p(x)$ in the following equation

$$\left(\Delta_f - \frac{\partial}{\partial t} \right) u + \mu(x)u + p(x)u^\alpha = 0, \quad \alpha < 1, \quad p(x) \not\equiv 0, \quad (1.7)$$

satisfy

- (1) $\mu^+|_{B(x_0,R)} = o(R^{-s})$ and $\sup_{B(x_0,R)} |\nabla\mu| = o(R^{-s})$, as $R \rightarrow \infty$;
- (2) $\sup_{B(x_0,R)} |p| = o[R^{-\kappa(1-\alpha)}]$ and $\sup_{B(x_0,R)} |\nabla p| = o[R^{-\kappa(1-\alpha)}]$, as $R \rightarrow \infty$.

Let $u(x, t)$ be a positive ancient solution to (1.7) such that

$$(r(x) + |t|)^{-\tilde{\kappa}} \leq u(x, t) \leq (r(x) + |t|)^\delta$$

for some $\tilde{\kappa} \in (0, \kappa)$ and $\delta > 0$ near infinity. Then $u(x, t) \equiv c^{\frac{1}{\alpha-1}}$ and $\mu(x) \equiv -cp(x)$ for some constant $c > 0$.

We also apply Theorem 1.1 to prove Liouville-type theorems for elliptic versions of Eq. (1.1); see for example Theorems 5.1 and 5.2 in Sect. 5. In particular, we apply Theorem 1.5 to study the problem about conformal deformation of the scalar curvature on complete manifolds.

Theorem 1.7. Let (M, g) be an n -dimensional ($n \geq 3$) complete (possibly non-compact) Riemannian manifold with $\text{Ric} \geq 0$ and $\sup_{B(x_0,R)} |\nabla S| = o(R^{-s})$ for some constant $s > 0$, as $R \rightarrow \infty$. For any $\kappa > 0$, there does not exist complete metric

$$\tilde{g} \in \left\{ u^{\frac{4}{n-2}} g \mid 0 < u \in C^\infty(M), u(x) = o(r^{\tilde{\kappa}}(x)) \right\}$$

for some $\tilde{\kappa} \in (0, \kappa)$, such that the scalar curvature \tilde{S} of \tilde{g} satisfies

$$\tilde{S}^+|_{B(x_0,R)} = o(R^{-\frac{4\kappa}{n-2}}) \quad \text{and} \quad \sup_{B(x_0,R)} |\nabla \tilde{S}| = o(R^{-\frac{4\kappa}{n-2}}), \quad \text{as } R \rightarrow \infty.$$

Remark 1.8. If \tilde{S} is nonpositive constant, the growth conditions of \tilde{S} and $\nabla\tilde{S}$ in Theorem 1.7 naturally hold and hence $u(x)$ can be relaxed to $u(x) = e^{o(r^{\frac{5}{3}}(x))}$. Compared with the work of [33,35], Theorem 1.7 is valid without any assumptions on sectional curvature, eigenvalue of the conformal operator $\Delta - \frac{n-2}{4(n-1)}S$, only assuming some conditions of the Ricci tensor and the growth of $u(x)$.

On a compact smooth metric measure space, Case [15] provided an example which shows that minimizers of the weighted Yamabe constant do not always exist. Using Theorem 1.1 we can prove

Theorem 1.9. *Let $(M, g, e^{-f} dv)$ be an n -dimensional ($n \geq 3$) complete smooth metric measure space with $\text{Ric}_f \geq 0$. For any $m > 0$, assume that there exist two constants $s > 0$ and $\kappa > 0$ such that*

- (1) $(S_f^m)^-|_{B(x_0,R)} = o(R^{-s})$ and $\sup_{B(x_0,R)} |\nabla S_f^m| = o(R^{-s})$, as $R \rightarrow \infty$;
- (2) $e^{\frac{f}{m}}|_{B(x_0,R)} = o[R^{\frac{-2\kappa}{m+n-2}}]$ and $\sup_{B(x_0,R)} |\nabla e^{\frac{f}{m}}| = o[R^{\frac{-2\kappa}{m+n-2}}]$, as $R \rightarrow \infty$.

Then there does not exist a minimizer of the weighted Yamabe constant $\Lambda \leq 0$ with $u(x) = o(r^{\tilde{\kappa}}(x))$ for some $\tilde{\kappa} \in (0, \kappa)$ near infinity.

When $\Lambda = 0$, we have a simple statement.

Theorem 1.10. *Let $(M, g, e^{-f} dv)$ be an n -dimensional ($n \geq 3$) complete smooth metric measure space with $\text{Ric}_f \geq 0$. For any $m > 0$, assume that*

$$(S_f^m)^-|_{B(x_0,R)} = o(R^{-1}) \quad \text{and} \quad \sup_{B(x_0,R)} |\nabla S_f^m| = o(R^{-\frac{3}{2}}), \quad \text{as } R \rightarrow \infty.$$

If the weighted Yamabe constant $\Lambda = 0$, there does not exist a critical point of the weighted Yamabe quotient $\mathcal{Q}(u)$ with $u(x) = e^{o(r^{1/2}(x))}$ near infinity.

On the other hand, we can apply Theorem 1.3 to give a new Liouville theorem for an elliptic case of Eq. (1.1), which is a supplement to Yang’s result [47].

Theorem 1.11. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space with $\text{Ric}_f^m \geq 0$. Then there does not exist any nontrivial positive solution $u(x)$ to the elliptic equation*

$$\Delta_f u + pu^\alpha = 0, \quad \alpha \leq 1, \tag{1.8}$$

where p is a nonnegative constant.

When $\Lambda = 0$, Theorem 1.11 indeed implies that

Corollary 1.12. *Let $(M, g, e^{-f} dv)$ be an n -dimensional ($n \geq 3$) complete smooth metric measure space with $\text{Ric}_f^m \geq 0$. Assume that the weighted scalar curvature S_f^m is nonpositive constant. If the weighted Yamabe constant $\Lambda = 0$, there does not exist a critical point of the weighted Yamabe quotient \mathcal{Q} .*

Remark 1.13. In view of Theorem 1.9, we may apply Theorem 1.3 to study the minimizer of the weighted Yamabe constant $\Lambda \leq 0$ (or $\Lambda \geq 0$). This enables us to determine many complicated assumptions so that we can apply the Li–Yau gradient estimate of Theorem 1.3 to achieve the Liouville-type theorem for Eq. (1.5). In the paper we do not describe this complicated case.

Inequalities in Theorems 1.1 and 1.3 are called local elliptic and parabolic gradient estimates, respectively (sometimes called Hamilton–Souplet–Zhang and Li–Yau gradient estimates, respectively), which are both proved by using the maximum principle in a locally supported set of the manifold. Similar inequalities have been obtained for the linear heat equation, e.g. [17, 26, 29, 30, 38, 42] and some nonlinear equations, e.g. [14, 18, 27, 46, 50, 51]. However, our case is more complicated due to the function coefficients of Eq. (1.1). To the best of our knowledge, the gradient estimate technique is originated by Yau [48] (see also Cheng–Yau [16]) in 1970s, who first proved a gradient estimate for the harmonic function on the manifold. In 1980s, this technique was developed by Li–Yau [29] for the heat equation on manifolds (though a precursory form of their estimate appeared in [1]). In 1990s, Hamilton [21] gave an elliptic gradient estimate for the heat equation. But this estimate is global which requires the equation defined on closed manifolds. In 2006, Souplet and Zhang [38] proved a local elliptic form by adding a logarithmic correction term. Recently, many authors extended the Li–Yau and Hamilton–Souplet–Zhang gradient estimates to the other heat-type equations; see for example [3, 11–13, 18, 42–44, 50, 51] and references therein.

The paper is organized as follows. In Sect. 2, we first give a useful lemma. Then we apply the lemma and the maximum principle to prove Theorem 1.1. In Sect. 3, we start to give a lemma, and then we apply the lemma to prove Theorem 1.3. In Sect. 4, we apply Theorem 1.1 to discuss Liouville-type theorems for some parabolic cases of Eq. (1.1), especially for Theorems 1.5 and 1.6. In Sect. 5, we apply Theorems 1.1 and 1.3 to study Liouville-type theorems for various elliptic versions of Eq. (1.1); see for example Theorems 1.11, 5.1 and 5.2. In particular, using these results, we study some Yamabe-type problems of complete manifolds and smooth metric measure spaces; see Theorems 1.7, 1.9, 1.10 and Corollary 1.12.

2. Elliptic gradient estimate

In this section, we first prove a lemma, which is a generalization of [38, 42]. Then we apply this lemma and the maximum principle to prove Theorem 1.1.

Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space. For any point $x_0 \in M$ and $R > 0$, assume that $0 < u(x, t) \leq D$ for some constant D is a smooth solution to Eq. (1.1) in $Q_{R,T}$, where

$$Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty), \quad t_0 \in \mathbb{R}, \quad T > 0.$$

Introduce a auxiliary function

$$h(x, t) := \ln \frac{u}{D}$$

in $Q_{R,T}$. Then $h \leq 0$ and h satisfies

$$\begin{aligned} & \left(\Delta_f - \frac{\partial}{\partial t} \right) h + |\nabla h|^2 + p(x, t)(De^h)^{\alpha-1} + q(x, t)(De^h)^{\beta-1} + \mu(x, t) \\ & = 0. \end{aligned} \quad (2.1)$$

Using (2.1) we have the following lemma, which will play an significant part in the proof of Theorem 1.1.

Lemma 2.1. *Let $(M, g, e^{-f} dv)$ be a complete smooth metric measure space. Assume that $\text{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$ in $B(x_0, R)$, where $x_0 \in M$ and $R > 0$. Let $h(x, t)$ is a nonpositive smooth function defined in $Q_{R,T}$ satisfying (2.1). Then the function*

$$\omega := |\nabla \ln(1-h)|^2 = \frac{|\nabla h|^2}{(1-h)^2} \quad (2.2)$$

satisfies

$$\begin{aligned} \frac{1}{2} \left(\Delta_f - \frac{\partial}{\partial t} \right) \omega & \geq \frac{h}{1-h} \langle \nabla h, \nabla \omega \rangle + (1-h)\omega^2 - (n-1)K\omega \\ & - \left(\alpha - 1 + \frac{1}{1-h} \right) p(De^h)^{\alpha-1} \omega - \frac{(De^h)^{\alpha-1}}{(1-h)^2} \langle \nabla p, \nabla h \rangle \\ & - \left(\beta - 1 + \frac{1}{1-h} \right) q(De^h)^{\beta-1} \omega - \frac{(De^h)^{\beta-1}}{(1-h)^2} \langle \nabla q, \nabla h \rangle \\ & - \frac{\mu}{1-h} \omega - \frac{1}{(1-h)^2} \langle \nabla \mu, \nabla h \rangle \end{aligned}$$

for all (x, t) in $Q_{R,T}$.

Proof. The proof is similar to that of Lemma 2.1 in [42], but is included for completeness. We shall apply local coordinates to conveniently compute these complicated evolution equations. Let e_1, e_2, \dots, e_n be a local orthonormal frame field at a point $x \in M^n$ and we adopt the notation that subscripts in i, j , and k , with $1 \leq i, j, k \leq n$, mean covariant differentiations in the e_i, e_j and e_k , directions respectively. We denote $h_i := \nabla_i h, h_{ii} = \nabla_i \nabla_i h = \Delta h$ and $h_{ijj} := \nabla_j \nabla_j \nabla_i h$, etc.

By the definition of ω in (2.2), we compute that

$$\begin{aligned} \omega_j & = \frac{2h_i h_{ij}}{(1-h)^2} + \frac{2h_i^2 h_j}{(1-h)^3}, \\ \langle \nabla f, \nabla \omega \rangle & = \frac{2h_{ij} h_i f_j}{(1-h)^2} + \frac{2h_i^2 h_j f_j}{(1-h)^3}, \end{aligned} \quad (2.3)$$

and

$$\Delta \omega = \frac{2|h_{ij}|^2}{(1-h)^2} + \frac{2h_i h_{ijj}}{(1-h)^2} + \frac{8h_i h_j h_{ij}}{(1-h)^3} + \frac{2h_i^2 h_{jj}}{(1-h)^3} + \frac{6h_i^2 h_j^2}{(1-h)^4}.$$

Hence,

$$\begin{aligned} \Delta_f \omega &= \Delta \omega - \langle \nabla f, \nabla \omega \rangle \\ &= \frac{2|h_{ij}|^2}{(1-h)^2} + \frac{2h_i h_{ijj}}{(1-h)^2} + \frac{8h_i h_j h_{ij}}{(1-h)^3} + \frac{2h_i^2 h_{jj}}{(1-h)^3} + \frac{6h_i^4}{(1-h)^4} \\ &\quad - \frac{2h_{ij} h_i f_j}{(1-h)^2} - \frac{2h_i^2 h_j f_j}{(1-h)^3}. \end{aligned}$$

Using the Ricci identity $h_{ijj} = h_{jji} + R_{ij}h_j$, the above inequality becomes

$$\begin{aligned} \Delta_f \omega &= \frac{2|h_{ij}|^2}{(1-h)^2} + \frac{2h_i(\Delta_f h)_i}{(1-h)^2} + \frac{2(R_{ij} + f_{ij})h_i h_j}{(1-h)^2} + \frac{8h_i h_j h_{ij}}{(1-h)^3} \\ &\quad + \frac{6h_i^4}{(1-h)^4} + \frac{2h_i^2 \cdot \Delta_f h}{(1-h)^3}. \end{aligned} \tag{2.4}$$

From (2.1) and (2.2), we obtain

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= \frac{2\nabla_i h \cdot \nabla_i (\Delta_f h + |\nabla h|^2 + p(De^h)^{\alpha-1} + q(De^h)^{\beta-1} + \mu)}{(1-h)^2} \\ &\quad + \frac{2|\nabla h|^2 (\Delta_f h + |\nabla h|^2 + p(De^h)^{\alpha-1} + q(De^h)^{\beta-1} + \mu)}{(1-h)^3} \\ &= \frac{2\nabla h \nabla \Delta_f h}{(1-h)^2} + \frac{4h_i h_j h_{ij}}{(1-h)^2} + \frac{2h_i^2 \Delta_f h}{(1-h)^3} + \frac{2|\nabla h|^4}{(1-h)^3} \\ &\quad + 2 \left(\alpha - 1 + \frac{1}{1-h} \right) p(De^h)^{\alpha-1} \omega + \frac{2p_i h_i (De^h)^{\alpha-1}}{(1-h)^2} \\ &\quad + 2 \left(\beta - 1 + \frac{1}{1-h} \right) q(De^h)^{\beta-1} \omega + \frac{2q_i h_i (De^h)^{\beta-1}}{(1-h)^2} \\ &\quad + \frac{2\mu}{1-h} \omega + \frac{2\mu_i h_i}{(1-h)^2}. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$\begin{aligned} \frac{1}{2} \left(\Delta_f - \frac{\partial}{\partial t} \right) \omega &= \frac{|h_{ij}|^2}{(1-h)^2} + \frac{(R_{ij} + f_{ij})h_i h_j}{(1-h)^2} + \frac{4h_i h_j h_{ij}}{(1-h)^3} \\ &\quad + \frac{3h_i^4}{(1-h)^4} - \frac{2h_i h_j h_{ij}}{(1-h)^2} - \frac{h_i^4}{(1-h)^3} \\ &\quad - \left(\alpha - 1 + \frac{1}{1-h} \right) p(De^h)^{\alpha-1} \omega - \frac{p_i h_i (De^h)^{\alpha-1}}{(1-h)^2} \\ &\quad - \left(\beta - 1 + \frac{1}{1-h} \right) q(De^h)^{\beta-1} \omega - \frac{q_i h_i (De^h)^{\beta-1}}{(1-h)^2} \\ &\quad - \frac{\mu}{1-h} \omega - \frac{\mu_i h_i}{(1-h)^2}. \end{aligned}$$

Since $\text{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$, we have

$$(\mathbf{R}_{ij} + f_{ij})h_i h_j \geq -(n-1)K h_i^2.$$

Since $1-h \geq 1$, we also have

$$\frac{|h_{ij}|^2}{(1-h)^2} + \frac{2h_i h_j h_{ij}}{(1-h)^3} + \frac{h_i^4}{(1-h)^4} \geq 0.$$

Using these two inequalities, the above equation can be simplified as

$$\begin{aligned} \frac{1}{2} \left(\Delta_f - \frac{\partial}{\partial t} \right) \omega &\geq -\frac{(n-1)K h_i^2}{(1-h)^2} + \frac{2h_i h_j h_{ij}}{(1-h)^3} + \frac{2h_i^4}{(1-h)^4} \\ &\quad - \frac{2h_i h_j h_{ij}}{(1-h)^2} - \frac{h_i^4}{(1-h)^3} \\ &\quad - \left(\alpha - 1 + \frac{1}{1-h} \right) p (De^h)^{\alpha-1} \omega - \frac{p_i h_i (De^h)^{\alpha-1}}{(1-h)^2} \\ &\quad - \left(\beta - 1 + \frac{1}{1-h} \right) q (De^h)^{\beta-1} \omega - \frac{q_i h_i (De^h)^{\beta-1}}{(1-h)^2} \\ &\quad - \frac{\mu}{1-h} \omega - \frac{\mu_i h_i}{(1-h)^2}. \end{aligned} \quad (2.6)$$

From (2.3), we know that

$$\omega_j h_j = \frac{2h_i h_j h_{ij}}{(1-h)^2} + \frac{2h_i^4}{(1-h)^3}.$$

Using this formula, (2.6) can be rewritten as

$$\begin{aligned} \frac{1}{2} \left(\Delta_f - \frac{\partial}{\partial t} \right) \omega &\geq -\frac{(n-1)K h_i^2}{(1-h)^2} + \frac{h}{1-h} \omega_j h_j + \frac{h_i^4}{(1-h)^3} \\ &\quad - \left(\alpha - 1 + \frac{1}{1-h} \right) p (De^h)^{\alpha-1} \omega - \frac{p_i h_i (De^h)^{\alpha-1}}{(1-h)^2} \\ &\quad - \left(\beta - 1 + \frac{1}{1-h} \right) q (De^h)^{\beta-1} \omega - \frac{q_i h_i (De^h)^{\beta-1}}{(1-h)^2} \\ &\quad - \frac{\mu}{1-h} \omega - \frac{\mu_i h_i}{(1-h)^2}. \end{aligned}$$

By the definition of ω , the desired inequality immediately follows. \square

In the rest of this section, we will apply Lemma 2.1 and the localized technique of Souplet–Zhang [38] and the author [42] to give an elliptic-type gradient estimate for positive smooth solutions to Eq. (1.1).

We first introduce a useful space-time cut-off function originated by Li–Yau [29] (see also [38, 42]) as follows.

Lemma 2.2. Fix $t_0 \in \mathbb{R}$ and $T > 0$. For any given $\tau \in (t_0 - T, t_0]$, there exists a smooth function $\tilde{\psi} : [0, \infty) \times [t_0 - T, t_0] \rightarrow \mathbb{R}$ satisfying following propositions:

- (1) $0 \leq \bar{\psi}(r, t) \leq 1$ in $[0, R] \times [t_0 - T, t_0]$, and it is supported in a open subset of $[0, R] \times [t_0 - T, t_0]$.
- (2) $\bar{\psi}(r, t) = 1$ and $\partial_r \bar{\psi}(r, t) = 0$ in $[0, R/2] \times [\tau, t_0]$ and $[0, R/2] \times [t_0 - T, t_0]$, respectively.
- (3) $|\partial_t \bar{\psi}| \leq \frac{C}{\tau - (t_0 - T)} \bar{\psi}^{\frac{1}{2}}$ in $[0, \infty) \times [t_0 - T, t_0]$ for some $C > 0$, and $\bar{\psi}(r, t_0 - T) = 0$ for all $r \in [0, \infty)$.
- (4) $-\frac{C_\epsilon}{R} \bar{\psi}^\epsilon \leq \partial_r \bar{\psi} \leq 0$ and $|\partial_r^2 \bar{\psi}| \leq \frac{C_\epsilon}{R^2} \bar{\psi}^\epsilon$ in $[0, \infty) \times [t_0 - T, t_0]$ for every $\epsilon \in (0, 1)$ with some constant C_ϵ depending on ϵ .

Then we apply Lemmas 2.1 and 2.2 to prove Theorem 1.1 via the maximum principle in a local space-time supported set. The proof mainly follows the arguments of [4,42], which is a little different from [38].

Proof of Theorem 1.1. Pick any number $\tau \in (t_0 - T, t_0]$ and choose a cutoff function $\bar{\psi}(r, t)$ satisfying the conditions of Lemma 2.2. Briefly, we will show that the inequalities in Theorem 1.1 hold at the point (x, τ) for all $x \in M$ such that $d(x, x_0) < R/2$. Since τ is arbitrary, the assertion of theorem will immediately follow. In the following we provide a detailed description.

Let $\psi : M \times [t_0 - T, t_0] \rightarrow \mathbb{R}$ be the cutoff function $\psi = \bar{\psi}(d(x, x_0), t) \equiv \bar{\psi}(r, t)$. Then $\psi(x, t)$ could be viewed as smooth cut-off function supported in $Q_{R,T}$. Our strategy is to estimate $(\Delta_f - \frac{\partial}{\partial t})(\psi\omega)$ and carefully analyze the result at a space-time point where the function $\psi\omega$ attains its maximum.

We apply Lemma 2.1 to conclude that

$$\begin{aligned}
 & \frac{1}{2} \left(\Delta_f - \frac{\partial}{\partial t} \right) (\psi\omega) - \left(\frac{h}{1-h} \nabla h + \frac{\nabla \psi}{\psi} \right) \cdot \nabla (\psi\omega) \\
 & \geq \psi(1-h)\omega^2 - \left(\frac{h}{1-h} \nabla h \cdot \nabla \psi \right) \omega - \frac{|\nabla \psi|^2}{\psi} \omega \\
 & \quad + \frac{1}{2} (\Delta_f \psi) \omega - \frac{1}{2} \psi_t \omega - (n-1)K\psi\omega \\
 & \quad - \left(\alpha - 1 + \frac{1}{1-h} \right) p(De^h)^{\alpha-1} \psi\omega - \frac{\psi p_i h_i (De^h)^{\alpha-1}}{(1-h)^2} \\
 & \quad - \left(\beta - 1 + \frac{1}{1-h} \right) q(De^h)^{\beta-1} \psi\omega - \frac{\psi q_i h_i (De^h)^{\beta-1}}{(1-h)^2} \\
 & \quad - \frac{\mu}{1-h} \psi\omega - \frac{\psi \mu_i h_i}{(1-h)^2}.
 \end{aligned} \tag{2.7}$$

Now let (x_1, t_1) be a maximum space-time point for $\psi\omega$ in the closed set

$$\{(x, t) \in M \times [t_0 - T, \tau] \mid d(x, x_0) \leq R\}.$$

We may assume $(\psi\omega)(x_1, t_1) > 0$; otherwise, $\omega(x, \tau) \leq 0$ and the conclusion naturally holds at (x, τ) whenever $d(x, x_0) < \frac{R}{2}$. Notice that $t_1 \neq t_0 - T$, since we assume $(\psi\omega)(x_1, t_1) > 0$. We may also assume that $\psi(x, t)$ is smooth at (x_1, t_1) due to the standard Calabi argument [8]. Since (x_1, t_1) is a maximum space-time point, at this point we have

$$\Delta_f(\psi\omega) \leq 0, \quad (\psi\omega)_t \geq 0 \quad \text{and} \quad \nabla(\psi\omega) = 0.$$

Using the above estimates at (x_1, t_1) , (2.7) can be simplified as

$$\begin{aligned}
 \psi(1-h)\omega^2 &\leq \left(\frac{h}{1-h} \nabla h \cdot \nabla \psi + \frac{|\nabla \psi|^2}{\psi} \right) \omega \\
 &\quad - \frac{1}{2} (\Delta_f \psi) \omega + \frac{1}{2} \psi_t \omega + (n-1)K \psi \omega \\
 &\quad + \left(\alpha - 1 + \frac{1}{1-h} \right) p u^{\alpha-1} \psi \omega + \frac{\psi p_i h_i u^{\alpha-1}}{(1-h)^2} \\
 &\quad + \left(\beta - 1 + \frac{1}{1-h} \right) q u^{\beta-1} \psi \omega + \frac{\psi q_i h_i u^{\beta-1}}{(1-h)^2} \\
 &\quad + \frac{\mu}{1-h} \psi \omega + \frac{\psi \mu_i h_i}{(1-h)^2}
 \end{aligned} \tag{2.8}$$

at (x_1, t_1) , where in the above estimates we have used the fact that $u = De^h$.

In the rest, we will use (2.8) at the maximum space-time point (x_1, t_1) to give the desired gradient estimate in Theorem 1.1. We will achieve it by two steps.

Case One: We assume the maximum space-point $x_1 \notin B(x_0, 1)$. Recall that, $\text{Ric}_f \geq -(n-1)K$ and $r(x_1, x_0) \geq 1$ in $B(x_0, R)$, $R \geq 2$. Hence by the f -Laplacian comparison theorem (Theorem 3.1 in [40]), we have

$$\Delta_f r(x_1) \leq \sigma + (n-1)K(R-1), \tag{2.9}$$

where $\sigma := \max_{\{x|d(x,x_0)=1\}} \Delta_f r(x)$, which will be used later. Below we will carefully estimate upper bounds for each term on the right hand side (RHS) of (2.8), similar to the arguments of Souplet–Zhang [38] and the author [42]. This will lead us to give the desired result. We remark that the Young’s inequality will be repeatedly used in the following estimates. Below we let c denote a constant depending only on n whose value may change from line to line.

First, we estimate the first term on the RHS of (2.8):

$$\begin{aligned}
 \left(\frac{h}{1-h} \nabla h \cdot \nabla \psi \right) \omega &\leq |h| \cdot |\nabla \psi| \cdot \omega^{3/2} \\
 &= \left[\psi(1-h)\omega^2 \right]^{3/4} \cdot \frac{|h| \cdot |\nabla \psi|}{[\psi(1-h)]^{3/4}} \\
 &\leq \frac{1}{3} \psi(1-h)\omega^2 + c \frac{(h|\nabla \psi|)^4}{[\psi(1-h)]^3} \\
 &\leq \frac{1}{3} \psi(1-h)\omega^2 + \frac{ch^4}{R^4(1-h)^3}.
 \end{aligned} \tag{2.10}$$

For the second term on the RHS of (2.8), we have

$$\begin{aligned} \frac{|\nabla\psi|^2}{\psi}\omega &= \psi^{1/2}\omega \cdot \frac{|\nabla\psi|^2}{\psi^{3/2}} \\ &\leq \frac{1}{18}\psi\omega^2 + c\left(\frac{|\nabla\psi|^2}{\psi^{3/2}}\right)^2 \\ &\leq \frac{1}{18}\psi\omega^2 + \frac{c}{R^4}. \end{aligned} \tag{2.11}$$

For the third term on the RHS of (2.8), since ψ is a radial function, then at (x_1, t_1) , using (2.9) we have

$$\begin{aligned} -\frac{1}{2}(\Delta_f\psi)\omega &= -\frac{1}{2}\left[(\partial_r\psi)\Delta_f r + (\partial_r^2\psi)|\nabla r|^2\right]\omega \\ &\leq -\frac{1}{2}\left[\partial_r\psi(\sigma + (n-1)K(R-1)) + \partial_r^2\psi\right]\omega \\ &\leq \left[|\partial_r^2\psi| + (\sigma^+ + (n-1)K(R-1))|\partial_r\psi|\right]\omega \\ &= \psi^{1/2}\omega\frac{|\partial_r^2\psi|}{\psi^{1/2}} + \psi^{1/2}\omega[\sigma^+ + (n-1)K(R-1)]\frac{|\partial_r\psi|}{\psi^{1/2}} \\ &\leq \frac{1}{18}\psi\omega^2 + c\frac{|\partial_r^2\psi|^2}{\psi} + c\frac{(\sigma^+)^2|\partial_r\psi|^2}{\psi} + c\frac{K^2(R-1)^2|\partial_r\psi|^2}{\psi} \\ &\leq \frac{1}{18}\psi\omega^2 + \frac{c}{R^4} + \frac{c(\sigma^+)^2}{R^2} + cK^2, \end{aligned} \tag{2.12}$$

where $\sigma^+ := \max\{\sigma, 0\}$, and in the last inequality we have used proposition (4) in Lemma 2.2.

For the fourth term on the RHS of (2.8), we have

$$\begin{aligned} \frac{1}{2}|\psi_t|\omega &= \frac{1}{2}\psi^{1/2}\omega\frac{|\psi_t|}{\psi^{1/2}} \\ &\leq \frac{1}{18}\left(\psi^{1/2}\omega\right)^2 + c\left(\frac{|\psi_t|}{\psi^{1/2}}\right)^2 \\ &\leq \frac{1}{18}\psi\omega^2 + \frac{c}{(\tau - t_0 + T)^2}. \end{aligned} \tag{2.13}$$

For the fifth term on the RHS of (2.8), we have

$$\begin{aligned} (n-1)K\psi\omega &= (n-1)\psi^{1/2}\omega \cdot \psi^{1/2}K \\ &\leq \frac{1}{18}\psi\omega^2 + cK^2. \end{aligned} \tag{2.14}$$

For the sixth term on the RHS of (2.8), we easily get

$$\begin{aligned} \left(\alpha - 1 + \frac{1}{1-h}\right)p &\leq (\alpha - 1)p + \frac{p^+}{1-h} \\ &\leq [(\alpha - 1)p]^+ + p^+, \end{aligned}$$

where in the above inequality we used the fact $\frac{1}{1-h} > 1$ due to $h \leq 0$. Hence we have

$$\begin{aligned} \left(\alpha - 1 + \frac{1}{1-h}\right) p u^{\alpha-1} \psi \omega &\leq \left([\alpha - 1]p^+ + p^+\right) u^{\alpha-1} \psi \omega \\ &\leq \frac{1}{18} \psi \omega^2 + c \left([\alpha - 1]p^+ + p^+\right)^2 \sup_{Q_{R,T}} \{u^{2(\alpha-1)}\}, \end{aligned} \quad (2.15)$$

where in the last inequality, we have used the fact $\psi \leq 1$.

For the seventh term on the RHS of (2.8), since $h < 0$, we have the following estimate

$$\begin{aligned} \frac{u^{\alpha-1}}{(1-h)^2} \psi p_i h_i &\leq \frac{u^{\alpha-1}}{(1-h)^2} \psi |p_i| \cdot |h_i| \leq u^{\alpha-1} \psi \omega^{1/2} |\nabla p(x_1, t_1)| \\ &\leq \frac{1}{18} (\psi^{1/4} \omega^{1/2})^4 + c \left(\psi^{3/4} u^{\alpha-1} |\nabla p(x_1, t_1)|\right)^{\frac{4}{3}} \\ &\leq \frac{1}{18} \psi \omega^2 + c \sup_{Q_{R,T}} |\nabla p|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\alpha-1)}\}. \end{aligned} \quad (2.16)$$

For the eighth and ninth terms on the RHS of (2.8), the estimates are very similar to the sixth and seventh terms. We summarize these estimates without providing the detailed proof.

$$\begin{aligned} \left(\beta - 1 + \frac{1}{1-h}\right) q u^{\beta-1} \psi \omega \\ \leq \frac{1}{18} \psi \omega^2 + c \left([\beta - 1]q^+ + q^+\right)^2 \sup_{Q_{R,T}} \{u^{2(\beta-1)}\} \end{aligned} \quad (2.17)$$

and

$$\frac{u^{\beta-1}}{(1-h)^2} \psi q_i h_i \leq \frac{1}{18} \psi \omega^2 + c \sup_{Q_{R,T}} |\nabla q|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\beta-1)}\}. \quad (2.18)$$

For the tenth term on the RHS of (2.8), similar to (2.15), we have the following estimate

$$\frac{\mu}{1-h} \psi \omega \leq \frac{1}{18} \psi \omega^2 + c(\mu^+)^2, \quad (2.19)$$

where $\mu^+ := \sup_{(x,t) \in Q_{R,T}} \{\mu^+(x, t), 0\}$ and $\mu^+(x, t) = \max\{\mu(x, t), 0\}$. For the eleventh term on the RHS of (2.8), similar to (2.16), we have the estimate

$$\frac{\psi \mu_i h_i}{(1-h)^2} \leq \frac{1}{18} \psi \omega^2 + c \sup_{Q_{R,T}} |\nabla \mu|^{\frac{4}{3}}. \quad (2.20)$$

In the following, we will apply the above estimates to prove the theorem. Substituting (2.10)–(2.20) into the RHS of (2.8), at (x_1, t_1) , we have that

$$\begin{aligned}
 \psi(1-h)\omega^2 &\leq \frac{1}{3}\psi(1-h)\omega^2 + \frac{ch^4}{R^4(1-h)^3} + \frac{10}{18}\psi\omega^2 \\
 &\quad + \frac{c}{R^4} + \frac{c(\sigma^+)^2}{R^2} + \frac{c}{(\tau-t_0+T)^2} + cK^2 \\
 &\quad + c(\mu^+)^2 + c \sup_{Q_{R,T}} |\nabla\mu|^{\frac{4}{3}} \\
 &\quad + c\left([\alpha-1]p^+ + p^+\right)^2 \sup_{Q_{R,T}} \{u^{2(\alpha-1)}\} \\
 &\quad + c \sup_{Q_{R,T}} |\nabla p|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\alpha-1)}\} \\
 &\quad + c\left[(\beta-1)q^+ + q^+\right]^2 \sup_{Q_{R,T}} \{u^{2(\beta-1)}\} \\
 &\quad + c \sup_{Q_{R,T}} |\nabla q|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\beta-1)}\}.
 \end{aligned}$$

Since $1-h \geq 1$, the above estimate implies

$$\begin{aligned}
 \psi\omega^2 &\leq \frac{ch^4}{R^4(1-h)^4} + \frac{c}{R^4} + \frac{c(\sigma^+)^2}{R^2} + \frac{c}{(\tau-t_0+T)^2} \\
 &\quad + cK^2 + c(\mu^+)^2 + c \sup_{Q_{R,T}} |\nabla\mu|^{\frac{4}{3}} \\
 &\quad + c\left([\alpha-1]p^+ + p^+\right)^2 \sup_{Q_{R,T}} \{u^{2(\alpha-1)}\} + c \sup_{Q_{R,T}} |\nabla p|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\alpha-1)}\} \\
 &\quad + c\left[(\beta-1)q^+ + q^+\right]^2 \sup_{Q_{R,T}} \{u^{2(\beta-1)}\} + c \sup_{Q_{R,T}} |\nabla q|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\beta-1)}\}
 \end{aligned}$$

at (x_1, t_1) . Moreover, since $\frac{h^4}{(1-h)^4} \leq 1$, the above inequality implies that

$$\begin{aligned}
 (\psi^2\omega^2)(x_1, t_1) &\leq (\psi\omega^2)(x_1, t_1) \\
 &\leq \frac{c}{R^4} + \frac{c(\sigma^+)^2}{R^2} + \frac{c}{(\tau-t_0+T)^2} + cK^2 \\
 &\quad + c(\mu^+)^2 + c \sup_{Q_{R,T}} |\nabla\mu|^{\frac{4}{3}} \\
 &\quad + c\left([\alpha-1]p^+ + p^+\right)^2 \sup_{Q_{R,T}} \{u^{2(\alpha-1)}\} \\
 &\quad + c \sup_{Q_{R,T}} |\nabla p|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\alpha-1)}\} \\
 &\quad + c\left[(\beta-1)q^+ + q^+\right]^2 \sup_{Q_{R,T}} \{u^{2(\beta-1)}\} \\
 &\quad + c \sup_{Q_{R,T}} |\nabla q|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\beta-1)}\}.
 \end{aligned}$$

Since $\psi(x, \tau) = 1$ when $d(x, x_0) < R/2$ by the proposition (2) in Lemma 2.2, from the above estimate, we in fact get

$$\begin{aligned} \omega(x, \tau) &= (\psi\omega)(x, \tau) \\ &\leq (\psi\omega)(x_1, t_1) \\ &\leq \frac{c}{R^2} + \frac{c\sigma^+}{R} + \frac{c}{\tau - t_0 + T} + cK + c\mu^+ + c \sup_{Q_{R,T}} |\nabla\mu|^{\frac{2}{3}} \\ &\quad + c\left([\alpha - 1]p^+ + p^+\right) \sup_{Q_{R,T}} \{u^{\alpha-1}\} + c \sup_{Q_{R,T}} |\nabla p|^{\frac{2}{3}} \sup_{Q_{R,T}} \{u^{\frac{2}{3}(\alpha-1)}\} \\ &\quad + c\left[(\beta - 1)q^+ + q^+\right) \sup_{Q_{R,T}} \{u^{\beta-1}\} + c \sup_{Q_{R,T}} |\nabla q|^{\frac{2}{3}} \sup_{Q_{R,T}} \{u^{\frac{2}{3}(\beta-1)}\} \end{aligned}$$

for all $x \in M$ such that $d(x, x_0) < R/2$. By the definition of $w(x, \tau)$ and the fact that $\tau \in (t_0 - T, t_0)$ was chosen arbitrarily, we get the estimate

$$\begin{aligned} \frac{|\nabla h|}{(1-h)}(x, t) &\leq \frac{c}{R} + c\sqrt{\frac{\sigma^+}{R}} + \frac{c}{\sqrt{t - t_0 + T}} + c\sqrt{K} + c\sqrt{\mu^+} + c \sup_{Q_{R,T}} |\nabla\mu|^{\frac{1}{3}} \\ &\quad + c\sqrt{[(\alpha - 1)p^+ + p^+]} \cdot \sup_{Q_{R,T}} \{u^{\frac{\alpha-1}{2}}\} + c \sup_{Q_{R,T}} |\nabla p|^{\frac{1}{3}} \sup_{Q_{R,T}} \{u^{\frac{\alpha-1}{3}}\} \\ &\quad + c\sqrt{[(\beta - 1)q^+ + q^+]} \cdot \sup_{Q_{R,T}} \{u^{\frac{\beta-1}{2}}\} + c \sup_{Q_{R,T}} |\nabla q|^{\frac{1}{3}} \sup_{Q_{R,T}} \{u^{\frac{\beta-1}{3}}\} \end{aligned}$$

for all $(x, t) \in Q_{R/2,T}$ with $t \neq t_0 - T$. Since $h = \ln(u/D)$, substituting this into the above estimate completes the proof of theorem when $x_1 \notin B(x_0, 1) \subset B(x_0, R)$, where $R \geq 2$.

Case Two: We assume the maximum space-point $x_1 \in B(x_0, 1)$. In this case, ψ is constant in space direction in $B(x_0, R/2)$ by our assumption, where $R \geq 2$. So by (2.8), we have

$$\begin{aligned} \psi\omega^2 &\leq \frac{1}{2}\psi_i\omega + (n-1)K\psi\omega + \frac{\mu}{1-h}\psi\omega + \frac{\psi\mu_i h_i}{(1-h)^2} \\ &\quad + \left(\alpha - 1 + \frac{1}{1-h}\right) p(De^h)^{\alpha-1}\psi\omega + \frac{\psi p_i h_i (De^h)^{\alpha-1}}{(1-h)^2} \\ &\quad + \left(\beta - 1 + \frac{1}{1-h}\right) q(De^h)^{\beta-1}\psi\omega + \frac{\psi q_i h_i (De^h)^{\beta-1}}{(1-h)^2} \end{aligned}$$

at (x_1, t_1) , where we have used $1 - h \geq 1$ on the left hand side of the above inequality. By (2.13)–(2.20), the above inequality can be estimated by

$$\begin{aligned} \psi\omega^2 &\leq \frac{8}{18}\psi\omega^2 + \frac{c}{(\tau - t_0 + T)^2} + cK^2 + c(\mu^+)^2 + c \sup_{Q_{R,T}} |\nabla\mu|^{\frac{4}{3}} \\ &\quad + c\left([\alpha - 1]p^+ + p^+\right)^2 \sup_{Q_{R,T}} \{u^{2(\alpha-1)}\} + c \sup_{Q_{R,T}} |\nabla p|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\alpha-1)}\} \\ &\quad + c\left[(\beta - 1)q^+ + q^+\right]^2 \sup_{Q_{R,T}} \{u^{2(\beta-1)}\} + c \sup_{Q_{R,T}} |\nabla q|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\beta-1)}\} \end{aligned}$$

at (x_1, t_1) . Since $\psi(x_1, t_1) = 1$, the above inequality can be written as

$$\begin{aligned} \omega^2(x_1, t_1) &\leq \frac{c}{(\tau - t_0 + T)^2} + cK^2 + c(\mu^+)^2 + c \sup_{Q_{R,T}} |\nabla\mu|^{\frac{4}{3}} \\ &\quad + c\left([\alpha - 1]p^+ + p^+\right)^2 \sup_{Q_{R,T}} \{u^{2(\alpha-1)}\} \\ &\quad + c \sup_{Q_{R,T}} |\nabla p|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\alpha-1)}\} \\ &\quad + c\left[(\beta - 1)q^+ + q^+\right]^2 \sup_{Q_{R,T}} \{u^{2(\beta-1)}\} \\ &\quad + c \sup_{Q_{R,T}} |\nabla q|^{\frac{4}{3}} \sup_{Q_{R,T}} \{u^{\frac{4}{3}(\beta-1)}\}. \end{aligned}$$

Since $\psi(x, \tau) = 1$ when $d(x, x_0) < R/2$ by the proposition (2) in Lemma 2.2, the above estimate indeed gives that

$$\begin{aligned} \omega(x, \tau) &= (\psi\omega)(x, \tau) \\ &\leq (\psi\omega)(x_1, t_1) \\ &\leq \omega(x_1, t_1) \\ &\leq \frac{c}{\tau - t_0 + T} + cK + c\mu^+ + c \sup_{Q_{R,T}} |\nabla\mu|^{\frac{2}{3}} \\ &\quad + c\left([\alpha - 1]p^+ + p^+\right) \sup_{Q_{R,T}} \{u^{\alpha-1}\} + c \sup_{Q_{R,T}} |\nabla p|^{\frac{2}{3}} \sup_{Q_{R,T}} \{u^{\frac{2}{3}(\alpha-1)}\} \\ &\quad + c\left[(\beta - 1)q^+ + q^+\right) \sup_{Q_{R,T}} \{u^{\beta-1}\} + c \sup_{Q_{R,T}} |\nabla q|^{\frac{2}{3}} \sup_{Q_{R,T}} \{u^{\frac{2}{3}(\beta-1)}\} \end{aligned}$$

for all $x \in M$ such that $d(x, x_0) < R/2$. By the definition of $w(x, \tau)$ and the fact that $\tau \in (t_0 - T, t_0]$ was chosen arbitrarily, we in fact prove that the estimate in theorem still holds when $x_1 \in B(x_0, 1)$. □

3. Parabolic gradient estimate

In this section, by adapting the arguments of [29,41], we first give a useful lemma. Then we apply the lemma to prove Theorem 1.3 by the maximum principle in a locally supported set of the manifold.

Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space. For any point $x_0 \in M$ and $R > 0$, assume that $u(x, t)$ is a positive smooth solution to Eq. (1.1) in $H_{2R,T}$, where $H_{2R,T} := B(x_0, 2R) \times [0, T]$, $T > 0$. Introduce a auxiliary function

$$h(x, t) := \ln u(x, t)$$

in $H_{2R,T}$. By Eq. (1.1), the function h satisfies

$$\left(\Delta_f - \frac{\partial}{\partial t}\right)h + |\nabla h|^2 + p(x, t)(e^h)^{\alpha-1} + q(x, t)(e^h)^{\beta-1} + \mu(x, t) = 0. \quad (3.1)$$

Then we have the following useful lemma, which will be important in the proof of Theorem 1.3.

Lemma 3.1. *Let $(M, g, e^{-f} dv)$ be a complete smooth metric measure space. Assume that $\text{Ric}_f^m \geq -(m+n-1)K$ for some constant $K \geq 0$ in $B(x_0, 2R)$, where $x_0 \in M$ and $R > 0$. Let $h(x, t)$ be a smooth function in $H_{2R, T}$ satisfying Eq. (3.1). Then for any $\lambda > 1$, the function*

$$F := t \left[|\nabla h|^2 - \lambda \left(h_t - p(e^h)^{\alpha-1} - q(e^h)^{\beta-1} - \mu \right) \right] \quad (3.2)$$

satisfies

$$\begin{aligned} \left(\Delta_f - \frac{\partial}{\partial t}\right)F &\geq -\frac{F}{t} - 2\langle \nabla h, \nabla F \rangle - 2(m+n-1)Kt|\nabla h|^2 \\ &\quad + \frac{2t}{m+n} \left[|\nabla h|^2 + p(e^h)^{\alpha-1} + q(e^h)^{\beta-1} + \mu - h_t \right]^2 \\ &\quad - (\alpha-1)p(e^h)^{\alpha-1}F + 2(\lambda\alpha-1)t(e^h)^{\alpha-1}\langle \nabla h, \nabla p \rangle \\ &\quad + (\alpha-1)(\lambda\alpha-1)tp(e^h)^{\alpha-1}|\nabla h|^2 + \lambda t(e^h)^{\alpha-1}\Delta_f p \\ &\quad - (\beta-1)q(e^h)^{\beta-1}F + 2(\lambda\beta-1)t(e^h)^{\beta-1}\langle \nabla h, \nabla q \rangle \\ &\quad + (\beta-1)(\lambda\beta-1)tq(e^h)^{\beta-1}|\nabla h|^2 + \lambda t(e^h)^{\beta-1}\Delta_f q \\ &\quad + 2(\lambda-1)t\langle \nabla h, \nabla \mu \rangle + \lambda t\Delta_f \mu \end{aligned}$$

for all (x, t) in $H_{2R, T}$.

Proof of Lemma 3.1. The proof follows by direct computations. Using (3.1) and (3.2), by the definition of F , we compute that

$$\Delta_f F = t \left[\Delta_f |\nabla h|^2 - \lambda \Delta_f h_t + \lambda \Delta_f \mathcal{D} \right],$$

where $\mathcal{D} := p(e^h)^{\alpha-1} + q(e^h)^{\beta-1} + \mu$. By the Bochner formula of the m -Bakry-Émery Ricci tensor and the assumption $\text{Ric}_f^m \geq -(m+n-1)K$, we have

$$\Delta_f |\nabla h|^2 \geq \frac{2(\Delta_f h)^2}{m+n} + 2\langle \nabla h, \nabla \Delta_f h \rangle - 2(m+n-1)K|\nabla h|^2.$$

Hence,

$$\begin{aligned} \Delta_f F &\geq t \left[\frac{2(\Delta_f h)^2}{m+n} + 2\langle \nabla h, \nabla \Delta_f h \rangle - 2(m+n-1)K|\nabla h|^2 \right. \\ &\quad \left. - \lambda \Delta_f h_t + \lambda \Delta_f \mathcal{D} \right]. \end{aligned} \quad (3.3)$$

Notice that by (3.1) and (3.2), we have the following equality:

$$\begin{aligned}
 -\lambda \Delta_f h_t + 2\langle \nabla h, \nabla \Delta_f h \rangle &= \frac{F_t}{t} - \frac{F}{t^2} + 2(\lambda - 1)\nabla h \nabla h_t + 2\langle \nabla h, \nabla \Delta_f h \rangle \\
 &= \frac{F_t}{t} - \frac{F}{t^2} + 2(\lambda - 1)\nabla h \nabla (\Delta_f h + |\nabla h|^2 + \mathcal{D}) \\
 &\quad + 2\langle \nabla h, \nabla \Delta_f h \rangle \\
 &= \frac{F_t}{t} - \frac{F}{t^2} - \frac{2}{t}\langle \nabla h, \nabla F \rangle + 2(\lambda - 1)\langle \nabla h, \nabla \mathcal{D} \rangle,
 \end{aligned} \tag{3.4}$$

where in the last equality we have used the following formulae

$$\Delta_f h = -|\nabla h|^2 + h_t - \mathcal{D} = -\frac{F}{\lambda t} - \left(1 - \frac{1}{\lambda}\right)|\nabla h|^2. \tag{3.5}$$

Substituting (3.4) into (3.3) yields

$$\begin{aligned}
 \left(\Delta_f - \frac{\partial}{\partial t}\right) F &\geq -\frac{F}{t} + \frac{2t}{m+n}(\Delta_f h)^2 - 2(m+n-1)Kt|\nabla h|^2 \\
 &\quad - 2\langle \nabla h, \nabla F \rangle + 2(\lambda - 1)t\langle \nabla h, \nabla \mathcal{D} \rangle + \lambda t \Delta_f \mathcal{D}.
 \end{aligned}$$

Further using (3.5), the above inequality becomes

$$\begin{aligned}
 \left(\Delta_f - \frac{\partial}{\partial t}\right) F &\geq -\frac{F}{t} + \frac{2t}{m+n}(|\nabla h|^2 + \mathcal{D} - h_t)^2 - 2(m+n-1)Kt|\nabla h|^2 \\
 &\quad - 2\langle \nabla h, \nabla F \rangle + 2(\lambda - 1)t\langle \nabla h, \nabla \mathcal{D} \rangle + \lambda t \Delta_f \mathcal{D}.
 \end{aligned} \tag{3.6}$$

In the following we will compute the last two terms in the inequality (3.6). We first notice that

$$\begin{aligned}
 &2(\lambda - 1)t \left\langle \nabla h, \nabla (p(e^h)^{\alpha-1}) \right\rangle + \lambda t \Delta_f (p(e^h)^{\alpha-1}) \\
 &= 2(\lambda\alpha - 1)t(e^h)^{\alpha-1} \nabla p \nabla h + (\alpha - 1)(\lambda\alpha + \lambda - 2)tp(e^h)^{\alpha-1} |\nabla h|^2 \\
 &\quad + \lambda t(e^h)^{\alpha-1} \Delta_f p + \lambda(\alpha - 1)tp(e^h)^{\alpha-1} \Delta_f h \\
 &= 2(\lambda\alpha - 1)t(e^h)^{\alpha-1} \nabla p \nabla h + \lambda t(e^h)^{\alpha-1} \Delta_f p - (\alpha - 1)p(e^h)^{\alpha-1} F \\
 &\quad + (\alpha - 1)(\lambda\alpha - 1)tp(e^h)^{\alpha-1} |\nabla h|^2,
 \end{aligned}$$

where in the last equality we have used the formulae (3.5). Similar to the above equality, we also have

$$\begin{aligned}
 &2(\lambda - 1)t \left\langle \nabla h, \nabla (q(e^h)^{\beta-1}) \right\rangle + \lambda t \Delta_f (q(e^h)^{\beta-1}) \\
 &= 2(\lambda\beta - 1)t(e^h)^{\beta-1} \nabla q \nabla h + \lambda t(e^h)^{\beta-1} \Delta_f q - (\beta - 1)q(e^h)^{\beta-1} F \\
 &\quad + (\beta - 1)(\lambda\beta - 1) tq(e^h)^{\beta-1} |\nabla h|^2.
 \end{aligned}$$

Combining the above two equalities, we have

$$\begin{aligned}
 & 2(\lambda - 1)t\langle \nabla h, \nabla \mathcal{D} \rangle + \lambda t \Delta_f \mathcal{D} \\
 &= 2(\lambda\alpha - 1)t(e^h)^{\alpha-1} \nabla p \nabla h + \lambda t(e^h)^{\alpha-1} \Delta_f p - (\alpha - 1)p(e^h)^{\alpha-1} F \\
 &\quad + (\alpha - 1)(\lambda\alpha - 1)t p(e^h)^{\alpha-1} |\nabla h|^2 \\
 &\quad + 2(\lambda\beta - 1)t(e^h)^{\beta-1} \nabla q \nabla h + \lambda t(e^h)^{\beta-1} \Delta_f q - (\beta - 1)q(e^h)^{\beta-1} F \\
 &\quad + (\beta - 1)(\lambda\beta - 1)t q(e^h)^{\beta-1} |\nabla h|^2 \\
 &\quad + 2(\lambda - 1)t \nabla \mu \nabla h + \lambda t \Delta_f \mu.
 \end{aligned}$$

Finally substituting this into (3.6) gives the proof of the lemma. □

In the following, we will apply Lemma 3.1 and the localized technique of Li-Yau [29] and the author [41] to give parabolic gradient estimates for the positive smooth solutions to Eq. (1.1) on smooth metric measure spaces.

Proof of Theorem 1.3. Firstly, we introduce an auxiliary cut-off function and its useful properties. This cut-off function is very important in the following proof.

We choose any C^2 cut-off function $\tilde{\varphi}$ on $[0, \infty)$ such that $\tilde{\varphi}(r) \equiv 1$ for $r \in [0, 1]$, $\tilde{\varphi}(r) = 0$ for $r \in [2, \infty)$, and $0 \leq \tilde{\varphi}(r) \leq 1$; meanwhile $\tilde{\varphi}$ satisfies

$$-c_1 \leq \frac{\tilde{\varphi}'(r)}{\tilde{\varphi}^{1/2}(r)} \leq 0 \quad \text{and} \quad \tilde{\varphi}''(r) \geq -c_1$$

for some universal positive constant c_1 . Let

$$\varphi(x) = \tilde{\varphi}\left(\frac{r(x)}{R}\right),$$

where $r(x)$ denotes the distance between x and x_0 in M . Then $\text{supp} \varphi \subseteq B(x_0, 2R)$ and $\varphi|_{B(x_0, R)} \equiv 1$. We shall consider the function φF in $H_{2R, T}$. By the argument of Calabi [8], by using approximation, we can assume without loss of generality that $\varphi(x) \in C^2(M)$ with support in $B(x_0, 2R)$. By a easy computation, we have

$$\frac{|\nabla \varphi|^2}{\varphi} \leq \frac{c_1^2}{R^2} \tag{3.7}$$

and

$$\Delta_f \varphi = \frac{\tilde{\varphi}' \Delta_f r}{R} + \frac{\tilde{\varphi}'' |\nabla r|^2}{R^2} \tag{3.8}$$

in $B(x_0, 2R)$. On the other hand, since $\text{Ric}_f^m \geq -(m + n - 1)K$ for some $K \geq 0$, the generalized Laplacian comparison theorem (see [40]) gives that

$$\Delta_f r \leq (m + n - 1)\sqrt{K} \coth(\sqrt{K} r).$$

Since \coth is decreasing, and $\tilde{\varphi}' = 0$ when $r(x) < R$, by (3.8), this implies

$$\begin{aligned}
 \Delta_f \varphi &\geq -\frac{c_1}{R}(m + n - 1)\sqrt{K} \coth(\sqrt{K} R) - \frac{c_1}{R^2} \\
 &\geq -\frac{(m + n)c_1(1 + R\sqrt{K})}{R^2},
 \end{aligned} \tag{3.9}$$

where we have used the inequality $\sqrt{K} \coth(\sqrt{K}R) \leq \frac{1}{R}(1 + \sqrt{K}R)$.

Secondly, we will apply the f -Laplacian operator Δ_f to the function φF and get a useful inequality. Then we apply the maximum principle argument to the inequality in a compactly supported set and obtain the Li–Yau gradient estimate.

For any $0 < \tau \leq T$, if $\varphi F \leq 0$ in $H_{2R,\tau}$, then the desired estimate follows. Now we assume $\max_{(x,t) \in H_{2R,\tau}}(\varphi F) > 0$. Let (x_1, t_1) be a point where φF achieves the positive maximum, where $x_1 \in B(x_0, 2R)$ and $0 < t_1 \leq \tau$. Clearly, at (x_1, t_1) , we have

$$\nabla(\varphi F) = 0, \quad F_t \geq 0 \quad \text{and} \quad \Delta_f(\varphi F) \leq 0. \tag{3.10}$$

From now on all calculations below will be at (x_1, t_1) . Applying Lemma 3.1 to the following equality

$$\Delta_f(\varphi F) = F(\Delta_f \varphi) + 2\langle \nabla \varphi, \nabla F \rangle + \varphi(\Delta_f F),$$

and using (3.7), (3.8), (3.9), (3.10) and the fact $e^h = u$, we get that

$$\begin{aligned} 0 &\geq \Delta_f(\varphi F) \\ &\geq -F \frac{(m+n)c_1(1+R\sqrt{K})}{R^2} - 2F \frac{|\nabla \varphi|^2}{\varphi} \\ &\quad + \varphi \left[-\frac{F}{t_1} - 2\langle \nabla h, \nabla F \rangle - (\alpha-1)pu^{\alpha-1}F - (\beta-1)qu^{\beta-1}F \right] \\ &\quad + \frac{2t_1\varphi}{m+n} \left[|\nabla h|^2 + pu^{\alpha-1} + qu^{\beta-1} + \mu - h_t \right]^2 \\ &\quad + t_1\varphi |\nabla h|^2 \left[(\alpha-1)(\lambda\alpha-1)pu^{\alpha-1} + (\beta-1)(\lambda\beta-1)qu^{\beta-1} - 2(m+n-1)K \right] \\ &\quad + 2t_1\varphi \left[(\lambda\alpha-1)u^{\alpha-1}\langle \nabla h, \nabla p \rangle + (\lambda\beta-1)u^{\beta-1}\langle \nabla h, \nabla q \rangle + (\lambda-1)\langle \nabla h, \nabla \mu \rangle \right] \\ &\quad + \lambda t_1\varphi \left[u^{\alpha-1}\Delta_f p + u^{\beta-1}\Delta_f q + \Delta_f \mu \right] \\ &\geq F \left[-\frac{(m+n)c_1(1+R\sqrt{K}) + 2c_1^2}{R^2} - \frac{\varphi}{t_1} - (\alpha-1)pu^{\alpha-1}\varphi - (\beta-1)qu^{\beta-1}\varphi \right] \\ &\quad + 2F\langle \nabla h, \nabla \varphi \rangle + \frac{2t_1\varphi}{m+n} \left[|\nabla h|^2 + pu^{\alpha-1} + qu^{\beta-1} + \mu - h_t \right]^2 \\ &\quad + t_1\varphi |\nabla h|^2 \left[(\alpha-1)(\lambda\alpha-1)pu^{\alpha-1} + (\beta-1)(\lambda\beta-1)qu^{\beta-1} - 2(m+n-1)K \right] \\ &\quad + 2t_1\varphi \left[(\lambda\alpha-1)u^{\alpha-1}\langle \nabla h, \nabla p \rangle + (\lambda\beta-1)u^{\beta-1}\langle \nabla h, \nabla q \rangle + (\lambda-1)\langle \nabla h, \nabla \mu \rangle \right] \\ &\quad + \lambda t_1\varphi \left[u^{\alpha-1}\Delta_f p + u^{\beta-1}\Delta_f q + \Delta_f \mu \right]. \end{aligned}$$

Multiplying both sides of the above inequality by $t_1\varphi$, using the assumptions of $p(x, t)$, $q(x, t)$, $\mu(x, t)$ and $\varphi(x)$ in Theorem 1.1, recalling that $0 \leq \varphi \leq 1$, we

in fact get that

$$\begin{aligned}
 0 \geq & -t_1\varphi F \left[\frac{(m+n)c_1(1 + R\sqrt{K}) + 2c_1^2}{R^2} + \frac{1}{t_1} + [(\alpha - 1)p]^+ u^{\alpha-1} \right. \\
 & + [(\beta - 1)q]^+ u^{\beta-1} \Big] - 2c_1R^{-1}t_1F|\nabla h|\varphi^{3/2} \\
 & + \frac{2t_1^2\varphi^2}{m+n} \left\{ [|\nabla h|^2 + pu^{\alpha-1} + qu^{\beta-1} + \mu - h_t]^2 \right. \\
 & + \frac{m+n}{2}|\nabla h|^2 [(\alpha-1)(\lambda\alpha-1)p]^- u^{\alpha-1} \\
 & \left. + [(\beta-1)(\lambda\beta-1)q]^- u^{\beta-1} - 2(m+n-1)K \right\} \\
 & - 2t_1^2\varphi^{1/2}|\nabla h| \left[|\lambda\alpha - 1|a_1u^{\alpha-1} + |\lambda\beta - 1|a_2u^{\beta-1} + (\lambda - 1)a_3 \right] \\
 & + \lambda t_1^2 \left[\inf_{H_{2R,T}} \left(u^{\alpha-1}b_1 + u^{\beta-1}b_2 \right) + b_3 \right].
 \end{aligned} \tag{3.11}$$

In the above inequality, we denote

$$p^+ := \sup_{(x,t) \in H_{R,T}} \{p^+(x, t), 0\} \quad \text{and} \quad p^- := \inf_{(x,t) \in H_{R,T}} \{p^-(x, t), 0\},$$

for any $p(x, t) \in C^\infty(H_{R,T})$, where

$$p^+(x, t) := \max\{p(x, t), 0\} \quad \text{and} \quad p^-(x, t) := \min\{p(x, t), 0\}.$$

We let

$$y := \varphi|\nabla h|^2$$

and

$$z := \varphi(h_t - pu^{\alpha-1} - qu^{\beta-1} - \mu).$$

Then (3.11) can be rewritten as

$$\begin{aligned}
 0 \geq & -\varphi F \left[\frac{(m+n)c_1(1 + R\sqrt{K}) + 2c_1^2}{R^2} t_1 + 1 \right] \\
 & - \varphi F \left[[(\alpha - 1)p]^+ \sup_{H_{2R,T}} \{u^{\alpha-1}\} t_1 + [(\beta - 1)q]^+ \sup_{H_{2R,T}} \{u^{\beta-1}\} t_1 \right] \\
 & + \frac{2t_1^2}{m+n} \left\{ (y - z)^2 - c_1(m+n)R^{-1}y^{1/2}(y - \lambda z) \right. \\
 & \quad \left. - (m+n)\tilde{K}y - (m+n)\gamma y^{1/2} \right\} \\
 & + \lambda t_1^2 \left[\inf_{H_{2R,T}} \left(u^{\alpha-1}b_1 + u^{\beta-1}b_2 \right) + b_3 \right],
 \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \tilde{K} := & (n + m - 1)K - \frac{1}{2}[(\alpha - 1)(\lambda\alpha - 1)p]^- \sup_{H_{2R,T}} \{u^{\alpha-1}\} \\ & - \frac{1}{2}[(\beta - 1)(\lambda\beta - 1)q]^- \sup_{H_{2R,T}} \{u^{\beta-1}\} \end{aligned}$$

and

$$\gamma := |\lambda\alpha - 1|a_1 \sup_{H_{2R,T}} \{u^{\alpha-1}\} + |\lambda\beta - 1|a_2 \sup_{H_{2R,T}} \{u^{\beta-1}\} + (\lambda - 1)a_3.$$

Inequality (3.12) is rather complicated and we want to simplify it so that the inequality can be estimated efficiently. Indeed, we can follow the Li–Yau’s arguments [29] to estimate the third line of inequality (3.12). The similar argument also appeared in [41]. That is to say, we can use the Cauchy–Schwarz inequality to get the following key inequality

$$\begin{aligned} & (y - z)^2 - c_1(m + n)R^{-1}y^{1/2}(y - \lambda z) - (m + n)\tilde{K}y - (m + n)\gamma y^{1/2} \\ & \geq \lambda^{-2}(y - \lambda z)^2 - \frac{(m + n)^2}{8}c_1^2\lambda^2(\lambda - 1)^{-1}R^{-2}(y - \lambda z) \\ & \quad - \frac{3}{4}4^{-\frac{1}{3}}(m + n)^{4/3}\gamma^{4/3}\left(\frac{\lambda}{\lambda - 1}\right)^{2/3}\varepsilon^{-1/3} - \frac{(m + n)^2\lambda^2\tilde{K}^2}{4(1 - \varepsilon)(\lambda - 1)^2} \end{aligned}$$

for any $0 < \varepsilon < 1$. Substituting this inequality into (3.12) and arranging the terms yields

$$\begin{aligned} 0 \geq & -\varphi F \left[\frac{(m + n)c_1(1 + R\sqrt{K}) + 2c_1^2t_1 + 1}{R^2} \right] \\ & - \varphi F \left[[(\alpha - 1)p]^+ \sup_{H_{2R,T}} \{u^{\alpha-1}\}t_1 + [(\beta - 1)q]^+ \sup_{H_{2R,T}} \{u^{\beta-1}\}t_1 \right] \\ & + \frac{2}{m + n} \left[\lambda^{-2}(\varphi F)^2 - \frac{(m + n)^2c_1^2\lambda^2}{8(\lambda - 1)R^2}t_1(\varphi F) \right] \\ & + \frac{t_1^2}{m + n} \left[\frac{-3}{2} \left(\frac{(m + n)^4\lambda^2}{4\varepsilon(\lambda - 1)^2} \right)^{\frac{1}{3}} \gamma^{\frac{4}{3}} - \frac{(m + n)^2\lambda^2\tilde{K}^2}{2(1 - \varepsilon)(\lambda - 1)^2} \right] \\ & + \lambda t_1^2 \left[\inf_{H_{2R,T}} (u^{\alpha-1}b_1 + u^{\beta-1}b_2) + b_3 \right] \\ = & \frac{2\lambda^{-2}}{m + n}(\varphi F)^2 - \Phi \cdot (\varphi F) - t_1^2\Psi, \end{aligned}$$

where

$$\begin{aligned} \Phi := & \frac{(m + n)c_1(1 + R\sqrt{K}) + 2c_1^2t_1 + 1}{R^2} + \frac{(m + n)c_1^2\lambda^2}{4(\lambda - 1)R^2}t_1 + 1 \\ & + [(\alpha - 1)p]^+ \sup_{H_{2R,T}} \{u^{\alpha-1}\}t_1 + [(\beta - 1)q]^+ \sup_{H_{2R,T}} \{u^{\beta-1}\}t_1 \end{aligned}$$

and

$$\Psi := \frac{3}{2} \left(\frac{(m+n)\lambda^2}{4\varepsilon(\lambda-1)^2} \right)^{\frac{1}{3}} \gamma^{\frac{4}{3}} + \frac{(m+n)\lambda^2 \tilde{K}^2}{2(1-\varepsilon)(\lambda-1)^2} - \lambda \left[\inf_{H_{2R,T}} \left(u^{\alpha-1} b_1 + u^{\beta-1} b_2 \right) + b_3 \right],$$

and where

$$\gamma := |\lambda\alpha - 1| a_1 \sup_{H_{2R,T}} \{u^{\alpha-1}\} + |\lambda\beta - 1| a_2 \sup_{H_{2R,T}} \{u^{\beta-1}\} + (\lambda - 1) a_3$$

and

$$\tilde{K} := (n+m-1)K - \frac{1}{2} [(\alpha-1)(\lambda\alpha-1)p]^- \sup_{H_{2R,T}} \{u^{\alpha-1}\} - \frac{1}{2} [(\beta-1)(\lambda\beta-1)q]^- \sup_{H_{2R,T}} \{u^{\beta-1}\}.$$

This implies

$$\begin{aligned} (\varphi F)(x_1, t_1) &\leq \frac{m+n}{4} \lambda^2 \left[\Phi + \left(\Phi^2 + \frac{8}{m+n} \lambda^{-2} t_1^2 \Psi \right)^{1/2} \right] \\ &\leq \frac{m+n}{4} \lambda^2 \left[\Phi + \Phi + \left(\frac{8}{m+n} \lambda^{-2} t_1^2 \Psi \right)^{1/2} \right] \\ &= \frac{m+n}{2} \lambda^2 \Phi + t_1 \lambda \left(\frac{m+n}{2} \Psi \right)^{1/2}, \end{aligned} \tag{3.13}$$

where Φ and Ψ are defined as above. Notice that on $B(x_0, R) \times [0, \tau]$, since $\varphi \equiv 1$ and (x_1, t_1) is a maximum point of function φF , we have

$$\sup_{B(x_0,R)} F(x, \tau) \leq (\varphi F)(x_1, t_1). \tag{3.14}$$

Substituting (3.13) into (3.14), and using a easy fact that $t_1 \leq \tau$, we indeed show that

$$\begin{aligned} &\tau \cdot \sup_{B(x_0,R)} \left[|\nabla h|^2 + \lambda p u^{\alpha-1} + \lambda q u^{\beta-1} + \lambda \mu - \lambda h_t \right] (x, \tau) \\ &\leq \tau \lambda \left(\frac{m+n}{2} \Psi \right)^{1/2} \\ &\quad + \frac{m+n}{2} \lambda^2 \left[\frac{(m+n)c_1(1+R\sqrt{K}) + 2c_1^2}{R^2} \tau + \frac{(m+n)c_1^2 \lambda^2}{4(\lambda-1)R^2} \tau \right] \\ &\quad + \frac{m+n}{2} \lambda^2 \\ &\quad + \frac{m+n}{2} \lambda^2 \left\{ [(\alpha-1)p]^+ \sup_{H_{2R,T}} \{u^{\alpha-1}\} \tau + [(\beta-1)q]^+ \sup_{H_{2R,T}} \{u^{\beta-1}\} \tau \right\}, \end{aligned}$$

which immediately implies the theorem because $\tau \in (0, T]$ is arbitrary. □

4. Parabolic Liouville theorem

In this section, we will apply Theorem 1.1 to give many sufficient conditions on the growth of solutions and coefficients that guarantee the parabolic Liouville theorems for various cases of Eq. (1.1).

First, we will prove Theorem 1.5 in the introduction. We consider the case: $\alpha > 1$, $\mu(x, t) \equiv \mu(x)$, $p(x, t) \equiv p(x) \neq 0$ and $q(x, t) \equiv 0$ in Eq. (1.1).

Proof of Theorem 1.5. Under the assumptions of Theorem 1.5, let $u(x, t)$ be a positive smooth ancient solution to Eq. (1.6). For any fixed space-time point (x_0, t_0) , since $\alpha > 1$ and $K = 0$, we apply Theorem 1.1 to $u(x_0, t_0)$ in the space-time set $B(x_0, R) \times (t_0 - R, t_0]$ (i.e., let $T = R$ in $Q_{R,T}$), and obtain that

$$\begin{aligned} & |\nabla \ln u(x_0, t_0)| \\ & \leq c(n) (1 + \ln(D(Q_{R,R})) - \ln u(x_0, t_0)) \\ & \quad \times \left[\frac{1 + \sqrt{\sigma^+}}{\sqrt{R}} + o(R^{-\frac{\alpha}{2}}) + o(R^{-\frac{\alpha}{3}}) + o[R^{(\tilde{\kappa}-\kappa)\frac{\alpha-1}{2}}] + o[R^{(\tilde{\kappa}-\kappa)\frac{\alpha-1}{3}}] \right] \end{aligned}$$

for sufficiently large $R \gg 2$, depending on $|t_0|$, where R has been chosen sufficiently large such that $R \geq |t_0|$. Since $u(x, t) = o([r(x) + |t|]^{\tilde{\kappa}})$ in $Q_{R,R}$, then we have $D(Q_{R,R}) = o(R^{\tilde{\kappa}})$. For the number $\tilde{\kappa} \in (0, \kappa)$ and the fixed value $\ln u(x_0, t_0)$, letting $R \rightarrow \infty$ in the above inequality, we immediately get

$$|\nabla u(x_0, t_0)| = 0.$$

Since (x_0, t_0) was chosen arbitrarily, we conclude that $u(x, t) = u(t)$ for all $x \in M$.

Case One: $\mu(x) \equiv 0$.

In this case (1.6) becomes

$$u'(t) = p(x)u^\alpha(t), \quad p(x) \neq 0, \quad \alpha > 1.$$

This equation implies $p(x) \equiv c$ for some constant $c < 0$ due to the growth assumption on $p(x)$. Therefore,

$$u^{1-\alpha}(t) = c(1 - \alpha)t + u^{1-\alpha}(0).$$

Since u is a positive ancient solution, from above we see that $u^{1-\alpha}(-\infty) < 0$ for $t \rightarrow -\infty$. This is a contradiction with the positivity of $u(x, t)$.

Case Two: $\mu(x) \neq 0$.

In this case, (1.6) reduces to

$$u'(t) = \mu(x)u(t) + p(x)u^\alpha(t), \quad \mu(x) \neq 0, \quad p(x) \neq 0, \quad \alpha > 1,$$

which can be rewritten as a first-order ODE by

$$[u^{1-\alpha}(t)]' = (1 - \alpha)p(x) + (1 - \alpha)\mu(x)u^{1-\alpha}.$$

This equation has a general solution

$$u^{1-\alpha}(t) = Ce^{(1-\alpha)\mu(x)t} - p(x)/\mu(x), \tag{4.1}$$

where C is a arbitrary constant, $\mu(x) \neq 0$ and $p(x) \neq 0$.

Since the left-hand side of (4.1) is independent of x , it must hold that $p(x)/\mu(x)$ is constant. Moreover, if $\mu(x) \equiv c < 0$ ($c > 0$ is impossible due to the growth of $\mu(x)$), then $p(x) \equiv c' < 0$ ($c' > 0$ is impossible due to the growth of $p(x)$). In this case, (4.1) becomes

$$u^{1-\alpha}(t) = \left(u^{1-\alpha}(0) + \frac{c}{c'}\right)e^{(1-\alpha)ct} - \frac{c}{c'}, \quad t < 0,$$

where $\alpha > 1$, $c < 0$, $u(0) > 0$ and $c/c' > 0$. Letting $t \rightarrow -\infty$, we get

$$u^{1-\alpha}(t) \rightarrow -\frac{c}{c'} < 0,$$

which is impossible since $u(x, t) > 0$. So $\mu(x)$ is not constant and from (4.1), we conclude that $C \equiv 0$ and $u^{1-\alpha}(t) = -p(x)/\mu(x)$ is constant. Therefore $\mu(x) \equiv -cp(x)$ for some constant $c > 0$ and $u(x, t) \equiv c^{\frac{1}{\alpha-1}}$. \square

In Theorem 1.5, if $\mu(x)$ and $p(x)$ are both negative constants, then they naturally satisfy the conditions (1) and (2). In this case we are able to improve the growth condition of $u(x, t)$ and get a simple statement, which was also proved by Dung, Khanh and Ngo (see Corollary 2.6 in [18]).

Corollary 4.1. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space with $\text{Ric}_f \geq 0$. There does not exist any positive ancient solution to equation*

$$\left(\Delta_f - \frac{\partial}{\partial t}\right)u + \mu u + p u^\alpha = 0, \quad \alpha > 1, \quad \mu < 0, \quad p < 0, \tag{4.2}$$

such that $u(x, t) = e^{o(r^{\frac{1}{2}}(x)+|t|^{\frac{1}{2}})}$ near infinity. Moreover, if f is identically constant, then the growth of u can be relaxed to $u(x, t) = e^{o(r(x)+|t|^{\frac{1}{2}})}$.

Proof of Corollary 4.1. Because $\mu(x)$ and $p(x)$ are both negative constants, we know that the conditions (1) and (2) in Theorem 1.5 naturally hold. Now let $u(x, t)$ be a positive smooth ancient solution to Eq. (4.2), such that

$$\ln u(x, t) = o(r^{\frac{1}{2}}(x) + |t|^{\frac{1}{2}})$$

near infinity. Similar to the proof of Theorem 1.5, for a fixed space-time point (x_0, t_0) , we apply Theorem 1.1 (i) to $u(x_0, t_0)$ in $Q_{R,R} = B(x_0, R) \times (t_0 - R, t_0]$,

$$|\nabla \ln u(x_0, t_0)| \leq c(n) \left(1 + o(\sqrt{R}) - \ln u(x_0, t_0)\right) \left[\frac{1}{\sqrt{R}} + \sqrt{\frac{\sigma^+}{R}}\right] \tag{4.3}$$

for sufficiently large $R \gg 2$, depending on $|t_0|$. Then letting $R \rightarrow \infty$, we have $|\nabla u(x_0, t_0)| = 0$. Since (x_0, t_0) is arbitrary, we get $u(x, t) = u(t)$ for all $x \in M$. Finally, the conclusion follows by the same argument of Theorem 1.5.

As for the case f is constant, we assume that

$$\ln u(x, t) = o(r(x) + |t|^{\frac{1}{2}})$$

near infinity. We apply Theorem 1.1 to $u(x_0, t_0)$ in $Q_{R, R^2} = B(x_0, R) \times (t_0 - R^2, t_0]$ and the proof is almost the same as before except the corresponding gradient estimate of (4.3) is replaced by

$$|\nabla \ln u(x_0, t_0)| \leq c(n) (1 + o(R) - \ln u(x_0, t_0)) \cdot \frac{1}{R}$$

for sufficiently large R , depending on $|t_0|$. □

Second, we consider the case: $\alpha = 1, \mu(x, t) \equiv \mu(x), p(x, t) \equiv p(x)$ and $q(x, t) \equiv 0$ in Eq. (1.1). In this case we prove that

Theorem 4.2. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space with $\text{Ric}_f \geq 0$. Assume that $\mu(x)$ in the following equation*

$$\left(\Delta_f - \frac{\partial}{\partial t} \right) u + \mu(x)u = 0 \tag{4.4}$$

satisfies

$$\mu^+|_{B(x_0, R)} = o(R^{-1}) \text{ and } \sup_{B(x_0, R)} |\nabla \mu| = o(R^{-\frac{3}{2}}), \text{ as } R \rightarrow \infty.$$

- (1) For $\mu(x) \not\equiv 0$, there does not exist any positive ancient solution to Eq. (4.4) such that $u(x, t) = e^{o(r^{\frac{1}{2}}(x) + |t|^{\frac{1}{2}})}$ near infinity;
- (2) for $\mu(x) \equiv 0$, there only exist constant positive ancient solution to Eq. (4.4) such that $u(x, t) = e^{o(r^{\frac{1}{2}}(x) + |t|^{\frac{1}{2}})}$ near infinity.

Remark 4.3. There indeed exist many functions $\mu(x)$ satisfying the growth of μ , such as $\mu(x) = -e^{-x}/(x^2 + 1)$ in \mathbb{R}^1 . If $\mu(x)$ is negative constant, it naturally satisfies the growth of μ . If $\mu(x) \equiv 0$, the theorem returns to a slight improvement of [42]. Notice that the growth condition of u is necessary. For example, let $u = e^{x+t}$, $f = -x$ and $\mu(x) = -1$ in \mathbb{R}^1 . Then u is a positive eternal solution to Eq. (4.4).

Proof of Theorem 4.2. Let $u(x, t)$ be a positive smooth ancient solution to Eq. (4.4), such that

$$\ln u(x, t) = o(r^{\frac{1}{2}}(x) + |t|^{\frac{1}{2}})$$

near infinity. For any point (x_0, t_0) , since $\alpha = 1$ and $K = 0$, applying Theorem 1.1 to $u(x_0, t_0)$ in the set $Q_{R, R} := B(x_0, R) \times (t_0 - R, t_0]$,

$$|\nabla \ln u(x_0, t_0)| \leq c(n) \left(1 + o(\sqrt{R}) - \ln u(x_0, t_0) \right) \left[\frac{1 + \sqrt{\sigma^+}}{\sqrt{R}} + o(R^{-\frac{1}{2}}) \right]$$

for sufficiently large $R \gg 2$, depending on $|t_0|$. Letting $R \rightarrow \infty$, $|\nabla u(x_0, t_0)| = 0$. Since (x_0, t_0) is arbitrary, we know that $u(x, t) = u(t)$ for all $x \in M$, which satisfies

$$u'(t) = \mu(x)u(t).$$

If $\mu(x) \equiv 0$, then $u(x, t) \equiv c$ is positive constant. If $\mu(x) \not\equiv 0$, this implies $\mu(x) = C$ for some constant $C < 0$ by the growth of $\mu(x)$. So we have

$$u(t) = u(0)e^{Ct}, \quad t < 0.$$

This contradicts the assumption of theorem $u(x, t) = e^{o(r^{\frac{1}{2}}(x)+|t|^{\frac{1}{2}})}$ near infinity. Hence the theorem follows. \square

In Theorem 4.2, if we further assume f is a constant, we can improve the growth assumptions on $\mu(x)$ and $u(x, t)$. This has also been obtained by Zhu [51].

Corollary 4.4. *Let (M, g) be an n -dimensional complete noncompact Riemannian manifold with $\text{Ric} \geq 0$. Assume that $\mu(x)$ in the following equation*

$$\left(\Delta - \frac{\partial}{\partial t}\right)u + \mu(x)u = 0 \tag{4.5}$$

satisfies

$$\mu^+|_{B(x_0, R)} = o(R^{-2}) \quad \text{and} \quad \sup_{B(x_0, R)} |\nabla \mu| = o(R^{-3}), \quad \text{as } R \rightarrow \infty.$$

- (1) For $\mu(x) \not\equiv 0$, there does not exist any positive ancient solution to Eq. (4.5) such that $u(x, t) = e^{o(r(x)+|t|^{\frac{1}{2}})}$ near infinity;
- (2) for $\mu(x) \equiv 0$, there only exist constant positive ancient solution to Eq. (4.5) such that $u(x, t) = e^{o(r(x)+|t|^{\frac{1}{2}})}$ near infinity.

Proof of Corollary 4.4. The proof is nearly the same as the proof Theorem 4.2 with the only difference is that we apply Theorem 1.1 to $u(x_0, t_0)$ in the new space-time set $Q_{R, R^2} = B(x_0, R) \times (t_0 - R^2, t_0]$, and get that

$$|\nabla \ln u(x_0, t_0)| \leq c(n) (1 + o(R) - \ln u(x_0, t_0)) \left[\frac{1}{R} + o(R^{-1}) \right]$$

for sufficiently large $R \gg 2$, depending on $|t_0|$. We would like to point out that the term $\sqrt{\frac{\sigma^+}{R}}$ in Theorem 1.1 does not exist in this case (see Remark 1.2). \square

Remark 4.5. The growth condition of $u(x, t)$ is sharp in the space direction. For example, let $u = e^{2x+t}$ and $\mu(x) = -3$ in \mathbb{R}^1 . Obviously, u is a positive eternal solution to Eq. (4.5).

Third, we prove Theorem 1.6 in the introduction. Let $\alpha < 1$, $\mu(x, t) \equiv \mu(x) \not\equiv 0$, $p(x, t) \equiv p(x) \not\equiv 0$ and $q(x, t) \equiv q(x) \equiv 0$ in (1.1), and we have

Proof of Theorem 1.6. Let $u(x, t)$ be a positive ancient solution to Eq. (1.7) such that

$$(r(x) + |t|)^{-\tilde{\kappa}} \leq u(x, t) \leq (r(x) + |t|)^\delta$$

for some $\tilde{\kappa} \in (0, \kappa)$ and $\delta > 0$ near infinity. For any point (x_0, t_0) , since $\alpha < 1$ and $K = 0$, applying Theorem 1.1 to $u(x_0, t_0)$ in $B(x_0, R) \times (t_0 - R, t_0]$, we get that

$$\begin{aligned} |\nabla \ln u(x_0, t_0)| &\leq c(n) \left(1 + c(\delta) \ln R - \ln u(x_0, t_0) \right) \\ &\times \left[\frac{1 + \sqrt{\sigma^+}}{\sqrt{R}} + o(R^{-\frac{\delta}{2}}) + o(R^{-\frac{\delta}{3}}) + o[R^{-\frac{\kappa}{2}(1-\alpha)}] (R^{-\tilde{\kappa}})^{\frac{\alpha-1}{2}} \right. \\ &\left. + o[R^{-\frac{\kappa}{3}(1-\alpha)}] (R^{-\tilde{\kappa}})^{\frac{\alpha-1}{3}} \right] \end{aligned} \tag{4.6}$$

for sufficiently large $R \gg 2$, depending on $|t_0|$, where R has been chosen sufficiently large such that $R \geq |t_0|$. We would like to point out, in the above complicated estimate, we have chosen $\min_{Q_{R,T}} u(x, t) = (3R)^{-\tilde{\kappa}}$ due to the fact: $r(x) \leq R$ and $|t| \leq |t_0| + R \leq 2R$.

Letting $R \rightarrow \infty$ in (4.6), since $\tilde{\kappa} \in (0, \kappa)$, we have

$$|\nabla u(x_0, t_0)| = 0.$$

Since (x_0, t_0) is arbitrary, we conclude that $u(x, t) = u(t)$ and it satisfies

$$u'(t) = \mu(x)u(t) + p(x)u^\alpha, \quad p(x) \not\equiv 0, \quad \alpha < 1. \tag{4.7}$$

Case One: $\mu(x) \equiv 0$.

In this case (4.7) becomes

$$u'(t) = p(x)u^\alpha(t), \quad p(x) \not\equiv 0, \quad \alpha < 1.$$

Similar to the Case One in the proof of Theorem 1.5, this is impossible.

Case Two: $\mu(x) \not\equiv 0$.

Equation (4.7) can be rewritten as a first-order ODE by

$$[u^{1-\alpha}(t)]' = (1 - \alpha)\mu(x)u^{1-\alpha} + (1 - \alpha)p(x),$$

which has a general solution

$$u^{1-\alpha}(t) = C e^{(1-\alpha)\mu(x)t} - p(x)/\mu(x), \quad \mu(x) \not\equiv 0, \quad p(x) \not\equiv 0,$$

where C is a arbitrary constant. Similar to the proof of Case Two in Theorem 1.5, we have $\mu(x) \equiv -cp(x)$ for some constant $c > 0$ and $u^{1-\alpha}(t) = 1/c$. \square

5. Elliptic Liouville theorem

In this section, we have two goals. One is that we apply Theorem 1.1 to discuss Liouville-type theorems for some elliptic versions of Eq. (1.1) on complete (not necessarily compact) manifolds and smooth metric measure spaces.

Firstly, we consider a special elliptic version of (1.1) for $\alpha < 1$ on a smooth metric measure space, which supplements Yang’s result [47].

Theorem 5.1. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space with $\text{Ric}_f \geq 0$. Assume that there exists a constant $\kappa > 0$ such that $p(x)$ in the following elliptic equation*

$$\Delta_f u + p(x)u^\alpha = 0, \quad \alpha < 1, \quad p(x) \not\equiv 0, \tag{5.1}$$

satisfies

$$\sup_{B(x_0, R)} |p| = o[R^{-\kappa(1-\alpha)}] \quad \text{and} \quad \sup_{B(x_0, R)} |\nabla p| = o[R^{-\kappa(1-\alpha)}], \quad \text{as } R \rightarrow \infty.$$

Then there does not exist any positive solution to (5.1) on M , such that

$$r^{-\tilde{\kappa}}(x) \leq u(x) \leq r^\delta(x)$$

for some $\tilde{\kappa} \in (0, \kappa)$ and $\delta > 0$ near infinity.

Proof of Theorem 5.1. The proof is similar to the argument of Theorem 1.6. Let $u(x, t)$ be a positive smooth solution to Eq. (5.1) such that

$$r^{-\tilde{\kappa}}(x) \leq u(x) \leq r^\delta(x)$$

for some $\tilde{\kappa} \in (0, \kappa)$ and $\delta > 0$ near infinity. For any fixed point x_0 , since $\alpha < 1$ and $K = 0$, we apply Theorem 1.1 to $u(x_0)$ in $B(x_0, R)$ (here function u is independent of time t), and get that

$$|\nabla \ln u(x_0)| \leq c(n) (1 + c(\delta) \ln R - \ln u(x_0)) \times \left[\frac{1 + \sqrt{\sigma^+}}{\sqrt{R}} + o[R^{-\frac{\kappa}{2}(1-\alpha)}] (R^{-\tilde{\kappa}})^{\frac{\alpha-1}{2}} + o[R^{-\frac{\kappa}{3}(1-\alpha)}] (R^{-\tilde{\kappa}})^{\frac{\alpha-1}{3}} \right]$$

for $R > 2$, where we have used the fact that $\min_{x \in B(x_0, R)} u(x) = R^{-\tilde{\kappa}}$. Letting $R \rightarrow \infty$ and using $\kappa > \tilde{\kappa} > 0$, we get

$$|\nabla u(x_0)| = 0.$$

Since point x_0 was chosen arbitrarily, we have that $u(x) \equiv c$ for some constant $c > 0$. Substituting $u(x) \equiv c$ into (5.1) we get $p(x) \equiv 0$ which is impossible due to the assumption of $p(x)$. Therefore we complete the proof. □

Secondly, we consider a static version of Eq. (1.1) for $\alpha > 1$ and $\beta > 1$ on a smooth metric measure space. Because the proof method of Theorem 1.5 is suitable to the elliptic version of (1.1), we only state the result without the proof.

Theorem 5.2. *Let $(M, g, e^{-f} dv)$ be an n -dimensional complete smooth metric measure space with $\text{Ric}_f \geq 0$. Assume that there exist three constants $s > 0$, $\kappa > 0$ and $k > 0$ such that $\mu(x)$, $p(x)$ and $q(x)$ in the following elliptic equation*

$$\Delta_f u + \mu(x)u + p(x)u^\alpha + q(x)u^\beta = 0, \quad \alpha > 1, \beta > 1, \tag{5.2}$$

satisfy

- (1) $\mu^+|_{B(x_0, R)} = o(R^{-s})$ and $\sup_{B(x_0, R)} |\nabla \mu| = o(R^{-s})$, as $R \rightarrow \infty$;
- (2) $p^+|_{B(x_0, R)} = o[R^{-\kappa(\alpha-1)}]$ and $\sup_{B(x_0, R)} |\nabla p| = o[R^{-\kappa(\alpha-1)}]$, as $R \rightarrow \infty$;
- (3) $q^+|_{B(x_0, R)} = o[R^{-k(\beta-1)}]$ and $\sup_{B(x_0, R)} |\nabla q| = o[R^{-k(\beta-1)}]$, as $R \rightarrow \infty$.

Let $u(x)$ be positive solution to the elliptic equation (5.2) on M such that

$$u(x) = o(r^{\tilde{\kappa}}(x))$$

for some $\tilde{\kappa} \in (0, l)$ near infinity, where $l := \min\{\kappa, k\}$. Then $u(x)$ is a positive constant.

Thirdly, we will apply Theorem 1.1 to discuss some Yamabe-type problems of complete Riemannian manifolds and smooth metric measure spaces.

We now prove Theorem 1.7 by applying Theorem 1.5 to Eq.1.4).

Proof of Theorem 1.7. In order to prove the theorem, we only need to discuss the nonexistence of positive smooth solutions $u(x)$ to Eq.1.4) on (M, g) . In Theorem 1.5, if we let

$$u(x, t) = u(x), \quad f = 0, \quad \alpha = \frac{n+2}{n-2},$$

and

$$\mu(x) = -\frac{n-2}{4(n-1)} S, \quad p(x) = \frac{n-2}{4(n-1)} \tilde{S},$$

then by the assumptions of Theorem 1.7, we know that $S \geq 0$ and such $u(x)$ does not exist and hence the theorem follows. □

Theorem 1.9 can be proved by applying Theorem 5.2 to Eq.1.5).

Proof of Theorem 1.9. Assume that u is a minimizer of the weighted Yamabe constant $\Lambda \leq 0$. By the proof of Proposition 4.1 in [15], u is a solution of the equation

$$\Delta_f u - \frac{m+n-2}{4(m+n-1)} S_f^m u - c_1 e^{\frac{f}{m}} u^{\frac{m+n}{m+n-2}} + c_2 u^{\frac{m+n+2}{m+n-2}} = 0,$$

where

$$c_1 = \frac{2m(m+n-1)\Lambda}{n(m+n-2)} \left(\int_M u^{\frac{2(m+n)}{m+n-2}} \right)^{\frac{2m+n-2}{n}} \left(\int_M u^{\frac{2(m+n-1)}{m+n-2}} e^{\frac{f}{m}} \right)^{-\frac{2m+n}{n}},$$

$$c_2 = \frac{(2m+n-2)(m+n)\Lambda}{n(m+n-2)} \left(\int_M u^{\frac{2(m+n)}{m+n-2}} \right)^{\frac{2m-2}{n}} \left(\int_M u^{\frac{2(m+n-1)}{m+n-2}} e^{\frac{f}{m}} \right)^{-\frac{2m}{n}}.$$

In order to prove the theorem, we only need to check the nonexistence of nonconstant positive solutions $u(x)$ to the above equation under the assumptions of Theorem 1.9. Notice that $c_1 \leq 0$ and $c_2 \leq 0$ due to $\Lambda \leq 0$. In Theorem 5.2, if we let

$$\mu(x) = -\frac{m+n-2}{4(m+n-1)}S_f^m, \quad p(x) = -c_1e^{\frac{f}{m}}, \quad q(x) = c_2,$$

and

$$\alpha = \frac{m+n}{m+n-2} > 1, \quad \beta = \frac{m+n+2}{m+n-2} > 1,$$

then the assumptions of Theorem 1.9 imply that all the conditions of Theorem 5.2 are satisfied and hence such $u(x)$ does not exist, which contradicts the existence of positive minimizer u . \square

When $\Lambda = 0$, we can prove Theorem 1.10 by applying Theorem 4.2 to Eq.1.5).

Proof of Theorem 1.10. Assume that u is a critical point of the weighted Yamabe quotient $\mathcal{Q}(u)$ with $u(x) = e^{o(r^{1/2}(x))}$ near infinity. Then such u satisfies Eq.1.5). Since $\Lambda = 0$, we have $c_1 = c_2 = 0$ and the critical point in fact is a minimizer. So (1.5) becomes

$$\Delta_f u - \frac{m+n-2}{4(m+n-1)}R_f^m u = 0.$$

In order to prove Theorem 1.10, we only need to check the nonexistence of positive solutions to the above equation under conditions of Theorem 1.10. Indeed, if we let

$$\mu(x) = -\frac{m+n-2}{4(m+n-1)}S_f^m \quad \text{and} \quad u(x, t) = u(x),$$

in Theorem 4.2, then the assumptions of Theorem 1.10 satisfy all the conditions of Theorem 4.2. Therefore, by Theorem 4.2, we know that there does not exist any positive solution $u(x)$ with $u(x) = e^{o(r^{1/2}(x))}$ near infinity. So the theorem follows. \square

The other goal of this section is that we apply Theorem 1.3 to study elliptic Liouville-type theorems for some elliptic versions of Eq.1.1) on a smooth metric measure space. Here, we mainly apply Theorem 1.3 to prove Theorem 1.11 and Corollary 1.12.

Proof of Theorem 1.11. Let $u(x)$ be a positive smooth function to the equation

$$\Delta_f u + pu^\alpha = 0, \quad \alpha \leq 1,$$

where p is a nonnegative constant. Since $\text{Ric}_f^m \geq 0$ and $a_i = b_i = 0$ ($i = 1, 2, 3$), applying Theorem 1.3 to this equation, for any $\lambda > 1$, we have the gradient estimate

$$\frac{|\nabla u|^2}{\lambda u^2} + pu^{\alpha-1} \leq \sqrt{\frac{m+n}{2}}\Psi^{\frac{1}{2}}$$

by letting $R \rightarrow \infty$, where

$$\Psi := \frac{(m+n)\lambda^2 \tilde{K}^2}{2(1-\varepsilon)(\lambda-1)^2} \quad \text{and} \quad \tilde{K} := -\frac{1}{2}[(\alpha-1)(\lambda\alpha-1)p]^- \sup_{H_{2R,T}} \{u^{\alpha-1}\}.$$

In the above estimate, if $\alpha = 1$, then $\tilde{K} \equiv 0$ and

$$\frac{|\nabla u|^2}{\lambda u^2} + pu^{\alpha-1} = 0,$$

which implies the theorem. So we only consider the case $\alpha < 1$. In this case, we choose $\lambda = \lambda_0 > 1$ such that $\lambda_0\alpha < 1$ and then $\tilde{K} \equiv 0$, hence $\Psi \equiv 0$, which also implies the theorem. □

Theorem 1.11 immediately implies Corollary 1.12 as follows.

Proof of Corollary 1.12. Assume that u is a critical point of the weighted Yamabe quotient $Q(u)$. Then u satisfies Eq.1.5). Since $\Lambda = 0$, we have $c_1 = c_2 = 0$ and hence (1.5) becomes

$$\Delta_f u - \frac{m+n-2}{4(m+n-1)} S_f^m u = 0.$$

In the following we only need to check the nonexistence of positive solutions to the above equation under the condition of Corollary 1.12. Indeed, since S_f^m is non-positive constant, we know that $-\frac{m+n-2}{4(m+n-1)} S_f^m$ is nonnegative constant. According to Theorem 1.11, we immediately conclude that there does not exist any nontrivial positive solution to the above equation. So our assumption does not hold and the theorem follows. □

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