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Group actions on 2-categories

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Abstract. We study actions of discrete groups on 2-categories. The motivating examples are actions on the 2-category of representations of finite tensor categories and their relation with the extension theory of tensor categories by groups. Associated to a group action on a 2-category, we construct the 2-category of equivariant objects. We also introduce the G-equivariant notions of pseudofunctor, pseudonatural transformation and modification. Our first main result is a coherence theorem for 2-categories with an action of a group. For a 2-category $\mathcal{B}$ with an action of a group G, we construct a braided G-crossed monoidal category $\mathcal{Z}_G(\mathcal{B})$ with trivial component the Drinfeld center of $\mathcal{B}$. We prove that, in the case of a G-action on the 2-category of representation of a tensor category $\mathcal{C}$, the 2-category of equivariant objects is biequivalent to the module categories over an associated G-extension of $\mathcal{C}$. Finally, we prove that the center of the equivariant 2-category is monoidally equivalent to the equivariantization of a relative center, generalizing results obtained in Gelaki et al. (Algebra Number Theory 3(8):959–990, 2009).

Introduction

The theory of 2-categories appears in a natural way in diverse contexts. For example, it was used by Rouquier to "categorify" certain algebraic objects [23] and appears in topological field theories [6,20]. The theory of representations of 2-categories has been initiated in a series of papers [15–17].

Our motivation for the study of 2-categories comes from the theory of tensor categories. For a tensor category $\mathcal{C}$, a representation of $\mathcal{C}$, or $\mathcal{C}$-module category, is a category $\mathcal{M}$ equipped with an associative action $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ satisfying certain conditions. Given two $\mathcal{C}$-module categories $\mathcal{M}, \mathcal{N}$, the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is the category whose objects are $\mathcal{C}$-module functor between $\mathcal{M}$ and $\mathcal{N}$, and morphisms are $\mathcal{C}$-module natural transformations. The 2-category of (left) $\mathcal{C}$-modules ${}_{\mathcal{C}}\mathrm{Mod}$ has as 0-cells $\mathcal{C}$-module categories, 1-cells $\mathcal{C}$-module functors between them and

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2-cells are $\mathcal{C}$-module natural transformations. This 2-category is a strong invariant of the tensor category $\mathcal{C}$.

Given a 2-category $\mathcal{B}$ and a 2-monad $T : \mathcal{B} \to \mathcal{B}$ on $\mathcal{B}$, in [18], the notion of the *equivariantization* 2-category $\mathcal{B}^T$ was presented. The equivariantization of a 2-category by a group was studied later in [13].

One of the purposes of the paper is to explicitly describe an action of a group G on a 2-category $\mathcal{B}$, and describe all ingredients of the resulting equivariantization 2-category $\mathcal{B}^G$. An action of a group G on a 2-category $\mathcal{B}$ consists of

- a family of pseudofunctors $F_g : \mathcal{B} \to \mathcal{B}$, $g \in G$,
- pseudonatural equivalences $\chi_{g,h} : F_g \circ F_h \to F_{gh}$,
- invertible modifications

$$\omega_{g,h,f} : \chi_{gh,f} \circ (\chi_{g,h} \otimes \mathrm{id}_{F_f}) \Rightarrow \chi_{g,hf} \circ (\mathrm{id}_{F_g} \otimes \chi_{h,f}),$$

for any $g, h, f \in G$, satisfying certain axioms. We also prove a coherence theorem for group action, stating that there exists another equivalent action of G on $\mathcal{B}$, such that all pseudofunctors F_g involved in the group action are 2-functors, $F_g \circ F_h = F_{gh}$, and $\chi_{g,h}$, $\omega_{g,h,f}$ are all the identity. As an application of the coherent theorem we prove that associated to every action of group G on a 2-category $\mathcal{B}$ there is a braided G-crossed monoidal category $\mathcal{Z}_G(\mathcal{B})$ such that the trivial component is $\mathcal{Z}(\mathcal{B})$, the Drinfeld center of $\mathcal{B}$.

An important example comes from the theory of tensor categories. We show that, if $\mathcal{D} = \oplus_{g \in G} \mathcal{D}_g$ is a G-graded tensor category, and $\mathcal{D}_1 = \mathcal{C}$, there is an action of the group G acts on ${}_{\mathcal{C}}\mathrm{Mod}$, the 2-category of representations of $\mathcal{C}$, and there is a biequivalence

$$({}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}})^G \simeq {}_{\mathcal{D}}\mathrm{Mod}.$$

The coherence theorem for group actions allows us to construct an associated strict braided crossed monoidal category and to prove that there is a monoidal equivalence between the center $\mathcal{Z}(\mathcal{B}^G)$ of the equivariantization and the monoidal category of pseudonatural transformations of the forgetful pseudofunctor $\Phi : \mathcal{B}^G \to \mathcal{B}$. When applied this result to the 2-category $({}_{\mathcal{C}}\mathrm{Mod})^G$, we recover the results from [8], on the center of graded tensor categories.

The contents of the paper are organized as follows. In Sect. 1 we recall the basics of 2-categories. For any pseudofunctor $\mathcal{H} : \mathcal{B} \to \mathcal{B}'$ we define the monoidal category $\mathcal{Z}(\mathcal{H})$ of pseudonatural transformations $\eta : \mathcal{H} \to \mathcal{H}$. When $\mathcal{H}$ is the identity pseudofunctor, $\mathcal{Z}(\mathrm{Id})$ is a braided monoidal category called the *center* of the 2-category.

In Sect. 2 we explicitly describe the notion of a group action on a 2-category. Given two 2-categories $\mathcal{B}, \mathcal{B}'$ equipped with an action of a group G, we define the notion of G-pseudofunctor between them. When a G-pseudofunctor is a biequivalence, we say that $\mathcal{B}, \mathcal{B}'$ are G-biequivalent. Also, we define the notions of G-pseudonatural transformation and G-modifications. All these data, turns out to be a 2-category, denoted by $\mathbf{2Cat}^G(\mathcal{B}, \mathcal{B}')$. The equivariant 2-category is $\mathcal{B}^G = \mathbf{2Cat}^G(\mathcal{I}, \mathcal{B})$, where $\mathcal{I}$ is the unit 2-category, where G acts trivially.

In Sect. 3 we prove that any 2-category with a group action is G-biequivalent to another one where the action is *strict*. Section 4 is devoted to explicitly describe all ingredients in the equivariant 2-category $\mathcal{B}^G$.

In Sect. 5 we show an example coming from graded tensor categories. If $\mathcal{D} = \oplus_{g\in G}\mathcal{D}_g$ is a G-graded tensor category, then the group G acts on the 2-category ${}_{\mathcal{D}_1}\mathrm{Mod}$ of left $\mathcal{D}_1$-modules. The resulting equivariant 2-category $({}_{\mathcal{D}_1}\mathrm{Mod})^G$ is biequivalent to ${}_{\mathcal{D}}\mathrm{Mod}$. In Sect. 6 we define the G-braided center of a 2-category with an action of a group G. In Sect. 7, we show that there is a monoidal equivalence $\mathcal{Z}(\mathcal{B}^G) \simeq \mathcal{Z}(\Phi)^G$, where $\Phi : \mathcal{B}^G \to \mathcal{B}$ is the forgetful pseudofunctor. When applied to the example $({}_{\mathcal{C}}\mathrm{Mod})^G$, we recover results from [8].

1. 2-categories

Let us briefly recall the notion of a 2-category. For more details, the reader is referred to [14,21]. For any 2-category $\mathcal{B}$, the set of objects, also called *0-cells*, will be denoted by $\mathrm{Obj}(\mathcal{B})$. The composition in each hom-category $\mathcal{B}(A, B)$, that is, the vertical composition of 2-cells, is denoted by juxtaposition fg, while the symbol $\circ$ is used to denote the horizontal composition functors

$$\circ : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C).$$

The identity of a 0-cell A is written as $I_A : A \to A$. For any 1-cell X the identity will be denoted id_X or sometimes simply as 1_X, when space saving is needed. For any 2-category $\mathcal{B}$, we shall denote by $\mathcal{B}^{\mathrm{op}}$ the 2-category that is obtained from $\mathcal{B}$ by reversing 1-cells.

Example 1.1. The unit 2-category $\mathcal{I}$ has a single 0-cell, named $\star$. The monoidal category $\mathcal{I}(\star, \star)$ is the unit monoidal category.

A *pseudofunctor* $(F, \alpha) : \mathcal{B} \to \mathcal{B}'$, consists of a function $F : \mathrm{Obj}(\mathcal{B}) \to \mathrm{Obj}(\mathcal{B}')$, a family of functors $F : \mathcal{B}(A, B) \to \mathcal{B}'(F(A), F(B))$, for each $A, B \in \mathrm{Obj}(\mathcal{B})$, a collection of isomorphisms $\phi_A : I_{F(A)} \to F(I_A)$, and a family of natural isomorphisms

$$\begin{tikzcd}
\mathcal{B}(B,C) \times \mathcal{B}(A,B) \arrow[rr, "\circ"] \arrow[d, "F\times F"'] & \arrow[d, phantom, "\Uparrow\alpha"] & \mathcal{B}(A,C) \arrow[d, "F"] \\
\mathcal{B}'(F(B),F(C)) \times \mathcal{B}'(F(A),F(B)) \arrow[rr, "\circ"] & & \mathcal{B}'(F(A),F(C)),
\end{tikzcd}$$

for 0-cells A, B, C, subject to the usual axioms. A pseudofunctor is called *unital* if $F(I_A) = I_{F(A)}$, for any 0-cell A, and the isomorphisms ϕ_A are the identities. A pseudofunctor is called a *2-functor* if the associativity isomorphisms α are the identities.

If F, G are pseudofunctors, a *pseudonatural transformation* $\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\downarrow\chi"] & \mathcal{B}' \end{tikzcd}$ consists of a family of 1-cells $\chi^0_A : F(A) \to G(A)$, $A \in \mathrm{Obj}(\mathcal{B})$ and isomorphisms

$$\begin{tikzcd} F(A) \arrow[r, "F(X)"] \arrow[d, "\chi^0_A"'] \arrow[dr, phantom, "\Downarrow\chi_X"] & F(B) \arrow[d, "\chi^0_B"] \\ G(A) \arrow[r, "G(X)"'] & G(B) \end{tikzcd}$$

natural in $X \in \mathcal{B}(A, B)$, subject to the usual axioms. If χ, θ are pseudonatural transformations, a *modification* from $\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\downarrow\chi"] & \mathcal{B}' \end{tikzcd}$ to $\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\downarrow\theta"] & \mathcal{B}' \end{tikzcd}$, consists of a family of 2-cells $\omega_A : \chi^0_A \to \theta^0_A$, such that the diagrams

$$\begin{tikzcd} \chi^0_B \circ F(X) \arrow[r, "\chi_X"] \arrow[d, "\omega_B \circ \mathrm{id}_{F(X)}"'] & G(X) \circ \chi^0_A \arrow[d, "\mathrm{id}_{G(X)} \circ \omega_A"] \\ \theta^0_B \circ F(X) \arrow[r, "\theta_X"] & G(X) \circ \theta^0_A \end{tikzcd}$$

commute for all $X \in \mathcal{B}(A, B)$. This modification will be denoted as $\omega : \chi \Rightarrow \theta$. Given pseudofunctors $F, G : \mathcal{B} \to \mathcal{B}$, we shall denote Pseu-Nat(F, G) the category where objects are pseudonatural transformations from F to G and arrows are modifications.

A 1-cell $X \in \mathcal{B}(A, B)$ is called an *equivalence* if there exists a 1-cell $Y \in \mathcal{B}(B, A)$ such that $X \circ Y \cong I_B$ and $Y \circ X \cong I_A$. We will say that an invertible 1-cell X is an *isomorphism* if there is $X^* \in \mathcal{B}(B, A)$ such that $X \circ X^* = I_B$ and $X^* \circ X = I_A$. The next result will be useful later to simplify some proofs.

Proposition 1.2. *Every 2-category (or bicategory) is biequivalent to a 2-category where every equivalence 1-cell is an isomorphism.*

Proof. The proof goes along the lines of [9, Theorem 1.4]. Since every category is equivalent to a skeletal one. Every bicategory $\mathcal{B}$ is biequivalent to a locally skeletal one $\mathcal{B}'$, that is, each of its hom-category is skeletal. Then in $\mathcal{B}'$, every 1-cell equivalence is an isomorphism. By Street's Yoneda lemma for bicategories [22, p.117], the Yoneda embedding

$$\mathcal{B}' \to \mathbf{Bicat}(\mathcal{B}', \mathbf{Cat}) : A \mapsto \mathcal{B}'^{\mathrm{op}}(A, -),$$

is locally an equivalence. Therefore, $\mathcal{B}'$ is biequivalent to $\mathcal{B}''$; the full sub-2category of $\mathbf{Bicat}(\mathcal{B}'^{\mathrm{op}}, \mathbf{Cat})$ determined by the contravariant representables. Since every equivalence in $\mathcal{B}'$ is an isomorphism, every equivalence in $\mathcal{B}''$ is an isomorphism and $\mathcal{B}$ is biequivalent to $\mathcal{B}''$. $\square$

1.1. The tricategory of 2-categories

Given a pair of 2-categories $\mathcal{B}$ and $\mathcal{B}'$, we can define the *functor 2-category*, $\mathbf{2Cat}(\mathcal{B}, \mathcal{B}')$, whose 0-cells are pseudofunctors $\mathcal{B} \to \mathcal{B}'$, whose 1-cells are pseudonatural transformations, and whose 2-cells are modifications. Given 2-categories $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$, we define a pseudofunctor

$$\otimes : \mathbf{2Cat}(\mathcal{B}', \mathcal{B}'') \times \mathbf{2Cat}(\mathcal{B}, \mathcal{B}') \to \mathbf{2Cat}(\mathcal{B}, \mathcal{B}''),$$

called the *tensor product*. The tensor product at the level of pseudofunctors is the composition. The tensor product of pseudonatural transformations is

$$\Big(\begin{tikzcd} \mathcal{B}' \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\downarrow\beta"] & \mathcal{B}'' \end{tikzcd} \Big)\Big(\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\downarrow\alpha"] & \mathcal{B}' \end{tikzcd} \Big) = \Big(\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "GF"] \arrow[r, bend right, "G'F'"'] \arrow[r, phantom, "\downarrow\beta\otimes\alpha"] & \mathcal{B}'' \end{tikzcd} \Big), \tag{1.1}$$

where

$$(\beta \otimes \alpha)_A = \beta_{F'(A)} \circ G(\alpha_A)$$
$$(\beta \otimes \alpha)_X = (\beta_{F'(X)} \circ \mathrm{id}_{G(\alpha^0_A)})(\mathrm{id}_{\beta^0_{F'(B)}} \circ G(\alpha_X)).$$

Here, the isomorphisms constraints of the pseudofunctors have been omitted as a space-saving measure. If $\beta' : G \to G'$ and $\alpha' : F \to F'$ are another pseudonatural transformations and $\omega : \beta \to \beta'$ and $\omega' : \alpha \to \alpha'$ are modifications, their tensor product is defined as $\omega \otimes \omega' : \beta\otimes\alpha \to \beta'\otimes\alpha'$, $(\omega \otimes \omega')_A := \omega_{F'(A)} \circ G(\omega'_A)$, for any 0-cell A.

If $\alpha : F \to F'$ and $\beta : H \to H'$ are pseudonatural transformations between pseudofunctors $F, F' \in \mathbf{2Cat}(\mathcal{B}', \mathcal{B}'')$, $H, H' \in \mathbf{2Cat}(\mathcal{B}, \mathcal{B}')$, then there is a modification

$$\begin{tikzcd} & F'H \arrow[dr, "\mathrm{id}_{F'}\otimes\beta"] & \\ FH \arrow[ur, "\alpha\otimes\mathrm{id}_H"] \arrow[dr, "\mathrm{id}_H\otimes\beta"'] \arrow[rr, phantom, "\Downarrow c_{\alpha,\beta}"] & & F'H' \\ & FH' \arrow[ur, "\alpha\otimes\mathrm{id}_{H'}"'] & \end{tikzcd}$$

given by

$$(c_{\alpha,\beta})_A := \alpha^{-1}_{\beta_A} : F'(\beta_A) \circ \alpha_{H(A)} \to \alpha_{H'(A)} \circ F(\beta_A). \tag{1.2}$$

This modification is called the *comparison constraint.*

The tensor product is associative only at the level of pseudofunctors, but not for pseudonatural transformations. There exists an associativity constraint

$$\begin{tikzcd} KHG \arrow[rr, bend left, "(\alpha\otimes\beta)\otimes\gamma"] \arrow[rr, bend right, "\alpha\otimes(\beta\otimes\gamma)"'] \arrow[rr, phantom, "\Downarrow a_{\alpha,\beta,\gamma}"] & & K'H'G' \end{tikzcd}$$

for pseudonatural transformations $\alpha : K \to K'$, $\beta : H \to H'$ and $\gamma : G \to G'$. The modification

$$(a_{\alpha,\beta,\gamma})_A : \alpha_{F'H'(A)} \circ G(\beta_{H'(A)}) \circ GF(\gamma_A) \to \alpha_{F'H'(A)} \circ G(\beta'_H(A) \circ F(\gamma_A))$$

is defined by $(a_{\alpha,\beta,\gamma})_A = \mathrm{id}_{\alpha_{F'H'(A)}} \circ G_2(\beta_{H'(A)}, F(\gamma_A))$. It is easy to see that a satisfies the pentagonal identity.

1.2. Finite tensor categories

A (strict) monoidal category is a 2-category with one single 0-cell. A *finite tensor category over* $\Bbbk$ is a finite $\Bbbk$-linear abelian rigid monoidal category $\mathcal{C}$ such that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is $\Bbbk$-linear in each variable. The reader is referred to [5].

Suppose $\mathcal{C}$ and $\mathcal{D}$ are strict tensor categories. A *monoidal functor* $(F, \xi, \phi) : \mathcal{C} \to \mathcal{D}$ is a pseudofunctor between the corresponding 2-categories. Explicitly, it consists of a functor $F : \mathcal{C} \to \mathcal{D}$, natural isomorphisms $\xi_{X,Y} : F(X)\otimes F(Y) \to F(X\otimes Y)$, $X, Y \in \mathcal{C}$, and isomorphism $\phi : \mathbf{1} \to F(\mathbf{1})$, satisfying certain axioms. If (F, ξ, ϕ), (F', ξ', ϕ') are monoidal functors , a *natural monoidal transformation* $\theta : (F, \xi, \phi) \to (F', \xi', \phi')$ is a natural transformation $\theta : F \to F'$, such that for any pair of objects X, Y

$$\theta_{\mathbf{1}}\phi = \phi', \quad \theta_{X\otimes Y}\xi_{X,Y} = \xi'_{X,Y}(\theta_X \otimes \theta_Y). \tag{1.3}$$

1.3. The endomorphism category of a pseudofunctor

If $\mathcal{B}$ is a 2-category, the monoidal category

$$\mathcal{Z}(\mathcal{B}) = \mathbf{2Cat}(\mathcal{B}, \mathcal{B})(\mathrm{Id}_{\mathcal{B}}, \mathrm{Id}_{\mathcal{B}})$$

is exactly the center of $\mathcal{B}$, *i.e.*, the obvious generalization of the center construction of a monoidal category. See [19].

Let $\mathcal{B}, \mathcal{B}'$ be two 2-categories and $(\mathcal{H}, \alpha) : \mathcal{B} \to \mathcal{B}'$ be a unital pseudofunctor. Denote $\mathcal{Z}(\mathcal{H}) = \mathbf{2Cat}(\mathcal{B}, \mathcal{B}')(\mathcal{H}, \mathcal{H})$; the category of pseudonatural transformations of the pseudofunctor $\mathcal{H}$. This is a monoidal category with tensor product described in the previous section. Explicitly, objects in $\mathcal{Z}(\mathcal{H})$ are pairs (V, σ), where

$$V = \left\{V_A \in \mathcal{B}'(\mathcal{H}(A), \mathcal{H}(A))\text{1-cells, for any} A \in \mathcal{B}\right\},$$
$$\sigma = \left\{\sigma_X : V_B \circ \mathcal{H}_{A,B}(X) \to \mathcal{H}_{A,B}(X) \circ V_A\right\},$$

where, for any $X \in \mathcal{B}(A, B)$, σ_X is a natural isomorphism 2-cell such that

$$\sigma_{I_A} = \mathrm{id}_{V_A}, (\alpha_{X,Y} \circ \mathrm{id}_{V_A})\sigma_{X\circ Y} = (\mathrm{id}_{\mathcal{H}(X)} \circ \sigma_Y)(\sigma_X \circ \mathrm{id}_{\mathcal{H}(Y)})(\mathrm{id}_{V_B} \circ \alpha_{X,Y}), \tag{1.4}$$

for any 0-cells $A, B, C \in \mathcal{B}$, and any pair of 1-cells $X \in \mathcal{B}(C, B)$, $Y \in \mathcal{B}(A, C)$.

If (V, σ), (W, τ) are two objects in $\mathcal{Z}(\mathcal{H})$, a morphism $f : (V, \sigma) \to (W, \tau)$ in $\mathcal{Z}(\mathcal{H})$ is a collection of 2-cells $f_A : V_A \Rightarrow W_A$, $A \in \mathcal{B}$ such that

$$(\mathrm{id}_{\mathcal{H}(X)} \circ f_A)\sigma_X = \tau_X(f_B \circ \mathrm{id}_{\mathcal{H}(X)}), \tag{1.5}$$

for any 1-cell $X \in \mathcal{B}(A, B)$. The category $\mathcal{Z}(\mathcal{H})$ has a monoidal product defined as follows. Let (V, σ), $(W, \tau) \in \mathcal{Z}(\mathcal{H})$ be two objects. Then $(V, \sigma)\otimes(W, \tau) = (V\otimes W, \sigma\otimes\tau)$, where for any 0-cells $A, B \in \mathcal{B}$, and $X \in \mathcal{B}(A, B)$

$$(V\otimes W)_A = V_A \circ W_A, \quad (\sigma\otimes\tau)_X = (\sigma_X \circ \mathrm{id}_{W_A})(\mathrm{id}_{V_B} \circ \tau_X). \tag{1.6}$$

If $(V, \sigma), (V', \sigma'), (W, \tau), (W', \tau') \in \mathcal{Z}(\mathcal{H})$ are objects, and $f : (V, \sigma) \to (V', \sigma')$, $f' : (W, \tau), (W', \tau')$ are morphisms in $\mathcal{Z}(\mathcal{H})$, then $f\otimes f' : (V, \sigma)\otimes (V', \sigma') \to (W, \tau)\otimes(W', \tau')$ is defined by

$$(f\otimes f')_A = f_A \circ f'_A,$$

for any 0-cell A. The unit $(\mathbf{1}, \iota) \in \mathcal{Z}(\mathcal{H})$ is the object

$$\mathbf{1}_A = I_A, \quad \iota_X = \mathrm{id}_X,$$

for any 0-cells A, B and any 1-cell $X \in \mathcal{B}(A, B)$. The center $\mathcal{Z}(\mathrm{Id}_{\mathcal{B}})$ of the identity pseudofunctor $\mathrm{Id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is denoted as $\mathcal{Z}(\mathcal{B})$, and it coincides with the definition presented in [19].

2. Group actions on 2-categories

Assume G is a group and $\mathcal{B}$ is a 2-category. We shall denote by $\underline{\underline{G}}$ the 2-category that has 0-cells the elements of the group G. For any pair $g, h \in G$

$$\underline{\underline{G}}(g, h) = \begin{cases} \text{the unit category, if} \quad g = h \\ \emptyset \ \text{ if} \quad g \neq h. \end{cases}$$

Moreover, $\underline{\underline{G}}$ is a monoidal 2-category, see [9]. Since $\mathbf{2Cat}(\mathcal{B}, \mathcal{B})$ is also a monoidal 2-category, we define an *action* of G on $\mathcal{B}$ as a weak monoidal homomorphism $(\mathcal{F}, \chi, \omega, \iota, \kappa, \zeta) : \underline{\underline{G}} \to \mathbf{2Cat}(\mathcal{B}, \mathcal{B})$. See for example [9].

Explicitly, an action of G on a 2-category $\mathcal{B}$ consists of the following data:

- A family of pseudofunctors $F_g : \mathcal{B} \to \mathcal{B}$, $g \in G$,
- pseudonatural equivalences $(\chi_{g,h}, \chi^0_{g,h}) : F_g \circ F_h \to F_{gh}$, $g, h \in G$,
- a pseudonatural equivalence $\iota : \mathrm{Id}_{\mathcal{B}} \to F_1$,
- for any $g, h, f \in G$ invertible modifications

$$\omega_{g,h,f} : \chi_{gh,f} \circ (\chi_{g,h}\otimes\mathrm{id}_{F_f}) \Rightarrow \chi_{g,hf} \circ (\mathrm{id}_{F_g}\otimes\chi_{h,f}),$$
$$\kappa_g : \chi_{1,g} \circ (\iota\otimes\mathrm{id}_{F_g}) \Rightarrow \mathrm{id}_{F_g}, \quad \zeta_g : \chi_{g,1} \circ (\mathrm{id}_{F_g}\otimes\iota) \Rightarrow \mathrm{id}_{F_g},$$

such that for any 0-cell A

$$
\begin{aligned}
&1_{(\chi^0_{g,f})_A} \circ F_g(\kappa_f)_A(\omega_{g,1,f})_A = 1_{(\chi^0_{g,f})_A} \circ (\zeta_g)_{F_f(A)}, &(2.1)\\
&\big(\mathrm{id}_3 \circ (F_g(\omega_{h,f,k})_A)\big)\big(\omega_{g,hf,k} \circ \mathrm{id}_2\big)\big(\mathrm{id}_{(\chi^0_{ghf,k})_A} \circ (\omega_{g,h,f})_{F_k(A)}\big) = \\
&\quad = \big((\omega_{g,h,fk})_A \circ \mathrm{id}_4)\big)\big(\mathrm{id}_5 \circ (\chi_{g,h})_{\chi^0_{f,k}}\big)\big((\omega_{gh,f,k})_A \circ \mathrm{id}_6\big), &(2.2)
\end{aligned}
$$

for any $g, h, f, k \in G$. Where,

$$
\begin{aligned}
&\mathrm{id}_2 = 1_{F_g(\chi^0_{h,f})_{F_k(A)}}, \quad \mathrm{id}_3 = 1_{(\chi^0_{g,hfk})_A}, \quad \mathrm{id}_4 = 1_{F_g F_f(\chi^0_{h,k})_A},\\
&\mathrm{id}_5 = 1_{(\chi^0_{gh,fk})_A}, \quad \mathrm{id}_6 = 1_{(\chi^0_{g,h})_{F_f F_k(A)}}.
\end{aligned}
$$

In Eq. (2.2), we are omitting the associativity isomorphisms of the pseudofunctors F_g. In the following diagrams we shall denote by $\overline{g}$ the pseudofunctor F_g, the composition of functors as juxtaposition and the tensor product of pseudonatural transformations also by juxtaposition. Diagrammatically, we have modifications

$$
\begin{tikzcd}
\overline{g}\,\overline{h}\,\overline{f} \arrow[r, "\chi_{g,h}\otimes 1_{\overline{f}}"] \arrow[d, "1_{\overline{g}}\otimes\chi_{h,f}"'] \arrow[dr, phantom, "\Downarrow\omega_{g,h,f}"] & \overline{gh}\,\overline{f} \arrow[d, "\chi_{gh,f}"] \\
\overline{g}\,\overline{hf} \arrow[r, "\chi_{g,hf}"'] & \overline{ghf},
\end{tikzcd}
$$

such that the next diagrams are equal for all $g, h, f, k \in G$,

$$
\begin{tikzcd}
 & \overline{gh}\,\overline{f}\,\overline{k} \arrow[rr, "\chi_{gh,f}\otimes 1_{\overline{k}}"] \arrow[d, phantom, "\Downarrow\omega_{g,h,f}\otimes 1_{\overline{k}}"] & & \overline{ghf}\,\overline{k} \arrow[dr, "\chi_{ghf,k}"] \arrow[d, phantom, "\Downarrow\omega_{g,hf,k}"] & \\
\overline{g}\,\overline{h}\,\overline{f}\,\overline{k} \arrow[ur, "\chi_{g,h}\otimes 1_{\overline{f}}\otimes 1_{\overline{k}}"] \arrow[rr, "1_{\overline{g}}\otimes\chi_{h,f}\otimes 1_{\overline{k}}"] \arrow[dr, "1_{\overline{g}}\otimes 1_{\overline{h}}\otimes\chi_{f,k}"'] & \arrow[d, phantom, "\Downarrow 1_{\overline{g}}\otimes\omega_{h,f,k}"] & \overline{g}\,\overline{hf}\,\overline{k} \arrow[ur, "\chi_{g,hf}\otimes 1_{\overline{k}}"] \arrow[dr, "1_{\overline{g}}\otimes\chi_{hf,k}"] & & \overline{ghfk} \\
 & \overline{g}\,\overline{h}\,\overline{fk} \arrow[rr, "1_{\overline{g}}\otimes\chi_{h,fk}"'] & & \overline{g}\,\overline{hfk} \arrow[ur, "\chi_{g,hfk}"'] &
\end{tikzcd}
$$

(2.3)

$$||$$

$$\begin{tikzcd}
 & \overline{gh}\,\overline{f}\,\overline{k} \arrow[rr, "\chi_{gh,f}\otimes 1_{\overline{k}}"] \arrow[dr, "1_{\overline{gh}}\otimes\chi_{f,k}"] & & \overline{ghf}\,\overline{k} \arrow[dr, "\chi_{ghf,k}"] & \\
\overline{g}\,\overline{h}\,\overline{f}\,\overline{k} \arrow[ur, "\chi_{g,h}\otimes 1_{\overline{f}}\otimes 1_{\overline{k}}"] \arrow[dr, "1_{\overline{g}}\otimes 1_{\overline{h}}\otimes\chi_{f,k}"'] & \Downarrow c_{\chi_{g,h},\chi_{f,k}} & \overline{gh}\,\overline{fk} \arrow[rr, "\chi_{gh,fk}"'] \arrow[r, phantom, shift left=8, "\Downarrow\omega_{gh,f,k}"] \arrow[r, phantom, shift right=8, "\Downarrow\omega_{g,h,fk}"] & & \overline{ghfk} \\
 & \overline{g}\,\overline{h}\,\overline{fk} \arrow[rr, "1_{\overline{g}}\otimes\chi_{h,fk}"'] \arrow[ur, "\chi_{g,h}\otimes 1_{\overline{fk}}"'] & & \overline{g}\,\overline{hfk} \arrow[ur, "\chi_{g,hfk}"'] &
\end{tikzcd}$$

We say that a group G acts *trivially* on $\mathcal{B}$ if the weak monoidal homomorphism $(\mathcal{F}, \chi, \omega, \iota, \kappa, \zeta) : \underline{\underline{G}} \to \mathbf{2Cat}(\mathcal{B}, \mathcal{B})$ is the trivial one. This means that for any $g, h \in G$, the pseudofunctors F_g are the identity, $\chi_{g,h}$ are the identity pseudonatural transformations and all the modifications are identities.

Remark 2.1. A definition of action over a topological group was given in [13]. S

Definition 2.2. An action $(\mathcal{F}, \chi, \omega, \iota, \kappa, \zeta) : \underline{\underline{G}} \to \mathbf{2Cat}(\mathcal{B}, \mathcal{B})$ is called *unital* if F_g is a unital pseudofunctor, $F_1 = \mathrm{Id}_{\mathcal{B}}$, and $\chi_{g,1} = \mathrm{id}_{F_g} = \chi_{1,g}$, $\kappa_g = \mathrm{id} = \zeta_g$ for any $g \in G$. A unital G-action will be denoted simply by $(\mathcal{F}, \chi, \omega)$.

Definition 2.3. An action $(\mathcal{F}, \chi, \omega, \iota, \kappa, \zeta) : \underline{\underline{G}} \to \mathbf{2Cat}(\mathcal{B}, \mathcal{B})$ is called *strict* if each pseudofunctor F_g is a 2-functor, and $F_g \circ F_h = F_{gh}$, and the pseudonatural transformations $\chi_{g,h}$ and the modifications $\omega_{g,h,f}$ are the identities for any $g, h, f \in G$.

A similar argument as in [7, Proposition 3.1] applied in this case, allows us to consider only unital actions. Assume that $\mathcal{B}, \mathcal{B}'$ are 2-categories equipped with unital actions of a group G via

$$(\mathcal{F}, \chi, \omega) : \underline{\underline{G}} \to \mathbf{2Cat}(\mathcal{B}, \mathcal{B}), \ \ (\widetilde{\mathcal{F}}, \widetilde{\chi}, \widetilde{\omega}) : \underline{\underline{G}} \to \mathbf{2Cat}(\widetilde{\mathcal{B}}, \widetilde{\mathcal{B}}).$$

Definition 2.4. A *G-pseudofunctor* between $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ is a triple $(\mathcal{H}, \gamma, \Pi)$, where

- $\mathcal{H} : \mathcal{B} \to \widetilde{\mathcal{B}}$ is a unital pseudofunctor,
- for any $g \in G$, pseudonatural equivalences $\gamma_g : \mathcal{H} \circ F_g \to \widetilde{F}_g \circ \mathcal{H}$,
- invertible modifications

$$\begin{tikzcd}
 & \widetilde{f}\mathcal{H}\overline{g} \arrow[r, "1_{\widetilde{f}}\otimes\gamma_g"] & \widetilde{f}\widetilde{g}\mathcal{H} \arrow[dr, "\widetilde{\chi_{f,g}}\otimes 1_{\mathcal{H}}"] & \\
\mathcal{H}\overline{f}\,\overline{g} \arrow[ur, "\gamma_f\otimes 1_{\overline{g}}"] \arrow[dr, "1_{\mathcal{H}}\otimes\chi_{f,g}"'] & & \Downarrow\Pi_{f,g} & \widetilde{fg}\mathcal{H} \\
 & \mathcal{H}\overline{fg} \arrow[urr, "\gamma_{fg}"'] & &
\end{tikzcd}$$

such that for all $f, g, h \in G$

$$\gamma_1 = \mathrm{id}_{\mathcal{H}}, \quad \Pi_{g,1} = \mathrm{id}_{\gamma_g} = \Pi_{1,g}, \tag{2.4}$$

$$\begin{tikzcd}[column sep=small]
 & & & \widetilde{f}\widetilde{g}\mathcal{H}\overline{h} \arrow[dd, phantom, "\Downarrow 1_{\widetilde{f}}\otimes\Pi_{g,h}"] \arrow[drr, "1_{\widetilde{f}}1_{\widetilde{g}}\gamma_h"] & & & \\
 & \widetilde{f}\mathcal{H}\overline{g}\overline{h} \arrow[urr, "1_{\widetilde{f}}\gamma_g 1_{\overline{h}}"] \arrow[dr, "1_{\widetilde{f}\mathcal{H}}\chi_{g,h}"] & & & & \widetilde{f}\widetilde{g}\widetilde{h}\mathcal{H} \arrow[dl, "1_{\widetilde{f}}\widetilde{\chi_{g,h}}1_{\mathcal{H}}"'] \arrow[dr, "\widetilde{\chi_{f,g}}1_{\widetilde{h}\mathcal{H}}"] & \\
\mathcal{H}\overline{f}\overline{g}\overline{h} \arrow[ur, "\gamma_f 1_{\overline{g}\,\overline{h}}"] \arrow[dr, "1_{\mathcal{H}\overline{f}}\chi_{g,h}"'] \arrow[rr, phantom, "c_{\gamma_f,\chi_{g,h}}"] & & \widetilde{f}\mathcal{H}\overline{gh} \arrow[rr, "1_{\widetilde{f}}\gamma_{gh}"] & & \widetilde{f}\widetilde{gh}\mathcal{H} \arrow[dr, "\widetilde{\chi_{f,gh}}1_{\mathcal{H}}"'] \arrow[rr, phantom, "\overset{\widetilde{\omega}_{f,g,h}}{\Leftarrow}"] & & \widetilde{fg}\widetilde{h}\mathcal{H} \arrow[dl, "\widetilde{\chi_{fg,h}}1_{\mathcal{H}}"] \\
 & \mathcal{H}\overline{f}\,\overline{gh} \arrow[ur, "\gamma_f 1_{\overline{gh}}"'] \arrow[drr, "\chi_{f,gh}"'] \arrow[rrrr, phantom, "\Downarrow\Pi_{f,gh}"] & & & & \widetilde{fgh}\mathcal{H} & \\
 & & & \mathcal{H}\overline{fgh} \arrow[urr, "\gamma_{fgh}"'] & & &
\end{tikzcd} \tag{2.5}$$

$$\|$$

$$\begin{tikzcd}[column sep=small]
 & & & \widetilde{f}\widetilde{g}\mathcal{H}\overline{h} \arrow[drr, "1_{\widetilde{f}}1_{\widetilde{g}}\gamma_h"] \arrow[ddr, "\widetilde{\chi_{f,g}}1_{\mathcal{H}\overline{h}}"'] & & & \\
 & \widetilde{f}\mathcal{H}\overline{g}\overline{h} \arrow[urr, "1_{\widetilde{f}}\gamma_g 1_{\overline{h}}"] \arrow[rr, phantom, "\Downarrow\Pi_{f,g}\otimes 1_{\overline{h}}"] & & & \arrow[r, phantom, "c_{\widetilde{\chi}_{f,g},\gamma_h}"] & \widetilde{f}\widetilde{g}\widetilde{h}\mathcal{H} \arrow[dr, "\widetilde{\chi_{f,g}}1_{\widetilde{h}\mathcal{H}}"] & \\
\mathcal{H}\overline{f}\overline{g}\overline{h} \arrow[ur, "\gamma_f 1_{\overline{gh}}"] \arrow[rr, "1_{\mathcal{H}}\chi_{f,g}1_{\overline{h}}"] \arrow[dr, "1_{\mathcal{H}\overline{f}}\chi_{g,h}"'] & & \mathcal{H}\overline{fg}\,\overline{h} \arrow[rr, "\gamma_{fg}1_{\overline{h}}"] \arrow[ddr, "1_{\mathcal{H}}\chi_{fg,h}"] \arrow[dl, phantom, "\overset{1_{\mathcal{H}}\otimes\omega_{f,g,h}}{\Leftarrow}"] & & \widetilde{fg}\mathcal{H}\overline{h} \arrow[rr, "1_{\widetilde{fg}}\gamma_h"] \arrow[d, phantom, "\Downarrow\Pi_{fg,h}"] & & \widetilde{fg}\widetilde{h}\mathcal{H} \arrow[dl, "\widetilde{\chi_{fg,h}}1_{\mathcal{H}}"] \\
 & \mathcal{H}\overline{f}\,\overline{gh} \arrow[drr, "1_{\mathcal{H}}\chi_{f,gh}"'] & & & & \widetilde{fgh}\mathcal{H} & \\
 & & & \mathcal{H}\overline{fgh} \arrow[urr, "\gamma_{fgh}"'] & & &
\end{tikzcd}$$

holds in $\mathbf{2Cat}(\mathcal{B}, \mathcal{B})$. In the above diagrams, we are using the comparison constraints c defined in (1.2).

Remark 2.5. A more general definition of G-functor, in the case G is a topological group, was given in [12].

Definition 2.6. Assume that $(\mathcal{H}, \gamma, \Pi)$, $(\mathcal{H}', \gamma', \Pi')$ are G-pseudofunctors. A G-*pseudonatural* transformation is a pair $(\theta, \{\theta_g\}_{g\in G})$, where $\theta : \mathcal{H} \to \mathcal{H}'$ is a pseudonatural transformation, and θ_g are invertible modifications

$$
\begin{tikzcd}
\mathcal{H}\overline{g} \arrow[r, "\gamma_g"] \arrow[d, "\theta\otimes 1_{\overline{g}}"'] \arrow[dr, phantom, "\Downarrow\theta_g"] & \widetilde{g}\mathcal{H} \arrow[d, "1_{\widetilde{g}}\otimes\theta"] \\
\mathcal{H}'\overline{g} \arrow[r, "\gamma'_g"] & \widetilde{g}\mathcal{H}'
\end{tikzcd}
$$

such that for all $g,\ f \in G$, the equation

$$
\begin{tikzcd}[column sep=small]
\mathcal{H}\overline{g}\,\overline{f} \arrow[rr, "\gamma_g 1_{\overline{f}}"] \arrow[d, "\theta 1_{\overline{g}} 1_{\overline{f}}"'] \arrow[rrd, phantom, "\Downarrow\theta_g 1_{\overline{f}}"] & & \widetilde{g}\mathcal{H}\overline{f} \arrow[rr, "1_{\widetilde{g}}\gamma_f"] \arrow[d, "1_{\widetilde{g}}\theta 1_{\overline{f}}"'] \arrow[rrd, phantom, "\Downarrow 1_{\widetilde{g}}\theta_f"] & & \widetilde{g}\widetilde{f}\mathcal{H} \arrow[rr, "\widetilde{\chi}_{g,f} 1_{\mathcal{H}}"] \arrow[d, "1_{\widetilde{g}} 1_{\widetilde{f}}\theta"'] \arrow[rrd, phantom, "\Downarrow c_{\theta,\widetilde{\chi_{g,f}}}"] & & \widetilde{gf}\mathcal{H} \arrow[d, "1_{\widetilde{gf}}\theta"] \\
\mathcal{H}'\overline{g}\,\overline{f} \arrow[rr, "\gamma'_g 1_{\overline{f}}"'] \arrow[rrrd, bend right=15, "1_{\mathcal{H}'}\chi_{g,f}"'] & & \widetilde{g}\mathcal{H}'\overline{f} \arrow[rr, "1_{\widetilde{g}}\gamma'_f"] \arrow[d, phantom, shift left=8, "\Downarrow\Pi'_{g,f}"] & & \widetilde{g}\widetilde{f}\mathcal{H}' \arrow[rr, "\widetilde{\chi_{g,f}} 1_{\mathcal{H}'}"'] & & \widetilde{gf}\mathcal{H}' \\
& & & \mathcal{H}'\overline{gf} \arrow[rrru, bend right=15, "\gamma'_{gf}"'] & & &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}[column sep=small]
\mathcal{H}\overline{g}\,\overline{f} \arrow[r, "\gamma_g 1_{\overline{f}}"'] \arrow[d, "\theta 1_{\overline{g}} 1_{\overline{f}}"'] \arrow[rrd, bend right=15, "1_{\mathcal{H}}\chi_{g,f}"] & \widetilde{g}\mathcal{H}\overline{f} \arrow[rr, "1_{\widetilde{g}}\gamma_f"] & \arrow[d, phantom, "\Downarrow\Pi_{g,f}"] & \widetilde{g}\widetilde{f}\mathcal{H}' \arrow[r, "\widetilde{\chi_{g,f}} 1_{\mathcal{H}}"'] & \widetilde{gf}\mathcal{H} \arrow[d, "1_{\widetilde{gf}}\theta"] \\
\mathcal{H}'\overline{g}\,\overline{f} \arrow[rrd, bend right=15, "1_{\mathcal{H}'}\chi_{g,f}"'] \arrow[rr, phantom, "\Downarrow c^{-1}_{\theta,\chi_{g,f}}"] & & \mathcal{H}\overline{gf} \arrow[rru, bend right=15, "\gamma_{gf}"'] \arrow[d, "\theta 1_{\overline{gf}}"'] \arrow[rr, phantom, "\Downarrow\theta_{gf}"] & & \widetilde{gf}\mathcal{H}' \\
& & \mathcal{H}'\overline{gf} \arrow[rru, bend right=15, "\gamma'_{gf}"'] & &
\end{tikzcd}
$$

holds in $\mathbf{2Cat}(\mathcal{B}, \mathcal{B})$.

Definition 2.7. Assume that $(\theta, \{\theta_g\}_{g\in G}), (\sigma, \{\sigma_g\}_{g\in G}) : (\mathcal{H}, \gamma, \Pi) \to (\widetilde{\mathcal{H}}, \widetilde{\gamma}, \widetilde{\Pi})$ are G-pseudonatural transformations. A *G-modification* $\alpha : (\theta, \{\theta_g\}_{g\in G}) \Rightarrow (\sigma, \{\sigma_g\}_{g\in G})$ is a modification $\alpha : \theta \Rightarrow \sigma$ such that

$$
\begin{tikzcd}
\mathcal{H}\overline{g} \arrow[r, "\gamma_g"] \arrow[d, bend right=60, "\sigma\otimes 1_{\overline{g}}"'] \arrow[d, bend left=60, "\theta\otimes 1_{\overline{g}}"] \arrow[d, phantom, "\overset{\alpha\otimes 1_{\overline{g}}}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\theta_g"] & \widetilde{g}\mathcal{H} \arrow[d, "1_{\widetilde{g}}\theta_g"] \\
\mathcal{H}'\overline{g} \arrow[r, "\gamma'_g"'] & \widetilde{g}\mathcal{H}'
\end{tikzcd}
$$

$$\|$$

$$\begin{tikzcd}
\mathcal{H}\overline{g} \arrow[r, "\gamma_g"] \arrow[d, "\sigma_g\otimes 1_{\overline{g}}"'] \arrow[dr, phantom, "\Downarrow\sigma_g"] & \widetilde{g}\mathcal{H} \arrow[d, bend right=60, "1_{\widetilde{g}}\otimes\sigma"'] \arrow[d, bend left=60, "1_{\widetilde{g}}\otimes\theta"] \arrow[d, phantom, "\overset{1_{\widetilde{g}}\otimes\alpha}{\Leftarrow}"] \\
\mathcal{H}'\overline{g} \arrow[r, "\gamma'_g"'] & \widetilde{g}\mathcal{H}'
\end{tikzcd}$$

Assume that $(\mathcal{H}^1, \gamma^1, \Pi^1)$, $(\mathcal{H}^2, \gamma^2, \Pi^2)$, $(\mathcal{H}^3, \gamma^3, \Pi^3)$ are G-pseudofunctors, and $(\theta, \{\theta_g\}_{g\in G}) : (\mathcal{H}^1, \gamma^1, \Pi^1) \to (\mathcal{H}^2, \gamma^2, \Pi^2)$, $(\sigma, \{\sigma_g\}_{g\in G}) : (\mathcal{H}^2, \gamma^2, \Pi^2) \to (\mathcal{H}^3, \gamma^3, \Pi^3)$ are G-pseudonatural transformations. The composition

$$(\sigma, \{\sigma_g\}_{g\in G}) \circ (\theta, \{\theta_g\}_{g\in G}) = (\rho, \{\rho_g\}_{g\in G})$$

is defined as follows. The pseudonatural transformation $\rho = \sigma \circ \theta$. For any 0-cell $A \in \mathcal{B}$ and any $g \in G$

$$(\rho_g)_A = \big((\sigma_g)_A \circ \mathrm{id}_{\theta^0_{F_g(A)}}\big)\big(\mathrm{id}_{\widetilde{F}_g(\sigma^0_A)} \circ (\theta_g)_A\big).$$

Here, we are also ommiting the associativity constraints of the pseudofunctor F_g. The composition of modifications of G-categories is the usual composition of modifications.

Definition 2.8. $\mathbf{2Cat}^G(\mathcal{B}, \widetilde{\mathcal{B}})$ is the 2-category in which 0-cells are pseudofunctors of G-categories, 1-cells are pseudonatural transformations of G-categories and 2-cells are modifications of G-categories.

The next result is a consequence of [9, Corollary 8.3].

Proposition 2.9. *$2Cat^G(\mathcal{B}, \widetilde{\mathcal{B}})$ is a 2-category.* □

Definition 2.10. We say that the 2-categories $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are *G-biequivalent* if there exists a G-pseudofunctor $\mathcal{H} : \mathcal{B} \to \widetilde{\mathcal{B}}$ that is also a biequivalence.

Lemma 2.11. (Transport of structure). *Let $\mathcal{B}$ be a 2-category with an action of G given by $(\mathcal{F}, \chi, \omega)$. Let $\mathcal{H} : \mathcal{B} \to \mathcal{B}'$ be a biequivalence,*

$$L_g : \mathcal{B}' \to \mathcal{B}', \;\; \gamma_g : \mathcal{H} \circ F_g \to L_g \circ \mathcal{H}$$

a G-indexed family of pseudofunctors and pseudonatural equivalences, respectively. Then, there is a way to endowed $\mathcal{B}'$ with a G-action (L, χ', ω') such that $(\mathcal{H}, \gamma, \Pi) : \mathcal{B} \to \mathcal{B}'$ is a G-biequivalence .

Proof. Since γ_g and $\chi_{f,g}$ are psedonatural equivalences, we can simultaneously provide the datum $\Pi_{f,g}$ and the pseudonatural equivalences $\chi'_{f,g} : L_f \circ L_g \to L_{fg}$, $f, g \in G$. Now, axiom 2.5 uniquely determines the modifications $\omega'_{f,g,h}$. Axiom 2.3 follows from the corresponding axioms of G-action via $(\mathcal{F}, \chi, \omega)$. The pseudofunctor $(\mathcal{H}, \gamma, \Pi) : \mathcal{B} \to \mathcal{B}'$ is a G-biequivalence by construction. □

Corollary 2.12. *Every 2-category with a G-action is G-biequivalent to a 2-category where G acts by 2-functors, that is, all F_g are 2-functors.*

Proof. By the coherence of theorem for pseudofunctor, see [11, Section 2.3], every bicategory $\mathcal{B}$ is biequivalent to a 2-category $\mathrm{st}(\mathcal{B})$ such that every pseudo-functor $F : \mathrm{st}(\mathcal{B}) \to \mathrm{st}(\mathcal{B})$ is pseudo-natural equivalent to a 2-functor. Then applying Lemma 2.11 we can transport the action of $\mathcal{B}$ to a G-biequivalent action on $\mathrm{st}(\mathcal{B})$ where G acts by 2-functors. □

3. Coherence for group actions on 2-categories

The main result of this section is to prove the following coherence theorem for a group action on a 2-category.

Theorem 3.1. (Coherence for group actions on 2-categories). *Let G be a group. Every 2-category with an action of G is G-biequivalent to a 2-category with a strict action of G.* □

Assume $\mathcal{B}$ is a 2-category equipped with a unital action of G, $(\mathcal{F}, \chi, \omega) : \underline{\underline{G}} \to \mathbf{2Cat}(\mathcal{B}, \mathcal{B})$. By Corollary 2.12 we can assume that F_g is a 2-functor for any $g \in G$. We shall first construct a 2-category $\mathcal{B}[G]$ with a strict action of G.

Objects of $\mathcal{B}[G]$ are triples (A, θ, α), where $A = \{A_g\}_g$ is a G-indexed family of objects, $\theta = \{\theta_{g,h} : F_g(A_h) \to A_{gh}\}_{g,h \in G}$ is a $G \times G$-indexed family of 1-cell equivalences and

$$\begin{tikzcd}
F_g F_h(A_f) \arrow[rr, "(\chi^0_{g,h})_{A_f}"] \arrow[d, "F_g(\theta_{h,f})"'] & & F_{gh}(A_f) \arrow[d, "\theta_{gh,f}"] \\
F_g(A_{hf}) \arrow[rr, "\theta_{g,hf}"'] & \Downarrow\alpha_{g,h,f} & A_{ghf},
\end{tikzcd}$$

a $G \times G \times G$-index family of isomorphism 2-cells, such

$$\theta_{1,g} = I_{A_g}, \quad \alpha_{1,h,f} = \mathrm{id}, \quad \alpha_{g,1,f} = \mathrm{id}$$

that for all g, h, f, k, and equation

$$\begin{tikzcd}
 & \overline{gh}\,\overline{f}\,A_k \arrow[rr, "\chi^0_{gh,f}"] \arrow[dr, phantom, "\Downarrow\omega_{g,h,f}"] & & \overline{ghf}\,A_k \arrow[dr, "\theta_{ghf,k}"] \arrow[d, phantom, "\Downarrow\alpha_{g,hf,k}"] & \\
\overline{g}\,\overline{h}\,\overline{f}\,A_k \arrow[ur, "\chi^0_{g,h}\otimes 1_{\overline{f}}"] \arrow[rr, "1_{\overline{g}}\otimes\chi^0_{h,f}"] \arrow[dr, "1_{\overline{g}}\otimes 1_{\overline{h}}\otimes\theta_{f,k}"'] & & \overline{g}\,\overline{hf}\,A_k \arrow[ur, "\chi^0_{g,hf}"] \arrow[dr, "1_{\overline{g}}\otimes\theta_{hf,k}"] & {} & A_{ghfk} \\
 & \overline{g}\,\overline{h}\,A_{fk} \arrow[rr, "1_{\overline{g}}\otimes\theta_{h,fk}"'] \arrow[ur, phantom, "\Downarrow 1_{\overline{g}}\otimes\alpha_{h,f,k}"] & & \overline{g}\,A_{hfk} \arrow[ur, "\theta_{g,hfk}"'] &
\end{tikzcd} \tag{3.1}$$

$$\Vert$$

$$\begin{tikzcd}
 & \overline{gh}\,\overline{f}\,A_k \arrow[rr, "\chi^0_{gh,f}"] \arrow[dr, "1_{\overline{gh}}\otimes\theta_{f,k}"] & & \overline{ghf}\,A_k \arrow[dr, "\theta_{ghf,k}"] \arrow[d, phantom, "\Downarrow\alpha_{gh,f,k}"] & \\
\overline{g}\,\overline{h}\,\overline{f}\,A_k \arrow[ur, "\chi^0_{g,h}\otimes 1_{\overline{f}}"] \arrow[dr, "1_{\overline{g}}\otimes 1_{\overline{h}}\otimes\theta_{f,k}"'] \arrow[rr, phantom, "\Downarrow(\chi_{g,h})_{\theta_{f,k}}"] & & \overline{gh}\,A_{fk} \arrow[rr, "\theta_{gh,fk}"'] & {} \arrow[d, phantom, "\Downarrow\alpha_{g,h,fk}"] & A_{ghfk} \\
 & \overline{g}\,\overline{h}\,A_{fk} \arrow[ur, "\chi^0_{g,h}"'] \arrow[rr, "1_{\overline{g}}\otimes\theta_{h,fk}"'] & & \overline{g}\,A_{hfk} \arrow[ur, "\theta_{g,hfk}"'] &
\end{tikzcd}$$

holds in $\mathcal{B}(F_g(F_h(F_f(A_k))), A_{ghfk})$. If (A, θ, α) is a 0-cell, the identity 1-cell $I_{(A,\theta,\alpha)}$ is defined as follows. $I_{(A,\theta,\alpha)} = (I_{A_g}, l)$, where $l_{g,h} = \mathrm{id}_{\theta_{g,h}}$, for any $g, h \in G$.

If (A, θ, α) and (B, ρ, β) are objects in $\mathcal{B}[G]$, a 1-cell is a pair (X, l), where $X = \{X_g : A_g \to B_g\}$ is a G-indexed family of 1-cells and

$$\begin{tikzcd}
F_g(A_h) \arrow[r, "F_g(X_h)"] \arrow[d, "\theta_{g,h}"'] \arrow[dr, phantom, "\Downarrow l_{g,h}"] & F_g(B_h) \arrow[d, "\rho_{g,h}"] \\
A_{gh} \arrow[r, "X_{gh}"'] & B_{gh},
\end{tikzcd}$$

is a $G \times G$-indexed family of isomorphism 2-cells, such that for all $f, g, h \in G$, $l_{1,g} = \mathrm{id}_{X_g}$ and equation

$$\begin{tikzcd}
\overline{f}\,\overline{g}(A_h) \arrow[r, "\overline{f}\,\overline{g}(X_h)"] \arrow[d, "\overline{f}(\theta_{g,h})"'] \arrow[dr, phantom, "\Downarrow\overline{f}(l_{g,h})"] & \overline{f}\,\overline{g}(B_h) \arrow[r, "\chi^0_{f,g}"] \arrow[d, "\overline{f}\rho_{g,h})"'] \arrow[dr, phantom, "\Downarrow\beta_{f,g,h}"] & \overline{fg}(B_h) \arrow[d, "\rho_{fg,h}"] \\
\overline{f}(A_{gh}) \arrow[r, "\overline{f}(X_{gh})"'] \arrow[dr, "\theta_{f,gh}"'] & \overline{f}(B_{gh}) \arrow[r, "\rho_{f,gh}"'] \arrow[d, phantom, "\Downarrow l_{f,gh}"] & B_{fgh} \\
 & A_{fgh} \arrow[ur, "X_{fgh}"'] &
\end{tikzcd} \tag{3.2}$$

$$\Vert$$

$$
\begin{tikzcd}
\overline{f}\,\overline{g}(A_h) \arrow[rr, "\overline{f}\,\overline{g}(X_h)"] \arrow[d, "\overline{f}(\theta_{g,h})"'] \arrow[drr, "\chi^0_{f,g}"'] & & \overline{f}\,\overline{g}(B_h) \arrow[rr, "\chi^0_{f,g}"] \arrow[d, phantom, "\Downarrow(\chi_{f,g})_{X_h}"] & & \overline{fg}(B_h) \arrow[d, "\rho_{fg,h}"] \\
\overline{f}(A_{gh}) \arrow[rr, phantom, "\Downarrow\alpha_{f,g,h}"] \arrow[drr, "\theta_{f,gh}"'] & & \overline{fg}(A_h) \arrow[urr, "\overline{fg}(X_h)"'] \arrow[d, "\theta_{fg,h}"'] \arrow[rr, phantom, "\Downarrow l_{fg,h}"] & & {} \\
& & A_{fgh} \arrow[urr, "X_{fgh}"'] & &
\end{tikzcd}
$$

holds in $\mathcal{B}(F_f(F_g(A_h)), B_{fgh})$. If (X, l), (Y, s) are 1-cells, a 2-cell $m : (X, l) \Rightarrow (Y, s)$ is a G-indexed family of 2-cells $m = \{m_g : X_g \to Y_g\}$ such that for all $g, f \in G$, equation

$$
\begin{tikzcd}
F_g(A_h) \arrow[r, bend left=40, "F_g(X_h)"] \arrow[r, phantom, "\Downarrow l_{g,h}"] \arrow[d, "\theta_{g,h}"'] & F_g(B_h) \arrow[d, "\rho_{g,h}"] \\
A_{gh} \arrow[r, bend left=40, "X_{gh}"] \arrow[r, bend right=40, "Y_{gh}"'] \arrow[r, phantom, "\Downarrow m_{gh}"] & B_{gh}
\end{tikzcd}
\tag{3.3}
$$

$$\Vert$$

$$
\begin{tikzcd}
F_g(A_h) \arrow[r, bend left=40, "F_g(X_h)"] \arrow[r, bend right=40, "F_g(Y_h)"'] \arrow[r, phantom, "\Downarrow F_g(m_h)"] \arrow[d, "\theta_{g,h}"'] & F_g(B_h) \arrow[d, "\rho_{g,h}"] \\
A_{gh} \arrow[r, bend right=40, "Y_{gh}"'] \arrow[r, phantom, "\Downarrow s_{g,h}"] & B_{gh}
\end{tikzcd}
$$

holds in $\mathcal{B}(F_g(A_h), B_{gh})$.

The (vertical) composition in each category $\mathcal{B}[G]((A, \theta, \alpha), (B, \rho, \beta))$ is defined pointwise.

Now, let us define the horizontal composition $\circ : \mathcal{B}[G]((A, \theta, \alpha), (B, \rho, \beta)) \times \mathcal{B}[G]((C, \kappa, \gamma), (A, \theta, \alpha)) \to \mathcal{B}[G]((C, \kappa, \gamma), (B, \rho, \beta))$. If (A, θ, α) and (B, ρ, β) are 0-cells, and

$$(X, l) \in \mathcal{B}[G]((A, \theta, \alpha), (B, \rho, \beta)), \;\; (Y, s) \in \mathcal{B}[G]((C, \kappa, \gamma), (A, \theta, \alpha))$$

are 1-cells, define

$$(X,l)\circ(Y,s)=(Z,t),$$

where $Z_g=X_g\circ Y_g$, and $t_{g,h}=\big(1_{X_{gh}}\circ s_{g,h}\big)\big(l_{g,h}\circ 1_{F_g(Y_h)}\big)$, for any $g,h\in G$. The horizontal composition of 2-cells in $\mathcal{B}[G]$ is just the horizontal composition of 2-cells in $\mathcal{B}$.

Lemma 3.2. *$\mathcal{B}[G]$ is a 2-category endowed with a strict action of G.*

Proof. The proof that $\mathcal{B}[G]$ is indeed a 2-category follows by a straightforward calculation. Let us define now a canonical strict action of G on the 2-category $\mathcal{B}[G]$. For any $g\in G$ define the 2-functors $L_g:\mathcal{B}[G]\to\mathcal{B}[G]$ as follows. If (A,θ,α) is a 0-cell, $g,x\in G$, then

$$L_g(A)_x=A_{xg},\quad Lg(\theta)_{x,y}=\theta_{x,yg},\quad L_g(\alpha)_{x,y,z}=\alpha_{x,y,zg}.$$

If $(X,l):(A,\theta,\alpha)\to(B,\rho,\beta)$ is a 1-cell,

$$L_g(X)_x=X_{xg},\quad L_g(l)_{x,y}=l_{xyg}.$$

If $m:(X,l)\Rightarrow(Y,s)$ is a 2-cell, then $L_g(m)_x=m_{xg}$, for any $x\in G$. Since the L_g are 2-functors such that $L_g\circ L_h=L_{gh}$ for all $g,h\in G$ and $L_e=\mathrm{Id}_{\mathcal{B}[G]}$, L defines a strict action of G on $\mathcal{B}[G]$. □

There is a pseudofunctor $\mathcal{H}:\mathcal{B}\to\mathcal{B}[G]$ defined as follows. If A is a 0-cell in $\mathcal{B}$, then

$$\mathcal{H}(A)=(\{F_g(A)\},(\chi^0_{g,h})_A,\omega_{g,h,f})_{f,g,h\in G},$$

if $X:A\to B$ is a 1-cell, then $\mathcal{H}(X)=(F_g(X),(\chi_{g,h})_X)$ and for 2-cells $m:X\to Y$, $\mathcal{H}(m)_g=F_g(m)$, where $f,g,h\in G$. The fact that ω are modifications implies that $\mathcal{H}(X)$ is indeed a 1-cell in $\mathcal{B}[G]$. The following proposition implies immediately Theorem 3.1

Proposition 3.3. *$\mathcal{H}:\mathcal{B}\to\mathcal{B}[G]$ is a G-biequivalence.*

Proof. If (A,θ,α) is an object in $\mathcal{B}[G]$, then the 1-equivalences $\theta_{g,e}:\mathcal{H}(A_e)_g\to A_g$ and the 2-cells

$$\begin{tikzcd}
F_g\mathcal{H}(A_e)_h \arrow[r, "\chi^0_{g,h}"] \arrow[d, "F_g(\theta_{h,e})"'] \arrow[dr, phantom, "\Downarrow\alpha_{g,h,e}"] & \mathcal{H}(A_e)_{gh} \arrow[d, "\theta_{gh,e}"] \\
F_g(A_h) \arrow[r, "\theta^0_{g,h}"'] & A_{ghf},
\end{tikzcd}$$

defines a 1-equivalence from $\mathcal{H}(A_1)$ to A, that is, $\mathcal{H}$ is bi-essentially surjective.

Let A and B be objects in $\mathcal{B}$, and $(X,l):\mathcal{H}(A)\to\mathcal{H}(B)$ be a 1-cell in $\mathcal{B}[G]$. The invertible 2-cells $l_{g,1}:\mathcal{H}(X_1)_g\to X_g$ define an invertible 2-cell from $\mathcal{H}(X_1)$ to X. Then $\mathcal{H}$ is locally essentially surjective.

If $X, Y, \in \mathcal{B}(A, B)$ and $f, f' : X \to Y$ such that $\mathcal{H}(f) = \mathcal{H}(f')$. Thus, $\mathcal{H}(f)_1 = \mathcal{H}(f')_1$, but since we are considering a unital action, $f = \mathcal{H}(f)_1 = \mathcal{H}(f')_1 = f'$, that is, $\mathcal{H}$ is locally faithful. Suppose $w : \mathcal{H}(X) \to \mathcal{H}(Y)$ is a 2-cell in $\mathcal{B}[G]$, condition (3.3) implies that $w_g = F_g(m_1)$, then $w = \mathcal{H}(w_1)$. Since, $\mathcal{H}$ is bi-essentially surjective and locally fully faithful, $\mathcal{H}$ is a biequivalence.

To see that $\mathcal{H}$ has a canonical structure of G-pseudofunctor, we note that

$$(\mathcal{H} \circ F_g)_x = F_x \circ F_g, \qquad (L_g \circ \mathcal{H})_x = F_{xg},$$

for any $x, g \in G$. Then, using the pseudonatural transformations $\chi_{x,g} : F_x \circ F_g \to F_{xg}$, we define a pseusonatural transformation

$$\gamma_g : \mathcal{H} \circ F_g \to L_g \circ \mathcal{H},$$

as follows. For any 0-cell $A \in \mathrm{Obj}\,(\mathcal{B})$ we have to define an equivalence 1-cell $\gamma_A^0 : \mathcal{H} \circ F_g(A) \to L_g \circ \mathcal{H}(A)$ in $\mathcal{B}[G]$. Set $\gamma_A^0 = (X, l)$, where, for any $x,\ f, h \in G$

$$X_x = (\chi_{x,g}^0)_A, \quad l_{f,h} = (\omega_{f,h,g}^{-1})_A.$$

Axiom (2.3) implies that morphisms $l_{f,h}$ fulfill condition (3.2). Thus, γ_A^0 is indeed a 1-cell in $\mathcal{B}[G]$. To complete the definition of of the pseudonatural equivalence γ_g, we have to define, 2-cells in $\mathcal{B}[G]$

$$(\gamma_g)_X : \gamma_B^0 \circ \mathcal{H}F_g(X) \to L_g\mathcal{H}(X) \circ \gamma_A^0,$$

for any 1-cell $X \in \mathcal{B}(A, B)$. Set $\big((\gamma_g)_X\big)_x = (\chi_{x,g})_X$, for any $x \in G$. The fact that ω are modifications, imply that 2-cells $\big((\gamma_g)_X\big)_x$ satisfy (3.3). To define the modifications

$$\begin{tikzcd}
 & L_f\mathcal{H}F_g \arrow[rr, "1_{L_f}\otimes\gamma_g"] & & L_fL_g\mathcal{H} \arrow[dr, "id"] & \\
\mathcal{H}F_f\ F_g \arrow[ur, "\gamma_f\otimes 1_{F_g}"] \arrow[drr, "1_{\mathcal{H}}\otimes\chi_{f,g}"'] & & \Downarrow\Pi_{f,g} & & L_{fg}\mathcal{H} \\
 & & \mathcal{H}F_{fg} \arrow[urr, "\gamma_{fg}"'] & &
\end{tikzcd}$$

we note that

$$[(1_{L_f\otimes\gamma_g}) \circ (\gamma_f \otimes 1_{F_g})]_x = \chi_{xf,g} \circ (\chi_{x,f} \otimes 1_{F_g}), \quad x, f, g \in G,$$

and

$$[(1_{\mathcal{H}} \otimes \chi_{f,g}) \circ (\gamma_{fg})]_x = \chi_{x,fg} \circ (1_{F_x} \otimes \chi_{f,g}), \quad x, f, g \in G.$$

Then we define $(\Pi_{f,g})_x = \omega_{x,f,g}$ for all $x, g, f \in G$.

Since $\omega_{x,f,g}$ are modifications, $\Pi_{g,h}$ turns out to be modifications for any $g, h \in G$. Condition described in diagram (2.5) is exactly diagram (2.3). □

4. The equivariant 2-category

Let G be a group. Denote by $\mathcal{I}$ the unit 2-category endowed with the trivial action of G, and assume that $\mathcal{B}$ is a 2-category with an action of G.

Definition 4.1. The *equivariant 2-category* is $\mathcal{B}^G = \mathbf{2Cat}^G(\mathcal{I}, \mathcal{B})$. 0-cells, 1-cells and 2-cells in $\mathcal{B}^G$ will be called *equivariant* 0-cells, 1-cells and 2-cells, respectively.

Proposition 4.2. *Assume $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are G-biequivalent. Then the 2-categories $\mathcal{B}^G$, $\widetilde{\mathcal{B}}^G$ are biequivalent.*

Proof. Straightforward. □

Lemma 4.3. *There exists a forgetfull 2-functor $\Phi : \mathcal{B}^G \to \mathcal{B}$.* □

Proof. If $(\mathcal{H}, \Pi, \gamma)$ is an equivariant 0-cell in $\mathcal{B}^G$, then $\Phi(\mathcal{H}, \Pi, \gamma) = \mathcal{H}(\star)$. If $(\theta, \{\theta_g\}_{g\in G})$ is an equivariant 1-cell, then $\Phi(\theta, \{\theta_g\}_{g\in G}) = \theta$. On 2-cells the functor Φ is the identity. □

4.1. Unpacking definition of equivariantization

We shall explicitly describe the 2-category $\mathcal{B}^G$. This would allows us to show concrete examples and obtain some results in Sect. 7.

We shall assume that there is a unital action of G on the 2-category $\mathcal{B}$ such that all pseudofunctors F_g are 2-functors. This is possible using Corollary 2.12. The 2-category $\mathcal{B}^G$ has 0-cells triples $(A, \{U_g\}_{g\in G}, \{\Pi_{g,h}\}_{g,h\in G})$, where

- A is a 0-cell in $\mathcal{B}$;
- U_g are invertible 1-cells in $\mathcal{B}(A, F_g(A))$;
- $\Pi_{g,h} : (\chi^0_{g,h})_A \circ F_g(U_h) \circ U_g \to U_{gh}$ are isomorphisms 2-cells in the category $\mathcal{B}(A, F_{gh}(A))$ such that

$$\begin{aligned} &U_1 = I_A,\ \Pi_{g,1} = \mathrm{id}_{U_g} = \Pi_{1,g},\\ &\Pi_{f,gh}\big(\mathrm{id}_{(\chi^0_{f,gh})_A} \circ F_f(\Pi_{g,h}) \circ \mathrm{id}_{U_f}\big)\big((\omega_{f,g,h})_A \circ \mathrm{id}_{F_f F_g(U_h) F_f(U_g) U_f}\big)\\ &\quad = \Pi_{fg,h}\big(\mathrm{id}_{(\chi^0_{fg,h})_A F_{fg}(U_h)} \circ \Pi_{f,g}\big)\big(\mathrm{id}_{(\chi^0_{fg,h})_A} \circ (\chi_{f,g})_{U_h} \circ \mathrm{id}_{F_f(U_g)U_f}\big) \end{aligned} \tag{4.1}$$

for all $g, h, f \in G$. For short, the collection $(A, \{U_g\}_{g\in G}, \{\Pi_{g,h}\}_{g,h\in G})$ will be denoted simply as (A, U, Π).

Given two equivariant 0-cells (A, U, Π), $(\widetilde{A}, \widetilde{U}, \widetilde{\Pi})$, an *equivariant 1-cell* is a pair $(\theta, \{\theta_g\}_{g\in G}) \in \mathcal{B}^G((A, U, \Pi), (\widetilde{A}, \widetilde{U}, \widetilde{\Pi}))$ where

- $\theta : \mathcal{B}(A, \widetilde{A})$ is a 1-cell,
- and for any $g \in G$, $\theta_g : F_g(\theta) \circ U_g \Rightarrow \widetilde{U}_g \circ \theta$, are invertible 2-cells such that $\theta_1 = \mathrm{id}_\theta$, and such that for any $g, f \in G$

$$\big(\widetilde{\Pi}_{g,f} \circ \mathrm{id}_{\theta}\big)\big(\mathrm{id}_{(\chi^0_{g,f})_A F_g(\widetilde{U}_f)} \circ \theta_g\big)\big(\mathrm{id}_{(\chi^0_{g,f})_A} \circ F_g(\theta_f) \circ \mathrm{id}_{U_g}\big)$$
$$= \theta_{gf}\big(\mathrm{id}_{F_{gf}(\theta)} \circ \Pi_{g,f}\big)\big((\chi_{g,f})_{\theta} \circ \mathrm{id}_{F_g(U_f)U_g}\big). \tag{4.2}$$

If $(\theta, \{\theta_g\}_{g\in G}), (\sigma, \{\sigma_g\}_{g\in G}) : (A, U, \mu) \to (\widetilde{A}, \widetilde{U}, \widetilde{\mu})$ are equivariant 1-cells, an *equivariant 2-cell* $\alpha : (\theta, \{\theta_g\}_{g\in G}) \to (\sigma, \{\sigma_g\}_{g\in G})$ is a 2-cell $\alpha : \theta \to \sigma$ such that for all $g \in G$

$$(\mathrm{id}_{\widetilde{U}_g} \circ \alpha)\theta_g = \sigma_g(F_g(\alpha) \circ \mathrm{id}_{U_g}). \tag{4.3}$$

Suppose that $(A, U, \mu), (\widetilde{A}, \widetilde{U}, \widetilde{\mu}), (A', U', \mu')$ are equivariant 0-cells, and

$$(\theta, \theta_g) : (A', U', \mu') \to (\widetilde{A}, \widetilde{U}, \widetilde{\mu}), (\sigma, \sigma_g) : (A, U, \mu) \to (A', U', \mu')$$

are equivariant 1-cells, then the composition $(\theta, \theta_g) \circ (\sigma, \sigma_g) : (A, U, \mu) \to (\widetilde{A}, \widetilde{U}, \widetilde{\mu})$ is defined as $(\theta, \theta_g) \circ (\sigma, \sigma_g) = (\theta \circ \sigma, (\theta \circ \sigma)_g)$, where for any $g \in G$

$$(\theta \circ \sigma)_g = (\theta_g \circ \mathrm{id}_{\sigma})(\mathrm{id}_{F_g(\theta)} \circ \sigma_g). \tag{4.4}$$

5. Group actions from graded tensor categories

Starting with a G-graded tensor category $\oplus_{g\in G}\mathcal{C}_g$, we shall construct a G-action on the 2-category of $\mathcal{C}_1$-representations.

5.1. Group actions on tensor categories

Let G be a finite group and $\mathcal{C}$ be a finite tensor category. An action of G on $\mathcal{C}$ consists of the following data:

- tensor autoequivalences $(g_*, \xi^g) : \mathcal{C} \to \mathcal{C}$ for any $g \in G$,
- a natural isomorphism $\zeta : \mathrm{Id}_{\mathcal{C}} \to (1)_*$,
- and monoidal natural isomorphisms $\nu_{g,h} : g_* \circ h_* \to (gh)_*$,

such that for all $X \in \mathcal{C}, g, h, f \in G$

$$(\nu_{gh,f})_X(\nu_{g,h})_{f_*(X)} = (\nu_{g,hf})_X g_*((\nu_{h,f})_X), \tag{5.1}$$
$$(\nu_{g,1})_X g_*(\zeta_X) = \mathrm{id}_X = (\nu_{1,g})_X \zeta_{g_*(X)}. \tag{5.2}$$

For simplicity, we shall assumed that $(1)_* = \mathrm{Id}_{\mathcal{C}}$, $\zeta = \mathrm{id}$ and $\mu_{g,1} = \mathrm{id} = \nu_{1,g}$ for all $g \in G$.

If a finite group G acts on a finite tensor category $\mathcal{C}$, there is associated a new finite tensor category $\mathcal{C}^G$ called the *equivariantization* of $\mathcal{C}$ by G. An object in $\mathcal{C}^G$ is a pair (X, s), where $X \in \mathcal{C}$ is an object together with isomorphisms $s_g : g_*(X) \to X$ satisfying

$$s_1 = \mathrm{id}_X, \quad s_{gh} \circ (\nu_{g,h})_X = s_g \circ g_*(s_h), \tag{5.3}$$

for all $g, h \in G$. A *G-equivariant morphism* $f : (V, s) \to (W, t)$ between G-equivariant objects (V, s) and (W, t), is a morphism $f : V \to W$ in $\mathcal{C}$ such that $f \circ s_g = t_g \circ g_*(f)$ for all $g \in G$. The category $\mathcal{C}^G$ has a monoidal product as follows. If $(V, s), (W, t) \in \mathcal{C}^G$, then $(V, s)\otimes(W, t) = (V\otimes W, r)$, where for any $g \in G$

$$r_g = (s_g \otimes t_g)(\xi^g_{V,W})^{-1}.$$

For more details we refer the reader to [1–3].

There is also associated the graded tensor category $\mathcal{C}[G]$, with underlying abelian category $\mathcal{C}[G] = \oplus_{g\in G}\mathcal{C}_g$, where $\mathcal{C}_g = \mathcal{C}$ for any $g \in G$. If $X \in \mathcal{C}$ is an object, the object in $\mathcal{C}_g$ is denoted by $[X, g]$. The tensor product is

$$[X, g]\otimes[Y, h] = [X\otimes g_*(Y), gh], \quad X, Y \in \mathcal{C}, g, h \in G.$$

The reader is refered to [24] for the complete monoidal structure of this tensor category.

5.2. *Representations of tensor categories*

A left $\mathcal{C}$-*module category* over a tensor category $\mathcal{C}$ is a finite $\Bbbk$-linear abelian category $\mathcal{M}$ equipped with

- a $\Bbbk$-bilinear bi-exact bifunctor $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$;
- natural associativity and unit isomorphisms $m_{X,Y,M} : (X \otimes Y)\overline{\otimes}M \to X\overline{\otimes}(Y\overline{\otimes}M)$, $\ell_M : \mathbf{1}\overline{\otimes}M \to M$, such that

$$m_{X,Y,Z\overline{\otimes}M}\, m_{X\otimes Y,Z,M} = (\mathrm{id}_X\overline{\otimes}m_{Y,Z,M})\, m_{X,Y\otimes Z,M}(a_{X,Y,Z}\overline{\otimes}\mathrm{id}_M), \tag{5.4}$$

$$(\mathrm{id}_X\overline{\otimes}l_M)m_{X,\mathbf{1},M} = \mathrm{id}_{X\overline{\otimes}M}. \tag{5.5}$$

A *module functor* between module categories $\mathcal{M}$ and $\mathcal{N}$ over a tensor category $\mathcal{C}$ is a pair (F, c), where

- $F : \mathcal{M} \to \mathcal{N}$ is a left exact functor;
- natural isomorphism: $c_{X,M} : F(X\overline{\otimes}M) \to X\overline{\otimes}F(M)$, $X \in \mathcal{C}$, $M \in \mathcal{M}$, such that for any $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$:

$$(\mathrm{id}_X\overline{\otimes}c_{Y,M})c_{X,Y\overline{\otimes}M}F(m_{X,Y,M}) = m_{X,Y,F(M)}\, c_{X\otimes Y,M} \tag{5.6}$$

$$\ell_{F(M)}\, c_{\mathbf{1},M} = F(\ell_M). \tag{5.7}$$

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{C}$-module categories. We denote by $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ the category whose objects are module functors (F, c) from $\mathcal{M}$ to $\mathcal{N}$. A morphism between (F, c) and $(G, d) \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is a natural transformation $\alpha : F \to G$ such that for any $X \in \mathcal{C}$, $M \in \mathcal{M}$:

$$d_{X,M}\alpha_{X\overline{\otimes}M} = (\mathrm{id}_X\overline{\otimes}\alpha_M)c_{X,M}. \tag{5.8}$$

We shall also say that $\alpha : F \to G$ is a $\mathcal{C}$-*module transformation*.

Let $(F, \xi, \phi) : \mathcal{C} \to \mathcal{C}$ be a tensor functor and let $(\mathcal{M}, \overline{\otimes}, m)$ be a $\mathcal{C}$-module category. We shall denote by $\mathcal{M}^F$ the $\mathcal{C}$-module category with the same underlying abelian category $\mathcal{M}$ and action, associativity and unit morphisms defined, respectively, by

$$X\overline{\otimes}^F M = F(X)\overline{\otimes}M,$$

$$m^F_{X,Y,M} = m_{F(X),F(Y),M}(\xi^{-1}_{X,Y}\overline{\otimes}\,\mathrm{id}_M), \quad l^F_M = l_M(\phi\overline{\otimes}\mathrm{id}_M),$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$. Right $\mathcal{C}$-module and $\mathcal{C}$-bimodule categories are defined in a similar way. For the complete definition see [10].

A $\mathcal{C}$-module category $\mathcal{M}$ is *exact* [5] if, for any projective object $P \in \mathcal{C}$, the object $P\overline{\otimes}M$ is projective in $\mathcal{M}$ for all $M \in \mathcal{M}$. If $\mathcal{M}$ is a left $\mathcal{C}$-module then $\mathcal{M}^{\mathrm{op}}$ is the right $\mathcal{C}$-module over the opposite Abelian category with action

$$\mathcal{M}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}^{\mathrm{op}}, (M, X) \mapsto X^*\overline{\otimes}M, \tag{5.9}$$

associativity isomorphisms $m^{\mathrm{op}}_{M,X,Y} = m_{Y^*,X^*,M}$ for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$. Analogously, if $\mathcal{M}$ is a right $\mathcal{C}$-module category, then $\mathcal{M}^{\mathrm{op}}$ is a left $\mathcal{C}$-module category. If $\mathcal{M}$ is a $\mathcal{C}$-bimodule category, we denote $\overline{\mathcal{M}}$ the opposite Abelian category, with left and right $\mathcal{C}$-module structure given as in (5.9).

5.3. 2-categories of representations of tensor categories

Suppose that $\mathcal{C}$ is a tensor category. The 2-category ${}_{\mathcal{C}}\mathrm{Mod}$ has as 0-cells, left $\mathcal{C}$-module categories, if $\mathcal{M}, \mathcal{N}$ are $\mathcal{C}$-module categories, then the category ${}_{\mathcal{C}}\mathrm{Mod}(\mathcal{M}, \mathcal{N}) = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$. Analogously we define the 2-category $\mathrm{Mod}_{\mathcal{C}}$ of right $\mathcal{C}$-module categories.

If $\mathcal{C}$ is a finite tensor category, the 2-category ${}_{\mathcal{C}}\mathrm{Mod}_{\mathrm{e}}$ of exact left $\mathcal{C}$-module categories is defined in a similar way as ${}_{\mathcal{C}}\mathrm{Mod}$, with 0-cells being exact left $\mathcal{C}$-module categories. It is known that ${}_{\mathcal{C}}\mathrm{Mod}_{\mathrm{e}}$ is 2-equivalent to ${}_{\mathcal{D}}\mathrm{Mod}_{e}$ if and only if $\mathcal{C}$ is Morita equivalent to $\mathcal{D}$.

5.4. G-Graded tensor categories

Let G be a finite group. A (faithful) G-grading on a finite tensor category $\mathcal{D}$ is a decomposition $\mathcal{D} = \oplus_{g\in G}\mathcal{C}_g$, where $\mathcal{C}_g$ are full abelian subcategories of $\mathcal{D}$ such that

- $\mathcal{C}_g \neq 0$;
- $\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$ for all $g, h \in G$.

In this case $\mathcal{C} = \mathcal{C}_1$ is a tensor subcategory of $\mathcal{D}$ and each $\mathcal{C}_g$ is an exact $\mathcal{C}$-bimodule category. We shall assume that $\mathcal{C}_g \neq 0$ for any $g \in G$. The tensor category $\mathcal{D}$ is called a *G-graded extension* of $\mathcal{C}$.

In [4] Etingof, Nikshych, and Ostrik studied fusion categories graded by a finite group. They reduce the classification problem of fusion categories graded by a group G to the classification (up to homotopy) of maps from BG to $B\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$, the classifying spaces of the monoidal bicategory where objects are invertible bimodules, 1-arrows are bimodule equivalences and 2-arrows are bimodule natural isomorphisms, see [4] for details. Since tricategories are algebraic models of homotopy 3-types, extension of a fusion categories are classified by monoidal pseudofunctors from $\underline{\underline{G}}$ to $\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$, where $\underline{\underline{G}}$ is the discrete monoidal 2-category with objects G, see

[4, Section 8]. Now, since the monoidal bicategory $\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$ can be interpreted as the monoidal bicategory of biequivalences of $\mathrm{Mod}_{\mathcal{C}}$, then it is natural to expect that every G-extension of $\mathcal{C}$ induces an action of G on $\mathrm{Mod}_{\mathcal{C}}$.

In this section, we explicitly present the action associated with a G-extension of any finite tensor category as well as some consequence of this fact.

If $\mathcal{M}$ is a left $\mathcal{C}$-module category, $X \in \mathcal{C}_g$, $M \in \mathcal{M}$, the functor $G_{X,M} : \overline{\mathcal{C}_g} \to \mathcal{M}$ defined by

$$G_{X,M}(Y) = ({}^*Y \otimes X)\overline{\otimes} M,$$

for any $Y \in \mathcal{C}_g$, is a $\mathcal{C}$-module functor. Moreover, the functor

$$\Phi : \mathcal{C}_g \boxtimes_{\mathcal{C}} \mathcal{M} \to \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{C}_g}, \mathcal{M}), \quad \Phi(X \boxtimes M) = G_{X,M},$$

is an equivalence of $\mathcal{C}$-module categories.This is a particular case of [10, Thm. 3.20].

5.5. *The relative center of a bimodule category*

The next definition appeared in [8].

Definition 5.1. Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ a $\mathcal{C}$-bimodule category. The *relative center* of $\mathcal{M}$ is the category $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ of $\mathcal{C}$-bimodule functors from $\mathcal{C}$ to $\mathcal{M}$.

Explicitly, objects of $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ are pairs (M, γ), where M is an objects of $\mathcal{M}$ and

$$\gamma = \{\gamma_X : X \overline{\otimes} M \xrightarrow{\sim} M \overline{\otimes} X\}_{X \in \mathcal{C}}$$

is a natural family of isomorphisms such that

$$\gamma_X \circ \alpha^{-1}_{X,M,Y} \circ \gamma_Y = \alpha^{-1}_{M,X,Y} \circ \gamma_{X \otimes Y} \circ \alpha^{-1}_{X,Y,M}, \tag{5.10}$$

where $\alpha_{X,M,Y} : (X \overline{\otimes} M) \overline{\otimes} Y \xrightarrow{\sim} X \overline{\otimes} (M \overline{\otimes} Y)$ are the associativity constraints in $\mathcal{M}$.

Let $\mathcal{D} = \oplus_{g \in G} \mathcal{C}_g$ be a G-graded tensor category, with $\mathcal{C} = \mathcal{C}_1$. The inclusion functor $\mathcal{C} \hookrightarrow \mathcal{D}$ induces the forgetful pseudofunctor $\mathcal{H} : {}_{\mathcal{D}}\mathrm{Mod} \to {}_{\mathcal{C}}\mathrm{Mod}$.

Proposition 5.2. *There is a monoidal equivalence* $\mathcal{Z}(\mathcal{H}) \backsimeq \mathcal{Z}_{\mathcal{C}}(\mathcal{D})$.

Proof. Let us define the functor $\mathcal{F} : \mathcal{Z}_{\mathcal{C}}(\mathcal{D}) \to \mathcal{Z}(\mathcal{H})$, as follows. For any $(V, \gamma) \in \mathcal{Z}_{\mathcal{C}}(\mathcal{D})$ set $\mathcal{F}(V, \gamma) = (W^V, \tau)$. Here, for each $\mathcal{M} \in {}_{\mathcal{D}}\mathrm{Mod}$, $W^V_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is the $\mathcal{C}$-module functor given by

$$W^V_{\mathcal{M}}(M) = V \overline{\otimes} M.$$

The isomorphisms endowing the functor $W^V_{\mathcal{M}}$ structure of $\mathcal{C}$-module functor are

$$c_{X,M} : W^V_{\mathcal{M}}(X \overline{\otimes} M) \to X \overline{\otimes} W^V_{\mathcal{M}}(M),$$

given by the following composition:

$$
\begin{aligned}
W^V_{\mathcal{M}}(X\overline{\otimes}M) = V\overline{\otimes}(X\overline{\otimes}M) &\xrightarrow{m^{-1}_{V,X,M}} \\
&(V\otimes X)\overline{\otimes}M \xrightarrow{\gamma_X^{-1}\overline{\otimes}\mathrm{id}\,_M} (X\otimes V)\overline{\otimes}M \\
\xrightarrow{m_{X,V,M}} X\overline{\otimes}(V\overline{\otimes}M) = X\overline{\otimes}W^V_{\mathcal{M}}(M),
\end{aligned}
$$

for any $X\in\mathcal{C}$, $M\in\mathcal{M}$. It follows that $(W^V_{\mathcal{M}}, c)$ is a $\mathcal{C}$-module functor.

Now, we shall explain the definition of τ. Take $\mathcal{M},\mathcal{N}\in {}_{\mathcal{D}}\mathrm{Mod}$, and $(G,d):\mathcal{M}\to\mathcal{N}$ a $\mathcal{D}$-module functor. Define

$$
\begin{aligned}
&\tau_{(G,d)}: W^V_{\mathcal{N}}\circ G\to G\circ W^V_{\mathcal{M}},\\
&\quad (\tau_{(G,d)})_M : V\overline{\otimes}G(M)\to G(V\overline{\otimes}M),\ (\tau_{(G,d)})_M = d^{-1}_{V,M},
\end{aligned}
$$

for any $M\in\mathcal{M}$. Then, $\tau_{(G,d)}$ is a $\mathcal{C}$-module natural isomorphism.

Now, we shall define the functor $\mathcal{F}$ on morphisms. Let (V,γ), (V',γ') be objects in $\mathcal{Z}_{\mathcal{C}}(\mathcal{D})$ and $f:(V,\gamma)\to(V',\gamma')$ be an arrow in $\mathcal{Z}_{\mathcal{C}}(\mathcal{D})$. Define $\mathcal{F}(f):(W^V,\tau)\to(W^{V'},\tau')$, as follows. For any $\mathcal{D}$-module $\mathcal{M}$, define the $\mathcal{C}$-module natural transformation

$$
\mathcal{F}(f)_{\mathcal{M}} : W^V_{\mathcal{M}}\to W^{V'}_{\mathcal{M}},\quad (\mathcal{F}(f)_{\mathcal{M}})_M = f\overline{\otimes}\mathrm{id}\,_M,
$$

for any $M\in\mathcal{M}$.

Now, we shall define a functor $\mathcal{G}:\mathcal{Z}(\mathcal{H})\to\mathcal{Z}_{\mathcal{C}}(\mathcal{D})$, that will be the inverse of $\mathcal{F}$. Any object $X\in\mathcal{C}$ induces a $\mathcal{D}$-module functor $J_X:\mathcal{D}\to\mathcal{D}$, $J_X(V)=V\otimes X$.

Let (W,τ) be an object in $\mathcal{Z}(\mathcal{H})$. For any $\mathcal{D}$-module category $\mathcal{M}$, $W_{\mathcal{M}}:\mathcal{M}\to\mathcal{M}$ is a $\mathcal{C}$-module functor. We shall denote it by $W_{\mathcal{M}}=(W_{\mathcal{M}},c^{\mathcal{M}})$. In particular, $W_{\mathcal{D}}(\mathbf{1})\in\mathcal{D}$. We have natural $\mathcal{C}$-module isomorphisms $(\tau_{\mathcal{D},\mathcal{D}})_{J_X}:W_{\mathcal{D}}\circ J_X\xrightarrow{\cong} J_X\circ W_{\mathcal{D}}$. In particular, we have isomorphisms

$$
((\tau_{\mathcal{D},\mathcal{D}})_{J_X})_{\mathbf{1}} : W_{\mathcal{D}}(X)\xrightarrow{\cong} W_{\mathcal{D}}(\mathbf{1})\otimes X.
$$

Using that $W_{\mathcal{D}}$ has a $\mathcal{C}$-module structure, there is a natural isomorphism

$$
c^{\mathcal{D}}_{X,\mathbf{1}} : X\otimes W_{\mathcal{D}}(\mathbf{1})\to W_{\mathcal{D}}(X).
$$

Let γ be the natural isomorphism defined as

$$
\gamma_X : X\otimes W_{\mathcal{D}}(\mathbf{1})\to W_{\mathcal{D}}(\mathbf{1})\otimes X,\quad \gamma_X = ((\tau_{\mathcal{D},\mathcal{D}})_{J_X})_{\mathbf{1}}\circ c_{X,\mathbf{1}}.
$$

The natural transformation γ satisfies 5.10 since $(\tau_{\mathcal{D},\mathcal{D}})_{J_X}$ is a $\mathcal{C}$-module natural transformation. Then $(W_{\mathcal{D}}(\mathbf{1}),\gamma)\in\mathcal{Z}_{\mathcal{C}}(\mathcal{D})$. Whence, we define $\mathcal{G}(W,\tau)=(W_{\mathcal{D}}(\mathbf{1}),\gamma)$.

Let $f:(W,\tau)\to(W',\tau')$ be a morphism in $\mathcal{Z}(\mathcal{H})$, then $(f_{\mathcal{D}})_{\mathbf{1}}$ is a morphism in $\mathcal{Z}_{\mathcal{C}}(\mathcal{D})$ since $f_{\mathcal{D}}$ is a $\mathcal{C}$-module natural transformation. Set $\mathcal{G}(f)=(f_{\mathcal{D}})_{\mathbf{1}}$. It follows straightforward that $\mathcal{G}$ is well-defined and that $\mathcal{F}$ and $\mathcal{G}$ are inverse of each other. $\square$

The center of the 2-category of representations of a tensor category $\mathcal{C}$ coincides with the Drinfeld center of $\mathcal{C}$.

Corollary 5.3. $\mathcal{Z}({}_{\mathcal{C}}\mathrm{Mod}\,) \simeq \mathcal{Z}(\mathcal{C})$.

Proof. Take $\mathcal{D} = \mathcal{C}$ and $\mathcal{H} : {}_{\mathcal{C}}\mathrm{Mod}\, \to {}_{\mathcal{C}}\mathrm{Mod}\,$ the identity pseudofunctor. □

5.6. Group actions coming from graded tensor categories

Throughout this section G will denote a finite group. Assume that $\mathcal{C}$ is a finite tensor category and $\mathcal{D} = \oplus_{g\in G}\mathcal{D}_g$ is a G-graded extension of $\mathcal{C}$. Set $\mathcal{D}_1 = \mathcal{C}$. We shall further assume that $\mathcal{D}$ is a strict monoidal category.

In this section we aim to prove the following result.

Theorem 5.4. *There is an action of G on the 2-category ${}_{\mathcal{C}}\mathrm{Mod}^{\,\mathrm{op}}$. Moreover, there are 2-equivalences*

$$({}_{\mathcal{C}}\mathrm{Mod}^{\,\mathrm{op}})^G \simeq {}_{\mathcal{D}}\mathrm{Mod}\,, \quad ({}_{\mathcal{C}}\mathrm{Mod}^{\,\mathrm{op}}_{e})^G \simeq {}_{\mathcal{D}}\mathrm{Mod}_{\,e}.$$

Proof. First, let us define an action of G on the 2-category $\mathcal{B} = {}_{\mathcal{C}}\mathrm{Mod}^{\,\mathrm{op}}$. For any $g \in G$ define the 2-functors $F_g : \mathcal{B} \to \mathcal{B}$ as follows. For any left $\mathcal{C}$-module category $\mathcal{M}$, set $F_g(\mathcal{M}) = \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{D}}_g, \mathcal{M})$. If $\mathcal{M}, \mathcal{N}$ are left $\mathcal{C}$-module categories, and $G : \mathcal{M} \to \mathcal{N}$ is a $\mathcal{C}$-module functor, then

$$F_g(G) : \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{D}}_g, \mathcal{M}) \to \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{D}}_g, \mathcal{N}), \quad F_g(G)(H) = G \circ H.$$

Now, we shall define the pseudonatural equivalences $\chi_{g,h} : F_g \circ F_h \to F_{gh}$, for any $g, h \in G$. For any left $\mathcal{C}$-module category $\mathcal{M}$

$$(\chi^0_{g,h})_{\mathcal{M}} : \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{D}}_{gh}, \mathcal{M}) \to \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{D}}_g, \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{D}}_h, \mathcal{M})),$$
$$(\chi^0_{g,h})_{\mathcal{M}}(H)(X)(Y) = H(X\otimes Y),$$

for any $H \in \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{D}}_{gh}, \mathcal{M})$, $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$. It follows that $(\chi^0_{g,h})_{\mathcal{M}}$ is a well-defined $\mathcal{C}$-module functor. For any $\mathcal{C}$-module functor $G : \mathcal{M} \to \mathcal{N}$ we have that $F_g(F_h(G)) \circ (\chi^0_{g,h})_{\mathcal{M}} = (\chi^0_{g,h})_{\mathcal{N}} \circ F_{gh}(G)$, whence, we can define

$$(\chi_{g,h})_G : F_g(F_h(G)) \circ (\chi^0_{g,h})_{\mathcal{M}} \to (\chi^0_{g,h})_{\mathcal{N}} \circ F_{gh}(G)$$

to be the identities. Since $\chi_{gh,f} \circ (\chi_{g,h}\otimes \mathrm{id}_{F_f}) = \chi_{g,hf} \circ (\mathrm{id}_{F_g}\otimes\chi_{h,f})$, for any $f, g, h \in G$, then we can choose $\omega_{g,h,f}$ to be the identities.

Now, we shall define a biequivalence $\Phi : \mathcal{B}^G \to {}_{\mathcal{D}}\mathrm{Mod}\,$. Assume $(\mathcal{M}, U, \Pi)$ is an equivariant 0-cell. This means that we have $\mathcal{C}$-module functors

$$U_g : \mathrm{Fun}_{\mathcal{C}}(\overline{\mathcal{D}}_g, \mathcal{M}) \to \mathcal{M},$$

together with $\mathcal{C}$-module natural isomorphisms

$$\Pi_{g,h} : U_g \circ F_g(U_h) \circ (\chi^0_{g,h})_{\mathcal{M}} \to U_{gh},$$

satisfying the required axioms. Recall the definition of the functors $G_{X,M}$ given in Sect. 5.4.

Claim 5.5. *Let be $g, h \in G$. If $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$, then, there exists a family of $\mathcal{C}$-module natural isomorphisms*

$$\beta_{X,Y,M} : F_g(U_h)\big((\chi^0_{g,h})_{\mathcal{M}}(G_{X\otimes Y,M})\big) \to G_{X,U_h(G_{Y,M})}.$$

Proof of Claim. If $Z \in \mathcal{C}_g$, then

$$G_{X,U_h(G_{Y,M})}(Z) = ({}^*Z\otimes X)\overline{\otimes} U_h(G_{Y,M}),$$
$$F_g(U_h)\big((\chi^0_{g,h})_{\mathcal{M}}(G_{X\otimes Y,M})\big)(Z) = U_h(G_{X\otimes Y,M}(Z\otimes -)).$$

Note that there are module natural isomorphisms

$$G_{X,M}(Z\otimes -) \simeq {}^*Z\overline{\otimes} G_{X,M}, \quad X\overline{\otimes} G_{Y,M} \simeq G_{X\otimes Y,M}.$$

Combining these two isomorphisms we get that

$$G_{X\otimes Y,M}(Z\otimes -) \simeq ({}^*Z\otimes X)\overline{\otimes} G_{Y,M}.$$

Using this isomorphism and the fact that U_h is a $\mathcal{C}$-module functor, we get that

$$U_h(G_{X\otimes Y,M}(Z\otimes -)) \simeq ({}^*Z\otimes X)\overline{\otimes} U_h(G_{Y,M}),$$

obtaining the desired isomorphisms. □

We define $\Phi(\mathcal{M}, U, \Pi) = \mathcal{M}$ as Abelian categories. We must endowed the category $\mathcal{M}$ with a structure of $\mathcal{D}$-module category. If $X \in \mathcal{C}_g$, $M \in \mathcal{M}$ set

$$X\overline{\otimes} M = U_g(G_{X,M}).$$

We have to define associativity isomorphisms

$$m_{X,Y,M} : (X\otimes Y)\overline{\otimes} M \to X\overline{\otimes}(Y\overline{\otimes} M).$$

Suppose that $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$, $M \in \mathcal{M}$. Then

$$(X\otimes Y)\overline{\otimes} M = U_{gh}(G_{X\otimes Y,M}), \quad X\overline{\otimes}(Y\overline{\otimes} M) = U_g(G_{X,U_h(G_{Y,M})}).$$

Hence, we define

$$m_{X,Y,M} = U_g(\beta_{X,Y,M})(\Pi_{g,h})^{-1}_{G_{X\otimes Y,M}}.$$

Axiom (5.4) is equivalent, in this case, to axiom (4.1). It is clear that Φ is a biequivalence and restricted to the category of exact modules (${}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}}_{e}$) gives the second biequivalence. □

6. Braided G-crossed tensor categories from G actions on 2-categories

In this section actions of groups on 2-categories are assumed to be strict. This does not lead to any loss of generality, since, in view of Theorem 3.1, all definitions and statements remain valid for non-strict actions after insertion of the suitable isomorphisms.

6.1. Strict braided G-crossed tensor categories

Braided G-crossed fusion categories play the same role in homotopy quantum field theory that braided fusion categories in the topological quantum field theory, see [25–27].

Definition 6.1. Let G be a groups and $\mathcal{C}$ a strict monoidal category. A *strict* braided G-crossed structure on $\mathcal{C}$ consist of the following data:

(1) a decomposition $\mathcal{C} = \coprod_{g\in G} \mathcal{C}_g$ (coproduct of categories) such that
- $\mathbf{1} \in \mathcal{C}_e$,
- $\mathcal{C}_g \otimes \mathcal{C}_h \subset \mathcal{C}_{gh}$ for all $g, h \in G$,

(2) a G-indexed family of strict monoidal functor $g_* : \mathcal{C} \to \mathcal{C}$, such that
- $g_*(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}, \quad g_* h_* = (gh)_*, \quad e_* = \mathrm{Id}_{\mathcal{C}}$,

(3) a family of natural isomorphisms

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C}_g \arrow[rr, "g_* \times \mathrm{Id}_{\mathcal{C}}"] & & \mathcal{C} \times \mathcal{C}_g \arrow[d, "\otimes"] \\
\mathcal{C}_g \times \mathcal{C} \arrow[u, "\mathrm{flip}"] \arrow[rr, "\otimes"'] \arrow[urr, phantom, "\Downarrow c"] & & \mathcal{C}
\end{tikzcd}$$

such that
- $g_*(c_{X,Z}) = c_{g_*(X), g_*(Z)}$
- $c_{X, Y\otimes Z} = (\mathrm{id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \mathrm{id}_Z)$
- $c_{X\otimes Y, Z} = (c_{X, h_*(Z)} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z})$

for all $X \in \mathcal{C}$, $Y \in \mathcal{C}_g$, $Z \in \mathcal{C}_h$, $g, h \in G$.

Even when the definition of strict braided G-crossed monoidal category is too restrictive, every *weak* braided G-crossed category is equivalent to a *strict* braided G-crossed category, see [7].

6.2. Center of a G-action

Let G be a group acting strictly on a 2-category $\mathcal{B}$, where $F_g : \mathcal{B} \to \mathcal{B}$, denotes the associated 2-functors. We shall introduce a G-graded monoidal category equipped with an action of G.

6.2.1. The G-graded monoidal category $\mathcal{Z}_G(\mathcal{B})$ Define the strict monoidal category $\mathcal{Z}_G(B) = \coprod_{g\in G} \mathcal{Z}_G(B)_g$, where $\mathcal{Z}_G(B)_g = \text{Pseu-Nat}(\mathrm{Id}_{\mathcal{B}}, F_g)$ and the product induced by the tensor product of pseudonatural transformation defined in (1.1). In other words, if $X \in \mathcal{Z}_G(B)_g$ and $Y \in \mathcal{Z}_G(B)_h$, we define $X \otimes Y \in$

$\mathcal{Z}_G(B)_{gh} =$ Pseu-Nat$(\mathrm{Id}_{\mathcal{B}}, F_{gh})$ as folows: for any object $A \in \mathcal{B}$, $(X \otimes B)_A = X_{F_h(A)} \circ Y_A$ and for any 1-cell $W \in \mathcal{B}(A, B)$

$$\begin{tikzcd}
A \arrow[r, "W"] \arrow[d, "Y_A"] \arrow[dd, bend right=60, "(X\otimes Y)_A"'] \arrow[dr, phantom, "\Downarrow Y_W"] & B \arrow[d, "Y_B"] \arrow[dd, bend left=60, "(X\otimes Y)_B"] \\
F_h(A) \arrow[r, "F_h(W)"] \arrow[d, "X_{F_h(A)}"] \arrow[dr, phantom, "\Downarrow X_{F_h(X)}"] & F_h(B) \arrow[d, "F_h(B)"] \\
F_{gh}(A) \arrow[r, "X_{F_h(W)}"] & F_{gh}(B)
\end{tikzcd}$$

The unit object is $1_{\mathrm{Id}_{\mathcal{B}}} \in$ Pseu-Nat$(\mathrm{Id}_{\mathcal{B}}, \mathrm{Id}_{\mathcal{B}})$.

6.2.2. The action of G on $\mathcal{Z}_G(\mathcal{B})$ Given $X \in \mathcal{Z}_G(B)_h$ and $g \in G$, we define $g_*(X) \in \mathcal{Z}_G(B)_{ghg^{-1}}$ as follows: for objects $A \in \mathcal{B}$, $g_*(X)_A = F_g(X_{F_{g^{-1}}(A)})$ and for any 1-arrow $W : A \to B$

$$\begin{tikzcd}
A \arrow[r, "W"] \arrow[d, "F_g(X_{F_{g^{-1}}(A)})"'] \arrow[dr, phantom, "\Downarrow F_g(X_{F_{g^{-1}}(W)})"] & B \arrow[d, "F_g(X_{F_{g^{-1}}(B)})"] \\
F_{ghg^{-1}}(A) \arrow[r, "F_{ghg^{-1}}(W)"] & F_{ghg^{-1}}(B).
\end{tikzcd}$$

Analogously, the functor g_* is defined for morphism in $\mathcal{Z}_G(B)$.

6.2.3. The G-braiding of $\mathcal{Z}_G(\mathcal{B})$ Let $X \in \mathcal{Z}_G(B)_g$ and $Y \in \mathcal{Z}_G(B)_h$. By the definition of pseudo-natural transformation we have

$$\begin{tikzcd}
A \arrow[r, "Y_A"] \arrow[d, "X_A"'] \arrow[dr, phantom, "\Downarrow X_{Y_A}"] & F_h(A) \arrow[d, "X_{F_h(A)}"] \\
F_g(A) \arrow[r, "F_g(Y_A)"] & F_{gh}(A),
\end{tikzcd}$$

but $(X \otimes Y)_A = X_{F_h(A)} \circ Y_A$ and $(g_*(Y) \otimes X)_A = F_g(Y_A) \circ X_A$, then the X_{Y_A} define natural isomorphism $c_{X,Y} := X_{Y_A} : X \otimes Y \to g_*(Y) \otimes X$.

Theorem 6.2. *Let G be a groups with a strcit action on a 2-categoy $\mathcal{B}$. Then the monoidal category $\mathcal{Z}_G(\mathcal{B})$ defined in 6.2.1 is a strict braided G-crossed monoidal category with action defined in 6.2.2 and G-braiding defined in 6.2.3. Moreover, the braided category $\mathcal{Z}_G(\mathcal{B})_e$ is exactly the Drinfeld center of $\mathcal{B}$.*

Proof. Since the action of G on $\mathcal{B}$ is strict, it follows by definition the equations

- $g_*(c_{X,Z}) = c_{g_*(X),g_*(Z)}$
- $c_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \mathrm{id}_Z)$
- $c_{X\otimes Y,Z} = (c_{X,h_*(Z)} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z})$.

□

6.3. Example

Let $\mathcal{D} = \oplus_{g\in G}\mathcal{D}_g$ be a faithfully G-graded fusion category.

Since every $\mathcal{D}_g$ is a $\mathcal{D}_e$-bimodule category, they define 2-functors

$$F_g(-) := \mathcal{D}_g \boxtimes_{\mathcal{D}_e} (-) : \mathcal{D}_e - \mathrm{Mod} \to \mathcal{D}_e - \mathrm{Mod},$$

the tensor products $\otimes : \mathcal{D}_g \times \mathcal{D}_h \to \mathcal{D}_{gh}$ induce pseudo-natural equivalences $\chi_{g,h} : F_g \circ F_h \to F_{gh}$ and the associator of $\mathcal{D}$ induce invertible modifications $\omega_{g,h,f} : \chi_{gh,f} \circ (\chi_{g,h} \otimes \mathrm{id}_{F_f}) \Rightarrow \chi_{g,hf} \circ (\mathrm{id}_{F_g} \otimes \chi_{h,f})$, that defines an action of G on $\mathcal{D}_e - \mathrm{Mod}$. See [4] for details.

In this case the category $\mathcal{Z}_G({}_{\mathcal{D}_e}\mathrm{Mod})_g$ is just $\mathrm{Fun}_{\mathcal{D}_e-\mathcal{D}_e}(\mathcal{D}_e, \mathcal{D}_g)$, the category of $\mathcal{D}_e$-bimodule functors and natural transformations from $\mathcal{D}_e$ to $\mathcal{D}_g$. The category $\mathcal{Z}_G({}_{\mathcal{D}_e}\mathrm{Mod})_g$ is canonically equivalent to the category $\mathcal{Z}_{\mathcal{D}_e}(\mathcal{D}_g)$ defined in [8, Definition 2.1] (use that $\mathrm{Fun}_{\mathcal{D}_e}(\mathcal{D}_e, \mathcal{D}_g) \to \mathcal{D}_g$, $F \mapsto F(\mathbf{1})$ is a category equivalence). Then the G-graded category $\mathcal{Z}_G({}_{\mathcal{D}_e}\mathrm{Mod})$ is equivalent to the monoidal category $\mathcal{Z}_{\mathcal{D}}(\mathcal{D}_e)$. The braided G-crossed category $\mathcal{Z}_G({}_{\mathcal{D}_e}\mathrm{Mod})$ is equivalent to the G-crossed category $\mathcal{Z}_{\mathcal{D}}(\mathcal{D}_e)$ defined in [8].

7. The center of the equivariant 2-category

This section is devoted to prove the following result. Let G be a finite group acting on a 2-category $\mathcal{B}$. Recall the forgetful 2-functor $\Phi : \mathcal{B}^G \to \mathcal{B}$ described in Lemma 4.3.

Theorem 7.1. *The group G acts on $\mathcal{Z}(\Phi)$ by monoidal autoequivalences, and there is a monoidal equivalence*

$$\mathcal{Z}(\mathcal{B}^G) \simeq \mathcal{Z}(\Phi)^G.$$

As a consequence, we have the following result.

Corollary 7.2. [8, Thm. 3.5] *Let $\mathcal{D} = \oplus_{g\in G}\mathcal{C}_g$ be a faithfully graded tensor category, with $\mathcal{C} = \mathcal{C}_1$. There is an action of the group G on the relative center $\mathcal{Z}_{\mathcal{C}}(\mathcal{D})$ and a monoidal equivalence*

$$\mathcal{Z}(\mathcal{D}) \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{D})^G.$$

Proof. Let $\mathcal{H} : {}_{\mathcal{D}}\mathrm{Mod} \to {}_{\mathcal{C}}\mathrm{Mod}$ be the forgetful pseudofunctor. Then

$$\mathcal{Z}(\mathcal{D}) \simeq \mathcal{Z}({}_{\mathcal{D}}\mathrm{Mod}) \simeq \mathcal{Z}\big(\big({}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}}\big)^{G}\big) \simeq \mathcal{Z}(\mathcal{H})^{G} \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{D})^{G}$$

The first equivalence follow from Corollary 5.3, the second one is Theorem 5.4, and the last one is Proposition 5.2. □

For the rest of this section we shall use the notation introduced in Sect. 4.1. There is no harm in assuming that the action is *unital* and *strict*, see definitions 2.2, 2.3. By Proposition 1.2, we can assume that any invertible 1-cell is an isomorphism. In particular, if (A, U, Π) is an equivariant 0-cell, for any $g \in G$, the 1-cell U_g is invertible. Thus, we can choose a 1-cell U^*_g such that

$$U_g \circ U^*_g = I_{F_g(A)}, \quad U^*_g \circ U_g = I_A.$$

If X, Y are 1-cells, we shall sometimes denote $X \circ Y = XY$, as a space saving measure.

7.1. A group action on $\mathcal{Z}(\Phi)$

For any $g \in G$, we shall define tensor autoequivalences $L_g : \mathcal{Z}(\Phi) \to \mathcal{Z}(\Phi)$ such that they define an action of G on $\mathcal{Z}(\Phi)$. First, let us explicitly describe objects in $\mathcal{Z}(\Phi)$. An object $(X, \sigma) \in \mathcal{Z}(\Phi)$ consists of

$$X = \{X_{(A,U,\Pi)} \in \mathcal{B}(A, A) \text{ a 1-cell}, (A, U, \Pi) \in \mathrm{Obj}\,(\mathcal{B}^G)\},$$
$$\sigma = \{\sigma_{(\theta,\theta_g)} : X_{(\widetilde{A},\widetilde{U},\widetilde{\Pi})} \circ \theta \Rightarrow \theta \circ X_{(A,U,\Pi)} \text{ isomorphisms 2-cells in } \mathcal{B}^G\},$$

where $(\theta, \theta_g) \in \mathcal{B}^G((A, U, \Pi), (\widetilde{A}, \widetilde{U}, \widetilde{\Pi}))$ is an equivariant 1-cell. The isomorphisms $\sigma_{(\theta,\theta_g)}$ satisfy (1.4). If $(X, \sigma), (Y, \tau) \in \mathcal{Z}(\Phi)$, a morphism $f : (X, \sigma) \to (Y, \tau)$ is a collection of 2-cells in $\mathcal{B}(A, A)$

$$f_{(A,U,\Pi)} : X_{(A,U,\Pi)} \Rightarrow Y_{(A,U,\Pi)},$$

such that for any equivariant 1-cell $(\theta, \theta_g) \in \mathcal{B}^G((A, U, \Pi), (\widetilde{A}, \widetilde{U}, \widetilde{\Pi}))$

$$\big(\mathrm{id}_{\theta} \circ f_{(A,U,\Pi)}\big)\sigma_{(\theta,\theta_g)} = \tau_{(\theta,\theta_g)}\big(f_{(\widetilde{A},\widetilde{U},\widetilde{\Pi})} \circ \mathrm{id}_{\theta}\big).$$

Lemma 7.3. *Suppose $g, h \in G$ and (A, U, Π) is an equivariant 0-cell. There are isomorphisms 2-cells*

$$\epsilon_{g,h,(A,U,\Pi)} : U^*_g \circ F_g(U^*_h) \Rightarrow U^*_{gh}$$

such that

$$\epsilon_{g,h,(A,U,\Pi)} \circ \Pi_{g,h} = \mathrm{id}_{I_A}, \quad \Pi_{g,h} \circ \epsilon_{g,h,(A,U,\Pi)} = \mathrm{id}_{I_{F_{gh}(A)}}, \tag{7.1}$$

$$\epsilon_{gh,f,(A,U,\Pi)}\big(\epsilon_{g,h,(A,U,\Pi)} \circ \mathrm{id}_{F_{gh}(U^*_f)}\big) = \epsilon_{g,hf,(A,U,\Pi)}\big(\mathrm{id}_{U^*_g} \circ F_g(\epsilon_{h,f,(A,U,\Pi)})\big), \tag{7.2}$$

for any $g, h, f \in G$.

Proof. Take $\epsilon_{g,h,(A,U,\Pi)} = \mathrm{id}_{U^*_g \circ F_g(U^*_h)} \circ \Pi^{-1}_{g,h} \circ \mathrm{id}_{U^*_{gh}}$. Equation (7.2) follow from (4.1). □

For any $g \in G$, let us define the functors $L_g : \mathcal{Z}(\Phi) \to \mathcal{Z}(\Phi)$, $L_g(X, \sigma) = (X^g, \sigma^g)$. Where, for any equivariant 0-cell (A, U, Π)

$$X^g_{(A,U,\Pi)} = U^*_g \circ F_g(X_{(A,U,\Pi)}) \circ U_g.$$

Remark 7.4. As a saving space measure, if (A, U, Π), $(\widetilde{A}, \widetilde{U}, \widetilde{\Pi})$ are equivariant 0-cells, we are going to denote $X = X_{(A,U,\Pi)}$, $\widetilde{X} = X_{(\widetilde{A},\widetilde{U},\widetilde{\Pi})}$. Also, we shall denote $\epsilon_{g,h} = \epsilon_{g,h,(A,U,\Pi)}$ and $\widetilde{\epsilon}_{g,h} = \epsilon_{g,h,(\widetilde{A},\widetilde{U},\widetilde{\Pi})}$ when no confusion arises.

If $(\theta, \theta_g) \in \mathcal{B}^G((A, U, \Pi), (\widetilde{A}, \widetilde{U}, \widetilde{\Pi}))$ is an equivariant 1-cell, then

$$\sigma^g_{(\theta,\theta_g)} = \big(1_{\widetilde{U}^*_g} \circ \theta_g \circ 1_{U^*_g F_g(X) U_g}\big)\big(1_{\widetilde{U}^*_g} \circ F_g(\sigma_{(\theta,\theta_g)}) \circ 1_{U_g}\big)\big(1_{\widetilde{U}_g F_g(\widetilde{X})} \circ \theta_g^{-1}\big).$$

If $f : (X, \sigma) \to (Y, \tau)$ is a morphism in $\mathcal{Z}(\Phi)$, then

$$L_g(f)_{(A,U,\Pi)} = \mathrm{id}_{U^*_g} \circ F_g(f_{(A,U,\Pi)}) \circ \mathrm{id}_{U_g}.$$

The proof of the next result follows straightforwardly.

Proposition 7.5. *The functors $L_g : \mathcal{Z}(\Phi) \to \mathcal{Z}(\Phi)$ are well-defined monoidal functors.* □

Now, for any $g, h \in G$, we shall define monoidal natural isomorphisms $\nu_{g,h} : L_g \circ L_h \to L_{gh}$ satisfying (5.1) and (5.2). Take $(X, \sigma) \in \mathcal{Z}(\mathcal{H})$, so we must define an arrow

$$(\nu_{g,h})_{(X,\sigma)} : L_g \circ L_h(X, \sigma) \to L_{gh}(X, \sigma).$$

For each equivariant 0-cell (A, U, Π) we define the map

$$\begin{aligned} &\big((\nu_{g,h})_{(X,\sigma)}\big)_{(A,U,\Pi)} : U^*_g F_g(U^*_h) F_{gh}(X_{(A,U,\mu)}) F_g(U_h) \\ &\quad U_g \to U_{gh} F_{gh}(X_{(A,U,\mu)}) U^*_{gh}, \\ &\quad \big((\nu_{g,h})_{(X,\sigma)}\big)_{(A,U,\mu)} = \epsilon_{g,h} \circ \mathrm{id}_{F_{gh}(X_{(A,U,\mu)})} \circ \Pi_{g,h}. \end{aligned}$$

Proposition 7.6. *For any $g, h, f \in G$, the following assertions holds.*

(i) $\nu_{g,h} : L_g \circ L_h \to L_{gh}$ *are well-defined natural isomorphisms in $\mathcal{Z}(\Phi)$.*
(ii) $\nu_{g,h} : L_g \circ L_h \to L_{gh}$ *are monoidal natural transformations.*
(iii) *For any $g, h, f \in G$ and any $(X, \sigma) \in \mathcal{Z}(\Phi)$, the following equation holds*

$$(\nu_{gh,f})_{(X,\sigma)}(\nu_{g,h})_{L_f(X,\sigma)} = (\nu_{g,hf})_{(X,\sigma)} L_g((\nu_{h,f})_{(X,\sigma)}). \tag{7.3}$$

Proof. (i). We must verify that $(\nu_{g,h})_{(X,\sigma)}$ are morphisms in the category $\mathcal{Z}(\Phi)$, that is, equation

$$\big(\mathrm{id}_{\theta}\circ\big((\nu_{g,h})_{(X,\sigma)}\big)_{(A,U,\mu)}\big)((\sigma^h)^g)_{(\theta,\theta_g)}=\sigma^{gh}_{(\theta,\theta_g)}\big(\big((\nu_{g,h})_{(X,\sigma)}\big)_{(\widetilde{A},\widetilde{U},\widetilde{\Pi})}\circ\mathrm{id}_{\theta}\big) \tag{7.4}$$

is fulfilled for any equivariant 1-cell $(\theta,\theta_g)\in\mathcal{B}^G((A,U,\Pi),(\widetilde{A},\widetilde{U},\widetilde{\Pi}))$. The left hand side of (7.4) equals to

$$\begin{aligned}
&=\big(\mathrm{id}_{\theta}\circ\epsilon_{g,h}\circ\mathrm{id}_{F_{gh}(X)}\circ\Pi_{g,h}\big)\big(\mathrm{id}_{\widetilde{U}^*_g}\circ\theta_g\circ\mathrm{id}_{U^*_gF_g(U^*_h)F_{gh}(X)F_g(U_h)U_g}\big)\\
&\quad\big(\mathrm{id}_{\widetilde{U}^*_g}\circ F_g(\sigma^h_{(\theta,\theta_g)})\circ\mathrm{id}_{U_g}\big)\big(\mathrm{id}_{\widetilde{U}^*_g}\circ\theta_g\circ\mathrm{id}_{U^*_gF_g(U^*_h)F_{gh}(X)F_g(U_h)U_g}\big)\\
&=\big(\mathrm{id}_{\theta}\circ\epsilon_{g,h}\circ\mathrm{id}_{F_{gh}(X)}\circ\Pi_{g,h}\big)\big(\mathrm{id}\circ\theta_g\circ\mathrm{id}\big)\big(\mathrm{id}_{\widetilde{U}^*_gF_g(\widetilde{U}^*_h)}\circ F_g(\theta_h)\circ\mathrm{id}\big)\\
&\big(\mathrm{id}_{\widetilde{U}^*_gF_g(\widetilde{U}^*_h)}\circ F_{gh}(\sigma_{(\theta,\theta_g)})\circ\mathrm{id}_{F_g(U_h)U_g}\big)\big(\mathrm{id}_{\widetilde{U}^*_gF_g(\widetilde{U}^*_h)}\circ F_g(\theta_h^{-1})\circ\mathrm{id}_{U_g}\big)\\
&\big(\mathrm{id}_{\widetilde{U}^*_g}\circ\theta_g\circ\mathrm{id}_{U^*_gF_g(U^*_h)F_{gh}(X)F_g(U_h)U_g}\big)\\
&=\big(\mathrm{id}\circ\epsilon_{g,h}\circ\mathrm{id}\big)\big(\mathrm{id}_{\widetilde{U}^*_gF_g(\widetilde{U}^*_h)}\circ(\mathrm{id}_{F_g(\widetilde{U}_h)}\\
&\circ\theta_g)(F_g(\theta_h)\circ\mathrm{id}_{U_g})\circ\mathrm{id}_{U^*_gF_g(U^*_h)F_{gh}(X)U_{gh}}\big)\\
&\big(\mathrm{id}_{\widetilde{U}^*_gF_g(\widetilde{U}^*_h)}\circ F_{gh}(\sigma_{(\theta,\theta_g)})\circ\mathrm{id}_{U_{gh}}\big)\\
&\big(\mathrm{id}_{\widetilde{U}^*_gF_g(\widetilde{U}^*_h)F_{gh}(\widetilde{X})}\circ(\mathrm{id}_{F_{gh}(\theta)}\circ\Pi_{g,h})(F_g(\theta_h^{-1})\circ\mathrm{id}_{U_g})(\mathrm{id}_{F_{gh}(\widetilde{U}_h)}\circ\theta_g^{-1})\big)\\
&=\big(\mathrm{id}\circ\epsilon_{g,h}\circ\mathrm{id}\big)\big(\mathrm{id}_{\widetilde{U}^*_gF_g(\widetilde{U}^*_h)}\circ(\mathrm{id}_{F_g(\widetilde{U}_h)}\circ\theta_g)(F_g(\theta_h)\circ\mathrm{id}_{U_g})\circ\mathrm{id}\big)\\
&\big(\mathrm{id}_{\widetilde{U}^*_gF_g(\widetilde{U}^*_h)}\circ F_{gh}(\sigma_{(\theta,\theta_g)})\circ\mathrm{id}_{U_{gh}}\big)\big(\mathrm{id}\circ\theta_{gh}^{-1}(\widetilde{\Pi}_{g,h}\circ\mathrm{id}_{\theta})\big)
\end{aligned}$$

The second equation follows from the definition of $\sigma^h_{(\theta,\theta_g)}$, the fourth equality follows from (4.2). The right hand side of (7.4) equals to

$$\begin{aligned}
&=\big(\mathrm{id}_{\widetilde{U}^*_{gh}}\circ\theta_{gh}\circ\mathrm{id}_{U^*_{gh}F_{gh}(X)U_{gh}}\big)\big(\mathrm{id}_{\widetilde{U}^*_{gh}}\circ F_{gh}(\sigma_{(\theta,\theta_g)})\circ\mathrm{id}_{U_{gh}}\big)\\
&\big(\mathrm{id}_{\widetilde{U}^*_{gh}F_{gh}(\widetilde{X})}\circ\theta_{gh}^{-1}\big)\big(\widetilde{\epsilon}_{g,h}\circ\mathrm{id}_{F_{gh}(\widetilde{X})}\circ\widetilde{\Pi}_{g,h}\circ\mathrm{id}_{\theta}\big)\\
&=\big(\widetilde{\epsilon}_{g,h}\circ\theta_{gh}\circ\mathrm{id}_{U^*_{gh}F_{gh}(X)U_{gh}}\big)\big(\mathrm{id}_{U^*_gF_g(U^*_h)}\circ F_{gh}(\sigma_{(\theta,\theta_g)})\circ\mathrm{id}_{U_{gh}}\big)\\
&\big(\mathrm{id}_{U^*_gF_g(U^*_h)F_{gh}(\widetilde{X})}\circ\theta_{gh}^{-1}(\widetilde{\Pi}_{g,h}\circ\mathrm{id}_{\theta})\big).
\end{aligned}$$

It follows from Eq. (7.1) that both sides are equal.

(ii). Let (X,σ), (Y,τ) be objects in $\mathcal{Z}(\Phi)$. Since the functors L_g are strict, this means that $L_g((X,\sigma)\otimes(Y,\tau))=L_g(X,\sigma)\otimes L_g(Y,\tau)$, we must prove that

$$(\nu_{g,h})_{(X,\sigma)\otimes(Y,\tau)}=(\nu_{g,h})_{(X,\sigma)}\otimes(\nu_{g,h})_{(X,\sigma)}. \tag{7.5}$$

Let (A,U,Π) be an equivariant 0-cell. The left hand side of (7.5) evaluated in (A,U,Π) equals to

$$\epsilon_{g,h}\circ\mathrm{id}_{F_{gh}(X_{(A,U,\Pi)})}\circ\Pi_{g,h}\circ\epsilon_{g,h}\circ\mathrm{id}_{F_{gh}(Y_{(A,U,\Pi)})}\circ\Pi_{g,h}.$$

The right hand side of (7.5) evaluated in (A,U,Π) equals to

$$\epsilon_{g,h}\circ\mathrm{id}_{F_{gh}(X_{(A,U,\Pi)}\circ Y_{(A,U,\Pi)})}\circ\Pi_{g,h}.$$

It follows from (7.1) that both sides are equal.

(iii). Let (A, U, Π) be an equivariant 0-cell. The left hand side of (7.3) evaluated in (A, U, Π) is equal to

$$
\begin{aligned}
&= \big(\epsilon_{gh,f} \circ \mathrm{id}_{F_{ghf}(X)} \circ \Pi_{gh,f}\big)\big(\epsilon_{g,h} \circ \mathrm{id}_{F_{gh}(U_f^* X U_f)} \circ \Pi_{g,h}\big) \\
&= \epsilon_{gh,f}(\epsilon_{g,h} \circ \mathrm{id}_{F_{gh}(U_f^*)}) \circ \mathrm{id}_{F_{ghf}(X)} \circ \Pi_{gh,f}(\mathrm{id}_{F_{gh}(U_f)} \circ \Pi_{g,h}).
\end{aligned}
$$

The right hand side of (7.3) evaluated in (A, U, Π) is equal to

$$
\begin{aligned}
&= \big(\epsilon_{g,hf} \circ \mathrm{id}_{F_{gh}(X)} \circ \Pi_{g,hf}\big)\big(\mathrm{id}_{U_g^*} \circ F_g(\epsilon_{h,f}) \circ \mathrm{id}_{F_{ghf}(X)} \circ F_g(\Pi_{h,f}) \circ \mathrm{id}_{U_g}\big) \\
&= \epsilon_{g,hf}(\mathrm{id}_{U_g^*} \circ F_g(\epsilon_{h,f})) \circ \mathrm{id}_{F_{ghf}(X)} \circ \Pi_{g,hf}(F_g(\Pi_{h,f}) \circ \mathrm{id}_{U_g}).
\end{aligned}
$$

Now, that both expressions are equal follow by (7.2) and (4.1). □

7.1.1. Proof of Theorem 7.1 Let us first describe an object in the equivariantization of the category $\mathcal{Z}(\Phi)$. An object in $\mathcal{Z}(\Phi)^G$ is a collection $((X, \sigma), s)$ where $(X, \sigma) \in \mathcal{Z}(\Phi)$, and $s_g : L_g(X, \sigma) \to (X, \sigma)$ is a morphism in the category, for any $g \in G$. This means, that $X_{(A,U,\Pi)} \in \mathcal{B}(A, A)$ is a 1-cell, for any equivariant 0-cell (A, U, Π), and for any equivariant 1-cell $(\tau, \tau_g) \in \mathcal{B}^G((A, U, \Pi), (\widetilde{A}, \widetilde{U}, \widetilde{\Pi}))$ there is an isomorphism $\sigma_{(\tau,\tau_g)} : X_{(\widetilde{A},\widetilde{U},\widetilde{\Pi})} \circ \tau \to \tau \circ X_{(A,U,\Pi)}$ such that Eq. (1.4) is fulfilled. Also, for any $g \in G$ and any equivariant 0-cell (A, U, Π) there are morphisms

$$
(s_g)_{(A,U,\Pi)} : U_g^* F_g(X_{(A,U,\Pi)}) U_g \to V_{(A,U,\Pi)},
$$

such that

$$
\big(\mathrm{id}_\tau \circ (s_g)_{(A,U,\Pi)}\big)\sigma^g_{(\tau,\tau^1)} = \sigma_{(\tau,\tau^1)}\big((s_g)_{(\widetilde{A},\widetilde{U},\widetilde{\Pi})} \circ \mathrm{id}_\tau\big), \tag{7.6}
$$

$$
(s_{gh})_{(A,U,\Pi)}(\nu_{g,h})_{(A,U,\Pi)} = (s_g)_{(A,U,\Pi)} L_g((s_h)_{(A,U,\Pi)}), \tag{7.7}
$$

for any equivariant 0-cells (A, U, Π), $(\widetilde{A}, \widetilde{U}, \widetilde{\Pi})$, any equivariant 1-cell $(\tau, \tau_g) \in \mathcal{B}^G((A, U, \Pi), (\widetilde{A}, \widetilde{U}, \widetilde{\Pi}))$, and any $g, h \in G$. Equation (7.6) follows from the fact that $s_g : L_g(V, \sigma) \to (V, \sigma)$ is a morphism in the category $\mathcal{Z}(\Phi)$, and Eq. (7.7) follows from (5.3).

Define the functor $\Psi : \mathcal{Z}(\Phi)^G \to \mathcal{Z}(\mathcal{B}^G)$ as follows. Let $((X, \sigma), s) \in \mathcal{Z}(\Phi)^G$, then $\Phi((X, \sigma), s) = (V, \widetilde{\sigma})$. For any equivariant 0-cell (A, U, Π), $V_{(A,U,\Pi)}$ must be an equivariant 1-cell in the category $\mathcal{B}^G((A, U, \Pi), (A, U, \Pi))$. Define $V_{(A,U,\Pi)} = (X_{(A,U,\Pi)}, \theta_g^{(A,U,\Pi)})$, where

$$
\begin{aligned}
\theta_g^{(A,U,\Pi)} &: F_g(X_{(A,U,\Pi)}) \circ U_g \Rightarrow U_g \circ X_{(A,U,\Pi)}, \\
\theta_g^{(A,U,\Pi)} &= \mathrm{id}_{U_g} \circ (s_g)_{(A,U,\Pi)}.
\end{aligned} \tag{7.8}
$$

If $(\tau, \tau_g) \in \mathcal{B}^G((A, U, \Pi), (\widetilde{A}, \widetilde{U}, \widetilde{\Pi}))$ is an equivariant 1-cell, then

$$
\begin{aligned}
&\widetilde{\sigma}_{(\tau,\tau_g)} : (X_{(\widetilde{A},\widetilde{U},\widetilde{\Pi})}, \theta_g^{(\widetilde{A},\widetilde{U},\widetilde{\Pi})}) \circ (\tau, \tau_g) \Rightarrow (\tau, \tau_g) \circ (X_{(A,U,\Pi)}, \theta_g^{(A,U,\Pi)}), \\
&\widetilde{\sigma}_{(\tau,\tau_g)} = \sigma_{(\tau,\tau_g)}.
\end{aligned}
$$

Claim 7.7. *The following statements hold.*

(i) $V_{(A,U,\Pi)} = (X_{(A,U,\Pi)}, \theta_g^{(A,U,\Pi)}) \in \mathcal{B}^G$, *for any equivariant 0-cell* (A, U, Π).
(ii) *The object* $(V, \widetilde{\sigma})$ *belongs to the category* $\mathcal{Z}(\mathcal{B}^G)$. *In particular, the functor* Ψ *is well-defined.*
(iii) *The functor* $\Psi : \mathcal{Z}(\Phi)^G \to \mathcal{Z}(\mathcal{B}^G)$ *is an equivalence of categories, and it has a monoidal structure.*

Proof of Claim. (i). We must check that the maps $\theta_g^{(A,U,\Pi)})$ satisfy (4.2). In this case, we must prove that for any $g, h \in G$

$$\big(\Pi_{g,h} \circ \mathrm{id}_{X_{(A,U,\Pi)}}\big)\big(\mathrm{id}_{F_g(U_h)} \circ \theta_g^{(A,U,\Pi)}\big)\big(F_g(\theta_h^{(A,U,\Pi)}) \circ \mathrm{id}_{U_g}\big)$$

is equal to

$$\theta_{gh}^{(A,U,\Pi)}\big(\mathrm{id}_{F_{gh}(X_{(A,U,\Pi)})} \circ \Pi_{g,h}\big).$$

Using the definition of $\theta_g^{(A,U,\Pi)}$, we get that the first expression is equal to

$$\begin{aligned}
&\big(\Pi_{g,h} \circ \mathrm{id}_{X_{(A,U,\Pi)}}\big)\big(\mathrm{id}_{F_g(U_h)U_g} \circ (s_g)_{(A,U,\Pi)}\big)\big(\mathrm{id}_{F_g(U_h)} \circ F_g((s_h)_{(A,U,\Pi)}) \circ \mathrm{id}_{U_g}\big)\\
&\quad = \big(\Pi_{g,h} \circ \mathrm{id}_{X_{(A,U,\Pi)}}\big)\big(\mathrm{id}_{F_g(U_h)U_g} \circ (s_g)_{(A,U,\Pi)}(\mathrm{id}_{U_g^* \circ F_g((s_h)_{(A,U,\Pi)})}) \circ \mathrm{id}_{U_g}\big)\\
&\quad = \big(\Pi_{g,h} \circ \mathrm{id}_{X_{(A,U,\Pi)}}\big)\big(\mathrm{id}_{F_g(U_h)U_g} \circ (s_{gh})_{(A,U,\Pi)}(\nu_{g,h})_{(A,U,\Pi)}\big)\\
&\quad = \big(\mathrm{id}_{U_{gh}} \circ (s_{gh})_{(A,U,\Pi)}\big)\big(\Pi_{g,h} \circ (\nu_{g,h})_{(A,U,\Pi)}\big)\\
&\quad = \theta_{gh}^{(A,U,\Pi)}\big(\mathrm{id}_{F_{gh}(X_{(A,U,\Pi)})} \circ \Pi_{g,h}\big).
\end{aligned}$$

The second equality follows from (7.7), and the last one follows from (7.1).

(ii). Since $\widetilde{\sigma}_{(\tau,\tau_g)} = \sigma_{(\tau,\tau_g)}$ for any equivariant 1-cell (τ, τ_g), then $\widetilde{\sigma}$ satisfy (1.4). We must verify only that $\widetilde{\sigma}_{(\tau,\tau_g)}$ is an equivariant 2-cell, that is (4.3) is satisfied. To simplify the notation, let us denote $\theta_g^{(A,U,\Pi)} = \theta_g$, $\theta^{(\widetilde{A},\widetilde{U},\widetilde{\Pi})} = \widetilde{\theta}_g$. In this particular case, using the composition of equivariant 1-cells given by (4.4), we have to prove that

$$\big(1_{\widetilde{U}_g} \circ \sigma_{(\tau,\tau_g)}\big)\big(\widetilde{\theta}_g \circ 1_\tau\big)\big(1_{F_g(\widetilde{X})} \circ \tau_g\big) = \big(\tau_g \circ 1_X\big)\big(1_{F_g(\tau)} \circ \theta_g\big)\big(F_g(\sigma_{(\tau,\tau_g)}) \circ 1_{U_g}\big). \tag{7.9}$$

The left hand side of Eq. (7.9) is equal to

$$\begin{aligned}
&= \big(1_{\widetilde{U}_g} \circ \sigma_{(\tau,\tau_g)}\big)\big(1_{\widetilde{U}_g} \circ (s_g)_{(A,U,\Pi)}\big)\big(1_{F_g(\widetilde{X})} \circ \tau_g\big)\\
&\quad = \big(1_{\widetilde{U}_g} \circ (1_\tau \circ (s_g)_{(A,U,\Pi)})\sigma^g_{(\tau,\tau_g)}\big)\big(1_{F_g(\widetilde{X})} \circ \tau_g\big)\\
&\quad = \big(1_{\widetilde{U}_g} \circ (s_g)_{(A,U,\Pi)}\big)\big(\tau_g \circ 1_{U_g^* F_g(X) U_g}\big)\big(F_g(\sigma_{(\tau,\tau_g)}) \circ 1_{U_g}\big)\\
&\quad = \big(\tau_g \circ 1_X\big)\big(1_{F_g(\tau)} \circ \theta_g\big)\big(F_g(\sigma_{(\tau,\tau_g)}) \circ 1_{U_g}\big).
\end{aligned}$$

The first equality follows by using the definition of $\theta_g^{(A,U,\Pi)}$ given in (7.8), the second equality follows from (7.6), and the third one follows from the definition of $\sigma^g_{(\tau,\tau_g)}$.

(iii). The fact that Ψ is an equivalence follows easily. A direct computation shows that

$$\Psi\big(((X,\sigma),s)\otimes((Y,\tau),t)\big) = \Psi((X,\sigma),s)\otimes\Psi((Y,\tau),t),$$

for any pair of objects $((X,\sigma),s)$, $((Y,\tau),t) \in \mathcal{Z}(\Phi)^G$. $\square$

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