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# 2-Nilpotent co-Higgs structures

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**Abstract.** A co-Higgs sheaf on a smooth complex projective variety $X$ is a pair of a torsion-free coherent sheaf $\mathcal{E}$ and a global section of $\mathcal{E}nd(\mathcal{E}) \otimes T_X$ with $T_X$ the tangent bundle. We construct 2-nilpotent co-Higgs sheaves of rank two for some rational surfaces and of rank three for $\mathbb{P}^3$, using the Hartshorne-Serre correspondence. Then we investigate the non-existence, especially over projective spaces.

## 1. Introduction

Let $X$ be a smooth projective variety with tangent bundle $T_X$. A co-Higgs bundle, i.e. a pair of an holomorphic bundle $\mathcal{E}$ and a morphism $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ with $\Phi \wedge \Phi = 0$ called the *co-Higgs field*, is a generalized holomorphic bundle over a smooth complex projective variety $X$, considered as a generalized complex manifold [11, 14]. It is observed that the existence of a stable co-Higgs bundle gives a constraint on the position of $X$ in the Kodaira spectrum. Indeed, there are no stable co-Higgs bundles with non-zero co-Higgs field on curves of genus at least two, K3 surfaces and surfaces of general type [20,21]. With the same philosophy, M. Corrêa has shown in [8] that the existence of stable co-Higgs bundle of rank two with a non-trivial nilpotent co-Higgs field, forces the base surface to be uniruled up to finite étale cover. In [1] we investigate the surfaces with $H^0(T_X) = H^0(S^2T_X) = 0$, which implies that co-Higgs fields are automatically nilpotent. The natural definition of stable co-Higgs bundles allows one to study their moduli and there have been several recent works on the description of the moduli spaces over projective spaces and a smooth quadric surface; see [6,19,21].

In this article our main concern is the existence and non-existence of a co-Higgs sheaf with a nilpotent co-Higgs field. The Hartshorne-Serre correspondence

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states that the construction of vector bundles of rank at least two is closely related with the structure of two-codimensional locally complete intersection subschemes. Using the correspondence we produce a nilpotent co-Higgs structure on bundles satisfying a certain condition over various varieties; see Condition 2.2. Assuming $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ for a very ample line bundle $\mathcal{O}_X(1)$, we define $x_{\mathcal{E}}$ for a reflexive sheaf $\mathcal{E}$ of rank two to be the maximal integer $x$ such that $H^0(\mathcal{E}(-x)) \neq 0$, to measure its instability. Then we observe that any nilpotent map associated to $\mathcal{E}$ is trivial if $x_{\mathcal{E}}$ is low. In case $X = \mathbb{P}^n$ and rank two, we get the following:

**Theorem 1.1.** *The set of nilpotent co-Higgs fields on a fixed stable reflexive sheaf $\mathcal{E}$ of rank two on $\mathbb{P}^n$ is identified with the total space of $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus h^0(\mathcal{E}(1))}$, with the zero section blown down to a point corresponding to the trivial field, only if $c_1(\mathcal{E}) + 2x_{\mathcal{E}} = -3$. In the other cases the set is trivial.*

All co-Higgs structures on $T_{\mathbb{P}^2}(t)$ are described in [19, Case $r = 2$ of 5.5] and [21, Theorem 5.9]. In case $X = \mathbb{P}^3$ we show the existence of some nilpotent co-Higgs structures on some rank three semistable bundles with trivial first Chern class.

**Theorem 1.2.** *For each positive integer $c_2$, there exist both strictly semistable indecomposable bundle and stable bundle of rank three on $\mathbb{P}^3$ with trivial first Chern class, on which there are nilpotent co-Higgs structures $\Phi$ with $\ker(\Phi) = \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$.*

We have examples of rank two semistable co-Higgs bundles of several Chern classes on some rational surfaces and the three-dimensional projective space with respect to various polarizations in Sect. 2. In Example 2.11 we show the existence of semistable co-Higgs bundles of rank two with nilpotent co-Higgs fields over the variety with no global tangent vector fields. In Example 2.13 we produce nilpotent co-Higgs structures over a smooth quadric surface and in particular we derive the existence part of [6, Theorem in page 2].

Then we turn our attention to the non-existence of nilpotent co-Higgs structures. As observed in Lemma 3.5, the existence of non-semistable reflexive sheaf of rank two with semistable co-Higgs structures forces $X$ to be a projective space. From Proposition 3.7 any reflexive sheaf of rank two with high stability and extra condition involving new invariant $y_{\mathcal{E}}$ turns out to have no non-trivial trace-free co-Higgs structures. So we are driven to focus on projective spaces, especially $\mathbb{P}^2$ and $\mathbb{P}^3$. Using Theorem 1.2 we show the existence of both of strictly semistable indecomposable reflexive sheaf and stable reflexive sheaf of rank two with nilpotent co-Higgs structures for each Chern numbers from the Bogomolov inequality; see Corollaries 3.12 and 3.13. On the other hand, this existence are not expected to hold for vector bundles due to the following:

**Proposition 1.3.** *If $\mathcal{E}$ is a non-splitting and strictly semistable bundle of rank two on $\mathbb{P}^3$ with the Chern numbers $(c_1, c_2)$ with a non-trivial nilpotent co-Higgs structure, then we have $4c_2 - c_1^2 > 32$.*

We also get similar result for stable vector bundles of rank two with the condition $4c_2 - c_1^2 > 28$; see Proposition 3.15. In case of $\mathbb{P}^2$ a general stable rank two bundle has no non-zero trace zero co-Higgs structures, except for very few integers $c_1^2 - 4c_2$. Indeed, we prove the following result.

**Theorem 1.4.** *If $\mathcal{E}$ is a general element in the moduli of stable sheaves of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) \in \{-1, 0\}$, equipped with a non-trivial trace-free co-Higgs structure, then we have $c_2(\mathcal{E}) < 5(c_1(\mathcal{E}) + 5)$.*

Then we suggest a condition to insure the non-existence of non-trivial trace-free co-Higgs structure on a reflexive sheaf of rank two on non-projective spaces in Proposition 4.2, using another newly introduced invariant $z_{\mathcal{E}}$.

Let us summarize here the structure of this article. In Sect. 2 we introduce the definition of semistable co-Higgs sheaves and suggest a condition to construct a nilpotent co-Higgs structure, using the Hartshorne-Serre correspondence. Then we play this construction over several rational surfaces and three-dimensional projective space. In Sect. 3, we introduce two invariants $x_{\mathcal{E}}$ and $y_{\mathcal{E}}$ associated to a rank two reflexive sheaf, with which we collect the criterion for the existence and non-existence of non-trivial nilpotent co-Higgs structures. We finish the article in Sect. 4 by dealing with a criterion of non-existence over non-projective spaces.

## 2. Definitions and examples

Throughout the article our base field is the field $\mathbb{C}$ of complex numbers. We will always assume that $X$ is a smooth projective variety of dimension $n$ with tangent bundle $T_X$. For a fixed ample line bundle $\mathcal{O}_X(1)$ and a coherent sheaf $\mathcal{E}$ on $X$, we denote $\mathcal{E} \otimes \mathcal{O}_X(t)$ by $\mathcal{E}(t)$ for $t \in \mathbb{Z}$. The dimension of cohomology group $H^i(X, \mathcal{E})$ is denoted by $h^i(X, \mathcal{E})$ and we will skip $X$ in the notation, if there is no confusion.

**Definition 2.1.** A *co-Higgs* sheaf on $X$ is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a torsion-free coherent sheaf on $X$ and $\Phi \in H^0(\mathcal{E}nd(\mathcal{E}) \otimes T_X)$ for which $\Phi \wedge \Phi = 0$ as an element of $H^0(\mathcal{E}nd(\mathcal{E}) \otimes \wedge^2 T_X)$. Here $\Phi$ is called the *co-Higgs field* of $(\mathcal{E}, \Phi)$ and the condition $\Phi \wedge \Phi = 0$ is an integrability condition originating in the work of Simpson [22].

Let $\mathcal{E}$ be a torsion-free sheaf on $X$ and $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ be a map of $\mathcal{O}_X$-sheaves. We say that $\Phi$ is 2-*nilpotent* if $\Phi$ is non-trivial and $\Phi \circ \Phi = 0$. Note that any 2-nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ satisfies $\Phi \wedge \Phi = 0$ and so it is a non-zero co-Higgs structure on $\mathcal{E}$, i.e. a nilpotent co-Higgs structure.

**Condition 2.2.** *For a fixed integer $r \geq 2$, a two-codimensional locally complete intersection $Z \subset X$ and $\mathcal{A} \in \mathrm{Pic}(X)$, we consider the following two conditions:*

(i) $H^0(T_X \otimes \mathcal{A}^\vee) \neq 0$;
(ii) *the general sheaf fitting into the following exact sequence is locally free,*

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \xrightarrow{u} \mathcal{E} \xrightarrow{v} \mathcal{I}_Z \otimes \mathcal{A} \to 0. \tag{1}$$

Our main object of interest is the middle term $\mathcal{E}$ in (1) with the additional property that it is reflexive. If $X$ is a smooth surface, then reflexivity is equivalent to local-freeness and in the Examples 2.8, 2.9, 2.10, 2.11, 2.12 and 2.13 we produce vector bundles. If $n$ is at least 3, there are many reflexive, but non-locally free sheaves of rank two. In Example 2.14 we produce such sheaves.

*Remark 2.3.* By [23] any smooth projective variety $X$ of dimension $n$ satisfying $H^0(T_X(-1)) \neq 0$ is isomorphic to $\mathbb{P}^n$. So Condition 2.2(i) with $\mathcal{A} \cong \mathcal{O}_X(1)$ implies that $X = \mathbb{P}^n$. Note that we always have $H^0(T_X(-2)) = 0$, except when $X = \mathbb{P}^1$.

**Definition 2.4.** For a fixed ample line bundle $\mathcal{H}$ on $X$, a co-Higgs sheaf $(\mathcal{E}, \Phi)$ is *$\mathcal{H}$-semistable* (resp. *$\mathcal{H}$-stable*) if

$$\frac{\det(\mathcal{F}) \cdot \mathcal{H}^{n-1}}{\text{rank } \mathcal{F}} \leq \text{ (resp. } <) \frac{\det(\mathcal{E}) \cdot \mathcal{H}^{n-1}}{\text{rank } \mathcal{E}}$$

for every coherent subsheaf $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ with $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes T_X$. In case $\mathcal{H} \cong \mathcal{O}_X(1)$ we will simply call it semistable (resp. stable) without specifying $\mathcal{H}$.

*Remark 2.5.* Take any torsion-free sheaf $\mathcal{E}$ fitting into (1) with $Z = \emptyset$ and $\mathcal{A}$ any numerically trivial line bundle. Then $\mathcal{E}$ is $\mathcal{H}$-semistable with respect to any polarization $\mathcal{H}$. By Lemma 2.6, $\mathcal{E}$ has a nonzero 2-nilpotent co-Higgs field.

**Lemma 2.6.** *Fix a torsion-free sheaf $\mathcal{E}$ fitting into (1) and assume Condition 2.2(i). Then there exists a 2-nilpotent co-Higgs structure on $\mathcal{E}$ with* $\ker(\Phi) \cong \mathcal{O}_X^{\oplus(r-1)}$.

*Proof.* Any non-zero section $\sigma \in H^0(T_X \otimes \mathcal{A}^\vee)$ induces a non-zero map $h : \mathcal{I}_Z \otimes \mathcal{A} \to T_X$. Then we may define $\Phi$ to be the following composite:

$$\mathcal{E} \xrightarrow{v} \mathcal{I}_Z \otimes \mathcal{A} \xrightarrow{h} T_X \xrightarrow{g} \mathcal{O}_X^{\oplus(r-1)} \otimes T_X \xrightarrow{u \otimes \text{id}} \mathcal{E} \otimes T_X,$$

where the map $g$ is induced by an inclusion $\mathcal{O}_X \to \mathcal{O}_X^{\oplus(r-1)}$. □

Note that the way of constructing a 2-nilpotent co-Higgs structure, used in Lemma 2.6, will be used throughout the whole article, specially when we prove the existence of a non-trivial co-Higgs structure.

*Example 2.7.* Take $n = \dim(X) \geq 3$ and assume $H^0(T_X(-D)) \neq 0$ for some effective divisor $D$. Lemma 2.6 with $\mathcal{A} \cong \mathcal{O}_X(D)$ gives pairs $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is a torsion-free sheaf and $\Phi$ is nonzero with $\Phi \circ \Phi = 0$. Note that $(\mathcal{E}, \Phi)$ is stable for any polarization on $X$. We take as $Z$ a smooth two-codimensional subvariety, not necessarily connected. By [13, Theorem 4.1] it is sufficient that $\omega_Z \otimes \omega_X(D)$ is globally generated. We may take as $Z$ a disjoint union of smooth complete intersections of an element of $|\mathcal{O}_X(a)|$ and an element of $|\mathcal{O}_X(b)|$ with $\omega_X(a+b)$ globally generated. In particular, there are plenty of non-locally free examples. Among the examples we may take as $X$ the Segre variety $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with as $D$ the pull-back of $\mathcal{O}_{\mathbb{P}^{n_i}}(1)$ by the projection $\pi_i : X \to \mathbb{P}^{n_i}$ on the $i$-th factor.

*Example 2.8.* Let $X$ be a smooth and connected projective surface with $H^0(T_X) \neq 0$. Fix an integer $r \geq 2$. In Lemma 2.6 we take $\mathcal{A} \cong \mathcal{O}_X$ and a general subset $Z$ of $X$ with cardinality $s \geq r - 1 + h^0(\omega_X)$. Since $Z$ is general and $s > h^0(\omega_X)$, we have $h^0(\omega_X \otimes \mathcal{I}_{S\setminus\{p\}}) = 0$ for each $p \in Z$ and so the Cayley–Bacharach condition is satisfied. Thus the middle term $\mathcal{E}$ in the general extension (1) is locally free. We have $\det(\mathcal{E}) \cong \mathcal{O}_X$ and $\mathcal{E}$ is strictly semistable for any polarization of $X$. Since

$H^0(T_X \otimes \mathcal{A}^\vee) > 0$, Lemma 2.6 gives the existence of a non-trivial 2-nilpotent $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$. From the long exact sequence of cohomology of

$$0 \to \mathcal{I}_Z \otimes \omega_X \to \omega_X \to \omega_X \otimes \mathcal{O}_Z \to 0,$$

we get $h^1(\mathcal{I}_Z \otimes \omega_X) \geq s - h^0(\omega_X) \geq r-1$ and so $\dim \operatorname{Ext}^1(\mathcal{I}_Z, \mathcal{O}_X) \geq r-1$. Hence there is $\mathcal{E}$ with no trivial factor. Now we check that any locally free $\mathcal{E}$ with no trivial factor is indecomposable. Assume $\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_2$ with $k = \operatorname{rank}(\mathcal{E}_1)$ and $1 \leq k \leq r-1$. Let $\mathcal{G}_i \subseteq \mathcal{E}_i$ for $i = 1, 2$, be the image of the evaluation map $H^0(\mathcal{E}_i) \otimes \mathcal{O}_X \to \mathcal{E}_i$. Since $Z$ is not empty, we have $h^0(\mathcal{E}) = r-1$ and $\mathcal{G} := u(\mathcal{O}_X^{\oplus(r-1)})$ is the image of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}$. Since $\mathcal{G}_1 \oplus \mathcal{G}_2 \cong \mathcal{G} \cong \mathcal{O}_X^{\oplus(r-1)}$ and $\mathcal{G}$ is saturated in $\mathcal{E}$, $\mathcal{G}_i$ is locally free and saturated in $\mathcal{E}_i$ for each $i$. Since $\operatorname{rank}(\mathcal{G}_1) + \operatorname{rank}(\mathcal{G}_2) + 1 = \operatorname{rank}(\mathcal{E}_1) + \operatorname{rank}(\mathcal{E}_2)$, there exists $i$ with $\mathcal{E}_i = \mathcal{G}_i$ and so $\mathcal{E}$ has a trivial factor, contradicting our assumptions.

In Condition 2.2, if $\mathcal{A}$ is negative with respect to a polarization $\mathcal{H}$, then the co-Higgs bundle $(\mathcal{E}, \Phi)$ in Lemma 2.6 is not $\mathcal{H}$-semistable, because $\ker(\Phi) = \mathcal{O}_X^{\oplus(r-1)}$. Note that if $\mathcal{E}$ is (semi)stable with respect to $\mathcal{H}$, then each co-Higgs structure on $\mathcal{F}$ has the same property. Thus it is necessary to check when $\mathcal{E}$ is (semi)stable and we will focus on the sheaves in Condition 2.2(ii) for a few cases such as

- Blow-ups of $\mathbb{P}^2$ at a finite set of points;
- A smooth quadric surface;
- The three-dimensional projective space $\mathbb{P}^3$.

*Example 2.9.* Let $X = \mathbb{P}^2$ and take $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^2}(1)$. Note that the Cayley–Bacharach condition is satisfied for any locally complete intersection zero-dimensional subscheme, or a finite set, $Z$ to get a locally free sheaf $\mathcal{E}$ with $c_1(\mathcal{E}) = 1$ and $c_2(\mathcal{E}) = \deg(Z)$. For $\mathcal{E}$ to have no trivial factor, it is sufficient to have $\deg(Z) \geq r-1$ and that the extension is general. If $r = 2$ and (1) does not split, then $\mathcal{E}$ is stable. Note that (1) does not split if $Z \neq \emptyset$ and $\mathcal{E}$ is locally free.

Now assume $r \geq 3$. Note that $\mathcal{E}$ is semistable if and only if it is stable i.e. there is no subsheaf $\mathcal{G} \subset \mathcal{E}$ with positive degree and rank less than $r$. We assume that $\mathcal{E}$ is locally free. Since $h^1(\mathcal{O}_{\mathbb{P}^2}) = 0$, we have $h^0(\mathcal{E}) = r - 1 + h^0(\mathcal{I}_Z(1))$. Assume that $\mathcal{E}$ is not semistable and so the existence of a subsheaf $\mathcal{G} \subset \mathcal{E}$ of rank $s < r$ with maximal positive degree among all subsheaves. Then $\mathcal{G}$ is saturated in $\mathcal{E}$, i.e. $\mathcal{E}/\mathcal{G}$ has no torsion. We take $s$ maximal with the previous properties, i.e. if $s \leq r-2$ we assume that no subsheaf of $\mathcal{E}$ with rank $\{s+1, \dots, r-1\}$ has positive degree. Since $\deg(\mathcal{G} \cap u(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)})) \leq 0$, we have $v(\mathcal{G}) \neq 0$ and $\deg(v(\mathcal{G})) > 0$. Thus we have $v(\mathcal{G}) \cong \mathcal{I}_W(1)$ for some zero-dimensional subscheme $W \supseteq Z$. From $h^0(\mathcal{I}_W(1)) \leq h^0(\mathcal{I}_Z(1))$, we get

$$h^0(\mathcal{G}) \leq h^0(\mathcal{I}_Z(1)) + h^0(\mathcal{G} \cap u(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)})).$$

Since $v(\mathcal{G})$ is not trivial, we have

$$\begin{cases} \operatorname{rank}(\mathcal{G} \cap u(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)})) = s-1, & \text{if } s > 1; \\ \mathcal{G} \cap u(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)}) = 0, & \text{if } s = 1. \end{cases}$$

Since $\mathcal{G} \cap u(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)})$ is a subsheaf of a trivial vector bundle, we have

$$\deg(\mathcal{G} \cap u(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)})) \leq 0\ ,\ h^0(\mathcal{G} \cap u(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)})) \leq s-1.$$

It implies that $h^0(\mathcal{E}/\mathcal{G}) > 0$ and so take a trivial subsheaf $\mathcal{O}_{\mathbb{P}^2} \subseteq \mathcal{E}/\mathcal{G}$. Let $\pi : \mathcal{E} \to \mathcal{E}/\mathcal{G}$ be the quotient map. If $s \leq r-2$, then $\pi^{-1}(\mathcal{O}_{\mathbb{P}^2})$ contradicts the maximality of $s$. Now assume $s = r-1$. Since $\mathcal{E}/\mathcal{G}$ is a torsion-free sheaf of rank one with a non-zero section, we have $\mathcal{E}/\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^2}$. Since $h^0(\mathcal{E}/\mathcal{G}) = 1$, we get $h^0(\mathcal{G} \cap u(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)})) = s-1 = r-2$. Then any element in

$$H^0\left(u\left(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)}\right)\right) \setminus H^0\left(\mathcal{G} \cap u\left(\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)}\right)\right)$$

induces a splitting of the surjection $\mathcal{E} \to \mathcal{E}/\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^2}$, contrary to the assumption that $\mathcal{E}$ has no trivial factor.

*Example 2.10.* Let $\pi : X \to \mathbb{P}^2$ be the blow-up at two points, say $p_1$ and $p_2$. Setting $D_i := \pi^{-1}(p_i)$ for $i = 1, 2$ and writing

$$\mathcal{O}_X(a; 0, 0) := \pi^*\mathcal{O}_{\mathbb{P}^2}(a), \mathcal{O}_X(0; b, 0) := \mathcal{O}_X(bD_1), \mathcal{O}_X(0; 0, c) := \mathcal{O}_X(cD_2),$$

we have $\omega_X \cong \mathcal{O}_X(-3; 1, 1)$. Let $D \subset X$ be the strict transform of the line through $p_1$ and $p_2$ and then we have $\{D\} = |\mathcal{O}_X(1; -1, -1)|$. Recall that for any smooth projective variety $Y$ the vector space $H^0(T_Y)$ is the tangent space at the identity of the scheme $\mathrm{Aut}(Y)$. So we have $h^0(T_X) = 4$, $h^0(T_X(-D_1)) = h^0(T_X(-D_2)) = 6$ and $h^0(T_X(-D_1 - D_2)) = h^0(T_X(-D_1 - D_2 - D)) = 4$. Set

$$\mathcal{S} := \{\mathcal{O}_X, \mathcal{O}_X(D_1), \mathcal{O}_X(D_2), \mathcal{O}_X(D), \mathcal{O}_X(D_1 + D_2), \mathcal{O}_X(D_1 + D_2 + D)\};$$
$$\mathcal{S}_1 := \{\mathcal{O}_X(B_1 + B_2 - B_3), \mathcal{O}_X(B_1 - B_2) \mid \{B_1, B_2, B_3\} = \{D_1, D_2, D\}\}.$$

If we take as $\mathcal{A}$ any element of $\mathcal{S} \cup S_1$, then we have $h^0(T_X \otimes \mathcal{A}^\vee) > 0$. Note that $h^0(\mathcal{A}) = 1$ if $\mathcal{A} \in \mathcal{S}$ and $h^0(\mathcal{A}) = 0$ if $\mathcal{A} \in \mathcal{S}_1$. Now fix an integer $r \geq 2$ and take as $Z$ a general subset of $X$ with cardinality $s$ in Condition 2.2(ii). Assume for the moment that the middle term $\mathcal{E}$ of (1) is locally free. If $\mathcal{A} \cong \mathcal{O}_X$, then $\mathcal{E}$ is strictly semistable for any polarization of $X$.

Assume $\mathcal{A} \in \mathcal{S} \setminus \{\mathcal{O}_X\}$ and fix a polarization $\mathcal{H}$ of $X$. If $\mathcal{L} \subset \mathcal{E}$ is a saturated subsheaf of rank one with positive $\mathcal{H}$-slope, then it is a line bundle. Since $\mathcal{L} \cdot \mathcal{H} > 0$, we have $\mathcal{L} \nsubseteq u(\mathcal{O}_X)$. Since $\mathrm{Im}(\Phi) \subseteq u(\mathcal{O}_X) \otimes T_X$, we have $\Phi(\mathcal{L}) \nsubseteq \mathcal{L} \otimes T_X$. Thus $(\mathcal{E}, \Phi)$ is $\mathcal{H}$-stable.

Now we check a criterion for $s$ with which $\mathcal{E}$ is locally free; moreover if $r > 2$, we also want $s$ so that $\mathcal{E}$ has no trivial factor. In the case $s = 0$, $\mathcal{E}$ is decomposable and so we may assume $s > 0$. First assume $r = 2$. In this case we only need to check the Cayley–Bacharach condition. Indeed this condition is satisfied, because $H^0(\omega_X) = 0$. Now assume $r > 2$ and then by the case $r = 2$ a general $\mathcal{E}$ fitting into (1) is locally free. To check that it has no trivial factor it is sufficient to have $\dim \mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{A}, \mathcal{O}_X) \geq r-1$, because (1) is induced by $r-1$ elements of $\mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{A}, \mathcal{O}_X)$ and a trivial factor of $\mathcal{E}$ would be a factor of the subsheaf $u(\mathcal{O}_X^{\oplus(r-1)})$ of $\mathcal{E}$, since we have $h^0(\mathcal{I}_Z \otimes \mathcal{A}) = 0$ due to generality of $Z$. Now for any $\mathcal{A} \in \mathcal{S}$, we have $\mathrm{Ext}^1(\mathcal{I}_Z \otimes \mathcal{A}, \mathcal{O}_X) \cong H^1(\mathcal{I}_Z \otimes \mathcal{A} \otimes \omega_X)$ whose dimension is always $s$ and so we may choose $s$ at least $r-1$.

*Example 2.11.* Let $\pi : X \to \mathbb{P}^2$ be the blow-up at three non-collinear points $p_1$, $p_2$ and $p_3$. Set $D_i := \pi^{-1}(p_i)$ for $i = 1, 2, 3$ and writing $\mathcal{O}_X(a; 0, 0, 0) := \pi^*\mathcal{O}_{\mathbb{P}^2}(a)$,

$$\mathcal{O}_X(0; b, 0, 0) := \mathcal{O}_X(bD_1), \mathcal{O}_X(0; 0, c, 0) := \mathcal{O}_X(cD_2),$$
$$\mathcal{O}_X(0; 0, 0, d) := \mathcal{O}_X(dD_3),$$

we have $\omega_X \cong \mathcal{O}_X(-3; 1, 1, 1)$. For any $h \in \{1, 2, 3\}$, let $T_h \subset X$ be the strict transform of the line through $p_i$ and $p_j$ with $\{h, i, j\} = \{1, 2, 3\}$. We have $\{T_1\} = |\mathcal{O}_X(1; 0, -1, -1)|$ and similar formulas hold for $T_2$ and $T_3$. As in Example 2.10 we have $h^0(T_X) = h^0(T_X(-D_1 - D_2 - D_3 - T_1 - T_2 - T_3)) = 2$.

Let $\mathcal{Z}$ be the collection of the line bundles $\mathcal{O}_X(D)$ with $D > 0$ and $D \subseteq D_1 \cup D_2 \cup D_3 \cup T_1 \cup T_2 \cup T_3$. As in Example 2.10, if $\mathcal{A} \cong \mathcal{O}_X$, then $\mathcal{E}$ is stable for any polarization, and if $\mathcal{A} \in \mathcal{Z}$, then $(\mathcal{E}, \Phi)$ is stable for any polarization. We may also take as $\mathcal{A}$ a line bundle $\mathcal{O}_X(B)$ with $B \neq 0$, $B$ a sum of some of the divisors $D_i$ and $T_j$ with sign. In this case $(\mathcal{E}, \Phi)$ is (semi)stable for some polarization, but not for all polarizations. Note that in any case we have $h^0(T_X \otimes \mathcal{A}^\vee) > 0$.

*Example 2.12.* Fix an integer $k \geq 3$ and a line $\ell \subset \mathbb{P}^2$. Let $\pi : X \to \mathbb{P}^2$ be the blow-up at $k$ points $p_1, \ldots, p_k \in \ell$. Set $D_i := \pi^{-1}(p_i)$ for $i = 1, \ldots, k$ and let $D \subset X$ be the strict transform of $\ell$. Then we have

$$(\pi^*\mathcal{O}_{\mathbb{P}^2}(1))(-D_1 - \cdots - D_k) \cong \mathcal{O}_X(D)\ ,\ \omega_X \cong (\pi^*\mathcal{O}_{\mathbb{P}^2}(-3))(D_1 + \cdots + D_k).$$

We also have $h^0(T_X) = h^0(T_X(-D_1 - \cdots - D_k)) > 0$.

Let $\mathcal{Z}$ be the collection of the line bundles $\mathcal{O}_X(T)$ with $T > 0$ and $T \subseteq D \cup D_1 \cup \cdots \cup D_k$. As in Examples 2.10 and 2.11, if $\mathcal{A} \cong \mathcal{O}_X$, then $\mathcal{E}$ is stable for any polarization, and if $\mathcal{A} \in \mathcal{Z}$, then $(\mathcal{E}, \Phi)$ is stable for any polarization. We may also take as $\mathcal{A}$ a line bundle $\mathcal{O}_X(B)$ with $B \neq 0$, $B$ a sum of some of the irreducible components of $D \cup D_1 \cup \cdots \cup D_k$ with sign. In this case $(\mathcal{E}, \Phi)$ is (semi)stable for some polarization, but not for all polarizations. Again in any case we have $h^0(T_X \otimes \mathcal{A}^\vee) > 0$.

*Example 2.13.* Let $X$ be a smooth quadric surface and take $\mathcal{A}$ from

$$\{\mathcal{O}_X, \mathcal{O}_X(1, 0), \mathcal{O}_X(2, 0), \mathcal{O}_X(0, 1), \mathcal{O}_X(0, 2)\}$$

In each case the Cayley–Bacharach condition is satisfied. If $\mathcal{A} \cong \mathcal{O}_X$, then for any $r \geq 2$ and integer $\deg(Z) \geq 0$ we get vector bundles which are strictly semistable for any polarization (see Example 2.9). Now assume $\mathcal{A} \ncong \mathcal{O}_X$ and let $\mathcal{H}$ be any polarization on $X$. We claim that $(\mathcal{E}, \Phi)$ is $\mathcal{H}$-stable. Take an integer $s \in \{1, \ldots, r-1\}$ and a subsheaf $\mathcal{G} \subset \mathcal{E}$ of rank $s$ with maximal $\mathcal{H}$-slope and with $\Phi(\mathcal{G}) \subset \mathcal{G} \otimes T_X$. We have $\mathrm{Im}(\Phi) \subset \mathcal{O}_X \otimes T_X$ and $\ker(\Phi) \cong \mathcal{O}_X^{\oplus(r-1)}$. Thus the $\mathcal{H}$-slope of $\mathcal{G} \cap \ker(\Phi)$ is at most zero. We have $\Phi(\mathcal{G}) \subset \mathcal{O}_X \otimes T_X$ and so $\mathcal{G} \subset \mathcal{O}_X^{\oplus(r-1)}$. In particular, we have $\deg_{\mathcal{H}}(\mathcal{G}) \leq 0$ and so $(\mathcal{E}, \Phi)$ is stable for any polarization. In many cases even $\mathcal{E}$ is stable for some or most polarizations.

Assume $\mathcal{A} \cong \mathcal{O}_X(1, 0)$. If $Z = \emptyset$, then $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{O}_X(1, 0)$ and so $\mathcal{E}$ is not semistable for any polarization. Assume $Z \neq \emptyset$ and that $(\mathcal{E}, \Phi)$ is not stable with respect to a polarization $\mathcal{H} \cong \mathcal{O}_X(a, b)$ with $b < 2a$. There is a saturated subsheaf $\mathcal{L} = \mathcal{O}_X(u, v) \subset \mathcal{E}$ of rank one with $av + bu \geq b/2$. In particular, at least one of the integers $u$ and $v$ is positive. Write $\mathcal{E}/\mathcal{L} \cong \mathcal{I}_W(1-u, 0-v)$ for some zero-dimensional scheme $W \subset X$. We have $c_2(\mathcal{E}) = \deg(W) + v(1-u) - uv$. Composing the inclusion $\mathcal{L} \subset \mathcal{E}$ with the surjection $v$ in (1), we get a non-zero map $f : \mathcal{O}_X(u, v) \to \mathcal{I}_Z(1, 0)$, and so we get $v \leq 0$ and $u = 1$.

First assume $v < 0$ and then we have $h^0(\mathcal{L}) = 0$. Since $H^0(\mathcal{E}) \neq 0$, we get $h^0(\mathcal{I}_W(0, -v)) > 0$. Since $b > 0$ and $0 < a < 2b$, we get $av + b < b/2$, a contradiction.

Now assume $v = 0$ and we get $h^0(\mathcal{L}) = 2$. Then we have $h^0(\mathcal{E}) \geq 2$. Since $Z$ is not empty, (1) implies that $Z$ is a single point and so $c_2(\mathcal{E}) = 1$. From $\mathcal{E}/\mathcal{L} \cong \mathcal{I}_W$, we get $c_2(\mathcal{E}) = \deg(W)$ and so $W$ is a single point. The map $u$ in (1) and the inclusion $\mathcal{L} \subset \mathcal{E}$ induce an injective map $j : \mathcal{O}_X(1, 0) \oplus \mathcal{O}_X \to \mathcal{E}$. Since $j$ is an injective map between vector bundles with the same rank and isomorphic determinant, it is an isomorphism. Thus we have $c_2(\mathcal{E}) = 0$, a contradiction. The same proof works for the case $\mathcal{A} \cong \mathcal{O}_X(0, 1)$ for the polarization $\mathcal{H} \cong \mathcal{O}_X(a, b)$ with $a < 2b$.

For $r = 2$ we recover most of the existence part in part (1) of [7, Theorem in page 2]. The advantage of the current argument is that we prove stability simultaneously with respect to many polarizations $\mathcal{H} \cong \mathcal{O}_X(a, b)$ and that we explicitly state that our co-Higgs fields are nilpotent. To be in the framework of part (2) of [7, Theorem in page 2] we need to modify the general set-up. Instead of vector bundles $\mathcal{E}$ fitting into the exact sequence (1) with $\mathcal{A}$ as above, we take vector bundles fitting into the exact sequence with $\mathcal{A} \cong \mathcal{O}_X(1, -1)$,

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_Z(1, -1) \to 0 \tag{2}$$

with $Z$ a zero-dimensional scheme, where we have $\det(\mathcal{E}) \cong \mathcal{O}_X(1, -1)$. By taking a twist by some $\mathcal{O}_X(\alpha, \beta)$ we get vector bundles of rank two with an arbitrary determinant $\mathcal{O}_X(\gamma, \delta)$ with both $\gamma$ and $\delta$ odd. But the twist may destroy the stability with respect to certain polarizations.

*Example 2.14.* Let $X = \mathbb{P}^3$ and take $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}(1)$. Then we have either

- $\omega_Z(3)$ is spanned, if $r \geq 3$;
- $\omega_Z \cong \mathcal{O}_Z(-3)$, if $r = 2$.

In case of $r = 2$, we get curvilinear reflexive sheaves $\mathcal{E}$ with $c_2(\mathcal{E}) = \deg(Z)$ and $c_3(\mathcal{E}) = \deg(\omega_Z) + 3\deg(Z)$; see [4]. We always assume $Z \neq \emptyset$, so that $\mathcal{E}$ is indecomposable. We claim that $\mathcal{E}$ is stable. Assume the existence of a line bundle $\mathcal{O}_X(t) \subset \mathcal{E}$ with $t > 0$. Composing with the surjection $v : \mathcal{E} \to \mathcal{I}_Z(1)$ we get the zero map, because $t > 0$ and $Z \neq \emptyset$. Thus we get $\mathcal{O}_X(t) \subseteq \mathcal{O}_X$, a contradiction.

Now we take $r \geq 3$ and $Z$ a non-empty disjoint union of smooth curves. Assume that $\mathcal{E}$ has no trivial factor, e.g. if $Z$ is large, and that $h^0(\mathcal{I}_Z(1)) = 0$, i.e. $Z$ is not planar. If $(\mathcal{E}, \Phi)$ is not stable, then there is a subsheaf $\mathcal{G} \subset \mathcal{E}$ of rank $s \in \{1, \ldots, r-1\}$ with $\deg(\mathcal{G}) > 0$ such that $\Phi(\mathcal{G}) \subset \mathcal{G} \otimes T_X$ and $s$ is the minimum among all subsheaves of $\mathcal{E}$ with the other properties. Since $\mathrm{Im}(\Phi) \subset T_X$ has rank one, so we get $\mathrm{Im}(\Phi)^{\vee\vee} \cong \mathcal{O}_X(1)$, i.e. $\mathrm{Im}(\Phi) \cong \mathcal{I}_W(1)$ for some $W \subset \mathbb{P}^3$ with

$\dim(W) \le 1$. $\mathcal{G}$ is saturated in $\mathcal{E}$, i.e. $\mathcal{E}/\mathcal{G}$ is torsion-free, and so $\mathcal{G}$ is a reflexive sheaf. Since $\mathcal{E}$ is assumed to be locally free, in the case $s = 1$ we get $\mathcal{G} \cong \mathcal{O}_X(1)$. We exclude this case, because $\mathcal{O}_X(1) \nsubseteq \mathcal{E}$.

Now assume $r = 3$ and $s = 2$. The map $\mathcal{G} \to \mathcal{I}_Z(1)$ induced by the surjection in (1) must be non-zero. Due to $s = 2$, we get $\mathcal{G} \ncong \mathcal{O}_X^{\oplus 2}$ and $\mathcal{O}_X^{\oplus 2}$ is the image of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}$, we have $h^0(\mathcal{G}) \le 1$ and so $h^0(\mathcal{E}/\mathcal{G}) > 0$. Since $\mathcal{E}/\mathcal{G}$ is a torsion-free sheaf of rank one, we get $\mathcal{E}/\mathcal{G} \cong \mathcal{O}_X$. Since $h^0(\mathcal{E}) > h^0(\mathcal{G})$, there is $\sigma \in H^0(\mathcal{E})$ whose image in $\mathcal{E}/\mathcal{G} \cong \mathcal{O}_X$. The map $1 \mapsto \sigma$ shows that $\mathcal{O}_X$ is a factor of $\mathcal{E}$, contradicting our assumption.

Now we assume $r = 3$ and list several $Z$ for which the middle term $\mathcal{E}$ of a general extension (1) with $\mathcal{A} \cong \mathcal{O}_X(1)$ has not $\mathcal{O}_X$ as a factor; in each case we certainly need that $\omega_Z(3)$ is spanned and that $h^0(\omega_Z(3)) \ge 2$. Assume $\mathcal{E} \cong \mathcal{O}_X \oplus \mathcal{G}$. Since $\mathcal{E}$ is locally free, so is $\mathcal{G}$. Since $h^0(\mathcal{G}) = 1$ and $h^0(\mathcal{G}(-1)) = h^0(\mathcal{E}(-1)) = 0$, $\mathcal{G}$ fits in an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{G} \to \mathcal{I}_W(1) \to 0, \tag{3}$$

where $W$ is a locally complete curve with $\omega_W(3) \cong \mathcal{O}_W$ and $h^0(\mathcal{I}_W(1)) = 0$. We obviously have that $W$ is not reduced. From $H^0(\mathcal{G}(-1)) = 0$, we get that $\mathcal{G}$ is a stable vector bundle of rank two on $\mathbb{P}^3$ with $c_1(\mathcal{G}) = 1$ and $c_2(\mathcal{G}) = \deg(W)$. The subsheaf $\mathcal{O}_X$ of $\mathcal{G}$ is the image of the evaluation map $H^0(\mathcal{G}) \otimes \mathcal{O}_X \to \mathcal{G}$. So the surjective maps in (1) and (3) induce a non-zero map $\mathcal{I}_W(1) \to \mathcal{I}_Z(1)$ and so we get $W \supseteq Z$. Since $c_2(\mathcal{F}) = c_2(\mathcal{G})$, we have $\deg(Z) = \deg(W)$ and so $Z = W$, which gives a contradiction each time we chose $Z$ with $\omega_Z \ncong \mathcal{O}_Z(-3)$, e.g. each time we chose as $Z$ a disjoint union of $d$ lines.

## 3. Existence and non-existence of co-Higgs structures

Let $X$ be a smooth projective variety of dimension $n$ with $\mathrm{Pic}(X) \cong \mathbb{Z}$, where the ample generator $\mathcal{O}_X(1)$ is very ample. We keep this assumption until Theorem 3.16, where we assume $\mathrm{Num}(X) \cong \mathbb{Z}$. Set $\delta := \deg(X)$ with respect to $\mathcal{O}_X(1)$. For any reflexive sheaf $\mathcal{E}$ of rank two on $X$, define $x_{\mathcal{E}}$ to be

$$\max\{x \in \mathbb{Z} \mid h^0(\mathcal{E}(-x)) > 0\}. \tag{4}$$

Then $\mathcal{E}$ fits into an exact sequence for a subscheme $Z$ with pure codimension two,

$$0 \to \mathcal{O}_X(x_{\mathcal{E}}) \to \mathcal{E} \to \mathcal{I}_Z(c_1 - x_{\mathcal{E}}) \to 0, \tag{5}$$

where $c_1 = c_1(\mathcal{E})$ and $c_2(\mathcal{E}) = \deg(Z) + x_{\mathcal{E}}(c_1 - x_{\mathcal{E}})\delta$. Note that we have $h^0(\mathcal{I}_Z(c_1 - x_{\mathcal{E}} - 1)) = 0$ by definition of $x_{\mathcal{E}}$.

**Proposition 3.1.** *Let $\mathcal{E}$ be a reflexive sheaf of rank two on $X$ with $c_1(\mathcal{E}) \in \{-1, 0\}$ and $x_{\mathcal{E}} \le -2$. Then any nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ is trivial.*

*Proof.* If $\Phi \neq 0$, then we have $\ker(\Phi) \cong \mathcal{O}_X(t)$ for some $t \leq x_{\mathcal{E}} \leq -2$. Since $\mathrm{Im}(\Phi)$ has rank one with no torsion, we have $\mathrm{Im}(\Phi) \cong \mathcal{I}_B(-t + c_1)$ for some closed scheme $B \subset X$ with $\dim(B) \leq n-2$. Since $\Omega^1_X(2)$ is globally generated and $\mathrm{Im}(\Phi)$ is a subsheaf of $\mathcal{E} \otimes T_X$, we may consider $\mathrm{Im}(\Phi)$ as a subsheaf of $\mathcal{E}(2)^{\oplus N}$ for some $N > 0$. In particular, we get $-t + c_1 - 2 \leq x_{\mathcal{E}}$, a contradiction. □

**Proposition 3.2.** *Assume $X \neq \mathbb{P}^n$. If $\mathcal{E}$ is a reflexive sheaf of rank two on $X$ with $c_1(\mathcal{E}) + 2x_{\mathcal{E}} = -3$, then any nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ is trivial.*

*Proof.* Up to a twist we may assume $c_1(\mathcal{E}) = -1$. Assume the existence of a nonzero nilpotent $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$. We have $\ker(\Phi) \cong \mathcal{O}_X(t)$ for some $t < 0$. By Proposition 3.1 we have $t = -1$. Since $\mathrm{Im}(\Phi)$ has rank one with no torsion, we have $\mathrm{Im}(\Phi) \cong \mathcal{I}_B$ for some closed scheme $B \subset X$ with $\dim(B) \leq \dim(X) - 2$. Since $\mathrm{Im}(\Phi) \subset \ker(\Phi) \otimes T_X$, we get $H^0(T_X(-1)) \neq 0$, and so $X = \mathbb{P}^n$ by [23], a contradiction. □

*Remark 3.3.* Let $\mathcal{E}$ be a stable reflexive sheaf of rank two on $X$ with $c_1(\mathcal{E}) = -1$. By the stability of $\mathcal{E}$, we have $x_{\mathcal{E}} \leq -1$. If $x_{\mathcal{E}} \leq -2$, then any nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ is trivial by Proposition 3.1. As an example, we may take as $\mathcal{E}$ the Horrocks–Mumford bundle; $X = \mathbb{P}^4$, $c_1 = -1$ and $c_2 = 4$. If $x_{\mathcal{E}} = -1$, then $\mathcal{E}$ fits in an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E}(1) \to \mathcal{I}_Z(1) \to 0$$

for some 2-codimensional scheme $Z \subset X$. Assume $H^0(T_X(-1)) \neq 0$ and so $X = \mathbb{P}^n$ by [23]. Then by Lemma 2.6 there exists a non-trivial nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_{\mathbb{P}^n}$ with $\ker(\Phi) = \mathcal{O}_{\mathbb{P}^n}$.

**Proposition 3.4.** *Let $\mathcal{E}$ be a stable reflexive sheaf of rank two on $\mathbb{P}^n$ with $c_1(\mathcal{E}) = 0$. Then there exists no non-trivial nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_{\mathbb{P}^n}$.*

*Proof.* Since $\mathcal{E}$ is stable, we have $\ker(\Phi) \cong \mathcal{O}_{\mathbb{P}^n}(t)$ for some $t \leq -1$ and the proof of Proposition 3.1 gives $t = -1$. Since $\mathrm{Im}(\Phi)$ has rank one with no torsion, we have $\mathrm{Im}(\Phi) \cong \mathcal{I}_B(1)$ for some closed subscheme $B \subsetneq \mathbb{P}^n$.

First assume $\dim B \leq n-2$. Since $\Phi \circ \Phi = 0$, we have $\mathrm{Im}(\Phi) \subset \ker(\Phi) \otimes T_{\mathbb{P}^n} \cong T_{\mathbb{P}^n}(-1)$. In particular, we get a nonzero map $h : \mathcal{I}_B(1) \to T_{\mathbb{P}^n}(-1)$. Since $T_{\mathbb{P}^n}(-2)$ is locally free and $\dim B \leq n-2$, we have

$$H^0(\mathbb{P}^n \setminus B, T_{\mathbb{P}^n}(-2))_{|\mathbb{P}^n \setminus B}) = H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-2))$$

by [13, Proposition 1.6], which is trivial. But the map $h$ gives $H^0(T_{\mathbb{P}^n}(-2)) \neq 0$, a contradiction.

Now assume that $B$ contains a hypersurface of degree $e$. We get $\mathrm{Im}(\Phi) \cong \mathcal{I}_Z(1-e)$ for some closed subscheme $Z$ with $\dim Z \leq n-2$. Since $c_1(\mathcal{E}) = 0$ and $e > 0$, $\mathcal{E}$ is not stable, a contradiction. □

*Proof of Theorem 1.1:*. Denote by $\mathcal{S}$ the set of all nilpotent maps and up to a twist we may assume $c_1(\mathcal{E}) \in \{-1, 0\}$. By Proposition 3.4 we can consider only the case of $c_1(\mathcal{E}) = -1$. By Proposition 3.1 we have $\mathcal{S} = \{0\}$, unless $x_{\mathcal{E}} = -1$. Thus we may assume $x_{\mathcal{E}} = -1$ and so $\mathcal{E}$ fits into an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\sigma} \mathcal{E} \to \mathcal{I}_Z \to 0 \tag{6}$$

with $Z$ of codimension 2. By Lemma 2.6 a cheap way to get a non-trivial $\Phi$ is to take the composition of the surjection in (6) with the inclusion $\mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^n}(-1) \otimes T_{\mathbb{P}^n}$. In this way we get an $(n+1)$-dimensional vector space contained in $\mathcal{S}$, isomorphic to $H^0(T_{\mathbb{P}^n}(-1))$. Conversely, choose any arbitrary nonzero map $\Phi \in \mathcal{S}$. The proof of Proposition 3.1 gives $\ker(\Phi) \cong \mathcal{O}_{\mathbb{P}^n}(-1)$ and so $\mathrm{Im}(\Phi) \cong \mathcal{I}_B$ for some closed subscheme $B \subsetneq \mathbb{P}^n$ of codimension two. Since $\Phi \circ \Phi = 0$, we have $\mathrm{Im}(\Phi) \subset \ker(\Phi) \otimes T_{\mathbb{P}^n} \cong T_{\mathbb{P}^n}(-1)$, and thus $\Phi$ is also obtain by the same way as in Lemma 2.6. Thus any such nilpotent map is represented by an element in $H^0(\mathcal{E}(1)) \times H^0(T_{\mathbb{P}^n}(-1))$ with an action of $\mathbb{C}^*$ defined by $c \cdot (\sigma, s) = (c\sigma, c^{-1}s)$. Thus the set of nilpotent maps is parametrized by

$$H^0(\mathcal{E}(1)) \times H^0(T_{\mathbb{P}^n}(-1)) /\!\!/ \mathbb{C}^*,$$

which is the total space of $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a}$ with $a = h^0(\mathcal{E}(1))$; see [21, Theorem 5.7] for a similar description. Now the assertion follows from the observation that non-proportional sections of $\mathcal{E}(1)$ have different zeros as in [13, Theorem 4.1] and that if $\sigma$ of $s$ is trivial, then the pair $(\sigma, s)$ corresponds to the trivial nilpotent map. □

We still assume that $X$ is a smooth projective variety with $\mathrm{Pic}(X) \cong \mathbb{Z}$ generated by an ample line bundle $\mathcal{O}_X(1)$ and $H^0(T_X(-2)) = 0$, which excludes the case $X \cong \mathbb{P}^1$ by [23]. Let $\mathcal{E}$ be a non-semistable reflexive sheaf of rank two on $X$ such that $(\mathcal{E}, \Phi)$ is semistable for a map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$. Without loss of generality we assume that $\mathcal{E}$ is initialized, i.e. $H^0(\mathcal{E}) \neq 0$ and $H^0(\mathcal{E}(-1)) = 0$. Since $\mathcal{E}$ is not semistable, we have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_Z(-b) \to 0 \tag{7}$$

with $b > 0$ and $\dim(Z) \le \dim(X) - 2$.

**Lemma 3.5.** *Let $\mathcal{E}$ be a non-semistable reflexive sheaf of rank two on $X$ with $(\mathcal{E}, \Phi)$ semistable. Then we have $X \cong \mathbb{P}^n$ with $n \ge 2$ and $b = 1$. Also we have either*

- *$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)$, or*
- *$n = 2$ and $Z$ is a point of $\mathbb{P}^2$.*

*Proof.* Since $\mathcal{E}$ is reflexive, either $Z = \emptyset$ or $Z$ has pure codimension 2. From (7) we get an exact sequence

$$0 \to \mathcal{O}_X \otimes T_X \to \mathcal{E} \otimes T_X \xrightarrow{v} \mathcal{I}_Z \otimes T_X(-b) \to 0. \tag{8}$$

Since $(\mathcal{E}, \Phi)$ is semistable, we have $\Phi(\mathcal{O}_X) \nsubseteq \mathcal{O}_X \otimes T_X$ and so $v \circ \Phi : \mathcal{O}_X \to \mathcal{I}_Z \otimes T_X(-b)$ is a non-zero map. Since $X \ncong \mathbb{P}^1$ by [23], we have $X \cong \mathbb{P}^n$ with $n \ge 2$ and $b = 1$. We also get $h^0(\mathcal{I}_Z \otimes T_X) > 0$. The zero-locus of each non-zero section of $T_X(-1)$ is a single point. Hence we have either $Z = \emptyset$, or $n = 2$ and $Z$ is a single point. If $Z = \emptyset$, then (7) gives $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)$. □

Recall that $x_{\mathcal{E}}$ depends only on the isomorphism class of $\mathcal{E}$; see (4). For any $\mathcal{E}$ fitting into (5) with $Z$ satisfying $h^0(\mathcal{I}_Z(c_1 - x_{\mathcal{E}} - 1)) = 0$, we know that $\mathcal{E}$ is stable (resp. semistable) if and only if $2x_{\mathcal{E}} < c_1$ (resp. $2x_{\mathcal{E}} \le c_1$). For a fixed $\mathcal{E}$, the same subscheme $Z \subset X$ may occur only by proportional sections in $H^0(\mathcal{E}(-x))$ by [12, Proposition 1.3]. Define $y_{\mathcal{E}}$ to be

$$\min\{y \ge 0 \mid h^0(\mathcal{I}_Z(c_1 - x_{\mathcal{E}} + y)) > 0\}.$$

Note that $y_{\mathcal{E}} = 0$ if and only if $\mathcal{E}$ has at least two non-proportional maps $\mathcal{O}_X(x) \to \mathcal{E}$ and so fits in at least two non-proportional sequences (5), with different subschemes $Z$. Thus in all cases the integer $y_{\mathcal{E}}$ is well-defined.

**Lemma 3.6.** *Let $\mathcal{E}$ be a reflexive sheaf of rank two on $X$ with $c_1 - 2x_{\mathcal{E}} > 0$. Then we have $h^0(\mathcal{E}nd(\mathcal{E})(z)) = h^0(\mathcal{O}_X(z))$ for $0 \le z < \min\{x_{\mathcal{E}} + y_{\mathcal{E}}, c_1 - 2x_{\mathcal{E}}\}$.*

*Proof.* Set $x := x_{\mathcal{E}}$ and $y := y_{\mathcal{E}}$, and assume that $\mathcal{E}$ fits in (5) for some $Z$. Fix $f \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}(z))$ and let $f_1 : \mathcal{E} \to \mathcal{I}_Z(c_1 - x + z)$ be the map obtained by composing $f$ with the map $\mathcal{E}(z) \to \mathcal{I}_Z(c_1 - x + z)$ twisted from (5) with $\mathcal{O}_X(z)$. From the assumption $z < x + y$, we have $f_1(\mathcal{O}_X(x)) = 0$ and so $f$ induces $f_2 : \mathcal{I}_Z(c_1 - x) \to \mathcal{I}_Z(c_1 - x + z)$. Now take $g \in H^0(\mathcal{O}_X(z))$ inducing $f_2$ and let $\gamma : \mathcal{E} \to \mathcal{E}(z)$ be obtained by the multiplication by $g$. Our claim is that $f = \gamma$. Taking $f - \gamma$ instead of $f$ we reduce to the case $g = 0$ and in this case we need to prove that $f = 0$, when we have $f(\mathcal{E}) \subseteq \mathcal{O}_X(x + z)$. Since $\mathcal{E}$ is reflexive of rank two, we have $\mathcal{E}^\vee \cong \mathcal{E}(-c_1)$. Thus $f : \mathcal{E} \to \mathcal{O}_X(x + z)$ is induced by a unique $a \in H^0(\mathcal{E}(x + z - c_1))$. Since $x + z - c_1 < -x$, we have $a = 0$ and so $f = 0$. □

**Proposition 3.7.** *If $\mathcal{E}$ is a reflexive sheaf of rank two on $X$ with*

$$\min\{x_{\mathcal{E}} + y_{\mathcal{E}}, c_1(\mathcal{E}) - 2x_{\mathcal{E}}\} \ge 3,$$

*then it has no non-zero trace-free co-Higgs field, not even a non-integrable one.*

*Proof.* Take any map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$. Since $\mathcal{O}_X(1)$ is very ample, $\Omega_X^1(2)$ is spanned and so $T_X$ is a subsheaf of $\mathcal{O}_X(2)^{\oplus N}$, where $N = h^0(\Omega_X^1(2))$. Thus $\Phi$ induces $N$ elements $\Phi_i : \mathcal{E} \to \mathcal{E}(2)$ with $i = 1, \ldots, N$. By Lemma 3.6 each $\Phi_i$ is induced by $f_i \in H^0(\mathcal{O}_X(2))$. Composing the trace map $\mathrm{Tr}(\Phi) : \mathcal{O}_X \to T_X$ of $\Phi$ with the inclusion $T_X \subset \mathcal{O}_X(2)^{\oplus N}$, we also get $N$ elements $g_i \in H^0(\mathcal{O}_X(2))$. Note that we have $2f_i = g_i$ for all $i$. If $\Phi$ is trace-free, then we get $g_i = 0$ and so $f_i = 0$ for all $i$. Thus $\Phi$ is trivial. □

### *3.1. Case $X = \mathbb{P}^2$*

For $(c_1, c_2) \in \mathbb{Z}^{\oplus 2}$, let $\mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$ denote the moduli space of stable vector bundles of rank two on $\mathbb{P}^2$ with Chern numbers $(c_1, c_2)$. Schwarzenberger proved that $\mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$ is non-empty if and only if $-4 \ne c_1^2 - 4c_2 < 0$; see [12, Lemma 3.2]. When non-empty, $\mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$ is irreducible; see [2,15,17,18]. For $\mathcal{E} \in \mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$ and any $t \in \mathbb{Z}$, we have

$$\begin{aligned} c_1(\mathcal{E}(t)) &= c_1 + 2t, \\ c_2(\mathcal{E}(t)) &= c_2 + t^2 + tc_1, \\ \chi(\mathcal{E}(t)) &= (c_1 + 2t + 2)(c_1 + 2t + 1)/2 + 1 - c_2 - t^2 - t(c_1 + t) \\ &= c_1(c_1 + 2t + 3)/2 + (t + 1)(t + 2) - c_2; \end{aligned}$$

see [5, page 469]. Up to a twist we may assume that $c_1 \in \{-1, 0\}$. Since $\mathcal{E}$ is stable, we have $h^0(\mathcal{E}) = 0$ and so $x_{\mathcal{E}} < 0$. Define an integer $\alpha(c_1, c_2)$ as

$$\alpha(c_1, c_2) := \min\{t \in \mathbb{Z}_{>0} \mid c_1(c_1 + 2t + 3)/2 + (t + 1)(t + 2) > c_2\}.$$

For any $\mathcal{E} \in \mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$, we have $\chi(\mathcal{E}(a)) > 0$ for all $a \geq \alpha(c_1, c_2)$, and $\alpha(c_1, c_2)$ is the minimal positive integer with this property; see [12, Proposition 7.1]. By [5, Theorem 5.1], a general bundle $\mathcal{E} \in \mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$ has $x_{\mathcal{E}} = -\alpha(c_1, c_2)$ and $h^1(\mathcal{E}(t)) = 0$ for all $t \geq \alpha(c_1, c_2)$. By Proposition 3.4, if $c_1$ is even, no bundle $\mathcal{E} \in \mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$ has a non-zero nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$. If $c_1$ is odd, we have the following.

**Proposition 3.8.** *Let $\mathcal{E}$ be s general element of $\mathbf{M}_{\mathbb{P}^2}(-1, c_2)$ with $c_2 \geq 4$. If $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_{\mathbb{P}^2}$ is a nilpotent map, then we have $\Phi = 0$.*

*Proof.* By Proposition 3.1 it is sufficient to prove that $x_{\mathcal{E}} \leq -2$, i.e. $h^0(\mathcal{E}(1)) = 0$. Note that $\chi(\mathcal{E}(1)) = 4 - c_2 \leq 0$ and so we may apply [5, Theorem 5.1]. □

For any $x \in \mathbb{Z}$, let $\mathbf{M}_{\mathbb{P}^2}(c_1, c_2, x)$ denote the set of all $\mathcal{E} \in \mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$ with $x_{\mathcal{E}} = x$. It is an irreducible family and we have a description of the nilpotent co-Higgs fields on each bundle in $\mathbf{M}_{\mathbb{P}^2}(c_1, c_2, x)$; see Theorem 1.1.

*Remark 3.9.* Any $\mathcal{E} \in \mathbf{M}_{\mathbb{P}^2}(-1, c_2)$ with $\mathcal{E}(-1)$ as in Lemma 2.6 and (1) for $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^2}(1)$ occurs in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{E} \to \mathcal{I}_Z \to 0 \tag{9}$$

with $Z$ a locally complete intersection scheme $Z \subset \mathbb{P}^2$ with $\deg(Z) = c_2(\mathcal{E})$, using that $\deg(Z) = c_2(\mathcal{E}(1))$ by [13, Corollary 2.2]. Since $Z$ is not empty, every vector bundle fitting into (9) is stable and so we have $x_{\mathcal{E}} = -1$. The general element of $\mathbf{M}_{\mathbb{P}^2}(-1, c_2, -1)$ admits an extension (9) with as $Z$ the general subset of $\mathbb{P}^2$ with cardinality $c_2$.

For a general stable vector bundle of rank two on $\mathbb{P}^2$, we have $y_{\mathcal{E}} \leq 1$ by [5] and so we cannot use Proposition 3.7 for it. We prove Theorem 1.4 using the following key observation.

*Remark 3.10.* Take an irreducible family $\Gamma$ of reflexive sheaves of rank two on $X$. Let $\mathcal{G}$ denote the general element of $\Gamma$. Assume the existence of some $\mathcal{E} \in \Gamma$ with $c_1(\mathcal{E}) - 2x_{\mathcal{E}} \geq 3$ and $y_{\mathcal{E}} + x_{\mathcal{E}} \geq 3$. By Lemma 3.6 we have $h^0(\mathcal{E}nd(\mathcal{E})(2)) =$

$h^0(\mathcal{O}_X(2))$, which is the minimum possibility for $h^0(\mathcal{E}nd(\mathcal{G})(2))$ with $\mathcal{G}$ reflexive of rank two on $X$, i.e. $H^0(\mathcal{E}nd(\mathcal{E})(2))$ has the minimal dimension among all reflexive sheaves of rank two on $X$. By the semicontinuity theorem we have $h^0(\mathcal{E}nd(\mathcal{G})(2)) = h^0(\mathcal{O}_X(2))$. Thus we may apply the proof of Proposition 3.7 to $\mathcal{G}$, even when $\mathcal{G}$ does not satisfy the assumptions of Proposition 3.7.

*Proof of Theorem 1.4:.* The proof of Proposition 3.7 shows that it is enough to prove $h^0(\mathcal{E}nd(\mathcal{E})(2)) = 6$. And by semicontinuity it is also sufficient to prove that $h^0(\mathcal{E}nd(\mathcal{G})(2)) = 6$ for some $\mathcal{G} \in \mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$. Furthermore, by Lemma 3.6 it is sufficient to find $\mathcal{G} \in \mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$ with $x_{\mathcal{G}} = -2$ and $y_{\mathcal{G}} \geq 5$.

Now take a general $S \subset \mathbb{P}^2$ with $\sharp(S) = c_2 + 4 + 2c_1$ and let $\mathcal{G}$ be a general sheaf fitting into

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{G} \to \mathcal{I}_S(c_1 + 2) \to 0.$$

By Bogomolov inequality we have $4c_2 > c_1^2$. We have $h^0(\mathcal{I}_{S\setminus\{p\}}(c_1 + 1)) = 0$ for $p \in S$ and so the Cayley–Bacharach condition is satisfied. Thus $\mathcal{G}$ is locally free. We also have $h^0(\mathcal{G}(1)) = 0$ from $h^0(\mathcal{I}_S(c_1 + 1)) = 0$, and so we have $x_{\mathcal{G}} = -2$. On the other hand, we have $\sharp(S) > \binom{c_1+8}{2}$, we have $h^0(\mathcal{I}_S(c_1 + 6)) = 0$ and so $y_{\mathcal{G}} \geq 5$. Now we may use Remark 3.10 and the irreducibility of $\mathbf{M}_{\mathbb{P}^2}(c_1, c_2)$. □

### *3.2. Case $X = \mathbb{P}^3$ and $r \geq 3$*

We look at locally free sheaves $\mathcal{E}$ of rank at least three on $\mathbb{P}^3$ fitting into (1) with either $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}$ or $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}(1)$. By Lemma 2.6 any such a sheaf $\mathcal{E}$ has a 2-nilpotent $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_{\mathbb{P}^3}$ with $\ker(\Phi) \cong \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-1)}$. If $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}$, then any torsion-free $\mathcal{E}$ fitting into (1) is strictly slope-semistable. Note also that if $Z$ is empty in (1), then $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^3}^{\oplus(r-1)} \oplus \mathcal{A}$ and that $\deg(Z) = c_2(\mathcal{E})$. In particular if $\mathcal{E}$ is not a direct sum of line bundles, then we have $c_2(\mathcal{E}) > 0$.

**Lemma 3.11.** *Let $\mathcal{E}$ be a reflexive sheaf of rank three fitting into (1) with $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}(1)$. Then the followings are equivalent.*

(i) *$\mathcal{E}$ is slope-semistable;*
(ii) *$\mathcal{E}$ is slope-stable;*
(iii) *$\mathcal{E}$ has no trivial factor.*

*Proof.* Assume that $\mathcal{E}$ has a saturated subsheaf $\mathcal{G}$ of rank $s < 3$ with $\deg(\mathcal{G})/s \geq 1/3$.

If $s = 1$, then we have $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^3}(t)$ for some $t > 0$, because $\mathcal{E}$ is reflexive and $\mathcal{E}/\mathcal{G}$ has no torsion (see [13, Propositions 1.1 and 1.9]). Then we have $\mathcal{G} \not\subseteq u(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2})$ and so $v(\mathcal{O}_{\mathbb{P}^3}(t))$ is a non-zero subsheaf of $\mathcal{I}_Z(1)$. In particular, we get $t = 1$ and $Z = \emptyset$. Thus we have $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$.

Now assume $s = 2$. Again $v(\mathcal{G})$ is a non-zero subsheaf of $\mathcal{I}_Z(1)$ and so we get $\deg(\mathcal{G}) = 1$ and that $\mathcal{G}$ is an extension of some $\mathcal{I}_W(1)$ with $W \supseteq Z$ by $\mathcal{O}_{\mathbb{P}^3}$. It implies that the torsion-free sheaf $\mathcal{E}/\mathcal{G}$ is a rank one sheaf of degree zero with $h^0(\mathcal{E}/\mathcal{G}) > 0$. Thus we have $\mathcal{E}/\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^3}$ and the map $u(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}) \to \mathcal{E}/\mathcal{G}$ is surjective.

Taking a section of $u(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2})$ with $1 \in H^0(\mathcal{E}/\mathcal{G})$ as its image, we get a map $\mathcal{E}/\mathcal{G} \to \mathcal{E}$ inducing a splitting $\mathcal{E} \cong \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}^3}$. Thus (iii) implies (ii). Clearly (ii) implies (i). Now assume that $\mathcal{E}$ has a trivial factor, i.e. $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{F}$ with $\mathcal{F}$ a bundle of rank two. Then the slope of $\mathcal{F}$ is $1/2$, which is greater than the slope of $\mathcal{E}$. Thus (i) implies (iii). □

*Proof of Theorem 1.2:.* For the strictly semistable bundle, we apply Lemma 2.6 with $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}$. Except the indecomposability, it is sufficient to find a locally Cohen-Macaulay curve $Z \subset \mathbb{P}^3$ of $\deg(Z) = c_2$ such that $\omega_Z(4)$ is spanned and there is a 2-dimensional linear subspace $V \subseteq H^0(\omega_Z(4))$ spanning $\omega_Z(4)$ at each point of $Z_{\mathrm{red}}$. We may even take a smooth $Z$. Note that for every smooth and connected curve $Z \subset \mathbb{P}^3$, $\omega_Z(4)$ is spanned and non-trivial, and so we get $h^0(\omega_Z(4)) \geq 2$. Since $\omega_Z(4)$ is a line bundle on a curve $Z$, a general 2-dimensional linear subspace of $H^0(\omega_Z(4))$ spans $\omega_Z(4)$.

Assume now that $\mathcal{E}$ is decomposable. Using the same argument in the proof of Lemma 3.11 to show that (iii) implies (ii), we get $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{G}$ for some vector bundle $\mathcal{G}$ of rank two. Since $Z$ is not empty, we have $\mathcal{G} \ncong \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$ and we see that $\mathcal{G}$ fits in (1) with the same $Z$ above and $r = 2$. Thus we get $\omega_Z \cong \mathcal{O}_Z(-4)$ by [12, Theorem 1.1], contradicting the assumption that $Z$ is a reduced curve.

For the stable bundle, we follow the argument above with $\omega_Z(3)$ instead of $\omega_Z(4)$. □

For any reflexive sheaf $\mathcal{G}$ of rank two on $\mathbb{P}^3$ we have $c_1(\mathcal{G}(t))^2 - 4c_2(\mathcal{G}(t)) = c_1(\mathcal{G})^2 - 4c_2(\mathcal{G})$ for all $t \in \mathbb{Z}$. Take $\mathcal{E}$ produced by Theorem 1.2 and consider its quotient by a subsheaf $\mathcal{O}_X \subset \mathcal{E}$, or use (1) for $r = 2$ and the Hartshorne-Serre correspondence in [13, Theorem 4.1]. Then we get the following results (for the " only if " part use [13, Corollary 3.3]).

**Corollary 3.12.** *For a fixed pair of integers $(c_1, c_2)$ with $c_1$ even, there are an indecomposable and strictly semistable reflexive sheaf $\mathcal{E}$ of rank two on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$, and a non-trivial nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_{\mathbb{P}^3}$ if and only if $c_1^2 - 4c_2 < 0$.*

**Corollary 3.13.** *For a fixed pair of integers $(c_1, c_2)$ with $c_1$ odd, there is a stable reflexive sheaf $\mathcal{E}$ of rank two on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$, equipped with a non-trivial nilpotent map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_{\mathbb{P}^3}$ if and only if $c_1^2 - 4c_2 < 0$.*

*Remark 3.14.* The interested reader may state and prove statements similar to Theorem 1.2 and Corollary 3.12 that involve Lemma 2.6 with $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}$, when $X$ is the three-dimensional smooth quadric $Q_3 \subset \mathbb{P}^4$, using $\omega_Z(-3)$ instead of $\omega_Z(-4)$.

*Proof of Proposition 1.3:.* Since $4c_2(\mathcal{E}(t)) - c_1(\mathcal{E}(t))^2$ is a constant function on $t$, we may reduce to the case $c_1 = 0$. By [9], [10] and [16, Appendix C], we see that $\mathcal{E}$ must be as in Lemma 2.6 and (1) with $r = 2$ and $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}$. Since we have $d := \deg(Z) = c_2(\mathcal{E})$, so we get $c_2(\mathcal{E}) = 0$ if and only if $Z = \emptyset$, i.e. $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^3}^{\oplus 2}$. For the conclusion, it is sufficient to exclude the Chern numbers $c_2$ with $1 \leq c_2 \leq 8$. If such $\mathcal{E}$ exists, then $Z$ is a locally complete intersection and $\omega_Z \cong \mathcal{O}_Z(-4)$. By the duality we have $2\chi(\mathcal{O}_Z) = \deg(\omega_Z) = -4d$, i.e. $\chi(\mathcal{O}_Z) = -2d$.

Macaulay proved that a polynomial $q(t)$ is the Hilbert function of a curve of degree $d$ in some $\mathbb{P}^n$, not necessarily locally a complete intersection, if and only if there is a non-negative integer $\alpha$ such that

$$q(t) = \sum_{i=0}^{d-1}(t+i-i) + \alpha = dt - (d-2)(d-3)/2 + \alpha;$$

see [9,10,16]; for locally Cohen-Macaulay space curves, one can also use [3]. If $p(t)$ is the Hilbert polynomial of the scheme $Z$, then we have $\chi(\mathcal{O}_Z) = p(0)$ and so $-(d-2)(d-3)/2 \leq -2d$, i.e. $(d-2)(d-3) \geq 4d$. But it is false if $1 \leq d \leq 8$. □

**Proposition 3.15.** *For a fixed pair of integers* $(c_1, c_2)$ *with* $c_1$ *odd and* $4c_2 - c_1^2 \leq 28$, *there is no pair* $(\mathcal{E}, \Phi)$, *where* $\mathcal{E}$ *is a stable vector bundle of rank two on* $\mathbb{P}^3$ *with* $c_1(\mathcal{E}) = c_1$ *and* $c_2(\mathcal{E}) = c_2$, *and* $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_{\mathbb{P}^3}$ *is a non-trivial nilpotent map.*

*Proof.* Since $c_1$ is odd, we get that $c_2$ is even by [12, Corollary 2.2]. As in the proof of Proposition 1.3 we first reduce to the case $c_1 = 1$ and then use that $\omega_Z \cong \mathcal{O}_Z(-3)$, implying $2\chi(\mathcal{O}_X) = 3c_2$, to exclude the cases $c_2 \in \{2, 4, 6\}$ by the inequality $(c_2 - 2)(c_2 - 3) \geq 3c_2$. □

### 3.3. *Case* $\mathrm{Num}(X) \cong \mathbb{Z}$

Now we drop the main assumption on $\mathrm{Pic}(X)$; let $\mathrm{Num}(X)$ be the quotient of $\mathrm{Pic}(X)$ by numerical equivalence. Note that if $\mathrm{Num}(X) \cong \mathbb{Z}$, then the notion of (semi)stability does not depend on the choice of a polarization. For $\mathcal{L} \in \mathrm{Pic}(X)$ we call $\deg(\mathcal{L})$ the numerical class of $\mathcal{L}$.

**Theorem 3.16.** *Assume that* $\mathrm{Num}(X) \cong \mathbb{Z}$ *and that* $X \neq \mathbb{P}^n$. *If* $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ *is a nilpotent map for a stable reflexive sheaf of rank two on* $X$, *then we have* $\Phi = 0$.

*Proof.* Assume $\Phi \neq 0$ and then $\mathcal{L} := \ker(\Phi)$ is a rank one saturated subsheaf of $\mathcal{E}$. Set $\mathcal{F} := \mathcal{E} \otimes \mathcal{L}^\vee$. Since $\mathcal{L}$ is saturated in $\mathcal{E}$, $\mathcal{F}$ fits in an exact sequence (1) with $r = 2$. Since $\mathcal{E}$ is stable, we have $\deg(\mathcal{A}) > 0$ and so $\mathcal{A}$ is ample. Call $\Psi : \mathcal{F} \to \mathcal{F} \otimes T_X$ the non-zero nilpotent map obtained from $\Phi$. Since $\Psi \circ \Psi = 0$, we have $\Psi(\mathcal{F}) \subset u(\mathcal{O}_X) \otimes T_X \cong T_X$. Thus $\Psi$ induces a non-zero map $\mathcal{A} \to T_X$. Since $\mathcal{A}$ is ample, we have $X = \mathbb{P}^n$ by [23], a contradiction. □

## 4. Arbitrary Picard groups

Now we drop the assumption $\mathrm{Pic}(X) \cong \mathbb{Z}$, but we fix a very ample line bundle $\mathcal{H} \cong \mathcal{O}_X(1)$ on $X$ and we use $\mathcal{H}$ to check the slope-(semi)stability of sheaves on $X$. We use that $\mathcal{O}_X(1)$ is very ample only to guarantee that $\Omega_X^1(2)$ is spanned. For any torsion-free sheaf of rank two on $X$, define $z_{\mathcal{E}} = z_{\mathcal{E},\mathcal{H}}$ to be

$$\max\{z \in \mathbb{Z} \mid H^0(\mathcal{E}(-z)) \text{ has a section not vanishing on a divisor of } X\}.$$

Then we have an exact sequence

$$0 \to \mathcal{O}_X(z) \to \mathcal{E} \to \mathcal{I}_Z \otimes \det(\mathcal{E})(-z) \to 0 \qquad (10)$$

with $z = z_{\mathcal{E},\mathcal{H}}$ and $Z \subset X$ of codimension 2. The integer $\rho_{2,\mathcal{H}}(\mathcal{E})$ is the minimal integer $t$ such that $h^0(\mathcal{I}_Z \otimes \det(\mathcal{E})(t-z)) > 0$ for some (10). Recall that $x_{\mathcal{E}}$ or $x_{\mathcal{E},\mathcal{H}}$ was defined to be the only integer $x$ such that $H^0(\mathcal{E}(-x)) \neq 0$ and $H^0(\mathcal{E}(-x-1)) = 0$. The following result is an adaptation of Lemma 3.6.

**Lemma 4.1.** *Let $\mathcal{E}$ be a reflexive sheaf of rank two on $X$. For $a \in \mathbb{Z}$ such that*

$$a < \min\{\rho_{2,\mathcal{H}}(\mathcal{E}) - z_{\mathcal{E}}, \max\{-x_{\mathcal{E}^\vee} - z_{\mathcal{E}}, -x_{\mathcal{E}} - x_{\det(\mathcal{E})} - z_{\mathcal{E}} - 1\}\},$$

*we have $h^0(\mathcal{E}nd(\mathcal{E})(a)) = h^0(\mathcal{O}_X(a))$.*

*Proof.* Since $\mathcal{O}_X$ is a factor of $\mathcal{E}nd(\mathcal{E})$, we have $h^0(\mathcal{E}nd(\mathcal{E})(a)) \geq h^0(\mathcal{O}_X(a))$ and so it is sufficient to prove the inequality $h^0(\mathcal{E}nd(\mathcal{E})(a)) \leq h^0(\mathcal{O}_X(a))$.

Set $z := z_{\mathcal{E},\mathcal{H}}$ and assume that $\mathcal{E}$ fits in the exact sequence (10) computing the integer $\rho_{2,\mathcal{H}}(\mathcal{E})$. For a fixed $f \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}(a))$, let

$$f_1 : \mathcal{E} \to \mathcal{I}_Z \otimes \det(\mathcal{E})(-z+a)$$

be the map obtained by composing $f$ with the map $\mathcal{E}(a) \to \mathcal{I}_Z \otimes \det(\mathcal{E})(-z+a)$ twisted from (10) with $\mathcal{O}_X(a)$. Since $a < \rho_{2,\mathcal{H}}(\mathcal{E}) - z_{\mathcal{E}}$, we have $f_1(\mathcal{O}_X(z)) = 0$ and so $f$ induces

$$f_2 : \mathcal{I}_Z \otimes \det(\mathcal{E})(-z) \to \mathcal{I}_Z \otimes \det(\mathcal{E})(-z+a).$$

Now take $g \in H^0(\mathcal{O}_X(a))$ inducing $f_2$ and let $\gamma : \mathcal{E} \to \mathcal{E}(a)$ be the map obtained by the multiplication by $g$. Then it is enough to prove that $f = \gamma$. Taking $f - \gamma$ instead of $f$, we reduce to the case $g = 0$ and in this case we need to prove that $f = 0$. From the assumption that $g = 0$, we have $f(\mathcal{E}) \subseteq \mathcal{O}_X(z+a)$, and so $f = 0$ if $-x_{\mathcal{E}^\vee} > z + a$. Note that $\mathcal{E}$ is reflexive of rank two and so we have $\mathcal{E}^\vee \cong \mathcal{E} \otimes \det(\mathcal{E})^\vee$. Thus $f : \mathcal{E} \to \mathcal{O}_X(z+a)$ is induced by a unique $b \in H^0(\mathcal{E}(z+a) \otimes \det(\mathcal{E})^\vee)$. If $z + a < -x_{\mathcal{E}} - x_{\det(\mathcal{E})^\vee} - 1$, we have $b = 0$, because $h^0(\mathcal{E}(-x_{\mathcal{E}} - 1)) = 0$ and $h^0(\det(\mathcal{E})^\vee(-x_{\det(\mathcal{E})^\vee} - 1)) = 0$. □

**Proposition 4.2.** *Let $\mathcal{E}$ be a reflexive sheaf of rank two on $X \neq \mathbb{P}^n$ with $\rho_{2,\mathcal{H}}(\mathcal{E}) \geq 3$ and either $-x_{\mathcal{E}^\vee} - z_{\mathcal{E}}$ or $-x_{\det(\mathcal{E})} + 1$ at least two. Then any trace-zero co-Higgs field for $\mathcal{E}$ is identically zero.*

*Proof.* Basically the same argument in the proof of Proposition 3.7 works with Lemma 3.6 replaced by Lemma 4.1. Since $T_X$ is a subsheaf of $\mathcal{O}_X(2)^{\oplus N}$ for $N = h^0(\Omega_X^1(2))$, any map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$ induces $N$ elements $\Phi_i : \mathcal{E} \to \mathcal{E}(2)$ with $i = 1, \dots, N$. Then by Lemma 4.1 each $\Phi_i$ is induced by $f_i \in H^0(\mathcal{O}_X(2))$. Now by composing the trace map of $\Phi$ with the inclusion $T_X \subset \mathcal{O}_X(2)^{\oplus N}$, we also get $N$ elements $g_i \in H^0(\mathcal{O}_X(2))$. We know that $2f_i = g_i$ for each $i$. If $\Phi$ is trace-free, then we get $g_i = 0$ and so $f_i =$ for each $i$. Thus $\Phi$ is trivial. □

## References


[1] Ballico, E., Huh, S.: A note on co-Higgs bundles. Taiwanese J. Math. **2**, 267–281 (2017)

[2] Barth, W.: Moduli of vector bundles on the projective plane. Invent. Math. **42**, 63–91 (1977)

[3] Beorchia, V.: Bounds for the genus of space curves. Math. Nachr. **184**, 59–71 (1997)

[4] Bolondi, G.: Smoothing curves by reflexive sheaves. Proc. Am. Math. Soc. **102**(4), 797–803 (1988)

[5] Brun, J.: Les fibrés de rang deux sur $\mathbf{P}^2$ et leurs sections. Bull. Soc. Math. Fr. **108**(4), 457–473 (1980)

[6] Colmenares, A.V.: Semistable rank 2 co-Higgs bundles over Hirzebruch surfaces, Ph.D. thesis, Waterloo, Ontario (2015)

[7] Colmenares, A.V.: Moduli spaces of semistable rank-2 co-Higgs bundles over P1*P1. Q. J. Math. **68**(4), 1139–1162 (2017)

[8] Corrêa, M.: Rank two nilpotent co-Higgs sheaves on complex surface. Geom. Dedicata **183**(1), 25–31 (2016)

[9] Gotzmann, G.: Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes. Math. Z. **158**, 61–70 (1978)

[10] Gotzmann, G.: Einige einfach-zusammenhängende Hilbertschemata. Math. Z. **180**, 291–305 (1982)

[11] Gualtieri, M.: Generalized complex geometry. Ann. Math. **174**(1), 75–123 (2011)

[12] Hartshorne, R.: Stable vector bundles of rank 2 on $\mathbb{P}^3$. Math. Ann. **238**, 229–280 (1978)

[13] Hartshorne, R.: Stable reflexive sheaves. Math. Ann. **254**(2), 121–176 (1980)

[14] Hitchin, N.: Generalized Calabi–Yau manifolds. Q. J. Math. **54**(3), 281–308 (2003)

[15] Hulek, K.: Stable rank-2 vector bundles on $\mathbf{P}_2$ with $c_1$ odd. Math. Ann. **242**(3), 241–266 (1979)

[16] Iarrobino, A., Kanev, V.: With an appendix by A. Iarrobino and S. L. Kleiman, Power sums, Gorenstein algebras, and determinantal loci. In: Lecture Notes in Mathematics, vol. 1721. Appendix C by A. Iarrobino and S. L. Kleiman. Springer, Berlin (1999)

[17] Le Potier, J.: Fibrés stables de rang 2 sur $\mathbf{P}_2(\mathbb{C})$. Math. Ann. **241**(3), 217–256 (1979)

[18] Maruyama, M.: Stable vector bundles on an algebraic surface. Nagoya Math. J. **58**, 25–68 (1975)

[19] Rayan, S.: Geometry of co-Higgs bundles, Ph.D. thesis (2011)

[20] Rayan, S.: Co-Higgs bundles on $\mathbb{P}^1$. N. Y. J. Math. **19**, 925–945 (2013)

[21] Rayan, S.: Constructing co-Higgs bundles on $\mathbb{CP}^2$. Q. J. Math. **65**(4), 1437–1460 (2014)

[22] Simpson, C.: Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. J. Am. Math. Soc. **1**(4), 867–918 (1988)

[23] Wahl, J.: A cohomological characterization of $\mathbb{P}_n$. Invent. Math. **72**, 315–322 (1983)