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Shu-Yu Hsu

Global behaviour of solutions of the fast diffusion equation

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Abstract. We will extend a recent result of Choi and Daskalopoulos [4]. For any $n \geq 3$, $0 < m < \frac{n-2}{n}$, $m \neq \frac{n-2}{n+2}$, $\beta > 0$ and $\lambda > 0$, we prove the higher order expansion of the radially symmetric solution $v_{\lambda,\beta}(r)$ of $\frac{n-1}{m}\Delta v^m + \frac{2\beta}{1-m}v + \beta x \cdot \nabla v = 0$ in $\mathbb{R}^n$, $v(0) = \lambda$, as $r \to \infty$. As a consequence for any $n \geq 3$ and $0 < m < \frac{n-2}{n}$ if u is the solution of the equation $u_t = \frac{n-1}{m}\Delta u^m$ in $\mathbb{R}^n \times (0, \infty)$ with initial value $0 \leq u_0 \in L^\infty(\mathbb{R}^n)$ satisfying $u_0(x)^{1-m} = \frac{2(n-1)(n-2-nm)}{(1-m)\beta|x|^2}\left(\log|x| - \frac{n-2-(n+2)m}{2(n-2-nm)}\log(\log|x|) + K_1 + o(1)\right)$ as $|x| \to \infty$ for some constants $\beta > 0$ and $K_1 \in \mathbb{R}$, then as $t \to \infty$ the rescaled function $\widetilde{u}(x,t) = e^{\frac{2\beta}{1-m}t}u(e^{\beta t}x, t)$ converges uniformly on every compact subsets of $\mathbb{R}^n$ to $v_{\lambda_1,\beta}$ for some constant $\lambda_1 > 0$.

1. Introduction

Recently there is a lot of interest in the following singular diffusion equation [1,7,15,16],

$$u_t = \frac{n-1}{m}\Delta u^m \quad \text{in } \mathbb{R}^n \times (0, T) \tag{1.1}$$

which arises in the study of many physical models and geometric flows. When $0 < m < 1$, (1.1) is called the fast diffusion equation. On the other hand as observed by Daskalopoulos et al. [6,9,10], the metric $g = u^{\frac{4}{n-2}}dy^2$ satisfies the Yamabe flow [2,3],

$$\frac{\partial g}{\partial t} = -Rg \tag{1.2}$$

on $\mathbb{R}^n$, $n \geq 3$, for $0 < t < T$, where R is the scalar curvature of the metric g, if and only if u satisfies (1.1) with

$$m = \frac{n-2}{n+2}.$$

For

$$n \geq 3 \quad \text{and} \quad 0 < m < \frac{n-2}{n}, \tag{1.3}$$

S.-Y. Hsu (✉): Department of Mathematics, National Chung Cheng University, 168 University Road, Min-Hsiung, Chia-Yi 621, Taiwan, ROC.
e-mail: shuyu.sy@gmail.com

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asymptotic behaviour of solution of (1.1) near finite extinction time was studied by Galaktionov and Peletier [11] and Daskalopoulos and Sesum [8]. Extinction profile for solutions of (1.1) for the case $m = \frac{n-2}{n+2}$, $n \geq 3$, was studied in [10] by del Pino and Sáez. Such extinction behaviour occurs for the solution of (1.1) when (1.3) holds and the solution $u \in L^\infty(\mathbb{R}^n \times (0,T))$ satisfies

$$u(x,t) \leq C|x|^{-\frac{2}{1-m}} \quad \forall |x| \geq R_1, \quad 0 \leq t < T$$

for some constants $C > 0$ and $R_1 > 0$ (cf. [8]). On the other hand if (1.3) holds and $0 \leq u_0 \in L^p_{loc}(\mathbb{R}^n)$ for some constant $p > \frac{n(1-m)}{2}$ which satisfies

$$\liminf_{R\to\infty} \frac{1}{R^{n-\frac{2}{1-m}}} \int_{|x|\leq R} u_0\,dx = \infty,$$

then Hsu [13] proved the existence and uniqueness of solutions of

$$\begin{cases} u_t = \frac{n-1}{m}\Delta u^m & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.4}$$

These results say that we will have either global existence of solution of (1.1) or extinction in finite time for solution of (1.1) depending on whether the growth rate of the solution is large enough at infinity. When (1.3) holds, existence, uniqueness and decay rate of self-similar solutions of (1.1) were also proved by Hsu in [12]. Interested reader can read the book [7] by Daskalopoulos and Kenig and the book [16] by Vazquez for the most recent results on (1.1).

In the recent paper [4], Choi and Daskalopoulos proved the higher order expansion of the radially symmetric solution $v_{\lambda,\beta}(r)$ of

$$\begin{cases} \dfrac{n-1}{m}\Delta v^m + \dfrac{2\beta}{1-m}v + \beta x\cdot\nabla v = 0, \quad v > 0, \text{ in } \mathbb{R}^n \\ v(0) = \lambda \end{cases} \tag{1.5}$$

for any constant $\lambda > 0$ where $m = \frac{n-2}{n+2}$, $n \geq 3$, as $r \to \infty$. They also proved that if u is the solution of (1.4) in $\mathbb{R}^n \times (0,\infty)$ with $m = \frac{n-2}{n+2}$, $n \geq 3$, and initial value $u_0 \geq 0$ satisfying

$$u_0(x)^{1-m} \approx \frac{(n-1)(n-2)}{\beta|x|^2}(\log|x| + K_1 + o(1)) \quad \text{as } r = |x| \to \infty$$

for some constants $\beta > 0$ and $K_1 \in \mathbb{R}$, then as $t \to \infty$ the rescaled function

$$\widetilde{u}(x,t) = e^{\frac{2\beta}{1-m}t}u(e^{\beta t}x,t) \tag{1.6}$$

converges uniformly on every compact subsets of $\mathbb{R}^n$ to $v_{\lambda_1,\beta}(x)$ for some constant $\lambda_1 > 0$. Note that for any solution u of (1.1) in $\mathbb{R}^n \times (0,\infty)$, $\widetilde{u}$ satisfies

$$\widetilde{u}_t = \frac{n-1}{m}\Delta\widetilde{u} + \frac{2\beta}{1-m}\widetilde{u} + \beta x\cdot\nabla\widetilde{u} \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{1.7}$$

This result of Choi and Daskalopoulos [4] shows that the asymptotic large time behaviour of the solution of (1.4) depends critically on the higher order expansion of the initial value of the solution. Moreover the asymptotic large time behaviour of the solution of (1.4) is after a rescaling similar to the solution of (1.5) with the same higher order expansion. Hence it is important to study the higher order expansion of the solution of (1.5).

In this paper we will extend the result of [4]. We will prove the higher order expansion of the radially symmetric solution $v_{\lambda,\beta}(r)$ of (1.5) as $r \to \infty$ for any $n \geq 3, 0 < m < \frac{n-2}{n}, m \neq \frac{n-2}{n+2}, \beta > 0$ and $\lambda > 0$. We will also prove that when $n \geq 3, 0 < m < \frac{n-2}{n}$, and

$$u_0(x)^{1-m} \approx \frac{2(n-1)(n-2-nm)}{(1-m)\beta|x|^2} \times \left(\log|x| - \frac{n-2-(n+2)m}{2(n-2-nm)}\log(\log|x|) + K_1 + 0(1)\right)$$

as $r = |x| \to \infty$ for some constants $\beta > 0$ and $K_1 \in \mathbb{R}$, then as $t \to \infty$ the rescaled solution $\widetilde{u}$ given by (1.6) will converges uniformly on every compact subsets of $\mathbb{R}^n$ to the radially symmetric solution $v_{\lambda_1,\beta}(x)$ of (1.5) for some constant $\lambda = \lambda_1 > 0$.

We first start with a definition. For any $0 \leq u_0 \in L^1_{loc}(\mathbb{R}^n)$, we say that a function u is a solution of (1.4) if $u > 0$ in $\mathbb{R}^n \times (0, \infty)$ is a classical solution of (1.1) in $\mathbb{R}^n \times (0, \infty)$ and

$$\|u(\cdot, t) - u_0\|_{L^1(E)} \to 0 \quad \text{as} \quad t \to 0$$

for any compact subset E of $\mathbb{R}^n$. For any $\lambda > 0$ and $\beta > 0$, we say that v is a solution of (1.5) if v is a positive classical solution of (1.5) in $\mathbb{R}^n$. When there is no ambiguity we will drop the subscript and write v for the radially symmetric solution $v_{\lambda,\beta}$ of (1.5). Let

$$w(s) = r^2 v(r)^{1-m} \quad \text{and} \quad s = \log r. \tag{1.8}$$

We will assume that $n \geq 3, 0 < m < \frac{n-2}{n}$, $\beta > 0$, $\lambda > 0$ and w be given by (1.8) for the rest of this paper. Unless stated otherwise we will also assume that $m \neq \frac{n-2}{n+2}$.

We obtain the following two main theorems in this paper.

Theorem 1.1. *Let $n \geq 3, 0 < m < \frac{n-2}{n}, m \neq \frac{n-2}{n+2}, \beta > 0$ and $\lambda > 0$. Let $v_{\lambda,\beta}(r)$ be the radially symmetric solution of (1.5) given by [12]. Then there exists a constant K_0 independent of β and λ and a constant $K(\lambda, \beta)$ such that*

$$v_{\lambda,\beta}(r)^{1-m} = \frac{2(n-1)(n-2-nm)}{(1-m)\beta r^2}\left\{\log r - \frac{n-2-(n+2)m}{2(n-2-nm)}\log(\log r)) + \frac{1-m}{2}\log\lambda + \frac{1}{2}\log\beta + K_0 + \frac{a_0}{\log r} + \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2} \times \frac{\log(\log r)}{\log r} + \frac{o(\log r)}{\log r}\right\} \tag{1.9}$$

as $r \to \infty$ *where*

$$a_0 = \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2} - \frac{(1-m)^2 a_1(1,1)}{4(n-1)(n-2-nm)^2} \tag{1.10}$$

and

$$a_1(\lambda,\beta) = \frac{2(1-2m)(n-1)(n-2-nm)}{(1-m)^2} + \frac{(n-1)(n-2-(n+2)m)^2}{(1-m)^2} + \frac{(n-2-(n+2)m)}{(1-m)} K(\lambda,\beta)\beta. \tag{1.11}$$

Theorem 1.2. *Let* $n \geq 3$, $0 < m < \frac{n-2}{n}$, $m \neq \frac{n-2}{n+2}$, $\beta > 0$, $0 \leq u_0 = \phi + \psi$, $\phi \in L^p_{loc}(\mathbb{R}^n)$, $\psi \in L^1(\mathbb{R}^n) \cap L^p_{loc}(\mathbb{R}^n)$, *for some constant* $p > \frac{(1-m)n}{2}$, *be such that*

$$\phi(x)^{1-m} = \frac{2(n-1)(n-2-nm)}{(1-m)\beta|x|^2} \left(\log|x| - \frac{n-2-(n+2)m}{2(n-2-nm)} \log(\log|x|) + K_1 + o(1)\right) \text{ as } |x| \to \infty \tag{1.12}$$

for some constant $K_1 \in \mathbb{R}$. *If* u *is the unique solution of* (1.1) *in* $\mathbb{R}^n \times (0,\infty)$ *given by Theorem 1.1 of* [13], *then as* $t \to \infty$, *the rescaled function* $\widetilde{u}(x,t)$ *given by* (1.6) *converges to* $v_{\lambda_1,\beta}$ *in* $L^1_{loc}(\mathbb{R}^n)$ *with* $\lambda_1 = \left(e^{2K_1/K_0}/\beta\right)^{\frac{1}{1-m}}$ *where the constant* K_0 *is given by Theorem* 1.1.

Moreover if u_0 *also satisfies* $u_0 = \phi \in L^\infty(\mathbb{R}^n)$, *then as* $t \to \infty$, $\widetilde{u}(x,t)$ *also converges to* $v_{\lambda_1,\beta}$ *uniformly in* $C^{2,1}(E)$ *for any compact subset* $E \subset \mathbb{R}^n$.

2. Proofs

In this section we will give the proof of Theorems 1.1 and 1.2. We first recall some results of [8,12,14].

Theorem 2.1. (Theorem 1.3 of [12] and its proof) *Let* v *be the unique radially symmetric solution of* (1.5) *and* w *be given by* (1.8). *Then*

$$\lim_{|x|\to\infty} \frac{|x|^2 v(x)^{1-m}}{\log|x|} = \lim_{s\to\infty} \frac{w(s)}{s} = \lim_{s\to\infty} w_s(s) = \frac{2(n-1)(n-2-nm)}{(1-m)\beta}. \tag{2.1}$$

Lemma 2.2. (cf. Corollary 2.2 of [8] and Lemma 2.2 of [14]) *Let* $0 \leq u_{0,1}, u_{0,2} \in L^1_{loc}(\mathbb{R}^n)$ *be such that* $u_{0,1} - u_{0,2} \in L^1(\mathbb{R}^n)$. *Suppose* u_1, u_2, *are solutions of* (1.1) *in* $\mathbb{R}^n \times (0,\infty)$ *with initial values* $u_{0,1}, u_{0,2}$ *respectively such that for any* $T > 0$ *there exist constants* $C_1 > 0$, $R_1 > 0$, *such that*

$$u_i(x,t) \geq C_1|x|^{-\frac{2}{1-m}} \quad \forall |x| \geq R_1, \quad 0 < t < T, \quad i = 1,2.$$

Then

$$\|u_1(\cdot,t)-u_2(\cdot,t)\|_{L^1(\mathbb{R}^n)}\le\|u_{0,1}-u_{0,2}\|_{L^1(\mathbb{R}^n)}\quad\forall t>0$$

and hence

$$\|\widetilde{u}_1(\cdot,t)-\widetilde{u}_2(\cdot,t)\|_{L^1(\mathbb{R}^n)}\le e^{-\frac{(n-2-nm)}{1-m}t}\|\widetilde{u}_{0,1}-\widetilde{u}_{0,2}\|_{L^1(\mathbb{R}^n)}\quad\forall t>0$$

where $\widetilde{u}_1(\cdot,t),\widetilde{u}_2(\cdot,t)$, *are the rescaled solutions of* u_1,u_2, *given by* (1.6).

By the computation of [12] we have

$$\begin{aligned}w_{ss}=&\frac{1-2m}{1-m}\cdot\frac{w_s^2}{w}-\frac{n-2-(n+2)m}{1-m}w_s\\&+\frac{\beta}{n-1}\left(\frac{2(n-1)(n-2-nm)}{(1-m)\beta}-w_s\right)w\quad\text{in }\mathbb{R}.\end{aligned}\tag{2.2}$$

Let

$$h(s)=w(s)-\frac{2(n-1)(n-2-nm)}{(1-m)\beta}s.\tag{2.3}$$

Then by (2.2),

$$\begin{aligned}h_{ss}=&\frac{1-2m}{1-m}\cdot\frac{w_s^2}{w}-\frac{(n-2-(n+2)m)}{1-m}\left(\frac{2(n-1)(n-2-nm)}{(1-m)\beta}+h_s\right)\\&-\frac{\beta}{n-1}\left(\frac{2(n-1)(n-2-nm)}{(1-m)\beta}s+h\right)h_s\quad\text{in }\mathbb{R}.\end{aligned}$$

Hence

$$\begin{aligned}&h_{ss}+\left(\frac{2(n-2-nm)}{(1-m)}s+\frac{\beta}{n-1}h+\frac{n-2-(n+2)m}{1-m}\right)h_s\\&=\frac{1-2m}{1-m}\cdot\frac{w_s^2}{w}-b_0\quad\text{in }\mathbb{R}\end{aligned}\tag{2.4}$$

where

$$b_0=\frac{2(n-1)(n-2-nm)(n-2-(n+2)m)}{(1-m)^2\beta}.$$

Lemma 2.3. *Let* $n\ge3$, $0<m<\frac{n-2}{n}$ *and* $m\ne\frac{n-2}{n+2}$. *Then* h *satisfies*

$$\lim_{s\to\infty}\frac{h(s)}{\log s}=\lim_{s\to\infty}s\,h_s(s)=-\frac{(n-1)[n-2-(n+2)m]}{(1-m)\beta}.\tag{2.5}$$

Proof. We first observe that by Theorem 2.1,

$$\lim_{s\to\infty}\frac{w_s^2(s)}{w(s)}=\frac{\lim_{s\to\infty}w_s^2(s)}{\lim_{s\to\infty}s\cdot(w(s)/s)}=0.\tag{2.6}$$

Then by (2.4) and (2.6) for any $0<\varepsilon<|b_0|/2$ there exists a constant $s_1\in\mathbb{R}$ such that

$$-b_0-\varepsilon\le h_{ss}+\left(\frac{2(n-2-nm)}{(1-m)}s+\frac{\beta}{n-1}h+\frac{n-2-(n+2)m}{1-m}\right)h_s$$
$$\le -b_0+\varepsilon\quad \forall s\ge s_1. \tag{2.7}$$

Let

$$f(s)=\exp\left(\frac{(n-2-nm)}{(1-m)}s^2+\frac{\beta}{n-1}\int_{s_1}^s h(z)\,dz+\frac{n-2-(n+2)m}{1-m}s\right). \tag{2.8}$$

By (2.7),

$$(-b_0-\varepsilon)f(s)\le (f(s)h_s(s))_s\le(-b_0+\varepsilon)f(s)\quad\forall s\ge s_1$$
$$\Rightarrow\ \frac{f(s_1)h_s(s_1)s+(-b_0-\varepsilon)s\int_{s_1}^s f(z)\,dz}{f(s)}\le sh_s(s)$$
$$\le\frac{f(s_1)h_s(s_1)s+(-b_0+\varepsilon)s\int_{s_1}^s f(z)\,dz}{f(s)}\quad\forall s\ge s_1. \tag{2.9}$$

By (2.1), $h_s(s)\to 0$ as $s\to\infty$. Hence $h(s)=o(s)$ and $h(s)/s\to 0$ as $s\to\infty$. Then by (2.8) $f(s)\to\infty$ as $s\to\infty$. Hence by the l'Hospital rule,

$$\lim_{s\to\infty}\frac{\int_{s_1}^s f(z)\,dz}{f(s)}=\lim_{s\to\infty}\frac{f(s)}{f(s)\left(\frac{2(n-2-nm)}{(1-m)}s+\frac{\beta}{n-1}h(s)+\frac{n-2-(n+2)m}{1-m}\right)}=0 \tag{2.10}$$

and

$$\lim_{s\to\infty}\frac{s}{f(s)}=\lim_{s\to\infty}\frac{1}{f(s)\left(\frac{2(n-2-nm)}{(1-m)}s+\frac{\beta}{n-1}h(s)+\frac{n-2-(n+2)m}{1-m}\right)}=0. \tag{2.11}$$

Hence by (2.10) and the l'Hospital rule,

$$\lim_{s\to\infty}\frac{s\int_{s_1}^s f(z)\,dz}{f(s)}=\lim_{s\to\infty}\frac{sf(s)+\int_{s_1}^s f(z)\,dz}{f(s)\left(\frac{2(n-2-nm)}{(1-m)}s+\frac{\beta}{n-1}h(s)+\frac{n-2-(n+2)m}{1-m}\right)}$$
$$=\frac{(1-m)}{2(n-2-nm)}. \tag{2.12}$$

Letting first $s\to\infty$ and then $\varepsilon\to 0$ in (2.9), by (2.11) and (2.12) we have

$$\lim_{s\to\infty}s\,h_s(s)=-\frac{(1-m)b_0}{2(n-2-nm)}=-\frac{(n-1)[n-2-(n+2)m]}{(1-m)\beta}. \tag{2.13}$$

Hence

$$h(s)\le-\frac{(n-1)[n-2-(n+2)m]}{2(1-m)\beta}\log s\quad\text{as } s\to\infty \tag{2.14}$$

and (2.5) follows from (2.13), (2.14) and the l'Hospital rule. □

Corollary 2.4. *Let $n \ge 3, 0 < m < \frac{n-2}{n}$ and $m \ne \frac{n-2}{n+2}$. Then there exists a constant $s_0 \in \mathbb{R}$ such that*

$$\begin{cases} h_s(s) < 0 \quad \forall s \ge s_0 \quad \text{if } 0 < m < \dfrac{n-2}{n+2} \\ h_s(s) > 0 \quad \forall s \ge s_0 \quad \text{if } \dfrac{n-2}{n+2} < m < \dfrac{n-2}{n}. \end{cases}$$

Let

$$h_1(s) = h(s) + \frac{(n-1)[n-2-(n+2)m]}{(1-m)\beta} \log s. \tag{2.15}$$

Then by Lemma 2.3, $h_1(s) = o(\log s)$ as $s \to \infty$. By (2.4) h_1 satisfies

$$\begin{aligned} &h_{1,ss} + \left(\frac{2(n-2-nm)}{(1-m)} s + \frac{\beta}{n-1} h + \frac{n-2-(n+2)m}{1-m} \right) h_{1,s} \\ &= \frac{1-2m}{1-m} \cdot \frac{w_s^2}{w} + a_2 \left[-\frac{1}{s^2} + \left(\frac{\beta}{n-1} h + \frac{n-2-(n+2)m}{1-m} \right) \frac{1}{s} \right] \quad \text{in } \mathbb{R} \end{aligned} \tag{2.16}$$

where

$$a_2 = \frac{(n-1)[n-2-(n+2)m]}{(1-m)\beta}.$$

Lemma 2.5. *Let $0 < m < \frac{n-2}{n}$, $m \ne \frac{n-2}{n+2}$, $\lambda > 0$ and $\beta > 0$. Then*

$$\lim_{s \to \infty} \frac{s^2 h_{1,s}(s)}{\log s} = -\frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta}. \tag{2.17}$$

Proof. Let

$$H(s) = \frac{1-2m}{1-m} \cdot \frac{w_s^2}{w} + a_2 \left[-\frac{1}{s^2} + \left(\frac{\beta}{n-1} h + \frac{n-2-(n+2)m}{1-m} \right) \frac{1}{s} \right].$$

Then by Theorem 2.1 and Lemma 2.3,

$$\lim_{s \to \infty} \frac{sH(s)}{\log s} = -\frac{(n-2-(n+2)m)}{1-m} a_2 = -a_3 \tag{2.18}$$

where

$$a_3 = \frac{(n-1)(n-2-(n+2)m)^2}{(1-m)^2 \beta}. \tag{2.19}$$

By (2.16) and (2.18) for any $0 < \varepsilon < a_3/2$ there exists a constant $s_1 > 1$ such that

$$\begin{aligned} (-a_3 - \varepsilon) \frac{\log s}{s} &\le h_{1,ss} + \left(\frac{2(n-2-nm)}{(1-m)} s + \frac{\beta}{n-1} h \right. \\ &\quad \left. + \frac{n-2-(n+2)m}{1-m} \right) h_{1,s} \le (-a_3 + \varepsilon) \frac{\log s}{s} \end{aligned} \tag{2.20}$$

holds for all $s \ge s_1$. Let f be given by (2.8). Multiplying (2.20) by f and integrating over (s_1, s),

$$\frac{s^2}{\log s}\left(\frac{f(s_1)h_{1,s}(s_1) + (-a_3 - \varepsilon)\int_{s_1}^s \frac{\log z}{z} f(z)\,dz}{f(s)}\right) \le \frac{s^2 h_{1,s}(s)}{\log s}$$
$$\le \frac{s^2}{\log s}\left(\frac{f(s_1)h_{1,s}(s_1) + (-a_3 + \varepsilon)\int_{s_1}^s \frac{\log z}{z} f(z)\,dz}{f(s)}\right) \tag{2.21}$$

holds for all $s \ge s_1$. By the l'Hospital rule and Lemma 2.3,

$$\lim_{s\to\infty} \frac{s^2\int_{s_1}^s \frac{\log z}{z} f(z)\,dz}{f(s)\log s}$$
$$= \lim_{s\to\infty} \frac{f(s)s\log s + 2s\int_{s_1}^s \frac{\log z}{z} f(z)\,dz}{f(s)\left(\frac{2(n-2-nm)}{(1-m)}s + \frac{\beta}{n-1}h(s) + \frac{n-2-(n+2)m}{1-m}\right)\log s + \frac{f(s)}{s}}$$
$$= \frac{1-m}{2(n-2-nm)} + \frac{1-m}{n-2-nm}\lim_{s\to\infty} \frac{\int_{s_1}^s \frac{\log z}{z} f(z)\,dz}{f(s)\log s} \tag{2.22}$$

Since

$$\left|\frac{\int_{s_1}^s \frac{\log z}{z} f(z)\,dz}{f(s)\log s}\right| \le \frac{\int_{s_1}^s f(z)\,dz}{f(s)} \quad \forall s \ge s_1,$$

by (2.10),

$$\lim_{s\to\infty} \frac{\int_{s_1}^s \frac{\log z}{z} f(z)\,dz}{f(s)\log s} = 0. \tag{2.23}$$

By (2.22) and (2.23),

$$\lim_{s\to\infty} \frac{s^2\int_{s_1}^s \frac{\log z}{z} f(z)\,dz}{f(s)\log s} = \frac{1-m}{2(n-2-nm)}. \tag{2.24}$$

Letting first $s \to \infty$ and then $\varepsilon \to 0$ in (2.21), by (2.24) we get (2.17) and the lemma follows. □

Corollary 2.6. *Let $n \ge 3$, $0 < m < \frac{n-2}{n}$, $m \ne \frac{n-2}{n+2}$, $\lambda > 0$ and $\beta > 0$. Then*

$$K(\lambda,\beta) := \lim_{s\to\infty} h_1(s) \in \mathbb{R} \quad \textit{exists} \tag{2.25}$$

and

$$h_1(s) = K(\lambda,\beta) + \frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta}\left(\frac{1+\log s}{s}\right)$$
$$+\,o\left(\frac{1+\log s}{s}\right) \quad \textit{as } s \to \infty. \tag{2.26}$$

Proof. By (2.17) there exist constants $C_1 > 0$ and $s_1 > 1$ such that

$$\left|\frac{s^2 h_{1,s}(s)}{\log s}\right| \le C_1 \quad \forall s \ge s_1. \tag{2.27}$$

Hence

$$\begin{aligned}|h_1(s) - h_1(s_1)| &\le \int_{s_1}^{s} |h_{1,s}(z)|\,dz \le C_1 \int_{s_1}^{s} \frac{\log z}{z^2}\,dz \le C_2 \quad \forall s \ge s_1\\ &\Rightarrow |h_1(s)| \le C_2 + |h_1(s_1)| \quad \forall s \ge s_1\end{aligned} \tag{2.28}$$

for some constant $C_2 > 0$. On the other hand by (2.17) there exists a constant $s_0 > s_1$ such that

$$h_{1,s}(s) < 0 \quad \forall s \ge s_0.$$

Then $h_1(s)$ is monotone decreasing in (s_0, ∞). This together with (2.28) implies that (2.25) holds. By (2.17) for any $0 < \varepsilon < \frac{(n-1)(n-2-(n+2)m)^2}{4(n-2-nm)(1-m)\beta}$ there exists a constant $s_2 > 1$ such that

$$\begin{aligned}&\left(-\frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta} - \varepsilon\right)\frac{\log s}{s^2} \le h_{1,s}(s)\\ &\quad\le \left(-\frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta} + \varepsilon\right)\frac{\log s}{s^2}\end{aligned} \tag{2.29}$$

holds for all $s \ge s_2$. Integrating (2.29) over (s, ∞), $s \ge s_2$,

$$\begin{aligned}&-\left(\frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta} + \varepsilon\right)\frac{(1+\log s)}{s} \le K(\lambda,\beta) - h_1(s)\\ &\quad\le -\left(\frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta} - \varepsilon\right)\frac{(1+\log s)}{s}\end{aligned}$$

for all $s \ge s_2$ and (2.26) follows. □

Let $K(\lambda, \beta)$ be given by (2.25) and

$$h_2(s) = h_1(s) - K(\lambda,\beta) - \frac{(n-1)(n-2-(n+2)m)^2}{2(n-2-nm)(1-m)\beta}\left(\frac{1+\log s}{s}\right). \tag{2.30}$$

Then

$$h_2(s) = o\left(\frac{1+\log s}{s}\right) \text{ as } s \to \infty \tag{2.31}$$

and by (2.15) and (2.16),

$$\begin{aligned}&h_{2,ss} + \left(\frac{2(n-2-nm)}{(1-m)}s + \frac{\beta}{n-1}h + \frac{n-2-(n+2)m}{1-m}\right)h_{2,s}\\ &= \frac{1-2m}{1-m}\cdot\frac{w_s^2}{w} + \frac{(n-1)(n-2-(n+2)m)^2}{(1-m)^2\beta}\cdot\frac{1}{s}\\ &\quad + \frac{(n-2-(n+2)m)}{1-m}\cdot\frac{h_1(s)}{s} - \frac{a_2}{s^2}\end{aligned}$$

$$+\frac{(n-1)(n-2-(n+2)m)^2}{2(1-m)(n-2-nm)\beta}\cdot\left[\frac{(1-2\log s)}{s^3}+\left(\frac{\beta}{n-1}h(s)\right.\right.$$
$$\left.\left.+\frac{n-2-(n+2)m}{1-m}\right)\cdot\frac{\log s}{s^2}\right]$$
$$=: H_1(s). \tag{2.32}$$

Lemma 2.7. *Let $n\geq 3$, $0<m<\frac{n-2}{n}$, $m\neq\frac{n-2}{n+2}$, $\lambda>0$ and $\beta>0$. Then h_2 satisfies*

$$\lim_{s\to\infty}s^2h_{2,s}(s)=\frac{(1-m)a_1(\lambda,\beta)}{2(n-2-nm)\beta} \tag{2.33}$$

where $a_1(\lambda,\beta)$ is given by (1.11) *with $K(\lambda,\beta)$ given by* (2.25).

Proof. By Theorem 2.1, Lemma 2.3 and (2.25),

$$\lim_{s\to\infty}sH_1(s)=\frac{a_1(\lambda,\beta)}{\beta} \tag{2.34}$$

where $a_1(\lambda,\beta)$ is given by (1.11) with $K(\lambda,\beta)$ given by (2.25). Then by (2.32) and (2.34) for any $0<\varepsilon<1$ there exists a constant $s_1>1$ such that

$$\left(\frac{a_1(\lambda,\beta)}{\beta}-\varepsilon\right)\frac{1}{s}$$
$$\leq h_{2,ss}+\left(\frac{2(n-2-nm)}{(1-m)}s+\frac{\beta}{n-1}h+\frac{n-2-(n+2)m}{1-m}\right)h_{2,s}$$
$$\leq\left(\frac{a_1(\lambda,\beta)}{\beta}+\varepsilon\right)\frac{1}{s} \tag{2.35}$$

holds for all $s\geq s_1$. Let f be given by (2.8). Multiplying (2.35) by f and integrating over (s_1,s),

$$\frac{f(s_1)h_2(s_1)s^2+\left(\frac{a_1(\lambda,\beta)}{\beta}-\varepsilon\right)s^2\int_{s_1}^s\frac{f(z)}{z}\,dz}{f(s)}\leq s^2h_{2,s}(s)$$
$$\leq\frac{f(s_1)h_2(s_1)s^2+\left(\frac{a_1(\lambda,\beta)}{\beta}+\varepsilon\right)s^2\int_{s_1}^s\frac{f(z)}{z}\,dz}{f(s)} \tag{2.36}$$

holds for all $s\geq s_1$. Since by the l'Hospital rule,

$$\lim_{s\to\infty}\frac{\int_{s_1}^s\frac{f(z)}{z}\,dz}{f(s)}=\lim_{s\to\infty}\frac{\frac{f(s)}{s}}{f(s)\left(\frac{2(n-2-nm)}{(1-m)}s+\frac{\beta}{n-1}h(s)+\frac{n-2-(n+2)m}{1-m}\right)}=0,$$

by Lemma 2.3 and the l'Hospital rule,

$$\lim_{s\to\infty}\frac{s^2\int_{s_1}^s\frac{f(z)}{z}\,dz}{f(s)}=\lim_{s\to\infty}\frac{sf(s)+2s\int_{s_1}^s\frac{f(z)}{z}\,dz}{f(s)\left(\frac{2(n-2-nm)}{(1-m)}s+\frac{\beta}{n-1}h(s)+\frac{n-2-(n+2)m}{1-m}\right)}$$

$$= \frac{(1-m)}{2(n-2-nm)} + \frac{(1-m)}{(n-2-nm)} \lim_{s\to\infty} \frac{\int_{s_1}^{s} \frac{f(z)}{z}\,dz}{f(s)}$$
$$= \frac{(1-m)}{2(n-2-nm)}. \tag{2.37}$$

Hence letting first $s \to \infty$ and then $\varepsilon \to 0$ in (2.36), by (2.8) and (2.37) we get (2.33) and the lemma follows. □

By (2.31) and Lemma 2.7 we have the following result.

Lemma 2.8. *Let $n \ge 3$, $0 < m < \frac{n-2}{n}$, $m \ne \frac{n-2}{n+2}$, $\lambda > 0$ and $\beta > 0$. Then*

$$h_2(s) = -\frac{(1-m)a_1(\lambda,\beta)}{2(n-2-nm)\beta s} + \frac{o(s)}{s} \quad as \quad s \to \infty \tag{2.38}$$

where $a_1(\lambda,\beta)$ is given by (1.11) with $K(\lambda,\beta)$ given by (2.25).

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Let a_0, $a_1(\lambda,\beta)$, be given by (1.10) and (1.11) with $K(\lambda,\beta)$ given by (2.25). Let

$$K_0 = \frac{(1-m)K(1,1)}{2(n-1)(n-2-nm)}.$$

By (1.8), (2.3), (2.15), (2.30) and (2.38),

$$\begin{aligned}
&v_{1,1}(r)^{1-m}\\
&= \frac{2(n-1)(n-2-nm)}{(1-m)r^2}\left\{\log r - \frac{(n-2-(n+2)m)}{2(n-2-nm)}\log(\log r) + K_0\right.\\
&\quad + \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2}\left(\frac{1+\log(\log r)}{\log r}\right)\\
&\quad \left. - \frac{(1-m)^2}{4(n-1)(n-2-nm)^2}\frac{a_1(1,1)}{\log r} + \frac{o(\log r)}{\log r}\right\}\\
&= \frac{2(n-1)(n-2-nm)}{(1-m)r^2}\left\{\log r - \frac{(n-2-(n+2)m)}{2(n-2-nm)}\log(\log r)\right.\\
&\quad \left. + K_0 + \frac{a_0}{\log r} + \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2}\cdot\frac{\log(\log r)}{\log r} + \frac{o(\log r)}{\log r}\right\} \quad \text{as } r \to \infty.
\end{aligned} \tag{2.39}$$

Then by (2.19) of [4] and (2.39),

$$\begin{aligned}
&v_{\lambda,\beta}(r)^{1-m}\\
&= \lambda^{1-m} v_{1,1}(\lambda^{\frac{1-m}{2}}\sqrt{\beta} r)^{1-m}\\
&= \frac{2(n-1)(n-2-nm)}{(1-m)\beta r^2}\left\{\log(\lambda^{\frac{1-m}{2}}\sqrt{\beta} r) - \frac{(n-2-(n+2)m)}{2(n-2-nm)}\right.\\
&\quad \times \log(\log(\lambda^{\frac{1-m}{2}}\sqrt{\beta} r)) + K_0 + \frac{a_0}{\log(\lambda^{\frac{1-m}{2}}\sqrt{\beta} r)} + \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2}
\end{aligned}$$

$$\times \frac{\log(\log(\lambda^{\frac{1-m}{2}}\sqrt{\beta}r))}{\log(\lambda^{\frac{1-m}{2}}\sqrt{\beta}r)} + \frac{o(\log(\lambda^{\frac{1-m}{2}}\sqrt{\beta}r))}{\log(\lambda^{\frac{1-m}{2}}\sqrt{\beta}r)}\Bigg\}$$
$$= \frac{2(n-1)(n-2-nm)}{(1-m)\beta r^2}\Bigg\{\log r - \frac{(n-2-(n+2)m)}{2(n-2-nm)}\log(\log r)) + \frac{1-m}{2}\log\lambda + \frac{1}{2}\log\beta + K_0 + \frac{a_0}{\log r} + \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2} \times \frac{\log(\log r)}{\log r} + \frac{o(\log r)}{\log r}\Bigg\}$$

as $r \to \infty$ and Theorem 1.1 follows. □

Remark 2.9. From (1.9),

$$h_1(s) = \frac{2(n-1)(n-2-nm)}{(1-m)\beta}\Bigg\{\frac{1-m}{2}\log\lambda + \frac{1}{2}\log\beta + K_0 + \frac{a_0}{s} + \frac{(n-2-(n+2)m)^2}{4(n-2-nm)^2}\cdot\frac{\log s}{s} + \frac{o(s)}{s}\Bigg\} \quad \text{as} \quad s \to \infty.$$

Hence

$$\lim_{s\to\infty} h_1(s) = \frac{2(n-1)(n-2-nm)}{(1-m)\beta}\left\{\frac{1-m}{2}\log\lambda + \frac{1}{2}\log\beta + K_0\right\}. \tag{2.40}$$

Thus by (2.25) and (2.40),

$$K(\lambda,\beta) = \frac{2(n-1)(n-2-nm)}{(1-m)\beta}\left\{\frac{1-m}{2}\log\lambda + \frac{1}{2}\log\beta + K_0\right\}.$$

Proof of Theorem 1.2. Since the proof is similar to the proof of Theorem 3.1 and Corollary 3.2 of [4] we will only sketch the proof here. Let K_0 be given by Theorem 1.1 and $\lambda_1 = \left(e^{2(K_1-K_0)}/\beta\right)^{\frac{1}{1-m}}$. Then for any $0 < \varepsilon < \lambda_1$ there exists a constant $R_\varepsilon > 0$ such that

$$u_{\lambda_1-\varepsilon,\beta}(x) \le \phi(x) \le u_{\lambda_1+\varepsilon,\beta}(x) \quad \forall |x| \ge R_\varepsilon$$
$$\Rightarrow u_{\lambda_1-\varepsilon,\beta}(x) + \psi(x) \le u_0(x) \le u_{\lambda_1+\varepsilon,\beta}(x) + \psi(x) \quad \forall |x| \ge R_\varepsilon. \tag{2.41}$$

For any $\delta > 0$, let $u_1, u_2, u_{1,\delta}$ and $w_{1,\delta}$ be the solution of (1.4) with initial value

$$\min(u_{\lambda_1-\varepsilon,\beta}(x) + \psi(x), u_0(x)),\ \max(u_{\lambda_1+\varepsilon,\beta}(x) + \psi(x), u_0(x)),$$
$$\min(u_{\lambda_1-\varepsilon,\beta}(x) + \psi(x), u_0(x)) + \delta,$$

and $u_{\lambda_1-\varepsilon,\beta}(x) + \delta$ respectively given by Theorem 1.1 of [13]. Let $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_{1,\delta}$ and $\widetilde{w}_{1,\delta}$ be given by (1.6) with u being replaced by $u_1, u_2, u_{1,\delta}$ and $w_{1,\delta}$ respectively. By (2.41) and the construction of solutions in [13],

$$u_1 \le u \le u_2, \quad u_{1,\delta} \ge \delta, \quad w_{1,\delta} \ge \delta \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \quad \forall \delta > 0 \tag{2.42}$$
$$\Rightarrow \quad \widetilde{u}_1 \le \widetilde{u} \le \widetilde{u}_2, \quad \text{in} \quad \mathbb{R}^n \times (0,\infty). \tag{2.43}$$

By (2.42) and Lemma 2.2,

$$\|\widetilde{u}_{1,\delta}(\cdot,t)-\widetilde{w}_{1,\delta}(\cdot,t)\|_{L^1(\mathbb{R}^n)}\le e^{-\frac{(n-2-nm)}{1-m}t}\|\min(u_{\lambda_1-\varepsilon,\beta}+\psi,u_0)-u_{\lambda_1-\varepsilon,\beta}\|_{L^1(\mathbb{R}^n)}\quad\forall t>0 \tag{2.44}$$

Since $\min(u_{\lambda_1-\varepsilon,\beta}(x)+\psi(x),u_0(x))+\delta$ and $u_{\lambda_1-\varepsilon,\beta}(x)+\delta$ decreases monotonically to

$$\min(u_{\lambda-\varepsilon,\beta}(x)+\psi(x),u_0(x))\quad\text{and}\quad u_{\lambda_1-\varepsilon,\beta}(x)$$

as $\delta\to 0$, $u_{1,\delta}$ and $w_{1,\delta}$ decreases monotonically to u_1 and $e^{-\frac{2\beta}{1-m}t}u_{\lambda_1-\varepsilon,\beta}(e^{-\beta t}x)$ as $\delta\to 0$. Hence letting $\delta\to 0$ in (2.44),

$$\|\widetilde{u}_1(\cdot,t)-u_{\lambda_1-\varepsilon,\beta}\|_{L^1(\mathbb{R}^n)}\le e^{-\frac{(n-2-nm)}{1-m}t}\|\min(u_{\lambda_1-\varepsilon,\beta}+\psi,u_0)-u_{\lambda_1-\varepsilon,\beta}\|_{L^1(\mathbb{R}^n)}\quad\forall t>0 \tag{2.45}$$

Similarly,

$$\|\widetilde{u}_2(\cdot,t)-u_{\lambda_1+\varepsilon,\beta}\|_{L^1(\mathbb{R}^n)}\le e^{-\frac{(n-2-nm)}{1-m}t}\|\max(u_{\lambda_1-\varepsilon,\beta}+\psi,u_0)-u_{\lambda_1+\varepsilon,\beta}\|_{L^1(\mathbb{R}^n)}\quad\forall t>0 \tag{2.46}$$

By (2.43), (2.45) and (2.46), and an argument similar to the proof of Theorem 3.1 of [4], the rescaled function $\widetilde{u}(x,t)$ given by (1.6) converges to $v_{\lambda_1,\beta}$ in $L^1_{loc}(\mathbb{R}^n)$ as $t\to\infty$.

Suppose now u_0 also satisfies $u_0=\phi\in L^\infty(\mathbb{R}^n)$. Then by an argument similar to the proof of Corollary 3.2 of [4], there exists a constant $\lambda_2>0$ such that

$$u_0\le v_{\lambda_2,\beta}\quad\text{in}\ \ \mathbb{R}^n$$

Hence by maximum principle for solutions of (1.1) in bounded domains (cf. Lemma 2.3 of [5]) and the construction of solution (1.4) in [13],

$$\begin{aligned}&u(x,t)\le e^{-\frac{2\beta}{1-m}t}v_{\lambda_2,\beta}(e^{-\beta t}x)\quad\forall x\in\mathbb{R}^n,t>0\\ \Rightarrow\ &\widetilde{u}(x,t)\le v_{\lambda_2,\beta}(x)\quad\forall x\in\mathbb{R}^n,t>0.\end{aligned} \tag{2.47}$$

Then by (1.7), (2.47), and an argument similar to the proof on P.10 of [4], the rescaled function $\widetilde{u}(x,t)$ converges to $v_{\lambda_1,\beta}$ uniformly in $C^{2,1}(E)$ for any compact subset $E\subset\mathbb{R}^n$ as $t\to\infty$. □

Finally by Theorem 1.1 and a similar argument as the proof of Theorem 3.6 and Proposition 3.9 of [4] we have the following result.

Theorem 2.10. *Let* $n\ge 3$, $0<m<\frac{n-2}{n}$, $m\ne\frac{n-2}{n+2}$, $\beta>0$, $0\le u_0=\phi+\psi$, $\phi\in L^p_{loc}(\mathbb{R}^n)$, $\psi\in L^1(\mathbb{R}^n)\cap L^p_{loc}(\mathbb{R}^n)$, *for some constant* $p>\frac{(1-m)n}{2}$, *such that*

$$K_2:=\limsup_{|x|\to\infty}\left[|x|^2u_0(x)^{1-m}-\frac{c_1}{\beta}\left(\log|x|-\frac{(n-2-(n+2)m)}{2(n-2-nm)}\log(\log|x|)\right)\right]<\infty.$$

holds and ϕ *satisfies* (1.12) *for some constant* $K_1\in\mathbb{R}$ *where* $c_1=2(n-1)(n-2-nm)/(1-m)$. *If* u *is the unique solution of* (1.1) *in* $\mathbb{R}^n\times(0,\infty)$ *given by*

Theorem 1.1 of [13], *then as* $t \to \infty$, *the rescaled function* $\widetilde{u}(x,t)$ *given by* (1.6) *converges to* $v_{\lambda_1,\beta}$ *uniformly in* $C^{2,1}(E)$ *for any compact subset* $E \subset \mathbb{R}^n$ *with* $\lambda_1 = \left(e^{2(K_1-K_0)}/\beta\right)^{\frac{1}{1-m}}$ *where the constant* K_0 *is given by Theorem* 1.1. *Moreover*

$$\limsup_{|x|\to\infty}\left[|x|^2u(x,t)^{1-m} - \frac{c_1}{\beta}\left(\log|x| - \frac{n-2-(n+2)m}{2(n-2-nm)}\log(\log|x|)\right)\right] \le K_2 - \frac{2(n-1)(n-2-nm)}{(1-m)}t \quad \forall t \ge 0.$$

3. Appendix

For the sake of completeness, in this appendix we will state and prove the analogue of Lemma 2.3 for the case $m = \frac{n-2}{n+2}$, $n \ge 3$.

Proposition 3.1. (Proposition 2.3 of [4]) *Let* $n \ge 3$, $m = \frac{n-2}{n+2}$, $\lambda > 0$ *and* $\beta > 0$. *Let* v *be the solution of* (1.5) *and* h *be given by* (2.3) *with* w *given by* (1.8). *Then* h *satisfies*

$$\lim_{s\to\infty} s^2 h_s(s) = \frac{(6-n)(n-1)}{4\beta}.$$

Proof. This proposition is stated and proved in [4]. For the sake of completeness we will give a simple different proof of the proposition here. We first observe that by Theorem 2.1,

$$\lim_{s\to\infty}\frac{sw_s(s)^2}{w(s)} = \frac{2(n-1)(n-2-nm)}{(1-m)\beta} = \frac{(n-1)(n-2)}{\beta}. \tag{3.1}$$

Let

$$a_4 = \frac{(1-2m)(n-1)(n-2)}{(1-m)\beta}.$$

Then by (2.4) and (3.1) for any $0 < \varepsilon < |a_4|/2$ there exists a constant $s_1 \in \mathbb{R}$ such that

$$(a_4-\varepsilon)s^{-1} \le h_{ss} + \left(\frac{2(n-2-nm)}{(1-m)}s + \frac{\beta}{n-1}h + \frac{n-2-(n+2)m}{1-m}\right)h_s \le (a_4+\varepsilon)s^{-1} \quad \forall s \ge s_1. \tag{3.2}$$

By (3.2) and an argument similar to the proof of Lemma 2.7,

$$\lim_{s\to\infty} s^2 h_s(s) = \frac{(1-m)a_4}{2(n-2-nm)} = \frac{(6-n)(n-1)}{4\beta}$$

and the proposition follows. □

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