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Exponentially harmonic maps between Finsler manifolds

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Abstract. Exponentially harmonic maps and harmonic maps are different. In this paper, we derive the first and second variations of the exponential energy of a smooth map between Finsler manifolds. We show that a non-constant exponentially harmonic map f from a unit m-sphere S^m ($m \geq 3$) into a Finsler manifold is stable in case $|df|^2 \geq m-2$, and is unstable in case $|df|^2 < m-2$.

1. Introduction

Exponentially harmonic maps between Riemannian manifolds were first explored by Eells and Lemaire [11] in 1990. Afterwards, Duc and Eells [10], Hong [14], Hong and Yang [15], Chiang et al. [3–8], Cheung and Leung [9], Zhang et al. [24], and others also investigated exponentially harmonic maps. In 2002, Kanfon et al. [16] discovered the applications of exponential harmonic maps on Friedmann–Lemaître universe, and considered some new models of exponentially harmonic maps which are coupled with gravity based on a generalization of Lagrangian for bosonic strings coupled with diatonic field. In 2011–2012, Omori [18,19] obtained results about Eells–Sampson's existence theorem and Sacks–Uhlenbeck's existence theorem for harmonic maps via exponentially harmonic maps.

Exponentially harmonic maps and harmonic maps (cf. [12]) are different. There are exponentially harmonic maps which are not harmonic maps, and there are harmonic maps which are not exponentially harmonic maps either (cf. [15]). Cheung and Leung [9] showed that the identity map from any compact manifold M^m into itself is always stable as an exponentially harmonic map, which is contrast to such an identity as a harmonic map [22] unstable. They also showed that an isometric and totally geodesic immersion of S^m into S^n is an unstable exponentially harmonic map if $m \neq n$, and is stable if $m = n$. Moreover, Chiang and Yang [8] proved that if f is an exponentially harmonic map from a Riemannian manifold into another Riemannian manifold with non-positive sectional curvature, then f is stable. Chiang [3] also obtained a theorem asserted as follows: If f is an exponentially harmonic

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map from a compact Riemannian manifold M into the unit n-sphere S^n $(n \geq 3)$ with $|df|^2 < n-2$, then f is unstable.

The structure of a Finsler manifold is different from the structure of a Riemannian manifold. Recently, Mo [17], He et al. [13,20,23], Shen [21] and others investigated harmonic maps of Finsler manifolds. In this paper, we study exponentially harmonic maps between Finsler manifolds. We derive the first and second variations of exponential energy between Finsler manifolds in Theorems 3.4 and 4.1. We then apply Theorem 4.1 to show that a non-constant exponentially harmonic map f from a unit m-sphere S^m $(m \geq 3)$ into a Finsler manifold is stable in case $|df|^2 \geq m-2$, and is unstable in case $|df|^2 < m-2$, in Theorem 5.4.

2. Finsler manifold

Let M be an m-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from its tangent bundle. Let (x, y) be a point in TM with $x \in M$, $y \in T_xM$ and let (x^i, y^i) be its local coordinates in TM with $y = y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, \infty)$ satisfying the following properties:

(i) Regularity: $F(x, y)$ is smooth in $TM\backslash\{0\}$;
(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$;
(iii) Strong convexity: The fundamental quadratic form $g = g_{ij} dx^i \otimes dx^j$ is positive definite, where $g_{ij} = \frac{1}{2}\partial^2(F^2)/\partial y^i \partial y^j$.

A *Finsler manifold* is a C^∞ manifold M with a Finsler metric F (cf. [1,2]). Let SM be the projective sphere bundle of M with canonical projection map $\pi : SM \to M$ given by $(x, y) \to x$, and let $S_xM = \pi^{-1}(x)$ be the projective sphere at x. Denote the pull-backs of TM and T^*M by π^*TM and π^*T^*M, respectively, and consider these as vector bundles (with m-dimensional fibres) over the $(2m-1)$-dimensional base SM.

Let (M, F) be an m-dimensional Finsler manifold. Given local coordinates (x^i) on M, we write any $y \in T_xM$ as $y^i \frac{\partial}{\partial x^i}$. This generates local coordinates (x^i, y^i) on SM. At each point of SM, the fiber of π^*TM has a basis $\{\frac{\partial}{\partial x^i}\}$. Therefore, F inherits the Hilbert form and the Cartan tensor as follows:

$$w = \frac{\partial F}{\partial y^i} dx^i, \;\; A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \;\; A_{ijk} = F\frac{\partial g_{ij}}{\partial y^k},$$

where $1 \leq i, j, k \leq m$. It is well-known that there exists a unique Chern connection ∇ (cf. [1]) on π^*TM with $\nabla_{\frac{\partial}{\partial x^i}} = w_i^j \frac{\partial}{\partial x^j}$ and $w_i^j = \Gamma_{ik}^j dx^k$ such that

$$dg_{ij} - g_{ik} w_j^k - g_{jk} w_i^k = 2A_{ijk}\frac{\delta y^k}{F}, \tag{2.1}$$

where $\delta y^i = dy^i + N_j^i dx^j$, $N_j^i = \gamma_{jk}^i y^k - \frac{1}{F} A_{jk}^i \gamma_{nl}^k y^n y^l$ and γ_{jk}^i are the Christoffel symbols of the second kind for g_{ij}. Since $\nabla e_m = \frac{\delta y^k}{F}\frac{\partial}{\partial x^k}$ with $e_m = \frac{y^i}{F}\frac{\partial}{\partial x^i}$, (2.1) is equivalent to

$$X(U, V) = (\nabla_X U, V) + (U, \nabla_X V) + 2C(U, V, \nabla_X(Fe_m)), \tag{2.2}$$

where $A_{ijk} = FC_{ijk}$ and $X, U, V \in \Gamma(\pi^*TM)$.

The curvature 2-forms of the Chern connection ∇ are

$$w^i_j - w^k_j \wedge w^j_k = \Omega^i_j = \frac{1}{2} R^i_{jkl} dx^k \wedge dx^l + \frac{1}{F} P^i_{jkl} dx^k \wedge \delta y^l. \tag{2.3}$$

Choose a g-orthonormal frame $\{e_i = u^j_i \frac{\partial}{\partial x^j}\}$ with $e_m = \frac{y^i}{F} \frac{\partial}{\partial x^i}$ for each fibre of π^*TM and $\{\omega^i\}$ its dual coframe. The collection $\{w^i, w^i_m\}$ forms an orthonormal basis for $T^*(TM\backslash\{0\})$ with respect to the Sasaki-type metric $g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j$. The pull-back of the Sasaki metric from $TM\backslash\{0\}$ to SM is a Riemannian metric

$$\bar{g} = g_{ij} dx^i \otimes dx^j + \delta_{ab} w^a_m \otimes w^b_m, \tag{2.4}$$

where $1 \leq a, b, c \leq m - 1$. Set $P_{abc} := P_{mabc}$. P_{abc} is called the Landberg curvature (cf. [21]). It follows from [2] that

$$P_{mab} = 0, \ \sum_a P_{aab} = -\sum_a \dot{A}_{aab},$$

where the dot denotes the covariant derivative along the Hilbert form.

Lemma 2.1. *For $\phi = \phi_i w^i \in \Gamma(\pi^* T^* M)$, we have*

$$\mathrm{div}_{\bar{g}} \phi = \sum_i \phi_{i|i} + \sum_{a,b} \phi_a P_{bba} = (\nabla_{e^H_i} \phi) e_i + \sum_{a,b} \phi_a P_{bba},$$

where $|$ denotes the horizontal covariant differential with respect to the Chern connection, $e^H_i = u^j_i \frac{\delta}{\delta x^j} = u^j_i (\frac{\partial}{\partial x^j} - N^k_j \frac{\partial}{\partial y^i})$ is the horizontal part of e_i and $P_{bba} = P^m_{bba}$ (cf. [2]).

For any fixed $x \in M$, $S_x M = \{y \in T_x M | F(y) = 1\}$ has a natural Riemannian metric

$$\hat{r}_x = \sum_a \theta^a_m \otimes \theta^a_m, \ \theta^a_m = w^a_m|_{S_x M}.$$

Lemma 2.2. *For $\psi = \nu \psi_i dy^i$, we have*

$$div_{\hat{r}_x} \psi = F^2 \nu g^{ij} [\psi_i]_{y^j} - m\nu \psi_i y^i,$$

where $\nu = \sqrt{\det(g_{ij})}$ and $[\psi_i]_{y^j} = \frac{\partial}{\partial y^j} \psi_i$ (cf. [13]).

3. Exponentially harmonic maps of Finsler manifolds

Let $f : M^m \to \hat{M}^n$ be a non-constant smooth map between two Finsler manifolds. The exponential energy density of f is the function from SM to $\mathbb{R}$ defined by

$$e_e(f)(x, y) = e^{\frac{1}{2}|df|^2} = e^{\frac{1}{2}g^{ij}(x,y)f_i^\alpha f_j^\beta \hat{g}_{\alpha\beta}(\hat{x},\hat{y})}, \tag{3.1}$$

where $df(\frac{\partial}{\partial x^i}) = f_i^\alpha \frac{\partial}{\partial \hat{x}^\alpha}$ and $\hat{y} = \hat{y}^\alpha \frac{\partial}{\partial \hat{x}^\alpha} = y^i f_i^\alpha \frac{\partial}{\partial \hat{x}^\alpha}$, $1 \le i,\ j \le m$, $1 \le \alpha,\ \beta \le n$. The exponential energy of f is given by

$$E_e(f) = \frac{1}{c}\int_{SM} e_e(f) dv_{SM}, \tag{3.2}$$

where $dv_{SM} = \Omega d\tau \wedge dx$, $\Omega = \det(g_{ij}/F)$, $d\tau = \sum_i (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots dy^m$, $dx = dx^1 \wedge \cdots \wedge dx^m$, and c denotes the volume of the unit sphere S^{m-1}.

Let $\tilde{\nabla}$ be the pullback Chern connection on $\pi^*(f^{-1}T\hat{M})$, and let $\tilde{\Omega}$ be the curvature form of the pullback connection $\tilde{\nabla}$.

Lemma 3.1. *We have*

$$\begin{aligned}
&(1)\ X\langle dfU,\ dfV\rangle = \langle \tilde{\nabla}_X(dfU),\ dfV\rangle + \langle dfU,\ \tilde{\nabla}_X(dfV)\rangle \\
&\quad + 2\hat{C}(dfU,\ dfV,\ \tilde{\nabla}_X(dfF_{e_m})), \qquad (3.3)\\
&(2)\ \tilde{\Omega}^\alpha_\beta(U, V) = \hat{R}^\alpha_\beta(dfU,\ dfV) + \frac{F}{\hat{F}}\hat{P}^\alpha_\beta(dfU,\ \tilde{\nabla}_V dfe_m) \\
&\quad - \frac{F}{\hat{F}}\hat{P}^\alpha_\beta(dfV,\ \tilde{\nabla}_U dfe_m), \qquad (3.4)
\end{aligned}$$

where $\hat{R}^\alpha_\beta = \hat{R}^\alpha_{\beta\gamma\sigma} d\hat{x}^\gamma \otimes d\hat{x}^\sigma$ *and* $\hat{P}^\alpha_\beta = \hat{P}^\alpha_{\beta\gamma\sigma} d\hat{x}^\gamma \otimes d\hat{x}^\sigma$.

Proof. (1) follows from (2.2) with $df(F_{e_m}) = \hat{F}\hat{e}_n$, and (2) follows from (2.3). □

Lemma 3.2. *We obtain*

$$\begin{aligned}
&\sum_i \int_{SM} \Big\langle \tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t},\ e^{\frac{1}{2}|df|^2} dfe_i\Big\rangle dv_{SM} \\
&= -\int_{SM} \Big\{ \sum_i \Big[\Big\langle df\frac{\partial}{\partial t},\ \Big(\tilde{\nabla}_{e_i^H} e^{\frac{1}{2}|df|^2} df\Big)e_i\Big\rangle \\
&\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}\Big(df\frac{\partial}{\partial t}, dfe_i,\ \tilde{\nabla}_{e_i^H} dfF_{e_m}\Big)\Big] \\
&\quad + \sum_{a,b} e^{\frac{1}{2}|df|^2}\Big\langle df\frac{\partial}{\partial t},\ dfe_a\Big\rangle P_{aab}\Big\} dv_{SM}.
\end{aligned}$$

Proof. Set $\phi = e^{\frac{1}{2}|df|^2}\langle df\frac{\partial}{\partial t}, df e_i\rangle w^i$. Lemma 2.1 implies that

$$\begin{aligned}
\mathrm{div}_{\bar{g}}\phi &= \left\langle \tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t}, e^{\frac{1}{2}|df|^2} df e_i\right\rangle + \left\langle df\frac{\partial}{\partial t}, \tilde{\nabla}_{e_i^H}(e^{\frac{1}{2}|df|^2} df e_i)\right\rangle \\
&\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}\left(df\frac{\partial}{\partial t}, df e_i, \tilde{\nabla}_{e_i^H} df F_{e_m}\right) + e^{\frac{1}{2}|df|^2}\langle df\frac{\partial}{\partial t}, df e_j\rangle(\tilde{\nabla}_{e_i^H} w^j)e_i \\
&\quad + \sum_{a,b} e^{\frac{1}{2}|df|^2}\left\langle df\frac{\partial}{\partial t}, df e_b\right\rangle P_{aab} \\
&= \left\langle \tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t}, e^{\frac{1}{2}|df|^2} df e_i\right\rangle + \langle df\frac{\partial}{\partial t}, \left(\tilde{\nabla}_{e_i^H} e^{\frac{1}{2}|df|^2} df\right)e_i\rangle \\
&\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}\left(df\frac{\partial}{\partial t}, df e_i, (\tilde{\nabla}_{e_i^H} df)F_{e_m}\right) + \sum_{a,b} e^{\frac{1}{2}|df|^2}\left\langle df\frac{\partial}{\partial t}, df e_b\right\rangle P_{aab}.
\end{aligned} \tag{3.5}$$

Integrating (3.5), we conclude the result. □

Likewise, let $\phi = e^{\frac{1}{2}|df|^2}\hat{C}(df e_i, df e_i, \frac{df}{dt})F w^m$ which is a global section on $T^*(S_x M)$. We arrive at the following lemma due to $P_{aam} = 0$.

Lemma 3.3.

$$\begin{aligned}
&\sum_i \int_{SM} e^{\frac{1}{2}|df|^2}\hat{C}\left(df e_i, df e_i, \tilde{\nabla}_{Fe_m^H}\frac{df}{dt}\right) dv_{SM} \\
&= -\sum_i \int_{SM}\Bigg[(\tilde{\nabla}_{Fe_m^H} e^{\frac{1}{2}|df|^2})\hat{C}\left(df e_i, df e_i, \frac{df}{dt}\right) + e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{Fe_m^H}\hat{C}) \\
&\quad \times \left(df e_i, df e_i, \frac{df}{dt}\right) \\
&\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}\left(\tilde{\nabla}_{Fe_m^H} df e_i, df e_i, \frac{df}{dt}\right)\Bigg] dv_{SM}.
\end{aligned} \tag{3.6}$$

A map $f : M \to \hat{M}$ is exponentially harmonic iff it is a critical point of the exponential energy. Let $\{f_t\}$ be a smooth variation of f with $f_0 = f$ and $f_t|_{\partial M} = f|_{\partial M}$. It induces a vector field V along f with

$$V = \frac{\partial f_t}{\partial t}|_{t=0} = V^\alpha\frac{\partial}{\partial \hat{x}^\alpha}, \quad V|_{\partial M} = 0.$$

Theorem 3.4. *A map $f : M \to \hat{M}$ between two Finsler manifolds is exponentially harmonic if and only if*

$$\int_{SM}\langle V, \tau_e(f)\rangle dv_{SM} = 0, \tag{3.7}$$

for any vector field $V \in \Gamma(f^{-1}T\hat{M})$.

Proof. Applying Lemmas 3.2 and 3.3, we obtain

$$\frac{d}{dt}E_e(f) = \frac{1}{c}\sum_i \int_{SM} e^{\frac{1}{2}|df|^2}\Big[\Big\langle(\tilde{\nabla}_{e_i} df)\frac{\partial}{\partial t},\, dfe_i\Big\rangle + \hat{C}\Big(dfe_i,\, dfe_i,\, (\tilde{\nabla}_{Fe_m^H} df)\frac{\partial f}{\partial t}\Big)\Big]dv_{SM}$$
$$= \frac{1}{c}\sum_i \int_{SM} e^{\frac{1}{2}|df|^2}\Big[\Big\langle\tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t},\, dfe_i\Big\rangle + \hat{C}(dfe_i,\, dfe_i,\, \Big(\tilde{\nabla}_{Fe_m^H} df)\frac{\partial f}{\partial t}\Big)\Big]dv_{SM}$$
$$= -\frac{1}{c}\int_{SM}\langle\frac{df}{dt},\, \tau_e(f)\rangle dv_{SM}, \tag{3.8}$$

where

$$\tau_e(f) = \sum_i(\tilde{\nabla}_{e_i^H} e^{\frac{1}{2}|df|^2} df)e_i + \sum_{i,\alpha}\Big[2e^{\frac{1}{2}|df|^2}\hat{C}(\hat{e}_\alpha, dfe_i,\, \tilde{\nabla}_{e_m^H} df Fe_m)\hat{e}_\alpha$$
$$+ (\tilde{\nabla}_{Fe_m^H} e^{\frac{1}{2}|df|^2})\hat{C}(dfe_i, dfe_i, \hat{e}_\alpha)\hat{e}_\alpha + e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{Fe_m^H}\hat{C})(dfe_i,\, dfe_i,\, \hat{e}_\alpha)\hat{e}_\alpha$$
$$+ 2e^{\frac{1}{2}|df|^2}\hat{C}(\tilde{\nabla}_{Fe_m^H} dfe_i, dfe_i, \hat{e}_\alpha)\hat{e}_\alpha\Big] + \sum_{a,b} e^{\frac{1}{2}|df|^2}\langle\hat{e}_\alpha,\, dfe_b\rangle\hat{e}_\alpha P_{aab}. \tag{3.9}$$

□

4. Second variation

In this section, we apply a few auxiliary lemmas to derive the second variation of the exponential energy of a smooth map between Finsler manifolds as follows.

Theorem 4.1. *If $f : M \to \hat{M}$ is a non-constant smooth map between two Finsler manifolds, then*

$$I_e(V,\, V) = \frac{d^2}{dt^2}E_e(f_t)|_{t=0} = A + B + C + D + E + G,$$

where

$$A = \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\langle\tilde{\nabla}_{e_i^H} V,\, dfe_i\rangle + \hat{C}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m} V)\Big]^2 dv_{SM}, \tag{4.1}$$

$$B = \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\langle\tilde{\nabla}_{e_i^H} V,\, \tilde{\nabla}_{e_i^H} V\rangle dv_{SM}, \tag{4.2}$$

$$C = \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[4\hat{C}(\tilde{\nabla}_{e_i^H} dV,\, dfe_i,\, \tilde{\nabla}_{Fe_m} V) + (\tilde{\nabla}_{V^H}\hat{C})(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m} V)$$
$$+ \hat{\mathbb{C}}(dfe_i,\, dfe_i\, \tilde{\nabla}_{Fe_m} V,\, \tilde{\nabla}_{Fe_m} V)\Big]dv_{SM} \text{ (see } \hat{\mathbb{C}} \text{ in Lemma 4.2)}, \tag{4.3}$$

$$D = \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[-\langle\hat{R}(dfe_i,\, V)V,\, dfe_i\rangle + \frac{F}{\hat{F}}\hat{P}(V, (\tilde{\nabla}_{e_i} df)e_m)V,\, dfe_i\rangle$$
$$- \frac{F}{\hat{F}}\langle\hat{P}(dfe_i,\, \tilde{\nabla}_{e_m^H} V)V,\, dfe_i\rangle\Big]dv_{SM}, \tag{4.4}$$

$$E = \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[-\hat{C}(dfe_i,\, dfe_i, \hat{R}(dfFe_m,\, V)V)$$

$$+\frac{F}{\hat{F}}\hat{C}(dfe_i,\, dfe_i,\, \hat{P}(V,\, (\tilde{\nabla}_{Fe_m} df)e_m)V)$$
$$-\frac{F}{\hat{F}}\hat{C}(dfe_i,\, dfe_i,\, \hat{P}(dfFe_m,\, \tilde{\nabla}_{e_i^H} V)V)\Big]dv_{SM}, \tag{4.5}$$

$$G = -\frac{1}{c}\int_{SM} \langle \tilde{\nabla}_V V, \tau_e(f)\rangle dv_{SM}. \tag{4.6}$$

Firstly, we obtain from Lemma 3.1 (2) that

$$\tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{e_i^H} df \frac{\partial}{\partial t} - \tilde{\nabla}_{e_i^H} \tilde{\nabla}_{\frac{\partial}{\partial t}} df \frac{\partial}{\partial t}$$
$$= -\hat{R}\Big(dfe_i,\, df\frac{\partial}{\partial t}\Big)df\frac{\partial}{\partial t} + \frac{F}{\hat{F}}\hat{P}\Big(df\frac{\partial}{\partial t},\, (\tilde{\nabla}_{e_i} df)e_m\Big)df\frac{\partial}{\partial t}$$
$$-\frac{F}{\hat{F}}\hat{P}\Big(dfe_i,\, \tilde{\nabla}_{e_m^H}\frac{df}{dt}\Big)df\frac{\partial}{\partial t}, \tag{4.7}$$

where $\hat{R} = \hat{R}^{\alpha}_{\beta\gamma\sigma}\frac{\partial}{\partial\hat{x}^{\alpha}} \otimes d\hat{x}^{\beta} \otimes d\hat{x}^{\gamma} \otimes d\hat{x}^{\sigma}$ and $\hat{P} = \hat{P}^{\alpha}_{\beta\gamma\sigma}\frac{\partial}{\partial\hat{x}^{\alpha}} \otimes d\hat{x}^{\beta} \otimes d\hat{x}^{\gamma} \otimes d\hat{x}^{\sigma}$.

In Lemma 2.1, setting $\phi = e^{\frac{1}{2}|df|^2}\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, dfe_i\rangle w^i$ we get

$$\mathrm{div}_{\tilde{g}}\phi = \sum_i e^{\frac{1}{2}|df|^2}\Big\langle \tilde{\nabla}_{e_i^H}\tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, dfe_i\Big\rangle + \Big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, (\tilde{\nabla}_{e_i^H} e^{\frac{1}{2}|df|^2} df)e_i\Big\rangle$$
$$+2\sum_i e^{\frac{1}{2}|df|^2}\hat{C}\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, dfe_i,\, \tilde{\nabla}_{e_i^H} dfFe_m\Big)$$
$$+\sum_{a,b} e^{\frac{1}{2}|df|^2}\Big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, dfe_b\Big\rangle P_{aab}. \tag{4.8}$$

It implies that

$$\sum_i \int_{SM}\sum_i e^{\frac{1}{2}|df|^2}\Big\langle \tilde{\nabla}_{e_i^H}\tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, dfe_i\Big\rangle dv_{SM}$$
$$= -\int_{SM}\Big\{\sum_i \Big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, (\tilde{\nabla}_{e_i^H} e^{\frac{1}{2}|df|^2} df)e_i\Big\rangle$$
$$+2\sum_i e^{\frac{1}{2}|df|^2}\hat{C}\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, dfe_i,\, \tilde{\nabla}_{e_i^H} dfFe_m\Big)$$
$$+\sum_{a,b} e^{\frac{1}{2}|df|^2}\Big\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, dfe_b\Big\rangle P_{aab}\Big\}dv_{SM}. \tag{4.9}$$

Similarly, we have

$$\int_{SM} e^{\frac{1}{2}|df|^2}\hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{e_m^H}\nabla_{\frac{\partial}{\partial t}} dfFe_m\Big)dv_{SM}$$
$$= -\int_{SM}\Big\{\Big(\tilde{\nabla}_{Fe_m^H} e^{\frac{1}{2}|df|^2}\Big)\hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t}\Big)$$

$$+e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{Fe_m^H}\hat{C})\Big(dfe_i, dfe_i, \tilde{\nabla}_{\frac{\partial}{\partial t}}df\frac{\partial}{\partial t}\Big)$$
$$+2e^{\frac{1}{2}|df|^2}\hat{C}\Big(\tilde{\nabla}_{Fe_m^H}dfe_i, dfe_i, \tilde{\nabla}_{\frac{\partial}{\partial t}}df\frac{\partial}{\partial t}\Big)\Big\}dv_{SM}. \tag{4.10}$$

Combining (4.9) and (4.10), we arrive at

$$\int_{SM}\Big\{e^{\frac{1}{2}|df|^2}\Big\langle\tilde{\nabla}_{e_i^H}\tilde{\nabla}_{\frac{\partial}{\partial t}}df\frac{\partial}{\partial t},\, dfe_i\Big\rangle + e^{\frac{1}{2}|df|^2}\hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{\frac{\partial}{\partial t}}df\frac{\partial}{\partial t}\Big)\Big\}dv_{SM}$$
$$= -\int_{SM}\Big\langle\tilde{\nabla}_{\frac{\partial}{\partial t}}df\frac{\partial}{\partial t},\, \tau_e(f)\Big\rangle dv_{SM} = -\int_{SM}\Big\langle\tilde{\nabla}_{df\frac{\partial}{\partial t}}df\frac{\partial}{\partial t},\, \tau_e(f)\Big\rangle dv_{SM}. \tag{4.11}$$

Lemma 4.2.

$$e^{\frac{1}{2}|df|^2}\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}}\hat{C}\Big)\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big) = e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{\frac{df}{dt}^H}\hat{C})\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)$$
$$+e^{\frac{1}{2}|df|^2}\hat{\mathbb{C}}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt},\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big),$$

where $\hat{\mathbb{C}} = \hat{\mathbb{C}}^{\alpha}_{\beta\gamma\sigma}\frac{\partial}{\partial\hat{x}^{\alpha}}\otimes d\hat{x}^{\beta}\otimes d\hat{x}^{\gamma}\otimes d\hat{x}^{\sigma}$, $\hat{\mathbb{C}}^{\alpha}_{\beta\gamma\sigma} = \frac{\partial\hat{C}^{\alpha}_{\beta\gamma}}{\partial\hat{y}^{\sigma}}$.

Proof. Due to $\frac{\partial\hat{y}^{\sigma}}{\partial t} = y^i\frac{\partial V^{\sigma}}{\partial x^i}$, we obtain

$$\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}}\hat{C}\Big)\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)$$
$$= V^{\sigma}\frac{\partial\hat{C}}{\partial x^{\sigma}}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}) + \hat{\mathbb{C}}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt},\, y^i\frac{\partial V^{\sigma}}{\partial x^i}\frac{\partial}{\partial\hat{x}^{\sigma}}\Big), \tag{4.12}$$

and

$$\Big(\tilde{\nabla}_{\frac{df}{dt}^H}\hat{C}\Big)\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big) = \tilde{\nabla}_{V^{\sigma}\left(\frac{\partial}{\partial\hat{x}^{\sigma}}-\hat{N}^{\sigma}_{\tau}\frac{\partial}{\partial\hat{y}^{\tau}}\right)}\hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)$$
$$= V^{\sigma}\frac{\partial\hat{C}}{\partial\hat{x}^{\sigma}}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big) - \hat{\mathbb{C}}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt},\, V^{\sigma}\hat{N}^{\sigma}_{\tau}\frac{\partial}{\partial\hat{x}^{\tau}}\Big). \tag{4.13}$$

Moreover, we have

$$\hat{\mathbb{C}}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt},\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}) = \hat{\mathbb{C}}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt},\, y^i\frac{\partial V^{\sigma}}{\partial x^i}\frac{\partial}{\partial\hat{x}^{\sigma}}$$
$$+V^{\sigma}\hat{N}^{\tau}_{\sigma}\frac{\partial}{\partial\hat{x}^{\tau}}\Big). \tag{4.14}$$

We conclude the result by applying (4.12), (4.13) and (4.14). □

Lemma 4.3.

$$\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\frac{d}{dt}\Big\langle\tilde{\nabla}_{e_i} df\frac{\partial}{\partial t},\, dfe_i\Big\rangle dv_{SM} = \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big\{\Big\langle\tilde{\nabla}_{e_i^H}\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t}\Big),\, dfe_i\Big\rangle$$
$$+2e^{\frac{1}{2}|df|^2}\bar{C}\Big(\tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t},\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big\}dv_{SM} + B + D.$$

Proof. It follows from (4.7) that

$$\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\frac{d}{dt}\langle\tilde{\nabla}_{e_i} df\frac{\partial}{\partial t},\, dfe_i\rangle dv_{SM} = \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\langle\tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t},\, dfe_i\rangle$$
$$+\Big\langle\tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t},\, \tilde{\nabla}_{\frac{\partial}{\partial t}} dfe_i\Big\rangle + 2\hat{C}\Big(\tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t},\, dfe_i,\, \tilde{\nabla}_{\frac{\partial}{\partial t}} dfFe_m\Big)\Big]dv_{SM}$$
$$= \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\Big\langle\tilde{\nabla}_{e_i^H}\tilde{\nabla}_{\frac{\partial}{\partial t}} df\frac{\partial}{\partial t},\, dfe_i\Big\rangle - \Big\langle\hat{R}(dfe_i,\, df\frac{\partial}{\partial t},\, dfe_i)df\frac{\partial}{\partial t},\, dfe_i\Big\rangle$$
$$+\frac{F}{\hat{F}}\Big\langle\hat{P}\Big(df\frac{\partial}{\partial t},\, (\tilde{\nabla}_{e_i} df)e_m\Big)df\frac{\partial}{\partial t},\, dfe_i\Big\rangle - \frac{F}{\hat{F}}\Big\langle\hat{P}\Big(dfe_i,\, (\tilde{\nabla}_{e_i^H} df)\frac{\partial}{\partial t},\, dfe_i\Big\rangle$$
$$+\Big\langle\tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t},\, \tilde{\nabla}_{e_i} df\frac{\partial}{\partial t}\Big\rangle + 2\hat{C}\Big(\tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t},\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big]dv_{SM}, \tag{4.15}$$

which completes the proof. □

Lemma 4.4.

$$\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\frac{d}{dt}\Big[\hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big]dv_{SM}$$
$$= \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{df}{dt}\Big)$$
$$-2\hat{C}\Big(\tilde{\nabla}_{e_i^H}\frac{df}{dt},\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big]dv_{SM} + C + E. \tag{4.16}$$

Proof. We have

$$\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\frac{d}{dt}\Big[\hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big]dv_{SM}$$
$$= \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}}\hat{C}\Big)\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)$$
$$+2\hat{C}\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}} dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big) + \hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big]dv_{SM}$$
$$= \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}}\hat{C}\Big)\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)$$
$$+2\hat{C}\Big(\tilde{\nabla}_{e_i^H}\frac{df}{dt},\, dfe_i,\, \tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big) + \hat{C}\Big(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}\tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{df}{dt}\Big)$$
$$-\hat{C}\Big(dfe_i,\, dfe_i,\, \hat{R}\Big(dfFe_m,\, df\frac{\partial}{\partial t}\Big)df\frac{\partial}{\partial t}\Big)$$
$$+\frac{F}{\hat{F}}\hat{C}\Big(dfe_i,\, dfe_i,\, \hat{P}\Big(df\frac{\partial}{\partial t},\, \Big(\tilde{\nabla}_{Fe_m} df\frac{\partial}{\partial t}\Big)e_m\Big)df\frac{\partial}{\partial t}\Big)$$
$$-\frac{F}{\hat{F}}\hat{C}\Big(dfe_i,\, dfe_i,\, \hat{P}\Big(dfFe_m,\, \tilde{\nabla}_{e_i^H} df\frac{\partial}{\partial t}\Big)df\frac{\partial}{\partial t}\Big)\Big]dv_{SM}. \tag{4.17}$$

The result follows from (4.17) and Lemma 4.2. □

Proof of Theorem4.1. Applying (4.11), Lemma 4.3 and Lemma 4.4, we obtain

$$\begin{aligned}
&\frac{d^2}{dt^2}E_e(f)\\
&=\frac{1}{c}\frac{d}{dt}\int_{SM}e^{\frac{1}{2}|df|^2}\Big[\Big\langle\tilde{\nabla}_{e_i^H}df\frac{\partial}{\partial t},\,dfe_i\Big\rangle+\hat{C}\Big(dfe_i,\,dfe_i,\,\tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big]dv_{SM}\\
&=\frac{1}{c}\int_{SM}e^{\frac{1}{2}|df|^2}\Big[\Big\langle\tilde{\nabla}_{e_i^H}df\frac{\partial}{\partial t},\,dfe_i\Big\rangle+\hat{C}\Big(dfe_i,\,dfe_i,\,\tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big]^2dv_{SM}\\
&\quad+\frac{1}{c}\int_{SM}e^{\frac{1}{2}|df|^2}\Big\{\frac{d}{dt}\Big\langle\tilde{\nabla}_{e_i}df\frac{\partial}{\partial t},\,dfe_i\Big\rangle+\frac{d}{dt}\Big[\hat{C}\Big(dfe_i,\,dfe_i,\,\tilde{\nabla}_{Fe_m}\frac{df}{dt}\Big)\Big]\Big\}dv_{SM}\\
&=A+B+C+D+E+G.
\end{aligned}$$

5. Stability

We consider that the domain is a unit m-sphere S^m in this section. Let $\{e_i\}$ be the orthonormal frame of S^m, and $\{E_1,\ldots,E_{m+1}\}$ be the constant orthonormal basis in $\mathbb{R}^{m+1}$. Set $V_\nu=\langle E_\nu,e_i\rangle e_i,\ \nu=1,\ldots,m+1$. Based on [2], we have

$$\nabla_X V_\nu=-\langle E_\nu,\,e_{m+1}\rangle X. \tag{5.1}$$

Applying Theorem4.1, the second variation of an exponentially harmonic map $f:S^m\to\hat{M}^n$ can be expressed as

$$\sum_\nu I_e(dfV_\nu,\,dfV_\nu)=A+B+C+D+E+G,$$

where

$$\begin{aligned}
A&=\sum_\nu\frac{1}{c}\int_{SM}e^{\frac{1}{2}|df|^2}\Big[\Big\langle\tilde{\nabla}_{e_i^H}dfV_\nu,\,dfe_i\Big\rangle+\hat{C}\Big(dfe_i,\,dfe_i,\,\tilde{\nabla}_{Fe_m}dfV_\nu\Big)\Big]^2dv_{SM},\\
B&=\sum_\nu\frac{1}{c}\int_{SM}e^{\frac{1}{2}|df|^2}\Big\langle\tilde{\nabla}_{e_i^H}dfV_\nu,\,\tilde{\nabla}_{e_i^H}dfV_\nu\Big\rangle dv_{SM},\\
C&=\sum_\nu\frac{1}{c}\int_{SM}e^{\frac{1}{2}|df|^2}\Big[4\hat{C}\Big(\tilde{\nabla}_{e_i^H}dfV_\nu,\,dfe_i,\,\tilde{\nabla}_{Fe_m}dfV_\nu\Big)\\
&\quad+(\tilde{\nabla}_{(dfV_\nu)^H}\hat{C})(dfe_i,\,dfe_i,\,\tilde{\nabla}_{Fe_m}dfV_\nu)\\
&\quad+\hat{\mathbb{C}}(dfe_i,\,dfe_i,\,\tilde{\nabla}_{Fe_m}dfV_\nu,\,\tilde{\nabla}_{Fe_m}dfV_\nu)\Big]dv_{SM},\\
D&=\sum_\nu\frac{1}{c}\int_{SM}e^{\frac{1}{2}|df|^2}\Big[-\langle\hat{R}(dfe_i,dfV_\nu)dfV_\nu,\,dfe_i\rangle\\
&\quad+\frac{F}{\hat{F}}\langle\hat{P}\Big(dfV_\nu,(\tilde{\nabla}_{e_i}df)e_m\Big)dfV_\nu,\,dfe_i\rangle\\
&\quad-\frac{F}{\hat{F}}\Big\langle\hat{P}(dfe_i,\,\tilde{\nabla}_{e_m^H}dfV_\nu)dfV_\nu,\,dfe_i\Big\rangle\Big]dv_{SM},\\
E&=\sum_\nu\frac{1}{c}\int_{SM}e^{\frac{1}{2}|df|^2}\Big[-\hat{C}(dfe_i,\,dfe_i,\,\hat{R}(dfFe_m,dfV_\nu)dfV_\nu)
\end{aligned} \tag{5.2}$$

$$+\frac{F}{\hat{F}}\hat{C}(dfe_i,\,dfe_i,\,\hat{P}(dfV_\nu,(\tilde{\nabla}_{Fe_m}df)e_m)dfV_\nu)$$
$$-\frac{F}{\hat{F}}\hat{C}(dfe_i,\,dfe_i,\,\hat{P}(dfFe_m,\tilde{\nabla}_{e_m^H}dfV_\nu)dfV_\nu)\Big]dv_{SM},$$
$$G=-\frac{1}{c}\int_{SM}\langle\tilde{\nabla}_{dfV}dfV,\tau_e(f)\rangle dv_{SM}.$$

Referring to [1], we obtain

$$\begin{aligned}(\tilde{\nabla}_{X^H}\tilde{\nabla}_Z df)Y = &-dfR(X,Y)Z+(\tilde{\nabla}_{Y^H}\tilde{\nabla}_Z df)X\\ &+(\tilde{\nabla}_Y df)(\tilde{\nabla}_{X^H}Z)-(\tilde{\nabla}_X df)(\tilde{\nabla}_{Y^H}Z)\\ &+\hat{R}(dfX,dfY)dfZ+\frac{F}{\hat{F}}\hat{P}(dfX,(\tilde{\nabla}_{e_m}df)Y)dfZ\\ &-\frac{F}{\hat{F}}\hat{P}(dfY,\,(\tilde{\nabla}_{e_m}df)X)dfZ.\end{aligned}\tag{5.3}$$

Letting $X=Z=V,\ Y=e_i$ in (5.3), we have

$$\begin{aligned}&-\langle\hat{R}(dfe_i,\,dfV)dfV,\,dfe_i\rangle+\frac{F}{\hat{F}}\langle\hat{P}(dfV,\,((\tilde{\nabla}_{e_i}df)e_m)dfV,\,dfe_i\rangle\\ &\quad-\frac{F}{\hat{F}}\langle\hat{P}(dfe_i,\,((\tilde{\nabla}_{e_i}df)V)dfV,\,dfe_i\rangle\\ &=-\langle dfR(e_i,\,V)V,\,dfe_i\rangle+\langle(\tilde{\nabla}_{V^H}\tilde{\nabla}_V df)e_i,\,dfe_i\rangle-\langle(\tilde{\nabla}_{e_i^H}\tilde{\nabla}_V df)V),\,dfe_i\rangle\\ &\quad-\langle(\tilde{\nabla}_{e_i}df)(\nabla_{V^H}V),\,dfe_i\rangle+\langle(\tilde{\nabla}_V df)(\nabla_{e_i^H}V),\,dfe_i\rangle.\end{aligned}\tag{5.4}$$

In order to prove Theorem5.4, we require the following useful lemmas.

Lemma 5.1.

$$\begin{aligned}D=\sum_\nu\frac{1}{c}\int_{SM}e^{\frac{1}{2}|df|^2}\Big[&-\langle df\hat{R}(e_i,\,V_\nu)V_\nu,\,dfe_i\rangle-\langle\tilde{\nabla}_{e_i^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu],\,dfe_i\rangle\\ &+\langle\tilde{\nabla}_{V_\nu^H}[(\tilde{\nabla}_{V_\nu}df)e_i],\,dfe_i\rangle-\langle(\tilde{\nabla}_{V_\nu}df)(\tilde{\nabla}_{V_\nu^H}e_i),\,dfe_i\rangle\Big]dv_{SM}.\end{aligned}$$

Proof. Due to the fact $\tilde{\nabla}_{\frac{\partial}{\partial y^i}}V_\nu=0$ and (5.1), we have

$$\sum_\nu\tilde{\nabla}_{V_\nu^H}V_\nu=\sum_\nu\tilde{\nabla}_{V_\nu}V_\nu=-\sum_\nu v_\nu^i v_\nu^{m+1}e_i=0,\tag{5.5}$$

where $v_\nu^i=\langle E_\nu,e_i\rangle$. Therefore,

$$\sum_\nu(\tilde{\nabla}_{V_\nu}df)(\nabla_{e_i^H}V_\nu)=\sum_\nu(\tilde{\nabla}_{V_\nu}df)(\nabla_{e_i}V_\nu)=-\sum_\nu v_\nu^j v_\nu^{m+1}(\nabla_{e_j}df)e_i=0,\tag{5.6}$$

$$\begin{aligned}\sum_\nu(\nabla_{e_i^H}\tilde{\nabla}_{V_\nu}df)V_\nu&=\sum_\nu\nabla_{e_i^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu]-(\tilde{\nabla}_{V_\nu}df)(\tilde{\nabla}_{e_i^H}V_\nu)\\ &=\tilde{\nabla}_{e_i^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu],\end{aligned}\tag{5.7}$$

$$\sum_{\nu}(\nabla_{V_\nu^H}\tilde{\nabla}_{V_\nu}df)e_i = \sum_{\nu}\nabla_{V_\nu^H}[(\tilde{\nabla}_{V_\nu}df)e_i] - (\tilde{\nabla}_{V_\nu}dfe_i)(\nabla_{V_\nu^H}e_i). \tag{5.8}$$

Hence, the result follows from (5.5)–(5.8). □

Likewise, we also have the following by $\nabla_V e_m = 0$:

$$E = \sum_{\nu}\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big\{ - \hat{C}(dfe_i,\, dfe_i,\, df\hat{R}(Fe_m,\, V_\nu)V_\nu) - \hat{C}(dfe_i,\, dfe_i, \tilde{\nabla}_{Fe_m^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu]) + \hat{C}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{V_\nu^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu])\Big\}dv_{SM}. \tag{5.9}$$

Lemma 5.2.

$$-\sum_{\nu}\int_{SM}\Big\{e^{\frac{1}{2}|df|^2}\langle\tilde{\nabla}_{e_i^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu],\, dfe_i\rangle + e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i,\, dfe_i, \tilde{\nabla}_{Fe_m^H}((\tilde{\nabla}_{V_\nu}df)V_\nu)\Big\}dv_{SM} + G = 0.$$

Proof. Set $\phi = e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i,\, dfe_i,\, (\tilde{\nabla}_{V_\nu}df)V_\nu)Fw^m$. It follows from $\nabla_{V^H}Fw^m = 0$ and Lemma 2.1 that

$$\begin{aligned}\mathrm{div}_{\bar{g}}\phi = &(\tilde{\nabla}_{Fe_m}e^{\frac{1}{2}|df|^2})\hat{C}(dfe_i,\, dfe_i, (\tilde{\nabla}_{V_\nu}df)V_\nu) + e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{Fe_m}\hat{C})\\ &\times(dfe_i, dfe_i, (\tilde{\nabla}_{V_\nu}df)V_\nu)\\ &+ 2e^{\frac{1}{2}|df|^2}\hat{C}(\tilde{\nabla}_{Fe_m}dfe_i, dfe_i, (\tilde{\nabla}_{V_\nu}df)V_\nu) + e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i,\, dfe_i,\\ &\tilde{\nabla}_{Fe_m^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu]).\end{aligned} \tag{5.10}$$

Integrating (5.10), we have

$$\begin{aligned}&\sum_{\nu}\int_{SM}e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu])dv_{SM}\\ &= -\sum_{\nu}\int_{SM}\Big[(\tilde{\nabla}_{Fe_m^H}e^{\frac{1}{2}|df|^2})\hat{C}(dfe_i,\, dfe_i, (\tilde{\nabla}_{V_\nu}df)V_\nu)\\ &\quad + e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{Fe_m^H}\hat{C})(dfe_i,\, dfe_i, (\tilde{\nabla}_{V_\nu}df)V_\nu)\\ &\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}(\tilde{\nabla}_{Fe_m^H}dfe_i, dfe_i, (\tilde{\nabla}_{V_\nu}df)V_\nu)\Big]dv_{SM}.\end{aligned} \tag{5.11}$$

Putting $\phi = e^{\frac{1}{2}|df|^2}\langle(\tilde{\nabla}_{V_\nu}df)V_\nu, dfe_i\rangle w^i$, we obtain

$$\begin{aligned}\mathrm{div}_{\bar{g}}\phi &= e^{\frac{1}{2}|df|^2}\langle\tilde{\nabla}_{e_i^H}[(\tilde{\nabla}_{V_\nu}df)],\, dfe_i\rangle + \langle(\tilde{\nabla}_{V_\nu}df)V_\nu,\, \tilde{\nabla}_{e_i^H}(e^{\frac{1}{2}|df|^2}dfe_i)\rangle\\ &\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}((\tilde{\nabla}_{V_\nu}df)V_\nu,\, dfe_i,\, \tilde{\nabla}_{e_m^H}dfFe_m) + e^{\frac{1}{2}|df|^2}\\ &\quad \times\langle(\tilde{\nabla}_{V_\nu}df)V_\nu,\, dfe_j\rangle(\tilde{\nabla}_{e_i^H}w^j)e_i\\ &= e^{\frac{1}{2}|df|^2}\langle\tilde{\nabla}_{e_i^H}[(\tilde{\nabla}_{V_\nu}df)V_\nu],\, dfe_i\rangle + \langle(\tilde{\nabla}_{V_\nu}df)V_\nu,\, (\tilde{\nabla}_{e_i^H}e^{\frac{1}{2}|df|^2}df)e_i\rangle\\ &\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}((\tilde{\nabla}_{V_\nu}df)V_\nu,\, dfe_i,\, \tilde{\nabla}_{e_i^H}dfFe_m).\end{aligned} \tag{5.12}$$

It implies that

$$\sum_{\nu}\int_{SM} e^{\frac{1}{2}|df|^2}\langle\tilde{\nabla}_{e_i^H}[(\tilde{\nabla}_{V_\nu} df)V_\nu],\ df e_i\rangle dv_{SM}$$
$$= -\sum_{\nu}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\langle(\tilde{\nabla}_{V_\nu} df)V_\nu,\ (\tilde{\nabla}_{e_m^H} e^{\frac{1}{2}|df|^2} df)e_i\rangle$$
$$+ 2\hat{C}((\tilde{\nabla}_{V_\nu} df)V_\nu,\ df e_i,\ \nabla_{e_i^H} df F e_m)\Big]dv_{SM}. \tag{5.13}$$

Since f is exponentially harmonic, it follows from (5.11) and (5.13) that

$$-\sum_{\nu}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\hat{C}(df e_i,\ df e_i,\ \tilde{\nabla}_{e_m^H}[(\tilde{\nabla}_{V_\nu} df)V_\nu])$$
$$+ \langle\tilde{\nabla}_{e_i^H}[(\tilde{\nabla}_{V_\nu} df)V_\nu],\ df e_i\rangle\Big]dv_{SM} + G$$
$$= \sum_{\nu}\int_{SM}\langle\tilde{\nabla}_V df V - \tilde{\nabla}_{dfV} df V,\ \tau_e(f)\rangle dv_{SM} = 0. \tag{5.14}$$

Using the fact $\langle R(e_m, e_i)e_i, e_j\rangle = \delta_{mj}$ and $df(Fe_m) = \hat{F}\hat{e}_n$, we get

$$\sum_{\nu}\hat{C}(df e_i,\ df e_i,\ df R(Fe_m, V_\nu)V_\nu) = 0. \tag{5.15}$$

We obtain from (5.9), (5.15) and Lemmas 5.1 and 5.2 that

$$D + E + G = \sum_{\nu}\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[-\langle df R(e_i, V_\nu)V_\nu, df e_i\rangle + \langle\tilde{\nabla}_{V_\nu^H}(\tilde{\nabla}_{V_\nu} df)e_i),\ df e_i\rangle$$
$$-\langle(\tilde{\nabla}_{V_\nu} df)(\tilde{\nabla}_{V_\nu^H} e_i),\ df e_i\rangle + \hat{C}(df e_i,\ df e_i,\ \tilde{\nabla}_{V_\nu^H}((\tilde{\nabla}_{V_\nu} df)Fe_m))\Big]dv_{SM}. \tag{5.16}$$

□

Lemma 5.3.

$$\sum_{\nu}\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\langle\tilde{\nabla}_{V_\nu^H}[(\tilde{\nabla}_{V_\nu} df)e_i],\ df e_i\rangle dv_{SM} + B$$
$$= \frac{1}{c}\int_{SM}\Big\{-\sum_{\nu}(\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2})\langle\tilde{\nabla}_{V_\nu^H}(\tilde{\nabla}_{V_\nu} df)e_i,\ df e_i\rangle + e^{\frac{1}{2}|df|^2}\langle df e_i,\ df e_i\rangle$$
$$-\sum_{\nu} e^{\frac{1}{2}|df|^2}\langle df(\tilde{\nabla}_{V_\nu^H} e_i),\ (\tilde{\nabla}_{V_\nu} df)e_i\rangle - \sum_{\nu} 2e^{\frac{1}{2}|df|^2}\hat{C}((\tilde{\nabla}_{V_\nu} df)e_i,$$
$$df e_i,\ \tilde{\nabla}_{V_\nu^H} df F e_m)\Big\}dv_{SM}.$$

Proof. Set $\phi = \sum_{\nu,i} e^{\frac{1}{2}|df|^2} \langle (\tilde{\nabla}_{V_\nu} df)V_\nu, dfe_i\rangle v_\nu^j w^j$ with $v_\nu^i = \langle E_\nu, e_i\rangle$. We obtain

$$\begin{aligned}
\mathrm{div}_{\tilde{g}}\phi = \sum_\nu \Big[& (\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2}) \langle (\tilde{\nabla}_{V_\nu} df)e_i, dfe_i\rangle + e^{\frac{1}{2}|df|^2} \langle (\tilde{\nabla}_{V_\nu^H}((\tilde{\nabla}_{V_\nu} df)e_i), dfe_i\rangle \\
& + e^{\frac{1}{2}|df|^2} \langle (\tilde{\nabla}_{V_\nu} df)e_i, \tilde{\nabla}_{V_\nu^H} dfe_i)\rangle + 2e^{\frac{1}{2}|df|^2} \hat{C}(\tilde{\nabla}_{V_\nu} df)e_i, dfe_i, \tilde{\nabla}_{V_\nu^H} dfFe_m) \\
& - e^{\frac{1}{2}|df|^2} \langle (\tilde{\nabla}_{V_\nu} df)e_i, dfe_i\rangle v_\nu^{m+1} \Big] \\
= \sum_\nu \Big[& (\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2}) \langle (\tilde{\nabla}_{V_\nu} df)e_i, dfe_i\rangle + e^{\frac{1}{2}|df|^2} \langle \tilde{\nabla}_{V_\nu^H} (\tilde{\nabla}_{V_\nu} df)e_i, dfe_i\rangle \\
& + e^{\frac{1}{2}|df|^2} \langle (\tilde{\nabla}_{V_\nu} df)e_i, (\tilde{\nabla}_{V_\nu^H} df)e_i\rangle + 2e^{\frac{1}{2}|df|^2} \hat{C}((\tilde{\nabla}_{V_\nu} df)e_i, dfe_i, \tilde{\nabla}_{V_\nu^H} dfFe_m) \Big].
\end{aligned} \tag{5.17}$$

Integrating (5.17), we arrive at

$$\begin{aligned}
& \sum_\nu \int_{SM} \langle \tilde{\nabla}_{V_\nu^H} [(\tilde{\nabla}_{V_\nu} df)e_i], dfe_i\rangle dv_{SM} \\
& = -\sum_\nu \int_{SM} \Big[(\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2}) \langle (\tilde{\nabla}_{V_\nu} df)e_i, dfe_i\rangle + e^{\frac{1}{2}|df|^2} \langle (\tilde{\nabla}_{V_\nu} df)e_i, \nabla_{V_\nu^H} dfe_i\rangle \\
& \quad + 2e^{\frac{1}{2}|df|^2} \hat{C}((\tilde{\nabla}_{V_\nu} df)e_i, dfe_i, \tilde{\nabla}_{V_\nu^H} dfFe_m) \Big] dv_{SM}.
\end{aligned} \tag{5.18}$$

Furthermore, we have

$$\begin{aligned}
B &= \sum_\nu \frac{1}{c} \int_{SM} e^{\frac{1}{2}|df|^2} \langle (\tilde{\nabla}_{e_i^H} df)V_\nu + df(\tilde{\nabla}_{e_i^H} V_\nu), (\tilde{\nabla}_{e_i} df)V_\nu + df(\nabla_{e_i} V_\nu)\rangle dv_{SM} \\
&= \sum_\nu \frac{1}{c} \int_{SM} e^{\frac{1}{2}|df|^2} \Big[\langle (\tilde{\nabla}_{e_i^H} df)V_\nu, (\tilde{\nabla}_{e_i} df)V_\nu\rangle + \langle dfe_i, dfe_i\rangle \Big] dv_{SM} \\
&= \sum_\nu \frac{1}{c} \int_{SM} e^{\frac{1}{2}|df|^2} \Big[\langle \tilde{\nabla}_{V_\nu^H} (dfe_i), (\tilde{\nabla}_{V_\nu} (dfe_i)\rangle \\
& \quad - \langle df(\tilde{\nabla}_{V_\nu^H} e_i), (\tilde{\nabla}_{V_\nu} df)e_i\rangle + \langle dfe_i, dfe_i\rangle \Big] dv_{SM}.
\end{aligned} \tag{5.19}$$

Hence, we conclude the result by (5.18) and (5.19). □

Theorem 5.4. *If $f : S^m \to N$ $(m \geq 3)$ is a non-constant exponentially harmonic map from the unit sphere S^m to a Finsler manifold, then it is stable in case $|df|^2 \geq m-2$, and is unstable in case $|df|^2 < m-2$.*

Proof. Applying Lemma 5.3 to (5.16), we obtain

$$\begin{aligned}
B + D + E + G = \frac{1}{c} \int_{SM} \Big[& -\sum_\nu e^{\frac{1}{2}|df|^2} \langle dfR(e_i, V_\nu)V_\nu, dfe_i\rangle \\
& + e^{\frac{1}{2}|df|^2} \langle dfe_i, dfe_i\rangle \\
& - \sum_\nu (\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2}) \langle (\tilde{\nabla}_{V_\nu} df)e_i, dfe_i\rangle
\end{aligned}$$

$$+\sum_{\nu} e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i,\ dfe_i,\ \tilde{\nabla}_{V_\nu^H}[(\tilde{\nabla}_{V_\nu}df)Fe_m])$$
$$-\sum_{\nu} 2e^{\frac{1}{2}|df|^2}\hat{C}((\tilde{\nabla}_{V_\nu}df)e_i,\ dfe_i,\ \tilde{\nabla}_{V_\nu^H}dfFe_m)\Big]dv_{SM}. \tag{5.20}$$

Setting $\phi = e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i, dfe_i, \tilde{\nabla}_{V_\nu}Fe_m)v_\nu^k w^k$, we get

$$\begin{aligned}
\mathrm{div}_{\bar{g}}\phi &= (\tilde{\nabla}_{e_k^H}\Big[e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i,\ dfe_i, (\tilde{\nabla}_{V_\nu}df)Fe_m\Big]v^k \\
&\quad + e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i, dfe_i, (\tilde{\nabla}_{V_\nu}df)Fe_m)[\tilde{\nabla}_{e_k}(v^l w^l)]e_k \\
&= (\tilde{\nabla}_{V_\nu^H}e^{\frac{1}{2}|df|^2})\hat{C}(dfe_i,\ dfe_i, (\tilde{\nabla}_{V_\nu}df)Fe_m) \\
&\quad + e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{V_\nu^H}\hat{C})(dfe_i,\ dfe_i, (\tilde{\nabla}_{V_\nu}df)Fe_m) \\
&\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}(\tilde{\nabla}_{V_\nu^H}dfe_i, dfe_i, (\tilde{\nabla}_{V_\nu}df)Fe_m) \\
&\quad + e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i,\ dfe_i, \tilde{\nabla}_{V_\nu^H}[(\tilde{\nabla}_{V_\nu}df)Fe_m]).
\end{aligned} \tag{5.21}$$

Integrating (5.21), we arrive at

$$\begin{aligned}
&\sum_{\nu}\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\hat{C}(dfe_i,\ dfe_i,\ \tilde{\nabla}_{V_\nu^H}[\tilde{\nabla}_{V_\nu}dfFe_m])dv_{SM} \\
&\quad = -\sum_{\nu}\frac{1}{c}\int_{SM}\Big[(\tilde{\nabla}_{V_\nu^H}e^{\frac{1}{2}|df|^2})\hat{C}(dfe_i,\ dfe_i,\ (\tilde{\nabla}_{V_\nu}df)Fe_m) \\
&\qquad + e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{V_\nu^H}\hat{C})(dfe_i,\ dfe_i,\ \tilde{\nabla}_{V_\nu}dfFe_m) \\
&\quad + 2e^{\frac{1}{2}|df|^2}\hat{C}(\tilde{\nabla}_{V_\nu^H}dfe_i, dfe_i, (\tilde{\nabla}_{V_\nu}df)Fe_m)\Big]dv_{SM}.
\end{aligned} \tag{5.22}$$

Moreover, it follows from (5.1) that

$$\begin{aligned}
&\sum_{\nu} e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{V_\nu^H}\hat{C})(dfe_i,\ dfe_i, (\tilde{\nabla}_{V_\nu}df)Fe_m) \\
&\quad = \sum_{\nu} e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{V_\nu^H}\hat{C})(dfe_i,\ dfe_i,\ \tilde{\nabla}_{Fe_m}(dfV_\nu) - df(\nabla_{Fe_m}V_\nu)) \\
&\quad = \sum_{\nu} e^{\frac{1}{2}|df|^2}(\tilde{\nabla}_{V_\nu^H}\hat{C})(dfe_i,\ dfe_i, \tilde{\nabla}_{Fe_m}(dfV_\nu)),
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
&2e^{\frac{1}{2}|df|^2}\hat{C}(\tilde{\nabla}_{V_\nu^H}dfe_i, dfe_i, (\tilde{\nabla}_{V_\nu}df)Fe_m) \\
&\quad = 2e^{\frac{1}{2}|df|^2}\hat{C}(\tilde{\nabla}_{V_\nu^H}dfe_i, dfe_i, (\tilde{\nabla}_{Fe_m}df)V_\nu).
\end{aligned} \tag{5.24}$$

Substituting (5.23) and (5.24) into (5.22), we obtain

$$\begin{aligned}
&\sum_{\nu} \frac{1}{c} \int_{SM} e^{\frac{1}{2}|df|^2} \hat{C}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{V_\nu^H}[(\tilde{\nabla}_{V_\nu} df)Fe_m]) dv_{SM} \\
&= -\sum_{\nu} \frac{1}{c} \int_{SM} \Big[(\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2}) \hat{C}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{V_\nu} df Fe_m) \\
&\quad + e^{\frac{1}{2}|df|^2} (\tilde{\nabla}_{V_\nu^H} \hat{C})(dfe_i,\, dfe_i,\, (\tilde{\nabla}_{Fe_m} df)V_\nu) \\
&\quad + 2e^{\frac{1}{2}|df|^2} \hat{C}(\tilde{\nabla}_{V_\nu^H} dfe_i, dfe_i, (\tilde{\nabla}_{Fe_m} df)V_\nu) \Big].
\end{aligned} \tag{5.25}$$

By a calculation, we have

$$\begin{aligned}
&\sum_{\nu} 2e^{\frac{1}{2}|df|^2} \hat{C}(\tilde{\nabla}_{V_\nu} dfe_i, dfe_i, (\tilde{\nabla}_{V_\nu^H} df)Fe_m) \\
&= \sum_{\nu} 2e^{\frac{1}{2}|df|^2} \hat{C}(\tilde{\nabla}_{e_i}(dfV_\nu),\, dfe_i,\, \tilde{\nabla}_{Fe_m}(dfV_\nu)).
\end{aligned} \tag{5.26}$$

Substituting (5.25) and (5.26) into (5.20), we arrive at

$$\begin{aligned}
B + D + E + G = \frac{1}{c} \int_{SM} \Big[&- \sum_{\nu} e^{\frac{1}{2}|df|^2} \langle dfR(e_i, V_\nu)V_\nu, dfe_i \rangle \\
&+ e^{\frac{1}{2}|df|^2} \langle dfe_i,\, dfe_i \rangle \\
&- \sum_{\nu} (\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2}) \langle (\tilde{\nabla}_{V_\nu} df)e_i,\, dfe_i \rangle \\
&- \sum_{\nu} (\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2}) \hat{C}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{V_\nu} df Fe_m) \\
&- \sum_{\nu} e^{\frac{1}{2}|df|^2} (\tilde{\nabla}_{V_\nu^H} \hat{C})(dfe_i,\, dfe_i,\, (\tilde{\nabla}_{Fe_m} df)V_\nu) \\
&- \sum_{\nu} 4e^{\frac{1}{2}|df|^2} \hat{C}((\tilde{\nabla}_{V_\nu} df)e_i,\, dfe_i,\, \tilde{\nabla}_{V_\nu^H} df Fe_m) \Big] dv_{SM}.
\end{aligned} \tag{5.27}$$

Lemma 4.2 implies that

$$\begin{aligned}
&\sum_{\nu} e^{\frac{1}{2}|df|^2} (\tilde{\nabla}_{V_\nu^H} \hat{C})(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m}(dfV_\nu)) \\
&= \sum_{\nu} \Big[e^{\frac{1}{2}|df|^2} (\tilde{\nabla}_{(dfV_\nu)^H} \hat{C})(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m} dfV_\nu) \\
&\quad + e^{\frac{1}{2}|df|^2} \hat{\mathbb{C}}(dfe_i,\, dfe_i,\, \tilde{\nabla}_{Fe_m} dfV_\nu) \Big].
\end{aligned} \tag{5.28}$$

Combining (5.27), (5.28) and (5.2), we obtain

$$
\begin{aligned}
B+C+D+E+G &= \frac{1}{c}\int_{SM}\Big[-\sum_{\nu} e^{\frac{1}{2}|df|^2}\langle df R(e_i, V_\nu)V_\nu, df e_i\rangle \\
&\quad + e^{\frac{1}{2}|df|^2}\langle df e_i, df e_i\rangle \\
&\quad - \sum_{\nu}(\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2})\langle(\tilde{\nabla}_{V_\nu^H} df)e_i, df e_i\rangle - \sum_{\nu}(\tilde{\nabla}_{V_\nu^H} e^{\frac{1}{2}|df|^2}) \\
&\quad \hat{C}(df e_i, df e_i, \tilde{\nabla}_{Fe_m} df V_\nu)\Big] dv_{SM} \\
&= \frac{1}{c}\int_{SM}\Big[-\sum_{\nu} e^{\frac{1}{2}|df|^2}\langle df R(e_i, V_\nu)V_\nu, df e_i\rangle + e^{\frac{1}{2}|df|^2}\langle df e_i, df e_i\rangle \\
&\quad - \sum_{\nu} e^{\frac{1}{2}|df|^2}[\langle(\tilde{\nabla}_{V_\nu^H} df)e_i + df(\nabla_{V_\nu^H} e_i), df e_i\rangle + \hat{C}(df e_i, df e_i, \\
&\quad \tilde{\nabla}_{Fe_m}(df V_\nu))][\langle(\tilde{\nabla}_{V_\nu})e_i, df e_i\rangle \\
&\quad + \hat{C}(df e_i, df e_i, \tilde{\nabla}_{Fe_m} df V_\nu)]\Big] dv_{SM} \\
&= \frac{1}{c}\int_{SM}\Big[-\sum_{\nu} e^{\frac{1}{2}|df|^2}\langle df R(e_i, V_\nu)V_\nu, df e_i\rangle + e^{\frac{1}{2}|df|^2}\langle df e_i, df e_i\rangle \\
&\quad - \sum_{\nu} e^{\frac{1}{2}|df|^2}[\langle(\tilde{\nabla}_{V_\nu^H} df)e_i, df e_i\rangle + \hat{C}(df e_i, df e_i, \tilde{\nabla}_{Fe_m}(df V_\nu))]^2\Big] dv_{SM}.
\end{aligned} \tag{5.29}
$$

Furthermore, we have

$$
\begin{aligned}
A &= \sum_{\nu}\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\langle\tilde{\nabla}_{e_i} df V_\nu + df(\nabla_{e_i} V_\nu), df e_i\rangle \\
&\qquad + \hat{C}(df e_i, df e_i, \tilde{\nabla}_{V_\nu^H}(df Fe_m))\Big]^2 dv_{SM} \\
&= \sum_{\nu}\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\langle\tilde{\nabla}_{e_i} df V_\nu, df e_i\rangle + \hat{C}(df e_i, df e_i, \tilde{\nabla}_{V_\nu^H}(df Fe_m))\Big]^2 dv_{SM} \\
&\quad + \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\langle -v_\nu^{m+1} df e_i, df e_i\rangle^2 dv_{SM} \\
&= \sum_{\nu}\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}\Big[\langle\tilde{\nabla}_{e_i} df V_\nu, df e_i\rangle + \hat{C}(df e_i, df e_i, \tilde{\nabla}_{V_\nu^H}(df Fe_m))\Big]^2 dv_{SM} \\
&\quad + \sum_{\nu}\frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}|df|^4 dv_{SM}.
\end{aligned} \tag{5.30}
$$

Applying (5.29) and (5.30), we obtain

$$\sum_{\nu} I_e(df V_\nu, df V_\nu) = \frac{1}{c}\int_{SM}\Big[-\sum_{\nu} e^{\frac{1}{2}|df|^2}\langle df R(e_i, V_\nu)V_\nu, df e_i\rangle + e^{\frac{1}{2}|df|^2}\langle df e_i, df e_i\rangle + e^{\frac{1}{2}|df|^2}|df|^4\Big] dv_{SM} = \frac{1}{c}\int_{SM} e^{\frac{1}{2}|df|^2}|df|^2((2-m)+|df|^2)dv_{SM}. \tag{5.31}$$

Consequently, we conclude the result. □

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