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Two classes of non-weight modules over the twisted Heisenberg–Virasoro algebra

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Abstract. We study two classes of non-weight modules $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ and $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ over the twisted Heisenberg–Virasoro algebra in this paper. We present conditions under which these modules are irreducible, the necessary and sufficient conditions under which modules in each class are isomorphic, and also show that the irreducible modules in these two classes are new. Finally, we construct non-weight modules $\mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS})$ and $\mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ})$ over the twisted Heisenberg–Virasoro algebra and then apply the established results to give irreducible conditions for these modules.

1. Introduction

The well-known *twisted Heisenberg–Virasoro algebra* $\mathcal{H}$, initially studied by Arbarello et al. in [1], is the universal central extension of the Lie algebra $\overline{L}$ of differential operators on a circle of order at most one:

$$\overline{L} := \left\{ f(t)\frac{d}{dt} + g(t) \,\middle|\, f(t), g(t) \in \mathbb{C}[t, t^{-1}] \right\}.$$

To be more precise, $\mathcal{H}$ is an infinite dimensional complex Lie algebra with basis $\{L_m, I_m, C_i \mid m \in \mathbb{Z},\ i = 1, 2, 3\}$ subject to the following Lie brackets:

$$\begin{aligned}
&[\mathcal{H}, C_1] = [\mathcal{H}, C_2] = [\mathcal{H}, C_3] = 0,\\
&[L_m, L_n] = (n-m)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C_1,\\
&[L_m, I_n] = nI_{m+n} + \delta_{m+n,0}(m^2+m)C_2,\\
&[I_m, I_n] = n\delta_{m+n,0}C_3 \quad \text{for } m, n \in \mathbb{Z}.
\end{aligned}$$

Clearly, the subspaces spanned by $\{I_m, C_3 \mid 0 \neq m \in \mathbb{Z}\}$ and by $\{L_m, C_1 \mid m \in \mathbb{Z}\}$ are respectively the Heisenberg algebra and the Virasoro algebra. Notice that the

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center of $\mathcal{H}$ is spanned by $\{C_0 := I_0, C_i \mid i = 1, 2, 3\}$. Moreover, the twisted Heisenberg–Virasoro algebra has a triangular decomposition:

$$\mathcal{H} = \mathcal{H}_- \oplus \mathfrak{h} \oplus \mathcal{H}_+,$$

where $\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{L_0, C_i \mid i = 0, 1, 2, 3\}$ is a Cartan subalgebra of $\mathcal{H}$ and

$$\mathcal{H}_- = \mathrm{span}_{\mathbb{C}}\{L_{-m}, I_{-m} \mid m \in \mathbb{N}\},\ \mathcal{H}_+ = \mathrm{span}_{\mathbb{C}}\{L_m, I_m \mid m \in \mathbb{N}\}.$$

The twisted Heisenberg–Virasoro algebra is one of the most important Lie algebras both in mathematics and in mathematical physics, whose structure theory has been extensively studied (see, e.g., [7,10,20]).

A fundamental problem in the representation theory of the twisted Heisenberg–Virasoro algebra is to classify all its irreducible modules. In fact, the theory of weight modules with all weight subspaces being finite dimensional (namely, Harish-Chandra modules) has been well developed. Irreducible weight modules over $\mathcal{H}$ with a nontrivial finite dimensional weight subspace were proved to be Harish-Chandra modules [22]. And irreducible Harish-Chandra $\mathcal{H}$-modules were classified in [15], each of which was shown to be either the highest (or lowest) weight module, or the module of the intermediate series, which is consistent with the well-known result for the Virasoro algebra [16]. While weight modules with an infinite dimensional weight subspace were also studied (see [6,19]).

Non-weight modules constitute the other important ingredients of the representation theory of $\mathcal{H}$, the study of which is definitely necessary and became popular in the last few years. A large class of new non-weight irreducible $\mathcal{H}$-modules were constructed in [3], which includes the highest weight modules and Whittaker modules. Non-weight $\mathcal{H}$-modules whose restriction to the universal enveloping algebra of the degree-0 part (modulo center) are free of rank 1 were studied in [4] (see also [8]). And by twisting the weight modules, we obtained a family of new non-weight irreducible $\mathcal{H}$-modules [6]. However, the theory of non-weight $\mathcal{H}$-modules is far more from being well developed.

As a continuation of [6], we still study the representation theory of $\mathcal{H}$ in this paper. But we shall be concerned with non-weight $\mathcal{H}$-modules. It is well known that an important way of constructing modules is to consider the linear tensor product of two modules, see for instance, [5,23–25] for such modules over the Virasoro algebra. All non-weight $\mathcal{H}$-modules studied in the present paper are obtained in this way, which can be divided into two classes: one class consists of tensor product modules and the other class consists of modules constructed from the linear tensor product of two modules.

We briefly give a summary of the paper below. In Sect. 2, we recall some known modules and construct a class of non-weight modules $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ over the twisted Heisenberg–Virasoro algebra $\mathcal{H}$, where V is a module over some Lie algebra which is a subquotient of $\mathcal{H}$. Section 3 is devoted to studying the irreducibilities of tensor product modules $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ and the reducibilities of $\mathcal{H}$-modules $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$, where $\mathrm{Ind}(M)$ is an $\mathcal{H}$-module induced from some subalgebra of $\mathcal{H}$. We prove that $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ is irreducible if both $\Omega(\lambda, \alpha, \beta)$ and $\mathrm{Ind}(M)$ are irreducible (in fact, here we use some sufficient conditions for the irreducibility of $\mathrm{Ind}(M)$) and that $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ is reducible if and only if V is a certain one-dimensional module. In Sect. 4, we give the necessary and sufficient conditions under which two irreducible $\mathcal{H}$-modules $\Omega(\lambda_1, \alpha_1, \beta_1) \otimes \mathrm{Ind}(M_1)$ and $\Omega(\lambda_2, \alpha_2, \beta_2) \otimes \mathrm{Ind}(M_2)$ are isomorphic. And we also determine the isomorphism classes of the modules $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$. In Sect. 5, we prove irreducible modules in the two classes are new by showing they are not isomorphic to any other known irreducible non-weight $\mathcal{H}$-module. In Sect. 6, some non-weight $\mathcal{H}$-modules are constructed and using Theorem 3.1 we present irreducible conditions for these modules. The main results of this paper are summarized in Theorems 3.1, 3.3, 4.1, 4.6, 6.2, 6.3 and Proposition 5.2.

Throughout this paper, we respectively denote by $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of complex numbers, nonzero complex numbers, integers, nonnegative integers and positive integers, and use $\mathcal{U}(\mathfrak{a})$ to denote the universal enveloping algebra of a Lie algebra $\mathfrak{a}$. All vector spaces are assumed to be over $\mathbb{C}$.

2. Preliminaries

In this section, we shall first recall some known $\mathcal{H}$-modules and then introduce a functor from the category of $\mathcal{H}$-modules to some of its subcategory, and finally we present some non-weight modules which will be studied in this paper.

2.1. Some $\mathcal{H}$-modules

Definition 2.1. An $\mathcal{H}$-module V is called a weight module if V has the decomposition: $V = \oplus_{\lambda\in\mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V \mid xv = \lambda(x)v,\ \forall x \in \mathfrak{h}\}$ for each $\lambda \in \mathfrak{h}^*$ (the dual space of $\mathfrak{h}$); and called a non-weight module otherwise.

Remark 2.2. More generally, we call an $\mathcal{H}$-module V a weight module with respect to L_0 if $V = \oplus_{\lambda\in\mathbb{C}} V_\lambda$ with $V_\lambda = \{v \in V \mid L_0 v = \lambda v\}$ for any $\lambda \in \mathbb{C}$.

For later use, we are going to recall some weight and non-weight $\mathcal{H}$-modules. For $\lambda, \alpha, \beta \in \mathbb{C}$, the (weight) $\mathcal{H}$-module $A(\lambda, \alpha, \beta)$ of the intermediate series has a basis $\{v_i \mid i \in \mathbb{Z}\}$ with trivial central actions and

$$L_m v_n = \big(\lambda + n + m\alpha\big) v_{m+n},\ \ I_m v_n = \beta v_{m+n} \quad \text{for } m, n \in \mathbb{Z}.$$

Note that $A(\lambda, \alpha, \beta)$ is reducible if and only if $\lambda \in \mathbb{Z}$, $\alpha \in \{0, 1\}$ and $\beta = 0$ (see [9,11]).

For $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, we recall the non-weight module

$$\Omega(\lambda, \alpha, \beta) := \mathbb{C}[t],$$

with the action of $\mathcal{H}$ defined, for $i = 1, 2, 3$, $f(t) \in \mathbb{C}[t]$ and $m \in \mathbb{Z}$, by

$$L_m f(t) = \lambda^m (t - m\alpha) f(t - m),\ \ I_m f(t) = \lambda^m \beta f(t - m),\ \ C_i f(t) = 0.$$

Then $\Omega(\lambda, \alpha, \beta)$ is irreducible if and only if $\alpha \in \mathbb{C}^*$ or $\beta \in \mathbb{C}^*$ (see [4]). Notice that this module reduces to a Virasoro module if $\beta = 0$ (see [13]).

Now let us recall a class of irreducible modules for the twisted Heisenberg–Virasoro algebra, which includes the known irreducible modules such as highest weight modules and Whittaker modules. For any $e \in \mathbb{Z}_+$, denote by $\mathcal{H}_e$ the subalgebra of $\mathcal{H}$:

$$\sum_{m \in \mathbb{Z}_+} (\mathbb{C}L_m \oplus \mathbb{C}I_{m-e}) \oplus \mathbb{C}C_1 \oplus \mathbb{C}C_2 \oplus \mathbb{C}C_3.$$

Take $M(c_0, c_1, c_2, c_3)$ to be an irreducible $\mathcal{H}_e$-module such that I_0, C_1, C_2 and C_3 act as scalars c_0, c_1, c_2, c_3 respectively. For convenience, we briefly denote $M(c_0, c_1, c_2, c_3)$ by M. Form the induced $\mathcal{H}$-module

$$\mathrm{Ind}(M) := \mathcal{U}(\mathcal{H}) \otimes_{\mathcal{U}(\mathcal{H}_e)} M. \tag{2.1}$$

The following theorem was obtained in [3].

Theorem 2.3. *Let $e \in \mathbb{Z}_+$ and M be an irreducible $\mathcal{H}_e$-module with $c_3 = 0$. Suppose that there exists $k \in \mathbb{Z}_+$ such that*

(1) $\begin{cases} \text{the action of } I_k \text{ on } M \text{ is injective} & \text{if } k \neq 0, \\ c_0 + (n-1)c_2 \neq 0 \quad \text{for all } n \in \mathbb{Z}\backslash\{0\} & \text{if } k = 0, \end{cases}$

(2) $I_n M = L_m M = 0$ *for all $n > k$ and $m > k + e$.*

Then

(i) $\mathrm{Ind}(M)$ *is an irreducible $\mathcal{H}$-module*;

(ii) *the actions of I_n, L_m on $\mathrm{Ind}(M)$ for all $n > k$ and $m > k + e$ are locally nilpotent.*

2.2. *The functor $\mathcal{W}$*

Inspired by [18] (see also [14]) we define a functor $\mathcal{W}$ from the category of $\mathcal{H}$-modules to the category of weight $\mathcal{H}$-modules with respect to L_0 (see Remark 2.2) in this subsection.

For any $c \in \mathbb{C}$, denote $\mathcal{I}_c$ by the (maximal) ideal of $\mathbb{C}[L_0]$ generated by $L_0 - c$. For an $\mathcal{H}$-module V and $n \in \mathbb{Z}$, set

$$V_n = V/\mathcal{I}_n V, \ \mathcal{W}(V) = \oplus_{n \in \mathbb{Z}} V_n.$$

Then the vector space $\mathcal{W}(V)$ carries the structure of a weight $\mathcal{H}$-module with respect to L_0 under the following given actions:

$$L_m(v + \mathcal{I}_n V) = L_m v + \mathcal{I}_{m+n} V,$$
$$I_m(v + \mathcal{I}_n V) = I_m v + \mathcal{I}_{m+n} V,$$
$$C_i(v + \mathcal{I}_n V) = 0 \quad \text{for } i = 1, 2, 3, \ m, n \in \mathbb{Z} \text{ and } v \in V.$$

Given any $\mathcal{H}$-module homomorphism $f : V \to W$ we define $\mathcal{W}(f) : \mathcal{W}(V) \to \mathcal{W}(W)$ by sending $v + \mathcal{I}_n V$ to $f(v) + \mathcal{I}_n W$ for any $v \in V$ and $n \in \mathbb{Z}$. Then it is easy to check that W becomes a functor from the category of $\mathcal{H}$-modules to the category of weight $\mathcal{H}$-modules with respect to L_0.

2.3. Construction of $\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))$

Based on the non-weight $\mathcal{H}$-modules $\Omega(\lambda, \alpha, \beta)$ (see Sect. 2.1) we shall consider two classes of non-weight $\mathcal{H}$-modules. Objects in one class consist of the tensor product modules $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ (see (2.1)) and objects in the other class are constructed in this subsection.

For $d \in \{0, 1\}, r \in \mathbb{Z}_+$, denote by $\mathcal{H}_{r,d}$ the ideal of $\mathcal{H}_{+,d} = \mathrm{span}_{\mathbb{C}}\{L_i, I_j \mid i \geq 0, j \geq d\}$ generated by L_i, I_j for all $i > r, j > r + d$. Now we write $\bar{\mathcal{H}}_{r,d}$ the quotient algebra $\mathcal{H}_{+,d}/\mathcal{H}_{r,d}$, and $\bar{L}_i, \bar{I}_{i+d}$ the respective images of L_i, I_{i+d} in $\bar{\mathcal{H}}_{r,d}$. Let V be an $\bar{\mathcal{H}}_{r,d}$-module. For any $\lambda, \alpha, \beta \in \mathbb{C}$, define an $\mathcal{H}$-action on the vector space $\mathcal{M}(V, \Omega(\lambda, \alpha, \beta)) := V \otimes \mathbb{C}[t]$ as follows (see [14]):

$$L_m(v \otimes f(t)) = v \otimes \lambda^m (t - m\alpha) f(t - m) + \sum_{i=0}^{r} \left(\frac{m^{i+1}}{(i+1)!} \bar{L}_i \right) v \otimes \lambda^m f(t - m), \tag{2.2}$$

$$I_m(v \otimes f(t)) = \sum_{i=0}^{r} \left(\frac{m^{i+d}}{(i+d)!} \bar{I}_{i+d} \right) v \otimes \lambda^m \beta f(t - m), \tag{2.3}$$

$$C_i(v \otimes f(t)) = 0 \quad \text{for } i = 1, 2, 3, m \in \mathbb{Z}, v \in V, f(t) \in \mathbb{C}[t]. \tag{2.4}$$

Proposition 2.4. *Let $d \in \{0, 1\}, r \in \mathbb{Z}_+$ and V be an $\bar{\mathcal{H}}_{r,d}$-module. Then $\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))$ is a non-weight $\mathcal{H}$-module under the actions given in* (2.2)–(2.4).

Proof. Define a series of operators x_m on $\mathbb{C}[t]$ as follows:

$$x_m f(t) = \lambda^m f(t - m) \quad \text{for } m \in \mathbb{Z} \text{ and } f(t) \in \mathbb{C}[t].$$

Then $I_n x_m f(t) = x_m I_n f(t) = I_{m+n} f(t)$ for any $m, n \in \mathbb{Z}$. It follows from [14, Section 3] that the relation $L_m L_n - L_n L_m = (n - m) L_{m+n}$ holds on $\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))$. By (2.2)–(2.4), we have

$$\begin{aligned}
&(L_m I_n - I_n L_m)(v \otimes f(t)) \\
&= \sum_{i=0}^{r} \left(\frac{n^{i+d}}{(i+d)!} \bar{I}_{i+d} \right) v \otimes (L_m I_n f(t)) \\
&\quad + \sum_{i=0}^{r} \left(\frac{m^{i+1}}{(i+1)!} \bar{L}_i \right) \sum_{i=0}^{r} \left(\frac{n^{i+d}}{(i+d)!} \bar{I}_{i+d} \right) v \otimes (x_m I_n f(t)) \\
&\quad - \sum_{i=0}^{r} \left(\frac{n^{i+d}}{(i+d)!} \bar{I}_{i+d} \right) v \otimes (I_n L_m f(t)) \\
&\quad - \sum_{i=0}^{r} \left(\frac{n^{i+d}}{(i+d)!} \bar{I}_{i+d} \right) \sum_{i=0}^{r} \left(\frac{m^{i+1}}{(i+1)!} \bar{L}_i \right) v \otimes (I_n x_m f(t)) \\
&= n \sum_{i=0}^{r} \left(\frac{n^{i+d}}{(i+d)!} \bar{I}_{i+d} \right) v \otimes (I_{m+n} f(t))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{r}\left(\frac{m^{i+1}}{(i+1)!}\bar{L}_i\right)\sum_{j=0}^{r}\left(\frac{n^{j+d}}{(j+d)!}\bar{I}_{j+d}\right)v\otimes\big(I_{m+n}f(t)\big)\\
&-\sum_{j=0}^{r}\left(\frac{n^{j+d}}{(j+d)!}\bar{I}_{j+d}\right)\sum_{i=0}^{r}\left(\frac{m^{i+1}}{(i+1)!}\bar{L}_i\right)v\otimes\big(I_{m+n}f(t)\big)\\
&=n\sum_{i=0}^{r}\left(\frac{n^{i+d}}{(i+d)!}\bar{I}_{i+d}\right)v\otimes\big(I_{m+n}f(t)\big)\\
&+\sum_{i,j=0}^{r}\left(\frac{m^{i+1}n^{j+d}}{(i+1)!(j+d)!}\big(\bar{L}_i\bar{I}_{j+d}-\bar{I}_{j+d}\bar{L}_i\big)\right)v\otimes\big(I_{m+n}f(t)\big)\\
&=n\sum_{i=0}^{r}\sum_{j=\delta_{d,0}}^{i+1}\left(\frac{m^{i+1-j}n^{j+d-1}}{(i+1-j)!(j+d-1)!}\bar{I}_{i+d}\right)v\otimes\big(I_{m+n}f(t)\big)\\
&=n\sum_{i=0}^{r}\left(\frac{(m+n)^{i+d}}{(i+d)!}\bar{I}_{i+d}\right)v\otimes\big(I_{m+n}f(t)\big)=nI_{m+n}\big(v\otimes f(t)\big).
\end{aligned}
$$

That is, $L_mI_n - I_nL_m = nI_{n+m}$ holds on $\mathcal{M}\big(V,\Omega(\lambda,\alpha,\beta)\big)$. Finally, the relation $I_mI_n - I_nI_m = 0$ on $\mathcal{M}\big(V,\Omega(\lambda,\alpha,\beta)\big)$ is trivial. Thus, the actions (2.2)–(2.4) make $\mathcal{M}\big(V,\Omega(\lambda,\alpha,\beta)\big)$ into a non-weight $\mathcal{H}$-module. □

Remark 2.5. Let $d\in\{0,1\}$, $r\in\mathbb{Z}_+$ and V be an irreducible $\bar{\mathcal{H}}_{r,d}$-module.

(1) V must be infinite dimensional if $\dim V>1$, since any irreducible finite dimensional module over the solvable Lie algebra $\bar{\mathcal{H}}_{r,d}$ is one-dimensional by Lie's Theorem.

(2) Consider now $V=\mathbb{C}v$ is one-dimensional. Then $\bar{L}_iv=\bar{I}_iv=0$ for any $i\in\mathbb{N}$, $\bar{L}_0v=\sigma v$, $\bar{I}_0v=\tau v$ for some $\sigma,\tau\in\mathbb{C}$. In this case V is denoted by $V_{\sigma,\tau}$ and it is clear that

$$\mathcal{M}\big(V_{\sigma,\tau},\Omega(\lambda,\alpha,\beta)\big)\cong\Omega(\lambda,\alpha-\sigma,\delta_{d,0}\beta\tau)\quad\text{for any }\lambda\in\mathbb{C}^*\text{ and }\alpha,\beta\in\mathbb{C},$$

where $\delta_{d,0}$ is the Kronecker delta.

(3) Note that if $\bar{I}_{j+d}V=0$ for all $0\le j\le r$, then $\mathcal{M}\big(V,\Omega(\lambda,\alpha,\beta)\big)$ reduces to a module over the Virasoro algebra. So until further notice we always assume that $\bar{I}_{j+d}V\neq0$ for some $0\le j\le r$. Choose such r' to be maximal such that $\bar{I}_{r'+d}V\neq0$. Then $\bar{I}_{r'+d}$ is a linear isomorphism of V by [6, Lemma 3.1] if V is irreducible.

3. Irreducibilities

Theorem 3.1. *Let* $(\lambda,\alpha)\in(\mathbb{C}^*)^2$ *or* $(\lambda,\beta)\in(\mathbb{C}^*)^2$. *Suppose that* $\mathrm{Ind}(M)$ *is an* $\mathcal{H}$*-module defined by* (2.1) *for which* M *satisfies the conditions in Theorem* 2.3. *Then the tensor product module* $\Omega(\lambda,\alpha,\beta)\otimes\mathrm{Ind}(M)$ *of* $\mathcal{H}$*-modules* $\Omega(\lambda,\alpha,\beta)$ *and* $\mathrm{Ind}(M)$ *is an irreducible* $\mathcal{H}$*-module.*

Proof. For any $v \in \mathrm{Ind}(M)$, there exists $K(v) \in \mathbb{Z}_+$ such that $I_m v = L_m v = 0$ for all $m \geq K(v)$ by Theorem 2.3. Suppose that P is a nonzero submodule of $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$. Choose a nonzero

$$w = \sum_{i=0}^{n} t^i \otimes v_i \in P \text{ with } 0 \neq v_n \in \mathrm{Ind}(M) \text{ and } n \in \mathbb{Z}_+ \text{ minimal.}$$

The case for $\alpha \in \mathbb{C}^*$ was proved in [23, Theorem 1], thus we only need to consider the case for $\beta \in \mathbb{C}^*$. □

Claim 1. $n = 0$.

Let $K = \max\{K(v_i) \mid i = 0, 1, \ldots, n\}$. Then we have

$$\lambda^{-m} I_m w = \sum_{i=0}^{n} \beta(t-m)^i \otimes v_i \in P \text{ for } m \geq K.$$

Note that the right-hand side of the above can also be written as

$$\sum_{i=0}^{n} m^i w_i \in P$$

for some $w_i \in \Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ (independent of the choice of m) with $w_n = \beta(-1)^n \otimes v_n \neq 0$. It follows from that $w_n \in P$. Thus, n must be zero by its minimality, proving the claim.

To complete the proof, it suffices to show the following claim.

Claim 2. $P = \Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$.

By Claim 1, we have $1 \otimes v_0 \in P$ for some nonzero $v_0 \in \mathrm{Ind}(M)$. Using

$$\begin{aligned} L_m(t^k \otimes v_0) &= (\lambda^m (t - m\alpha)(t-m)^k) \otimes v_0 \\ &= \lambda^m (t-m)^{k+1} \otimes v_0 - \lambda^m m(\alpha - 1)(t-m)^k \otimes v_0 \end{aligned}$$

for $m \geq K(v_0)$, $k \in \mathbb{Z}_+$ and by induction on k, we deduce that $t^k \otimes v_0 \in P$ for $k \in \mathbb{Z}_+$, i.e., $\Omega(\lambda, \alpha, \beta) \otimes v_0 \subseteq P$. It follows that $\Omega(\lambda, \alpha, \beta) \otimes \mathcal{U}(\mathcal{H}) v_0 \subseteq P$. Thus, $P = \Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$, since the nonzero $\mathcal{H}$-submodule $\mathcal{U}(\mathcal{H}) v_0$ of $\mathrm{Ind}(M)$ generated by v_0 is equal to $\mathrm{Ind}(M)$ by the irreducibility of $\mathrm{Ind}(M)$. □

Now we describe the following two examples of the modules in Theorem 3.1, which will be discussed in detail in Sect. 6.

Example 3.2. (i) Let $h \in \mathbb{C}$, $\underline{d} = (d_0, d_1, d_2, d_3) \in \mathbb{C}^4$ with $d_3 = 0$. Let J_1 be the left ideal of $\mathcal{U}(\mathfrak{h} \oplus \mathcal{H}_+)$ generated by $L_m, I_m, L_0 - h$ and $C_i - d_i$ for $i = 0, 1, 2, 3, m \in \mathbb{Z}_+$. Denote $\bar{M} := \mathcal{U}(\mathfrak{h} \oplus \mathcal{H}_+)/J_1$. Then $V = \mathrm{Ind}(\bar{M})$ is the classical Verma module (see, e.g., [2,21]). By Theorem 2.3 (cf. [2, Theorem 1]), we obtain that if $d_0 + (n-1)d_2 \neq 0$ for $n \in \mathbb{Z} \backslash \{0\}$, then V is both an irreducible $\mathcal{H}$-module and a locally nilpotent module over $\mathcal{H}_+$. From Theorem 3.1, we obtain that $\Omega(\lambda, \alpha, \beta) \otimes V$ is an irreducible $\mathcal{H}$-module if $d_0 + (n-1)d_2 \neq 0$ for $n \in \mathbb{Z} \backslash \{0\}$ and either $(\lambda, \alpha) \in (\mathbb{C}^*)^2$ or $(\lambda, \beta) \in (\mathbb{C}^*)^2$.

(ii) Let $(\lambda_1, \lambda_2, \mu_1) \in \mathbb{C}^3$, $\underline{e} = (e_0, e_1, e_2, e_3) \in \mathbb{C}^4$ with $e_3 = 0$. Let J_2 be the left ideal of $\mathcal{U}(\mathcal{H}_+)$ generated by

$$\{L_1 - \lambda_1, L_2 - \lambda_2, L_i, I_1 - \mu_1, I_j, C_k - e_k \mid i \geq 3, j \geq 2, k = 0, 1, 2, 3\}.$$

Denote $\tilde{M} := \mathcal{U}(\mathcal{H}_+)/J_2$. Then $V = \operatorname{Ind}(\tilde{M})$ is the classical Whittaker module (see, e.g., [3,12]). By Theorem 2.3 (cf. [3, Example 10]), we obtain that if $e_0 + (n-1)e_2 \neq 0$ for all $n \in \mathbb{Z}\backslash\{0\}$ and $\mu_1 \neq 0$, then V is both an irreducible $\mathcal{H}$-module and a locally nilpotent module over $\mathcal{H}_+^{(2)} = \operatorname{span}_{\mathbb{C}}\{L_m, I_m \mid m > 2\}$. From Theorem 3.1, we obtain that $\Omega(\lambda, \alpha, \beta) \otimes V$ is an irreducible $\mathcal{H}$-module if $e_0 + (n-1)e_2 \neq 0$ for all $n \in \mathbb{Z}\backslash\{0\}$ and either $(\lambda, \alpha, \mu_1) \in (\mathbb{C}^*)^3$ or $(\lambda, \beta, \mu_1) \in (\mathbb{C}^*)^3$.

Next we are going to characterise the reducibility of $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$. For any $m \in \mathbb{Z}_+, n \in \mathbb{Z}$, denote

$$J_n^0 = 1 \text{ and } J_n^m = \prod_{j=n+1}^{n+m} (t - j) \quad \text{for } m > 0.$$

Note that $\{J_n^m \mid m \in \mathbb{Z}_+\}$ forms a basis of $\Omega(\lambda, \alpha, \beta)$ for any $n \in \mathbb{Z}$. By the action of $\mathcal{H}$ on $\Omega(\lambda, \alpha, \beta)$, it is easy to check that

$$\begin{aligned} L_m J_n^k &= \lambda^m (t - m\alpha) J_{m+n}^k \\ \text{and } I_m J_n^k &= \lambda^m \beta J_{m+n}^k \quad \text{for } m, n \in \mathbb{Z}, k \in \mathbb{Z}_+. \end{aligned} \tag{3.1}$$

Now we are ready to state the other main result of this section.

Theorem 3.3. *Let $\lambda \in \mathbb{C}^*$ and V be an irreducible $\bar{\mathcal{H}}_{r,d}$-module. Then $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ is reducible if and only if $V \cong V_{\alpha, \delta_{d,0}\tau}$ for some $\tau \in \mathbb{C}$ such that $\delta_{d,0}\beta\tau = 0$.*

Proof. Consider first that V is finite dimensional. Then $V \cong V_{\sigma,\tau}$ for some $\sigma, \tau \in \mathbb{C}$ and $\mathcal{M}\big(V_{\sigma,\tau}, \Omega(\lambda, \alpha, \beta)\big) \cong \Omega(\lambda, \alpha - \sigma, \delta_{d,0}\beta\tau)$ by Remark 2.5(2). But we know that $\Omega(\lambda, \alpha - \sigma, \delta_{d,0}\beta\tau)$ is reducible if and only if $\alpha = \sigma$ and $\delta_{d,0}\beta\tau = 0$. So in this case the statement is true.

Assume that V is infinite dimensional. To complete the proof, it suffices to show that $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ is irreducible. For this, let M be a nonzero submodule of $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$. Without loss of generality, we may assume that $\lambda = 1$. If $\beta = 0$, then $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ reduces to a module over the Virasoro algebra, which is irreducible by [14, Theorem 3.2].

Consider now the case $\beta \neq 0$. Let r' be the nonnegative integer as in Remark 2.5(3) and $u = \sum_{m=0}^{p} v_m \otimes J_0^m$ a nonzero element in M with $v_p \neq 0$. It follows from (2.3) and the second relation in (3.1) that

$$I_n(v_m \otimes J_0^m) = \sum_{i=0}^{r'} \left(\frac{n^{i+d}}{(i+d)!} \bar{I}_{i+d} \right) v_m \otimes \beta J_n^m \quad \text{for } n \in \mathbb{Z}, m \in \mathbb{Z}_+.$$

Then one can check that

$$\begin{aligned}&I_k I_{n-k}(v_m \otimes J_0^m)\\&=\sum_{i=0}^{r'}\left(\frac{k^{i+d}}{(i+d)!}\bar{I}_{i+d}\right)\sum_{i=0}^{r'}\left(\frac{(n-k)^{i+d}}{(i+d)!}\bar{I}_{i+d}\right)v_m\otimes\beta^2 J_n^m\\&\text{for } k,n\in\mathbb{Z}, m\in\mathbb{Z}_+.\end{aligned}$$

Since k is arbitrary, we can view k as a variable. Observe that the coefficient of $k^{2r'+2d}$ in $I_k I_{n-k}u$ is

$$\frac{\beta^2}{((r'+d)!)^2}\sum_{m=0}^{p}\bar{I}_{r'+d}^2 v_m\otimes J_n^m\in M\quad\text{for } n\in\mathbb{Z}.$$

Similarly, viewing n as a variable we get $\bar{I}_{r'+d}^2 v_p\otimes 1\in M$. Set $v=\bar{I}_{r'+d}^2 v_p$. Now by $L_0^n\big(v\otimes 1\big)=v\otimes t^n$ for $n\in\mathbb{Z}$ and the injectivity of $\bar{I}_{r'+d}$ (see Remark 2.5(3)), $v\otimes\Omega(\lambda,\alpha,\beta)$ is nonzero subspace of M. It follows from

$$M\ni L_n(v\otimes J_{k-n}^m)=v\otimes(t-n\alpha)J_k^m+\sum_{i=0}^{r}\left(\frac{n^{i+1}}{(i+1)!}\bar{L}_i\right)v\otimes J_k^m$$

$$\text{and}\quad M\ni I_n(v\otimes J_{k-n}^m)=\sum_{i=0}^{r'}\left(\frac{n^{i+d}}{(i+d)!}\bar{I}_{i+d}\right)v\otimes\beta J_k^m$$

that both $\sum_{i=0}^{r}\big(\frac{n^{i+1}}{(i+1)!}\bar{L}_i\big)v\otimes J_k^m$ and $\sum_{i=0}^{r'}\big(\frac{n^{i+d}}{(i+d)!}\bar{I}_{i+d}\big)v\otimes J_k^m$ lie in M for $k,m\in\mathbb{Z}, n\in\mathbb{Z}_+$. In particular, $\bar{L}_i v\otimes\Omega(\lambda,\alpha,\beta)\in M$ and $\bar{I}_{j+d}v\otimes\Omega(\lambda,\alpha,\beta)\in M$ for $i=0,1,\ldots,r$ and $j=0,1,\ldots,r'$. Then by the irreducibility of V we have $M=\mathcal{M}\big(V,\Omega(\lambda,\alpha,\beta)\big)$, proving the irreducibility of $\mathcal{M}\big(V,\Omega(\lambda,\alpha,\beta)\big)$. □

4. Isomorphism classes

Theorem 4.1. *Let* $\alpha_1,\beta_1\in\mathbb{C}$, $(\lambda_1,\lambda_2,\alpha_2)\in(\mathbb{C}^*)^3$ *or* $(\lambda_1,\lambda_2,\beta_2)\in(\mathbb{C}^*)^3$. *Suppose that* $\mathrm{Ind}(M_1)$ *and* $\mathrm{Ind}(M_2)$ *are* $\mathcal{H}$*-modules for which* M_1 *and* M_2 *satisfy the conditions in Theorem* 2.3. *Then* $\Omega(\lambda_1,\alpha_1,\beta_1)\otimes\mathrm{Ind}(M_1)$ *and* $\Omega(\lambda_2,\alpha_2,\beta_2)\otimes\mathrm{Ind}(M_2)$ *are isomorphic as* $\mathcal{H}$*-modules if and only if* $(\lambda_1,\alpha_1,\beta_1)=(\lambda_2,\alpha_2,\beta_2)$ *and* $\mathrm{Ind}(M_1)\cong\mathrm{Ind}(M_2)$ *as* $\mathcal{H}$*-modules.*

Proof. The "if" part is trivial. Now we prove the "only if" part. Let ψ be an $\mathcal{H}$-module isomorphism from $\Omega(\lambda_1,\alpha_1,\beta_1)\otimes\mathrm{Ind}(M_1)$ to $\Omega(\lambda_2,\alpha_2,\beta_2)\otimes\mathrm{Ind}(M_2)$.

Choose a nonzero element $1\otimes v\in\Omega(\lambda_1,\alpha_1,\beta_1)\otimes\mathrm{Ind}(M_1)$. Assume

$$\psi(1\otimes v)=\sum_{i=0}^{n}t^i\otimes v_i,\ \text{where } v_i\in\mathrm{Ind}(M_2)\text{ with } v_n\neq 0.\tag{4.1}$$

There exists a positive integer K such that $I_m v=I_m v_i=L_m v=L_m v_i=0$ for all integers $m\geq K$ and $0\leq i\leq n$ by the condition (2) of Theorem 2.3.

Now we consider the following two cases.

Case 1. $\beta_2 \in \mathbb{C}^*$.

For any $m_1, m_2 \geq K$, it follows from $(\lambda_1^{-m_1} I_{m_1} - \lambda_1^{-m_2} I_{m_2})(1 \otimes v) = 0$ that

$$\begin{aligned} 0 &= \big(\lambda_1^{-m_1} I_{m_1} - \lambda_1^{-m_2} I_{m_2}\big)\psi(1 \otimes v) \\ &= \big(\lambda_1^{-m_1} I_{m_1} - \lambda_1^{-m_2} I_{m_2}\big)\sum_{i=0}^{n} t^i \otimes v_i \\ &= \sum_{i=0}^{n} \beta_2\bigg(\Big(\frac{\lambda_2}{\lambda_1}\Big)^{m_1}(t-m_1)^i \otimes v_i - \Big(\frac{\lambda_2}{\lambda_1}\Big)^{m_2}(t-m_2)^i \otimes v_i\bigg). \end{aligned} \tag{4.2}$$

In particular, we have

$$\bigg(\Big(\frac{\lambda_2}{\lambda_1}\Big)^{m_1} - \Big(\frac{\lambda_2}{\lambda_1}\Big)^{m_2}\bigg)(t^n \otimes v_n) = 0 \quad \text{for all } m_1, m_2 \geq K,$$

which forces $\lambda_1 = \lambda_2$. Whence (4.2) can be rewritten as

$$\sum_{i=0}^{n}\bigg((t-m_1)^i \otimes v_i - (t-m_2)^i \otimes v_i\bigg) = 0 \quad \text{for all } m_1, m_2 \geq K.$$

Note from the above formula that $n = 0$, since otherwise the coefficient $(-1)^n(1 \otimes v_n)$ of m_1^n would be zero, yielding a contradiction $v_n = 0$. Thus, by (4.1) there exists a linear isomorphism $\tau : \operatorname{Ind}(M_1) \to \operatorname{Ind}(M_2)$ such that

$$\psi(1 \otimes v) = 1 \otimes \tau(v) \quad \text{for all } v \in \operatorname{Ind}(M_1). \tag{4.3}$$

From $\lambda_1 = \lambda_2$ and $\psi\big(I_m(1 \otimes v)\big) = I_m\psi(1 \otimes v)$ for all $m \geq K$, it is easy to get $\beta_1\psi(1 \otimes v) = \beta_2(1 \otimes \tau(v))$ and therefore $\beta_1 = \beta_2$. Since $\psi\big(I_m(1 \otimes v)\big) = I_m\psi(1 \otimes v)$ for all $m \in \mathbb{Z}$, we have $\psi\big(1 \otimes (I_m v)\big) = 1 \otimes \big(I_m\tau(v)\big)$. Clearly,

$$\tau(I_m v) = I_m\tau(v) \quad \text{for all } m \in \mathbb{Z}, \quad v \in \operatorname{Ind}(M_1). \tag{4.4}$$

For any $m_1, m_2 \geq K$ and $m_1 \neq m_2$, we can deduce from

$$\psi\big((\lambda_1^{-m_1} L_{m_1} - \lambda_1^{-m_2} L_{m_2})(1 \otimes v)\big) = (\lambda_1^{-m_1} L_{m_1} - \lambda_1^{-m_2} L_{m_2})\psi(1 \otimes v)$$

that $(m_2 - m_1)\alpha_1\psi(1 \otimes v) = (m_2 - m_1)\alpha_2(1 \otimes \tau(v))$, which implies $\alpha_1 = \alpha_2$. Using $\psi\big(L_m(1 \otimes v)\big) = L_m\psi(1 \otimes v)$ for all $m \geq K$, we can conclude that $\psi(t \otimes v) = t \otimes \tau(v)$ for $v \in \operatorname{Ind}(M_1)$. Combining this with (2.3) gives immediately $\psi\big((L_m 1) \otimes v\big) = (L_m 1) \otimes \tau(v)$ for all $m \in \mathbb{Z}$ and $v \in \operatorname{Ind}(M_1)$. Now it follows from $\psi\big(L_m(1 \otimes v)\big) = L_m\psi(1 \otimes v)$ that $\psi\big(1 \otimes (L_m v)\big) = 1 \otimes \big(L_m\tau(v)\big)$, which and (4.3) force

$$\tau(L_m v) = L_m\tau(v) \quad \text{for all } m \in \mathbb{Z}, \ v \in \operatorname{Ind}(M_1). \tag{4.5}$$

It is clear that $\psi\big(C_i(1 \otimes v)\big) = C_i\psi(1 \otimes v)$, which implies $\tau(C_i v) = C_i\tau(v)$ for $i = 1, 2, 3,\ v \in \operatorname{Ind}(M_1)$. These together with (4.4) and (4.5) show that τ is an $\mathcal{H}$-module isomorphism if $\beta_2 \in \mathbb{C}^*$.

Case 2. $\alpha_2 \in \mathbb{C}^*$.

By the similar arguments as in the proof of [23, Theorem 2], we obtain that $\lambda_1 = \lambda_2$, $\alpha_1 = \alpha_2$ and there exists a linear bijection $\tau : \text{Ind}(M_1) \to \text{Ind}(M_2)$ such that $\psi(1 \otimes v) = 1 \otimes \tau(v)$ for all $v \in \text{Ind}(M_1)$. Meanwhile we get that $\tau(L_m v) = L_m \tau(v)$ for all $m \in \mathbb{Z}$, $v \in \text{Ind}(M_1)$. Since $\psi\big(I_m(1 \otimes v)\big) = I_m \psi(1 \otimes v)$ for all $m \geq K$, it is easy to see that $\beta_1 = \beta_2$. Then from $\psi\big(I_m(1 \otimes v)\big) = I_m \psi(1 \otimes v)$ and $\psi\big(C_i(1 \otimes v)\big) = C_i \psi(1 \otimes v)$ we see that $\tau(I_m v) = I_m \tau(v)$ and $\tau(C_i v) = C_i \tau(v)$ for $i = 1, 2, 3$, $m \in \mathbb{Z}$, respectively. Thus, $\text{Ind}(M_1) \cong \text{Ind}(M_2)$ as $\mathcal{H}$-modules for $\alpha_2 \in \mathbb{C}^*$.

To sum up, in either case we obtain $(\lambda_1, \alpha_1, \beta_1) = (\lambda_2, \alpha_2, \beta_2)$ and $\text{Ind}(M_1) \cong \text{Ind}(M_2)$ as $\mathcal{H}$-modules. □

The rest of this section is to establish isomorphisms among the modules $\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))$. We need to make some preparations first, for this case is more complicated than the former one.

Definition 4.2. Let $d \in \{0, 1\}$, $r \in \mathbb{Z}_+$, $\alpha \in \mathbb{C}$ and V, W be two $\bar{\mathcal{H}}_{r,d}$-modules. Denote by V^α the $\bar{\mathcal{H}}_{r,d}$-module obtained from V by modifying the $\bar{L}_0$-action as $\bar{L}_0 - \alpha \text{id}_V$. A linear isomorphism $\psi : V \to W$ is called an α-isomorphism if $\psi(\bar{L}_i v) = \bar{L}_i \psi(v)$ and $\psi(\bar{I}_{i+d} v) = \alpha \bar{I}_{i+d} \psi(v)$ for any $v \in V$ and $i \in \mathbb{Z}$.

It follows from the similar proof of [6, Theorem 4.1] that we have the following result.

Theorem 4.3. *Let $d_i \in \{0, 1\}$, $r_i \in \mathbb{Z}_+$, $\lambda_i, \alpha_i, 0 \neq \beta_i \in \mathbb{C}$ and V_i be an irreducible $\bar{\mathcal{H}}_{r_i,d_i}$-module for $i = 1, 2$. Then $\mathcal{M}\big(V_1, A(\lambda_1, \alpha_1, \beta_1)\big)$ and $\mathcal{M}\big(V_2, A(\lambda_2, \alpha_2, \beta_2)\big)$ are isomorphic as $\mathcal{H}$-modules if and only if $\lambda_1 - \lambda_2 \in \mathbb{Z}$, $d_1 = d_2$ and $V_1^{\alpha_1} \cong V_2^{\alpha_2}$ are $\beta_1^{-1}\beta_2$-isomorphic as $\bar{\mathcal{H}}_{\max\{r_1,r_2\},d_1}$-modules.*

Next we are going to make use of the functor $\mathcal{W}$ introduced in Sect. 2.2 to derive some useful results.

Lemma 4.4. *As $\mathcal{H}$-modules, $\mathcal{W}\big(\Omega(\lambda, \alpha, \beta)\big) \cong A(0, 1 - \alpha, \beta)$.*

Proof. By the definition of $\mathcal{I}_n$, $\dim\big(\Omega(\lambda, \alpha, \beta)/\mathcal{I}_n \Omega(\lambda, \alpha, \beta)\big) = 1$ for any $n \in \mathbb{Z}$. Take $w_n = 1 + \mathcal{I}_n(\Omega(\lambda, \alpha, \beta)) \in \Omega(\lambda, \alpha, \beta)/\mathcal{I}_n(\Omega(\lambda, \alpha, \beta))$. Then

$$L_m w_n = \lambda^m (t - m\alpha) + \mathcal{I}_{m+n}(\Omega(\lambda, \alpha, \beta)) = \lambda^m (n + m(1 - \alpha)) w_{m+n},$$
$$\text{and} \quad I_m w_n = \lambda^m \beta + \mathcal{I}_{m+n}(\Omega(\lambda, \alpha, \beta)) = \lambda^m \beta w_{m+n} \quad \text{for any } m, n \in \mathbb{Z}.$$

That is, $L_m v_n = (n + m(1 - \alpha)) v_{m+n}$ and $I_m w_n = \beta v_{m+n}$, where $v_n = \lambda^n w_n$. This completes the proof of this lemma. □

Let $d \in \{0, 1\}$, $r \in \mathbb{Z}_+$ and V be an $\bar{\mathcal{H}}_{r,d}$-module. Define the action of $\mathcal{H}$ on $V \otimes A(\lambda, \alpha, \beta)$ as follows

$$L_m\big(u \otimes v_n\big) = \left(n + \lambda + \alpha m + \sum_{i=0}^{r} \left(\frac{m^{i+1}}{(i + 1)!} \bar{L}_i\right)\right) u \otimes v_{m+n},$$

$$I_m(u \otimes v_n) = \sum_{i=0}^{r} \left(\frac{\beta m^{i+d}}{(i+d)!} \bar{I}_{i+d} \right) u \otimes v_{m+n},$$
$$C_1(u \otimes v_n) = C_2(u \otimes v_n) = C_3(u \otimes v_n) = 0,$$

where $m, n \in \mathbb{Z}$ and $u \in V$. Then one can check that under the given actions as above, $\mathcal{M}(V, A(\lambda, \alpha, \beta))$ becomes a weight $\mathcal{H}$-module, which is denoted by $\mathcal{M}(V, A(\lambda, \alpha, \beta))$.

Proposition 4.5. *We have the following isomorphism of $\mathcal{H}$-modules*

$$\mathcal{W}\Big(\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))\Big) \cong \mathcal{M}\big(V, A(0, 1-\alpha, \beta)\big).$$

Proof. Using (2.2), we have

$$\begin{aligned}\mathcal{I}_n\Big(\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))\Big) &= \mathcal{I}_n\Big(V \otimes \Omega(\lambda, \alpha, \beta)\Big) \\ &= V \otimes \mathcal{I}_n(\Omega(\lambda, \alpha, \beta)) \quad \text{for } n \in \mathbb{Z}.\end{aligned}$$

Then it follows from this and Lemma 4.4 that

$$\begin{aligned}\mathcal{W}\Big(\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))\Big) &= \bigoplus_{n \in \mathbb{Z}} \mathcal{M}(V, \Omega(\lambda, \alpha, \beta))_n \\ &= \bigoplus_{n \in \mathbb{Z}} \Big(\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))/\mathcal{I}_n\big(\mathcal{M}(V, \Omega(\lambda, \alpha, \beta))\big)\Big) \\ &= \bigoplus_{n \in \mathbb{Z}} \Big(V \otimes \Omega(\lambda, \alpha, \beta)/V \otimes \mathcal{I}_n\big(\Omega(\lambda, \alpha, \beta)\big)\Big) \quad \text{(by (2.2))} \\ &\cong V \otimes \bigoplus_{n \in \mathbb{Z}} \Big(\Omega(\lambda, \alpha, \beta)/\mathcal{I}_n\big(\Omega(\lambda, \alpha, \beta)\big)\Big) = V \otimes \mathcal{W}\big(\Omega(\lambda, \alpha, \beta)\big) \\ &\cong V \otimes A(0, 1-\alpha, \beta) = \mathcal{M}\big(V, A(0, 1-\alpha, \beta)\big).\end{aligned}$$

□

Since $\mathcal{W}$ is a functor, an $\mathcal{H}$-module isomorphism between $\mathcal{M}\big(V_1, \Omega(\lambda_1, \alpha_1, \beta_1)\big)$ and $\mathcal{M}\big(V_2, \Omega(\lambda_2, \alpha_2, \beta_2)\big)$ would induce an isomorphism between $\mathcal{M}\big(V_1, A(0, 1-\alpha_1, \beta_1)\big)$ and $\mathcal{M}\big(V_2, A(0, 1-\alpha_1, \beta_2)\big)$ by Proposition 4.5. Then it follows from Theorem 4.3 that $d_1 = d_2$ and that $V_1^{\alpha_1} \cong V_2^{\alpha_2}$ are $\beta_1^{-1}\beta_2$-isomorphic as $\bar{\mathcal{H}}_{\max\{r_1, r_2\}, d_1}$-modules. So it is reasonable to include these into sufficient conditions for $\mathcal{M}\big(V_1, \Omega(\lambda_1, \alpha_1, \beta_1)\big)$ and $\mathcal{M}\big(V_2, \Omega(\lambda_2, \alpha_2, \beta_2)\big)$ being isomorphic. In fact, one more condition $\lambda_1 = \lambda_2$ will be enough, as stated in the following result.

Theorem 4.6. *Let $d_i \in \{0, 1\}, r_i \in \mathbb{Z}_+, \lambda_i, \beta_i \in \mathbb{C}^*, \alpha_i \in \mathbb{C}$ and V_i be an irreducible $\bar{\mathcal{H}}_{r_i, d_i}$-module such that $\mathcal{M}\big(V_i, \Omega(\lambda_i, \alpha_i, \beta_i)\big)$ is irreducible for $i = 1, 2$. Then*

$$\mathcal{M}\big(V_1, \Omega(\lambda_1, \alpha_1, \beta_1)\big) \cong \mathcal{M}\big(V_2, \Omega(\lambda_2, \alpha_2, \beta_2)\big)$$

as $\mathcal{H}$-modules if and only if $d_1 = d_2$, $\lambda_1 = \lambda_2$ *and* $V_1^{\alpha_1} \cong V_2^{\alpha_2}$ *are* $\beta_1^{-1}\beta_2$-*isomorphic as* $\bar{\mathcal{H}}_{\max\{r_1,r_2\},d_1}$*-modules.*

Proof. Let

$$\phi : \mathcal{M}\big(V_1, \Omega(\lambda_1, \alpha_1, \beta_1)\big) \to \mathcal{M}\big(V_2, \Omega(\lambda_2, \alpha_2, \beta_2)\big)$$

be an isomorphism of $\mathcal{H}$-modules. By the remark before this theorem we know that $d_1 = d_2$ and that the linear map $\varphi : V_1^{\alpha_1} \to V_2^{\alpha_2}$ induced from ϕ is a $\beta_1^{-1}\beta_2$-isomorphism of $\bar{\mathcal{H}}_{\max\{r_1,r_2\},d_1}$-modules. It remains to show $\lambda_1 = \lambda_2$.

Take any $0 \neq w \in V_1$ and assume that $\phi(w \otimes 1) = \sum_i u_i \otimes t^i \in \mathcal{M}\big(V_2, \Omega(\lambda_2, \alpha_2, \beta_2)\big)$. Note on the one hand that ϕ induces an $\mathcal{H}$-module isomorphism

$$\phi_A : \mathcal{M}\big(V_1, A(0, 1-\alpha_1, \beta_1)\big) \to \mathcal{M}\big(V_2, A(0, 1-\alpha_2, \beta_2)\big)$$

sending $w \otimes v_n$ to $\sum_i (\frac{\lambda_1}{\lambda_2})^n n^i u_i \otimes v_n$ for any $n \in \mathbb{Z}$ by Lemma 4.4 and Proposition 4.5, and on the other hand that $\phi_A(w \otimes v_n) = \varphi(w) \otimes v_n$ for any $n \in \mathbb{Z}$ (see [6, Theorem 4.1]). Thus, $\varphi(w) = \sum_i (\frac{\lambda_1}{\lambda_2})^n n^i u_i$, which implies $\lambda_1 = \lambda_2$ and $u_i = 0$ if $i \neq 0$.

Conversely, let $\varphi : V_1^{\alpha_1} \to V_2^{\alpha_2}$ be a $\beta_1^{-1}\beta_2$-isomorphism of $\bar{\mathcal{H}}_{\max\{r_1,r_2\},d_1}$-modules. One can check the linear map $\phi : \mathcal{M}\big(V_1, \Omega(\lambda, \alpha_1, \beta_1)\big) \to \mathcal{M}\big(V_2, \Omega(\lambda, \alpha_2, \beta_2)\big)$ sending $v \otimes f(t)$ to $\varphi(v) \otimes f(t)$ is an isomorphism of $\mathcal{H}$-modules. □

5. New irreducible modules

In this section, we shall show that any one of $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ and $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ is not isomorphic to $\mathrm{Ind}(M)$ or the irreducible non-weight $\mathcal{H}$-modules defined in [6] and that $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ is not isomorphic to $\mathcal{M}\big(V, \Omega(\lambda', \alpha', \beta')\big)$.

For any $l, m \in \mathbb{Z}$, $s \in \mathbb{Z}_+$, define a sequence of operators $T_{l,m}^{(s)}$ as follows

$$T_{l,m}^{(s)} = \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} I_{l-m-i} I_{m+i}.$$

Lemma 5.1. *Let* $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ *and* V *an irreducible* $\bar{\mathcal{H}}_{r,d}$*-module. Suppose that* M *is an irreducible* $\mathcal{H}_e$*-module satisfying the conditions in Theorem 2.3. Assume that* r' *is the maximal nonnegative integer such that* $\bar{I}_{r'+d}V \neq 0$. *Then*

(i) *the action of* L_m *for* m *sufficiently large is not locally nilpotent on* $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$;
(ii) *the action of* L_m *for each* $m \in \mathbb{Z}$ *on* $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ *is not locally nilpotent*;
(iii) $T_{l,m}^{(2r'+2d)}$ *is a linear automorphism of* $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ *and* $\widetilde{\mathcal{M}}(V, \gamma(t))$ *for* $l, m \in \mathbb{Z}$ *and* $\gamma(t) \in \mathbb{C}[t, t^{-1}]$;

(iv) $T^{(s)}_{l,m}$ $(s \geq 1)$ *is locally nilpotent on* $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ *whenever* $m \gg 0$ *and* $l \gg m$;
(v) $T^{(1)}_{l,m}$ *acts nontrivially on* $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ *whenever* $m \ll 0$ *and* $l \ll m$.

Proof. (i) follows from the local nilpotency of L_m on $\mathrm{Ind}(M)$ by Theorem 2.3 for m sufficiently large and its non-local nilpotency on $\Omega(\lambda, \alpha, \beta)$. (ii) and (iii) follows from [6, Lemma 3.3]. (iv) follows from an easy observation that

$$T^{(s)}_{l,m}\Omega(\lambda, \alpha, \beta) = 0 \quad \text{for } l, m \in \mathbb{Z}$$

and $T^{(s)}_{l,m}$ is locally nilpotent on $\mathrm{Ind}(M)$ when $m \gg 0$ and $l \gg m$. Note when $m \ll 0$ and $l \ll m$ that $I_{l-m} \notin \mathcal{H}_e$ and $I_m \notin \mathcal{H}_e$. It follows from this and a direct computation that for $0 \neq 1 \otimes 1 \otimes v \in \Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ we have

$$\begin{aligned} T^{(1)}_{l,m}(1 \otimes 1 \otimes v) = 1 \otimes \big(&- \lambda^m \beta I_{l-m} - \lambda^{l-m} \beta I_m - I_{l-m} I_m \\ &+ \lambda^{m+1} \beta I_{l-m-1} + \lambda^{l-m-1} \beta I_{m+1} + I_{l-m-1} I_{m+1} \big) \otimes v \neq 0. \end{aligned}$$

So $T^{(1)}_{l,m}\big(\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)\big) \neq 0$, proving (v). □

Let $d \in \{0, 1\}, r \in \mathbb{Z}_+$ and V be an $\bar{\mathcal{H}}_{r,d}$-module. For any fixed $\gamma(t) = \sum_i c_i t^i \in \mathbb{C}[t, t^{-1}]$, define the action of $\mathcal{H}$ on $V \otimes \mathbb{C}[t, t^{-1}]$ as follows

$$\begin{aligned} L_m(v \otimes t^n) &= \left(L_m + \sum_i c_i I_{m+i} \right) (v \otimes t^n), \\ I_m(v \otimes t^n) &= I_m(v \otimes t^n), \\ C_i(v \otimes t^n) &= 0 \quad \text{for } m, n \in \mathbb{Z}, v \in V \text{ and } i = 1, 2, 3. \end{aligned}$$

Then $V \otimes \mathbb{C}[t, t^{-1}]$ carries the structure of an $\mathcal{H}$-module under the above given actions, which is denoted by $\widetilde{\mathcal{M}}(V, \gamma(t))$. Note that $\widetilde{\mathcal{M}}(V, \gamma(t))$ is a weight $\mathcal{H}$-module if and only if $\gamma(t) \in \mathbb{C}$ and also that the $\mathcal{H}$-module $\widetilde{\mathcal{M}}(V, \gamma(t))$ for $\gamma(t) \in \mathbb{C}[t, t^{-1}]$ is irreducible if and only if V is irreducible (see [6]).

We are now ready to state the main result of this section.

Proposition 5.2. *Let* $d \in \{0, 1\}$, $r, e \in \mathbb{Z}_+$, $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ *and* V *an irreducible* $\bar{\mathcal{H}}_{r,d}$*-module. Suppose that* M *is an irreducible* $\mathcal{H}_e$*-module satisfying the conditions in Theorem 2.3. Then any of* $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ *and* $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ *is not isomorphic to* $\mathrm{Ind}(M')$ *for any irreducible* $\mathcal{H}_e$*-module* M' *satisfying the conditions in Theorem 2.3 or* $\Omega(\lambda', \alpha', \beta')$ *for any* $\lambda' \in \mathbb{C}^*$, $\alpha', \beta' \in \mathbb{C}$, *or* $\widetilde{\mathcal{M}}(W, \gamma(t))$ *for any* $\bar{\mathcal{H}}_{r',d'}$*-module* W *and* $\gamma(t) = \sum_i c_i t^i \in \mathbb{C}[t, t^{-1}]$; *and* $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M)$ *is not isomorphic to* $\mathcal{M}\big(W, \Omega(\lambda', \alpha', \beta')\big)$ *for any* $\bar{\mathcal{H}}_{r',d'}$*-module* W *and* $\lambda' \in \mathbb{C}^*$, $\alpha', \beta' \in \mathbb{C}$.

Proof. $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M) \ncong \mathrm{Ind}(M')$ follows from Lemma 5.1(i) and Theorem 2.3; $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big) \ncong \mathrm{Ind}(M')$ follows from Lemma 5.1(ii) and Theorem 2.3; $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M) \ncong \mathcal{M}\big(W, \Omega(\lambda', \alpha', \beta')\big)$ and $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M) \ncong \widetilde{\mathcal{M}}(W, \gamma(t))$ follows from Lemma 5.1(iii) and (iv); $\Omega(\lambda, \alpha, \beta) \otimes \mathrm{Ind}(M) \ncong$

$\Omega(\lambda', \alpha', \beta')$ follows from Lemma 5.1(v) and the fact $T^{(1)}_{l,m}\Omega(\lambda, \alpha, \beta) = 0$ for $l, m \in \mathbb{Z}$. Finally, on the one hand, note that the restriction of $L_0 - \sum_i c_i I_i$ on $W \otimes t^n$ is the scalar n, namely, $L_0 - \sum_i c_i I_i$ is semisimple on $\widetilde{\mathcal{M}}(W, \gamma(t))$; on the other hand, $L_0 - \sum_i c_i I_i$ has no eigenvector in $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$. Thus, $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big) \not\cong \widetilde{\mathcal{M}}\big(W, \gamma(t)\big)$. □

6. Applications

Inspired by [17,23], we first construct two classes of non-weight modules and then apply Theorem 3.1 to give certain conditions for these modules being irreducible.

For $\lambda \in \mathbb{C}^*$, we denote $\mathcal{H}^{(0)}_\lambda = \operatorname{span}_{\mathbb{C}}\{L_m - \lambda^m L_0, I_0, I_m \mid m \geq 1\}$ and $\mathcal{H}^{(1)}_\lambda = \operatorname{span}_{\mathbb{C}}\{L_m - \lambda^{m-1} L_1, I_n \mid m \geq 2, n \geq 1\}$. It is easy to check that both $\mathcal{H}^{(0)}_\lambda$ and $\mathcal{H}^{(1)}_\lambda$ are Lie subalgebras of $\mathcal{H}$. For fixed $RS = (r_1, r_2, s_0, s_1) \in \mathbb{C}^4$ and $PQ = (p_2, p_3, p_4, q_1, q_2) \in \mathbb{C}^5$, we define an $\mathcal{H}^{(0)}_\lambda$-action on $\mathbb{C}$ by

$$
\begin{aligned}
(L_m - \lambda^m L_0)1 &= r_m \quad \text{for } m = 1, 2;\\
(L_m - \lambda^m L_0)1 &= \lambda^{m-2}(m-1)r_2 - \lambda^{m-1}(m-2)r_1 \quad \text{for } m > 2;\\
I_m 1 &= s_m \quad \text{for } m = 0, 1;\\
I_m 1 &= \lambda^{m-1} s_1 \quad \text{for } m > 1,
\end{aligned}
\tag{6.1}
$$

and an $\mathcal{H}^{(1)}_\lambda$-action on $\mathbb{C}$ by

$$
\begin{aligned}
(L_m - \lambda^{m-1} L_1)1 &= p_m \quad \text{for } m = 2, 3, 4;\\
(L_m - \lambda^{m-1} L_1)1 &= \lambda^{m-4}(m-3)p_4 - \lambda^{m-3}(m-4)p_3 \quad \text{for } m > 4;\\
I_m 1 &= q_m \quad \text{for } m = 1, 2;\\
I_m 1 &= \lambda^{m-2} q_2 \quad \text{for } m > 2.
\end{aligned}
\tag{6.2}
$$

It is straightforward to verify that under the given actions $\mathbb{C}$ is an $\mathcal{H}^{(0)}_\lambda$-module and also an $\mathcal{H}^{(1)}_\lambda$-module, denoted by $\mathbb{C}_{RS}$ and $\mathbb{C}_{PQ}$, respectively. For a fixed $\underline{y} = (y_1, y_2, y_3) \in \mathbb{C}^3$ and $\underline{z} = (z_0, z_1, z_2, z_3) \in \mathbb{C}^4$, form modules $\operatorname{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS})$ and $\operatorname{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ})$ as follows:

$$
\operatorname{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS}) = \mathcal{U}(\mathcal{H}) \otimes_{\mathcal{U}(\mathcal{H}^{(0)}_\lambda)} \mathbb{C}_{RS} \Big/ \sum_{i=1}^{3} (C_i - y_i)\mathcal{U}(\mathcal{H}) \otimes_{\mathcal{U}(\mathcal{H}^{(0)}_\lambda)} \mathbb{C}_{RS}, \tag{6.3}
$$

$$
\operatorname{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ}) = \mathcal{U}(\mathcal{H}) \otimes_{\mathcal{U}(\mathcal{H}^{(1)}_\lambda)} \mathbb{C}_{PQ} \Big/ \sum_{i=0}^{3} (C_i - z_i)\mathcal{U}(\mathcal{H}) \otimes_{\mathcal{U}(\mathcal{H}^{(1)}_\lambda)} \mathbb{C}_{PQ}. \tag{6.4}
$$

The following lemma is the key to proving the main results of this section, which generalizes [23, Lemma 6].

Lemma 6.1. *Suppose that V is a cyclic $\mathcal{H}$-module with a basis*

$$\left\{I_{i-m}^{k_{i-m}}\cdots I_i^{k_i}L_{j-n}^{l_{j-n}}\cdots L_j^{l_j}v \mid m,n,k_i,\ldots,k_{i-m},l_j,\ldots,l_{j-n}\in\mathbb{Z}_+\right\},$$

where $0\neq v\in V$ is a fixed vector, i, j are fixed integers and $I_pv\in\mathbb{C}v$, $L_qv\in\mathbb{C}v$ for all integers $p>i$, $q>j$. Then for $(\lambda,\alpha)\in(\mathbb{C}^*)^2$ or $(\lambda,\beta)\in(\mathbb{C}^*)^2$, $\Omega(\lambda,\alpha,\beta)\otimes V$ is a cyclic $\mathcal{H}$-module with a generator $1\otimes v$ and a basis

$$\begin{aligned}\mathcal{B}=\Big\{&I_{i-m}^{k_{i-m}}\cdots I_i^{k_i}L_{j-n}^{l_{j-n}}\cdots L_j^{l_j}L_{j+1}^{l_{j+1}}(1\otimes v)\mid\\ &m,n,k_i,\ldots,k_{i-m},l_{j+1},l_j,\ldots,l_{j-n}\in\mathbb{Z}_+\Big\}.\end{aligned}$$

Proof. Observe from (2.1) that $\Omega(\lambda,\alpha,\beta)\otimes V$ has a basis

$$\begin{aligned}\mathcal{B}'=\Big\{&t^{l_{j+1}}\otimes I_{i-m}^{k_{i-m}}\cdots I_i^{k_i}L_{j-n}^{l_{j-n}}\cdots L_j^{l_j}v\mid\\ &m,n,k_i,\ldots,k_{i-m},l_{j+1},l_j,\ldots,l_{j-n}\in\mathbb{Z}_+\Big\}.\end{aligned}$$

Now we define the following partial order "$\prec$" on $\mathcal{B}'$ by decreeing

$$t^{l_{j+1}}\otimes I_{i-m_1}^{k_{i-m_1}}\cdots I_i^{k_i}L_{j-n_1}^{l_{j-n_1}}\cdots L_j^{l_j}v\prec t^{q_{j+1}}\otimes I_{i-m_2}^{p_{i-m_2}}\cdots I_i^{p_i}L_{j-n_2}^{q_{j-n_2}}\cdots L_j^{q_j}v$$

if and only if

$$\begin{aligned}&\Big(l_j,\ldots,l_{j-n_1},k_i,\ldots,k_{i-m_1},\underbrace{0,\ldots,0}_{m_2+n_2},l_{j+1}\Big)\\ &<\Big(q_j,\ldots,q_{j-n_2},p_i,\ldots,p_{i-m_2},\underbrace{0,\ldots,0}_{m_1+n_1},q_{j+1}\Big).\end{aligned}$$

Here the order "$<$" is the lexicographical order, which is defined

$$\begin{aligned}&(a_1,\ldots,a_\ell)<(b_1,\ldots,b_\ell)\Longleftrightarrow\exists k>0\text{ such that }a_i=b_i\text{ for all }i<k\\ &\quad\text{and }a_k<b_k.\end{aligned}$$

Note that each element of $\mathcal{B}$ can be written as a linear combination of elements in $\mathcal{B}'$:

$$\begin{aligned}&I_{i-m}^{k_{i-m}}\cdots I_i^{k_i}L_{j-n}^{l_{j-n}}\cdots L_j^{l_j}L_{j+1}^{l_{j+1}}(1\otimes v)\\ &\quad=\lambda^{(j+1)l_{j+1}}t^{l_{j+1}}\otimes I_{i-m}^{k_{i-m}}\cdots I_i^{k_i}L_{j-n}^{l_{j-n}}\cdots L_j^{l_j}v+\text{lower terms }(w.r.t\ \prec).\end{aligned}$$

This shows that the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$ is upper triangular with diagonal entries nonzero. Thus, $\mathcal{B}$ is a basis of $\Omega(\lambda,\alpha,\beta)\otimes V$ and the lemma follows. □

Now we are ready to give some conditions under which $\mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS})$ is irreducible.

Theorem 6.2. *Let $\lambda\in\mathbb{C}^*$, $\underline{y}=(y_1,y_2,y_3)\in\mathbb{C}^3$, $RS=(r_1,r_2,s_0,s_1)\in\mathbb{C}^4$ with $y_3=0$. Then*

(i) $\mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS}) \cong \Omega(\lambda,\alpha,\beta)\otimes V$, *where* V *is the classical Verma module described in Example* 3.2(i) *and* α, β, h, d_i *for* $i = 0, 1, 2, 3$ *are defined as*

$$\begin{aligned} &\alpha = \lambda^{-2}(\lambda r_1 - r_2), \beta = \lambda^{-1}s_1, h = \lambda^{-2}(r_2 - 2\lambda r_1), \\ &d_0 = s_0 - \lambda^{-1}s_1, d_3 = y_3 = 0, \quad d_i = y_i \quad \text{for } i = 1, 2. \end{aligned} \tag{6.5}$$

(ii) $\mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS})$ *is irreducible if* $s_0 - \lambda^{-1}s_1 + (n-1)y_2 \neq 0$ *for all* $n \in \mathbb{Z}\backslash\{0\}$, *and either* $r_2 \neq \lambda r_1$ *or* $s_1 \neq 0$.

Proof. (i) Let $\alpha, \beta, h, d_i \in \mathbb{C}$ for $i = 0, 1, 2, 3$ as in (6.5). Then

$$\begin{aligned} &r_1 = -\lambda(\alpha + h), r_2 = -\lambda^2(2\alpha + h), s_1 = \lambda\beta, \\ &s_0 = d_0 + \beta, y_3 = d_3 = 0, y_i = d_i \quad \text{for } i = 1, 2. \end{aligned}$$

Denote $v = 1 + J_1 \in V$. By Lemma 6.1 and the structure of V, $\Omega(\lambda,\alpha,\beta)\otimes V$ is a cyclic module with a generator $1\otimes v$ and has a basis

$$\begin{aligned} \mathcal{B}_1 = \{&I_{-n}^{l_{-n}}\cdots I_{-1}^{l_{-1}}L_{-m}^{k_{-m}}\cdots L_{-1}^{k_{-1}}L_0^{k_0}(1\otimes v) \mid \\ &m, n, k_{-m}, \ldots, k_{-1}, k_0, l_{-n}, \ldots, l_{-1} \in \mathbb{Z}_+\}. \end{aligned}$$

By Theorem 3.1 and the fact that $d_3 = 0$, $\Omega(\lambda,\alpha,\beta)\otimes V$ is irreducible if $d_0 + (n-1)d_2 \neq 0$ for all $n \in \mathbb{Z}\backslash\{0\}$ and either $\alpha \in \mathbb{C}^*$ or $\beta \in \mathbb{C}^*$.

In $\Omega(\lambda,\alpha,\beta)\otimes V$, we have that

$$\begin{aligned} (L_m - \lambda^m L_0)(1\otimes v) &= -\lambda^m(m\alpha + h)(1\otimes v) = r_m(1\otimes v) \quad \text{for } m = 1, 2; \\ (L_m - \lambda^m L_0)(1\otimes v) &= -\lambda^m(m\alpha + h)(1\otimes v) \\ &= \big(\lambda^{m-2}(m-1)r_2 - \lambda^{m-1}(m-2)r_1\big)(1\otimes v) \quad \text{for } m > 2; \\ I_m(1\otimes v) &= (\lambda^m\beta + \delta_{m,0}d_0)(1\otimes v) = s_m(1\otimes v) \quad \text{for } m = 0, 1; \\ I_m(1\otimes v) &= \lambda^m\beta(1\otimes v) = \lambda^{m-1}s_1(1\otimes v) \quad \text{for } m > 1 \end{aligned}$$

and that $C_i(1\otimes v) = d_i(1\otimes v) = y_i(1\otimes v)$ for $i = 1, 2, 3$. It follows from these and (6.1) that there exists an $\mathcal{H}$-module epimorphism

$$\tau : \mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS}) \to \Omega(\lambda,\alpha,\beta)\otimes V,$$

which is uniquely determined by $\tau(\bar{1}) = 1\otimes v$, where

$$\bar{1} := 1\otimes 1 + \sum_{i=1}^{3}(C_i - y_i)\mathcal{U}(\mathcal{H})\otimes_{\mathcal{U}(\mathcal{H}_\lambda^{(0)})}\mathbb{C}_{RS} \in \mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS}).$$

Clearly, $\mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS})$ has a basis

$$\begin{aligned} \mathcal{B}_2 = \Big\{&I_{-n}^{l_{-n}}\cdots I_{-1}^{l_{-1}}L_{-m}^{k_{-m}}\cdots L_{-1}^{k_{-1}}L_0^{k_0}\bar{1} \mid \\ &m, n, k_{-m}, \ldots, k_{-1}, k_0, l_{-n}, \ldots, l_{-1} \in \mathbb{Z}_+\Big\}. \end{aligned}$$

Since $\tau|_{\mathcal{B}_2} : \mathcal{B}_2 \to \mathcal{B}_1$ is a bijection, $\tau : \mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS}) \to \Omega(\lambda,\alpha,\beta)\otimes V$ is an isomorphism.

(ii) By (i) and Theorem 3.1, $\mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS})$ is irreducible if and only if $\Omega(\lambda,\alpha,\beta)\otimes V$ is irreducible. But $\Omega(\lambda,\alpha,\beta)\otimes V$ is irreducible if $d_0+(n-1)d_2\neq 0$ for all $n\in\mathbb{Z}\backslash\{0\}$ and either $\alpha\neq 0$ or $\beta\neq 0$ by noting $d_3=0$, as pointed out in Examples 3.2(i). Thus by (6.5), $\mathrm{Ind}_{\underline{y},\lambda}(\mathbb{C}_{RS})$ is irreducible if $s_0-\lambda^{-1}s_1+(n-1)y_2\neq 0$ for all $n\in\mathbb{Z}\backslash\{0\}$ and either $r_2\neq\lambda r_1$ or $s_1\neq 0$. □

While the irreducible conditions for $\mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ})$ can be given as follows.

Theorem 6.3. *Let* $\lambda\in\mathbb{C}^*$, $\underline{z}=(z_0,z_1,z_2,z_3)\in\mathbb{C}^4$, $PQ=(p_2,p_3,p_4,q_1,q_2)\in\mathbb{C}^5$ *with* $z_3=0$. *Then*

(i) $\mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ})\cong\Omega(\lambda,\alpha,\beta)\otimes V$, *where* V *is the classical Whittaker module described in Example* 3.2(ii) *and* $\alpha,\beta,\lambda_1,\lambda_2,\mu_1,e_i$ *for* $i=0,1,2,3$ *are defined as*

$$\begin{aligned}&\alpha=\lambda^{-4}(\lambda p_3-p_4),\ \beta=\lambda^{-2}q_2,\ \lambda_1=\lambda^{-3}(2p_4-3\lambda p_3),\\&\lambda_2=\lambda^{-2}(p_4-2\lambda p_3+\lambda^2p_2),\ \mu_1=q_1-\lambda^{-1}q_2,\\&e_3=z_3=0,\ e_0=z_0-\lambda^{-2}q_2,\ e_i=z_i\quad\text{for } i=1,2;\end{aligned}\tag{6.6}$$

(ii) $\mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ})$ *is irreducible if* $z_0+(n-1)z_2\neq 0$ *for all* $n\in\mathbb{Z}\backslash\{0\}$, $\lambda q_1\neq q_2$ *and either* $p_4\neq\lambda p_3$ *or* $q_2\neq 0$.

Proof. (i) Let $\alpha,\beta,\lambda_1,\lambda_2,\mu_1\in\mathbb{C}$ be as in (6.6). Then we have

$$\begin{aligned}&p_2=\lambda_2-\lambda\lambda_1-\lambda^2\alpha,\ p_3=-\lambda^2(\lambda_1+2\lambda\alpha),\ p_4=-\lambda^3(\lambda_1+3\lambda\alpha),\\&q_1=\mu_1+\lambda\beta,\ q_2=\lambda^2\beta,\ z_3=e_3=0,\ z_0=e_0+\beta,\ z_i=e_i\quad\text{for } i=1,2.\end{aligned}$$

Denote $v=1+J_2\in V$. Clearly, $\mathcal{H}_+v\in\mathbb{C}v$. Since V has a basis

$$\{I_{-n}^{l_{-n}}\cdots I_{-1}^{l_{-1}}L_{-m}^{k_{-m}}\cdots L_{-1}^{k_{-1}}L_0^{k_0}v\mid m,n,k_{-m},\ldots,k_0,l_{-n},\ldots,l_{-1}\in\mathbb{Z}_+\},$$

using Lemma 6.1, we see that $\Omega(\lambda,\alpha,\beta)\otimes V$ is cyclic with a generator $1\otimes v$ and has a basis

$$\begin{aligned}\mathcal{B}_1=\{&I_{-n}^{l_{-n}}\cdots I_{-1}^{l_{-1}}L_{-m}^{k_{-m}}\cdots L_0^{k_0}L_1^{k_1}(1\otimes v)\mid\\&m,n,k_{-m},\ldots,k_0,k_1,l_{-n},\ldots,l_{-1}\in\mathbb{Z}_+\}.\end{aligned}$$

By Theorem 3.1 and the fact that $e_3=0$, $\Omega(\lambda,\alpha,\beta)\otimes V$ is irreducible if $e_0+(n-1)e_2\neq 0$ for all $n\in\mathbb{Z}\backslash\{0\}$, $\mu_1\neq 0$ and either $\alpha\in\mathbb{C}^*$ or $\beta\in\mathbb{C}^*$.

In W, we have that

$$\begin{aligned}(L_m-\lambda^{m-1}L_1)(1\otimes v)&=\big(\lambda^m(1-m)\alpha-\lambda^{m-1}\lambda_1+\delta_{2-m,0}\lambda_2\big)(1\otimes v)\\&=p_m(1\otimes v)\quad\text{for } m=2,3,4;\\(L_m-\lambda^{m-1}L_1)(1\otimes v)&=\big(\lambda^m(1-m)\alpha-\lambda^{m-1}\lambda_1\big)(1\otimes v)\\&=\big(\lambda^{m-4}(m-3)p_4-\lambda^{m-3}(m-4)p_3\big)(1\otimes v)\\&\quad\text{for } m>4;\\I_m(1\otimes v)&=(\lambda^m\beta+\delta_{1-m,0}\mu_1)(1\otimes v)=q_m(1\otimes v)\\&\quad\text{for } m=1,2;\\I_m(1\otimes v)&=\lambda^m\beta(1\otimes v)=\lambda^{m-2}q_2(1\otimes v)\quad\text{for } m>2;\end{aligned}$$

and that $C_i(1 \otimes v) = (e_i + \delta_{i,0}\beta)(1 \otimes v) = z_i(1 \otimes v)$ for $i = 0, 1, 2, 3$. It follows from these and (6.2) that there exists an $\mathcal{H}$-module epimorphism

$$\tau : \mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ}) \to \Omega(\lambda, \alpha, \beta) \otimes V,$$

which uniquely determined by $\tau(\bar{1}) = 1 \otimes v$, where

$$\bar{1} := 1 \otimes 1 + \sum_{i=0}^{3}(C_i - z_i)\mathcal{U}(\mathcal{H}) \otimes_{\mathcal{U}(\mathcal{H}_\lambda^{(1)})} \mathbb{C}_{PQ} \in \mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ}).$$

It is clear that $\mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ})$ has a basis

$$\mathcal{B}_2 = \{I_{-n}^{l_{-n}} \cdots I_{-1}^{l_{-1}} L_{-m}^{k_{-m}} \cdots L_0^{k_0} L_1^{k_1} \bar{1} \mid m, n, k_{-m}, \ldots, k_0, k_1, l_{-n}, \ldots, l_{-1} \in \mathbb{Z}_+\}.$$

Since $\tau\,|_{\mathcal{B}_2}: \mathcal{B}_2 \to \mathcal{B}_1$ is a bijection, $\tau : \mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ}) \to \Omega(\lambda, \alpha, \beta) \otimes V$ is an isomorphism.

(ii) By (i) and Theorem 3.1, $\mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ})$ is irreducible if and only if $\Omega(\lambda, \alpha, \beta) \otimes V$ is irreducible. But $\Omega(\lambda, \alpha, \beta) \otimes V$ is irreducible if $e_0 + (n-1)e_2 \neq 0$ for all $n \in \mathbb{Z}\backslash\{0\}$, $\mu_1 \neq 0$ and either $\alpha \neq 0$ or $\beta \neq 0$ by noting $e_3 = 0$, as pointed out in Example 3.2(ii), Thus by (6.6), $\mathrm{Ind}_{\underline{z},\lambda}(\mathbb{C}_{PQ})$ is irreducible if $z_0 + (n-1)z_2 \neq 0$ for all $n \in \mathbb{Z}\backslash\{0\}$, $\lambda q_1 \neq q_2$ and either $p_4 \neq \lambda p_3$ or $q_2 \neq 0$. □

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