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Static perfect fluid spacetime with half conformally flat spatial factor

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Abstract. The aim of this paper is to investigate the static perfect fluid spacetime $M^4 \times_f \mathbb{R}$ such that (M^4, g) is a half conformally flat Riemannian manifold. We prove that (M^4, g) is, in fact, locally isometric to a warped product manifold $I \times_\phi N^3$ where $I \subset \mathbb{R}$ and N^3 is a space form. Consequently, we make an analysis of the Fischer-Marsden conjecture for a 4-dimensional Riemannian manifold.

1. Introduction

The Einstein equation with perfect fluid as a matter field is given by

$$\hat{R}ic - \frac{\hat{R}}{2}\hat{g} = (\mu + \rho)\eta \otimes \eta + \rho\hat{g}, \tag{1.1}$$

where $\hat{R}ic$ and $\hat{R}$ are, respectively, the Ricci tensor and the scalar curvature for the metric $\hat{g}$. Moreover, η is a 1-form with $\hat{g}(\eta, \eta) = -1$ whose associated vector field represents the flux of the fluid, μ, ρ are smooth functions, namely the energy density and pressure, respectively (see [9–11]). The equation (1.1) was presented by Einstein in 1915, and shows the relationship between matter and spacetime. Some solutions of (1.1) provide models for galaxies, stars and black holes (cf. [9]). Therefore, it is natural to study Einstein's equation. One way to do this is to consider some special cases of $\hat{g}$.

In this work we will explore the static spacetime $(\widehat{M}^{n+1}, \hat{g}) = M^n \times_f \mathbb{R}$ such that

$$\hat{g} = -f^2dt^2 + g \tag{1.2}$$

where (M^n, g) is an open, connected and oriented Riemannian manifold, and $f : M^n \to (0, +\infty)$ is a smooth warped function.

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Therefore, from (1.1) and (1.2) we know that the *static perfect fluid equation* is equivalent to

$$f\mathring{Ric} = \mathring{\nabla}^2 f \tag{1.3}$$

and

$$2\mu = R \quad \text{and} \quad f\rho = \frac{n-1}{n}\left[\Delta f - \frac{(n-2)}{2(n-1)}Rf\right],$$

where $\mathring{Ric} = Ric - \frac{R}{n}g$ and $\mathring{\nabla}^2 f = \nabla^2 f - \frac{\Delta f}{n}g$ are, respectively, the Ricci and the Hessian traceless tensors, Δ is the Laplacian and R is the scalar curvature for g (cf. [10,11]).

An n-dimensional Riemannian manifold with $n \geq 4$ is locally conformally flat if and only if the Weyl tensor W is equal to zero. For $n = 3$ the Weyl tensor is always zero. Then, a 3-dimensional Riemannian manifold is locally conformally flat if and only if the Cotton tensor C vanishes. A paper published by Kobayashi and Obata [11] reveals that a complete locally conformally flat Riemannian manifold (M^n, g, f) in which (g, f) satisfies (1.3) has the property that, for a connected component M_0 of the open submanifold $\{\nabla f \neq 0\} \subset M$, M_0 is isometric to the warped product $I \times_\phi N$ of (I, dr^2) and $(N, \overline{g})$, i.e., $g|_{M_0} = dr^2 + \phi(r)^2\overline{g}$, where I is an open interval in $\mathbb{R}$, ϕ is a positive function on I, and $(N, \overline{g})$ is an $(n-1)$-dimensional complete space with constant sectional curvature (cf. Lemma B in [10]).

Here, we will prove that a half locally conformally flat (i.e., $W^{\pm} = 0$) complete Riemannian manifold satisfying the static perfect fluid equation (1.3) will be, in fact, locally conformally flat (see Section 2.1). A four oriented Riemannian manifold with compatible complex structures has a natural almost complex structure which is integrable if, and only if, $W^{\pm}$ vanishes (cf. Chapter 13 in [3]). Moreover, we provided a different proof that M_0 is isometric to the warped product mentioned earlier. It is worth to mention that in [1] they consider the metric (1.3) under the additional conditions that M^4 is compact with constant scalar curvature such that

$$(n-1)\Delta f + Rf = -n.$$

Since we are considering that (M^4, g) satisfies Equation (1.3) without any condition on the Laplacian (cf. Equation (2.2) in [1] or Equation (1.5) below) and that R is not necessarily constant, our result is quite different from the result provided in [1].

In [11], the following result was demonstrated for a locally conformally flat Riemannian manifold satisfying (1.3). In this paper, we prove such result for $n = 4$ and metric half locally conformally flat.

Without further ado, we state our main result.

Theorem 1. *Let (M^4, g, f) be a complete half locally conformally flat Riemannian manifold satisfying (1.3). Then, (M^4, g) is locally conformally flat. Moreover, for any connected component M_0 of the open submanifold $\{\nabla f \neq 0\} \subset M$, M_0 is isometric to the warped product*

$$(I, dr^2) \times_\phi (N^3, \bar{g}),$$

where $\phi(r) = e^{\frac{1}{3}\int_{r_0}^{r} H(t)dt}$ *and* H *is the mean curvature for the level set* $\Sigma_{r_0} = f^{-1}(r_0)$ *for any regular value* r_0 *of the function* f*. Furthermore,* I *is an open interval in* $\mathbb{R}$ *and* $(N, \overline{g})$ *is a* 3*-dimensional Einstein manifold (i.e., is a space form).*

Now, we will consider a special case of (1.3). Let (M^n, g) be a compact Riemannian manifold, for a while. Remember that from the linearization of the scalar curvature equation (cf. [3,7,10]) we can take from the Stokes formula that the adjoint operator of dR_g has non trivial kernel. Hence, we can conclude that

$$-g\Delta f + \nabla^2 f - fRic = 0. \tag{1.4}$$

Contracting (1.4) we obtain

$$-\Delta f = \frac{Rf}{n-1}. \tag{1.5}$$

Moreover, taking the divergent of (1.4), from the contracted Bianchi identity it is easy to see that R is constant (cf. [7]). And if we consider M compact, we have from (1.5) that $R \geq 0$.

So, from (1.3) and (1.5) we obtain (1.4) (cf. [10]). It means that (1.4) is equivalent to (1.3) with constant energy density and under the additional condition (1.5). Thus, the next result is a special case of Theorem 1.

Theorem 2. *Let* (M^4, g, f) *be a complete half locally conformally flat Riemannian manifold satisfying* (1.4)*. Then,* (M^4, g) *is locally conformally flat. In addition, for any connected component* M_0 *of the open submanifold* $\{\nabla f \neq 0\} \subset M$*,* M_0 *is isometric to the warped product*

$$(I, ds^2) \times_{\phi} (N^3, \bar{g}),$$

where I *is an open interval in* $\mathbb{R}$ *and* $(N, \overline{g})$ *is a* 3*-dimensional Einstein manifold (i.e., is a space form).*

An immediately consequence of Theorems 2 and 3.1 in [10] is that a complete half conformally flat Riemannian manifold satisfying (1.4) is isometric to a space form or it is a special case demonstrated in Section 3 in [10]. Further, from Theorems 1 and 4.1 in [11] we have that if the spatial factor M^4 of (1.2) is half conformally flat and Einstein, then $\widehat{M}^5$ is conformally flat.

Further, from Theorem 1 and Theorem 4.1 in [11] we have that if the spatial factor M^4 of (1.2) is half conformally flat and Einstein, then $\widehat{M}^5$ is conformally flat

2. Background

2.1. Four Dimensional Manifolds

The world of 4-dimensional manifolds is quite unusual. This environment is bigger than any other set of n-dimensional spaces. For example, some 4-dimensional manifolds have no smooth structure and others admit a variety of such structures.

Furthermore, we remember that $\mathbb{C}P2$ is an example of manifold that is half conformally flat but it is not conformally flat. On the other hand, any 4-dimensional conformally flat manifold is half conformally flat. This shows us that the set of the half conformally spaces is larger that the set of conformally flat spaces (see more in [3,6]).

In what follows, M^4 will denote an oriented 4-dimensional manifold and g a Riemannian metric on M^4. As it was previously stated, 4-manifolds are fairly special. For instance, following the notations used in [6], given any local orthogonal frame $\{e_1, e_2, e_3, e_4\}$ in an open set of M^4 with dual basis $\{e^1, e^2, e^3, e^4\}$, there is a unique bundle morphism $*$ called *Hodge star* (acting on bivectors), where

$$*(e^1 \wedge e^2) = e^3 \wedge e^4.$$

This implies that $*$ is an involution, that is, $*^2 = Id$. In particular, the bundle of 2-forms in a 4-dimensional oriented Riemannian manifold can be invariantly decomposed as a direct sum $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. From this, it follows that the Weyl tensor W is an endomorphism of $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ for which

$$W = W^+ \oplus W^-.$$

A four dimensional manifold is half conformally flat if it is self or anti-self dual, i.e., if $W^- = 0$ or $W^+ = 0$.

Observe that $dim_{\mathbb{R}}(\Lambda^2) = 6$ and $dim_{\mathbb{R}}(\Lambda^\pm) = 3$. Moreover, these spaces are generated by

$$\Lambda^+ = span\left\{\frac{e^1 \wedge e^2 + e^3 \wedge e^4}{\sqrt{2}}, \frac{e^1 \wedge e^3 + e^4 \wedge e^2}{\sqrt{2}}, \frac{e^3 \wedge e^2 + e^4 \wedge e^1}{\sqrt{2}}\right\} \quad (2.1)$$

and

$$\Lambda^- = span\left\{\frac{e^1 \wedge e^2 - e^3 \wedge e^4}{\sqrt{2}}, \frac{e^1 \wedge e^3 - e^4 \wedge e^2}{\sqrt{2}}, \frac{e^3 \wedge e^2 - e^4 \wedge e^1}{\sqrt{2}}\right\}. \quad (2.2)$$

Therefore, the bundles Λ^+ and Λ^- carry natural orientations such that the bases (2.1) and (2.2) are both positive-oriented.

Thus, from the Hodge star we have $W^\pm = \frac{1}{2}(W \pm W*)$, i.e.,

$$W^\pm_{pqls} = \frac{1}{2}\big(W_{pqls} \pm W_{\overline{p}\,\overline{q}ls}\big),$$

where $(\overline{p}\,\overline{q})$ stands for the dual of $(p\,q)$, that is, $(\overline{p}\,\overline{q}\,l\,s) = \sigma(1234)$ for some even permutation σ in the set $\{1, 2, 3, 4\}$. For more details refer to [6] and [12].

2.2. The Warped Product Structure

Here we state some well known warped products formulas that we will need to prove our main results.

Consider the warped product manifold

$$(M^n, g) = (I, dr^2) \times_\phi (N^{n-1}, \bar{g}), \tag{2.3}$$

where $ds^2 = dr^2 + \phi(r)^2\bar{g}$. Choose any local coordinates system $\theta = (\theta_2, \dots, \theta_n)$ on N^{n-1}, and pick $(x_1, x_2, \dots, x_n) = (r, \theta_2, \dots, \theta_n)$. In what follows a, b, c, d range from 2 to n. Also, curvature tensors with bar are the curvature tensors of $(N, \bar{g})$. It is well known that the Riemann curvature tensor of (M^n, g) is given by (cf. [3,4])

$$R_{1a1b} = -\phi\phi''\bar{g}_{ab}, \quad R_{1abc} = 0 \tag{2.4}$$

and

$$R_{abcd} = \phi^2\bar{R}_{abcd} - (\phi\phi')^2(\bar{g}_{ac}\bar{g}_{bd} - \bar{g}_{ad}\bar{g}_{bc}). \tag{2.5}$$

From the above equations, the Ricci tensor formulas of (M^n, g) and $(N^{n-1}, \bar{g})$ are related by

$$R_{11} = -(n-1)\frac{\phi''}{\phi}, \quad R_{1a} = 0 \quad (2 \le a \le n)$$

and

$$R_{ab} = \bar{R}_{ab} - [(n-2)(\phi')^2 + \phi\phi'']\bar{g}_{ab} \quad (2 \le a, b \le n).$$

Further,

$$R = \phi^{-2}\bar{R} - (n-1)(n-2)\left(\frac{\phi'}{\phi}\right)^2 - 2(n-1)\frac{\phi''}{\phi}.$$

2.3. Fundamental Equations

In a system of local coordinates, (1.3) is equivalent to

$$f\left(R_{ij} - \frac{R}{n}g_{ij}\right) = \nabla_i\nabla_j f - \frac{\Delta f}{n}g_{ij}. \tag{2.6}$$

It is easy to see from (2.6) that

$$f\mathring{R}_{ij}\nabla^j f = \nabla_i\nabla_j f\nabla^j f - \frac{\Delta f}{n}g_{ij}\nabla^j f.$$

Then, since $\nabla_i\nabla_j f\nabla^j f = \frac{1}{2}\nabla_i|\nabla f|^2$ and $g_{ij}\nabla^j f = \nabla_i f$ we get:

$$f\mathring{R}_{ij}\nabla^j f = \frac{1}{2}\nabla_i|\nabla f|^2 - \frac{\Delta f}{n}\nabla_i f. \tag{2.7}$$

Taking now the covariant derivative of (2.6) we have

$$\nabla_i f \mathring{R}_{jk} + f\nabla_i \mathring{R}_{jk} = \nabla_i\nabla_j\nabla_k f - \frac{1}{n}\nabla_i \Delta f g_{jk} \tag{2.8}$$

and

$$\nabla_j f \mathring{R}_{ik} + f\nabla_j \mathring{R}_{ik} = \nabla_j\nabla_i\nabla_k f - \frac{1}{n}\nabla_j \Delta f g_{ik}. \tag{2.9}$$

Then, subtracting (2.8) and (2.9) we obtain

$$\begin{aligned} f(\nabla_i \mathring{R}_{jk} - \nabla_j \mathring{R}_{ik}) + (\mathring{R}_{jk}\nabla_i f - \mathring{R}_{ik}\nabla_j f) &= \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f \\ &+\frac{1}{n}(\nabla_j(\Delta f)g_{ik} - \nabla_i(\Delta f)g_{jk}). \end{aligned} \tag{2.10}$$

Then, using the Ricci equation

$$\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = R_{ijkl}\nabla^l f$$

in (2.10) we can show that

$$\begin{aligned} f(\nabla_i \mathring{R}_{jk} - \nabla_j \mathring{R}_{ik}) + (\mathring{R}_{jk}\nabla_i f - \mathring{R}_{ik}\nabla_j f) &= R_{ijkl}\nabla^l f \\ &+\frac{1}{n}(\nabla_j(\Delta f)g_{ik} - \nabla_i(\Delta f)g_{jk}). \end{aligned} \tag{2.11}$$

Contracting Equation (2.11) over i and k and remembering that $\mathring{R}ic$ is trace free, we have

$$\begin{aligned} fg^{ik}\nabla_i \mathring{R}_{jk} + \mathring{R}_{jl}\nabla^l f &= R_{jl}\nabla^l f \\ &+\nabla_j(\Delta f) - \frac{1}{n}\nabla_j(\Delta f). \end{aligned}$$

Considering the contracted second Bianchi identity $g^{ik}\nabla_i \mathring{R}_{jk} = \frac{n-2}{2n}\nabla_j R$ we obtain from the above equation

$$\frac{1}{2(n-1)}[(n-2)f\nabla_j R - 2R\nabla_j f] = \nabla_j(\Delta f). \tag{2.12}$$

The next result has the same conclusion of Lemma 2 in [1]. However, the Miao-Tam metric has constant scalar curvature. Here, we are considering that the scalar curvature is not necessarily constant. To prove the following Lemma, Equation (2.12) will be essential.

Lemma 1. *Let (M^n, g, f) be a Riemannian manifold satisfying (1.3). Then,*

$$\begin{aligned} fC_{ijk} &= W_{ijkl}\nabla^l f - \frac{1}{n-2}(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) \\ &+\frac{n-1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{R}{n-2}(\nabla_i f g_{jk} - \nabla_j f g_{ik}), \end{aligned}$$

where C and W are, respectively, the Cotton tensor and the Weyl tensor.

Proof. Then, from (2.11) and (2.12) we obtain

$$f(\nabla_i \mathring{R}_{jk} - \nabla_j \mathring{R}_{ik}) + (\mathring{R}_{jk}\nabla_i f - \mathring{R}_{ik}\nabla_j f) = R_{ijkl}\nabla^l f + \frac{n-2}{2n(n-1)} f(\nabla_j R g_{ik} - \nabla_i R g_{jk}) + \frac{R}{n(n-1)}(\nabla_i f g_{jk} - \nabla_j f g_{ik}).$$

Further, since $\mathring{R}ic = Ric - \frac{R}{n} g$ we have

$$f(\nabla_i R_{jk} - \nabla_j R_{ik}) + (R_{jk}\nabla_i f - R_{ik}\nabla_j f) = R_{ijkl}\nabla^l f + \frac{1}{2(n-1)} f(\nabla_i R g_{jk} - \nabla_j R g_{ik}) + \frac{R}{(n-1)}(\nabla_i f g_{jk} - \nabla_j f g_{ik}). \quad (2.13)$$

Moreover, we know the Cotton tensor is given by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}). \quad (2.14)$$

Then, from (2.14), (2.13) becomes

$$f C_{ijk} = R_{ijkl}\nabla^l f + (R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{R}{n-1}(\nabla_i f g_{jk} - \nabla_j f g_{ik}). \quad (2.15)$$

Remember the Weyl formula

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}\big(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\big) - \frac{R}{(n-1)(n-2)}\big(g_{jl}g_{ik} - g_{il}g_{jk}\big). \quad (2.16)$$

Therefore, from (2.16) we have

$$R_{ijkl}\nabla^l f = W_{ijkl}\nabla^l f + \frac{1}{n-2}\big(R_{ik}\nabla_j f + R_{jl}\nabla^l f g_{ik} - R_{il}\nabla^l f g_{jk} - R_{jk}\nabla_i f\big) - \frac{R}{(n-1)(n-2)}\big(\nabla_j f g_{ik} - \nabla_i f g_{jk}\big). \quad (2.17)$$

From (2.15) and (2.17) we get

$$f C_{ijk} = W_{ijkl}\nabla^l f - \frac{1}{n-2}(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) + \frac{n-1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{R}{n-2}(\nabla_i f g_{jk} - \nabla_j f g_{ik}).$$

□

Namely,

$$T_{ijk} = -\frac{1}{n-2}(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) + \frac{n-1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{R}{n-2}(\nabla_i f g_{jk} - \nabla_j f g_{ik}). \quad (2.18)$$

That is, from Lemma 1 we have

$$fC_{ijk} = W_{ijkl}\nabla^l f + T_{ijk}. \tag{2.19}$$

It is important to highlight that T is skew-symmetric and trace free, i.e.,

$$T_{ijk} = -T_{jik}$$

and

$$g^{ij}T_{ijk} = g^{jk}T_{ijk} = g^{ik}T_{ijk} = 0.$$

The demonstration for the next result follows the same steps of [2,5].

Lemma 2. *Let (M^4, g, f) be a half conformally flat Riemannian manifold satisfying (1.3) in a neighborhood of a regular point q. Then, $T|_q \equiv 0$.*

Proof. Consider from now on an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ where e_1, e_2, e_3, e_4 are eigenvectors of Ric at q and $R_{ij}(q) = R_{ii}\delta_{ij}(q)$, where R_{ii} represents the eigenvalue of e_i.

Let's now prove that $T|_q = 0$. From the skew-symmetry property, it is easy to see that $T_{iik} = 0$ for all $i, k \in \{1, 2, 3, 4\}$. Furthermore, from (2.18) we can see that $T_{ijk} = 0$ for all different $i,\ j,\ k \in \{1, 2, 3, 4\}$.

Now, remember that $C_{ijk} = -\frac{n-2}{n-3}\nabla^l W_{ijkl}$. Thus,

$$C_{ijk} = -2\nabla^l W_{ijkl}.$$

Then, from (2.19) we have

$$T_{ijk} = -\left[2f\nabla^l W_{ijkl} + W_{ijkl}\nabla^l f\right].$$

On the other hand, we already know that

$$2W^+ = W + W*,$$

where $*$ is the Hodge star. By hypothesis, we have that $W^+ = 0$, it implies

$$W_{12kl} + W_{34kl} = 0; \quad W_{13kl} + W_{42kl} = 0 \quad \text{and} \quad W_{14kl} + W_{23kl} = 0. \tag{2.20}$$

Hence, we can infer

$$T_{12k} + T_{34k} = -\left[2f\nabla^l(W_{12kl} + W_{34kl}) + (W_{12kl} + W_{34kl})\nabla^l f\right] = 0.$$

Analogously, we obtain

$$T_{13k} + T_{42k} = 0 \quad \text{and} \quad T_{14k} + T_{23k} = 0.$$

Then, we have $T_{iji} = 0$ for all $i \neq j \in \{1, 2, 3, 4\}$. Finally, we have proof that $T|_q = 0$. □

Our following result can be compared with other well known results such as [1,2,5].

Proposition 1. *Let (M^4, g, f) be a half locally conformally flat Riemannian manifold satisfying (1.3) where Σ_c is a connected component of $f^{-1}(c) = \{q \in M^4;\ f(q) = c\}$, the level surface with respect to regular value c of f. Then, for any local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ with $e_1 = \frac{\nabla f}{|\nabla f|}$ and $\{e_2, e_3, e_4\}$ tangent to Σ_c, we have:*

(1) *∇f is an eigenvector of the Ricci operator. Furthermore,*
  (i) *$Ric(e_1, e_1) = R_{11}$;*
  (ii) *$Ric(e_1, e_a) = 0$, $a = 2, 3, 4$;*
  (iii) *$Ric(e_a, e_b) = R_{aa}\delta_{ab}$, $a, b = 2, 3, 4$, where $R_{11} = \ldots = R_{44} = \lambda$ or $R_{22} = R_{33} = R_{44} = \lambda$ and R_{11} has multiplicity 1;*

(2) *$|\nabla f|^2$ is constant on Σ_c;*
(3) *The second fundamental form of Σ_c is*

$$h_{ab} = \frac{H}{3} g_{ab},$$

where H is the constant mean curvature of Σ_c;
(4) *(M^4, g) is locally conformally flat manifold, i.e., $W = 0$.*

Proof. Consider an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ diagonalizing Ric at $q \in \Sigma_c$ with associated eigenvalues $R_{kk}(q)$, $k = 1, \ldots, 4$, respectively. That is, $R_{ij}(q) = R_{ii}\delta_{ij}(q)$. From Lemma 2 we can see that $T(q) = 0$. Then, from (2.18), $T_{ijj} = 0$, i.e.,

$$\frac{1}{2}\nabla_j f[R_{jj} + 3R_{ii} - R] = 0, \quad \forall i \neq j. \tag{2.21}$$

Assume that, for a fixed j, $\nabla_j f(p) \neq 0$ and $\nabla_i f(p) = 0$ for all $i \neq j$; $i, j \in \{1, 2, 3, 4\}$. Then, we have $Ric(\nabla f) = R_{jj}\nabla f$, i.e., ∇f is a eigenvector for Ric. Moreover, from (2.21) we know that R_{jj} has multiplicity 1 and R_{ii} has multiplicity 3, for all $i \neq j$. In the other case, if $\nabla_i f \neq 0$ for at least two distinct directions, from (2.21) we concluded that $\lambda = R_{11} = \cdots = R_{44}$ and we also have ∇f an eigenvector for Ric (see this discussion in [2]).

Therefore, in any case we perceive that ∇f is an eigenvector for Ric at q. From the above analysis we can take $\{e_1 = \frac{\nabla f}{|\nabla f|}, e_2, e_3, e_4\}$ an orthonormal frame for Σ_c diagonalizing Ric at $q \in \Sigma_c$.

We know that ∇f is an eigenvector for Ric. Then, (2.7) gives

$$f\mathring{R}ic(e_a, \nabla f) = \frac{1}{2}\nabla_a|\nabla f|^2 - \frac{\Delta f}{4}\nabla_a f.$$

Hence, $|\nabla f|^2$ is constant on Σ_c.

By definition of the second fundamental form we have

$$h_{ab} = \frac{\nabla_a\nabla_b f}{|\nabla f|} \quad \text{in} \quad \Sigma_c.$$

Now, since $R_{ab} = \lambda g_{ab}$ for all $a, b \in \{2, 3, 4\}$, from (1.3) we get

$$\begin{aligned}\nabla_a \nabla_b f &= \left[f \mathring{R}_{ab} + \frac{\Delta f}{4} g_{ab} \right] \\ &= \left[f \left(\lambda g_{ab} - \frac{R}{4} g_{ab} \right) + \frac{\Delta f}{4} g_{ab} \right] \\ &= \left[f \left(\lambda - \frac{R}{4} \right) + \frac{\Delta f}{4} \right] g_{ab}.\end{aligned}$$

Therefore,

$$h_{ab} = \frac{1}{|\nabla f|} \left[f \left(\lambda - \frac{R}{4} \right) + \frac{\Delta f}{4} \right] g_{ab}.$$

Contracting the above equation in a and b we obtain the mean curvature of Σ_c

$$H = \frac{3}{|\nabla f|} \left[f \left(\lambda - \frac{R}{4} \right) + \frac{\Delta f}{4} \right]. \tag{2.22}$$

Which implies that

$$h_{ab} = \frac{H}{3} g_{ab}. \tag{2.23}$$

Contracting the Codazzi equation

$$R_{1cab} = \nabla_a h_{bc} - \nabla_b h_{ac}$$

over c and b gives

$$R_{1a} = \nabla_a (H) - \frac{1}{3} \nabla_a (H) = \frac{2}{3} \nabla_a (H).$$

On the other hand, since $R_{1a} = 0$ we know that H is constant in Σ_c. Then, we can see from the Codazzi equation that $R_{1abc} = 0$. Since the frame that we consider diagonalizes the Ricci tensor, from the Weyl equation (2.16) we have that

$$\begin{aligned}0 = W_{1abc} &+ \frac{1}{2} \big(R_{1b} g_{ac} + R_{ac} g_{1b} - R_{1c} g_{ab} - R_{ab} g_{1c} \big) \\ &- \frac{R}{6} \big(g_{ac} g_{1b} - g_{1c} g_{ab} \big).\end{aligned}$$

Therefore, from the above equation we conclude that $W_{1abc} = 0$, and since $W + W* = 0$ we get from (2.20) that $W = 0$. □

3. Proof of Theorem 1

Now that we are familiarized with Proposition 1 we can move on to Theorem 1.

Proof of Theorem 1:. We started the demonstration saying that there is no open subset Ω of M^4 where $\{\nabla f = 0\}$ is dense. In fact, if f is constant in Ω, since M^4 is complete we have f analytic, which implies f constant everywhere (cf. Lemma 2.5 in [5]). Thus, consider Σ_c a connected component of the level surface $f^{-1}(c)$ (possibly disconnected) where c is any regular value of the function f. Suppose that I is an open interval containing c such that f has no critical points in the open neighborhood $U_I = f^{-1}(I)$ of Σ_c. For sake of simplicity, let U_I be a connected component of $f^{-1}(I)$. Therefore, we can extend smoothly $g|_{U_I}$ to a smooth metric $g|_{M_0}$.

From Proposition 1 we know that $|\nabla f|^2$ is constant on Σ_c. Then, we can express the metric g in the form

$$ds^2 = \frac{1}{|\nabla f|^2}df^2 + g_{ab}(f,\theta)d\theta_a d\theta_b,$$

where $g_{ab}(f,\theta)d\theta_a d\theta_b$ is the induced metric and $(\theta_2, \theta_3, \theta_4)$ is a local coordinate system on Σ_c. Since $|\nabla f|^2$ is a function of f only we can make a change of variables

$$r(x) = \int \frac{df}{|\nabla f|}$$

such that the metric g in U_I can be expressed by

$$ds^2 = dr^2 + g_{ab}(r,\theta)d\theta_a d\theta_b.$$

Let $\nabla r = \frac{\partial}{\partial r}$, then $|\nabla r| = 1$ and $\nabla f = f'(r)\frac{\partial}{\partial r}$ on U_I. Observe that $f'(r)$ does not change sign on U_I. Moreover, we have $\nabla_{\partial r}\partial r = 0$. From (2.23) the second fundamental formula on Σ_c is given by

$$h_{ab} = \frac{\nabla_a \nabla_b f}{|\nabla f|} = \frac{H}{3}g_{ab},$$

where $H = H(r)$, since H is constant in Σ_c.

For what follows, we fix a local coordinates system

$$(x_1, x_2, x_3, x_4) = (r, \theta_2, \theta_3, \theta_4)$$

in U_I, where $(\theta_2, \theta_3, \theta_4)$ is any local coordinates system on the level surface Σ_c. Admit that $a, b, c, \ldots \in \{2, 3, 4\}$, we have

$$h_{ab} = -g(\partial_r, \nabla_a \partial_b) = -g(\partial_r, \Gamma^l_{ab}\partial_l) = -\Gamma^1_{ab}.$$

Now, by definition

$$\Gamma^1_{ab} = \frac{1}{2}g^{11}\left(-\frac{\partial}{\partial r}g_{ab}\right) = -\frac{1}{2}\frac{\partial}{\partial r}g_{ab}.$$

Then,

$$\frac{2}{3}H(r)g_{ab} = \frac{\partial}{\partial r}g_{ab}.$$

Hence, we can infer that

$$g_{ab}(r,\theta) = \phi(r)^2 g_{ab}(r_0,\theta),$$

where $\phi(r) = e^{\frac{1}{3}\left(\int_{r_0}^{r} H(s)ds\right)}$ and the level set $\{r = r_0\}$ corresponds to the connected component Σ_c of $f^{-1}(c)$ (this c was defined at the beginning of the proof).

Moreover, from (2.4) and (2.5) the Weyl tensor W for an arbitrary warped product manifold (2.3) is given by (see [3,4]):

$$\begin{aligned} W_{1a1b} &= -\frac{1}{2}\bar{R}_{ab} + \frac{\bar{R}}{6}\bar{g}_{ab}, \\ W_{1abc} &= 0, \end{aligned} \tag{3.1}$$

and

$$W_{abcd} = \phi^2 \overline{W}_{abcd},$$

where $\overline{W}$ denotes the Weyl tensor of $(N^{n-1}, \bar{g})$. Since, from Proposition 1, the warped product manifold (2.3) is locally conformally flat (i.e., $W = 0$), from (3.1) we obtain

$$\bar{R}_{ab} = \frac{\bar{R}}{3}\bar{g}_{ab}.$$

□

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