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Sandro Zagatti

# Strong compactness in Sobolev spaces

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**Abstract.** We prove a strong compactness criterion in Sobolev spaces: given a sequence $(u_n)$ in $W^{1,p}_{\rm loc}(\mathbb{R}^d)$, converging in $L^p_{\rm loc}$ to a map $u \in W^{1,p}_{\rm loc}(\mathbb{R}^d)$ and such that $|\nabla u_n| \le f$ almost everywhere, for some $f \in L^p_{\rm loc}(\mathbb{R}^d)$, we provide a necessary and sufficient condition under which $(u_n)$ converges strongly to $u$ in $W^{1,p}_{\rm loc}(\mathbb{R}^d)$. In addition we prove a pointwise version of the criterion, according to which, given $(u_n)$ and $u$ as above, but with no boundedness assumptions on the sequence of gradients, we have $\nabla u_n \to \nabla u$ pointwise almost everywhere.

## 1. Introduction

The work exposed in this paper concerns properties which allow to extract strongly converging sequences out of weakly converging ones in Sobolev spaces, and is mainly motivated by the investigation of the minimum problem for non (weakly lower) semicontinuous functionals of the Calculus of variations of the form

$$\mathcal{F}(u) = \int_\Omega f(x, u, Du)\, dx,$$

where $\Omega$ is an open subset of $\mathbb{R}^n$, the integrand $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ is a Carathéodory function and the competing functions belong to some Sobolev space $W^{1,p}(\Omega)$ and satisfy prescribed boundary conditions.

The classical approach, in order to prove existence of minimum points for the functional $\mathcal{F}$, is the Direct Method, according to which, if the functional is coercive and sequentially weakly lower semicontinuous, then it admits at least one minimizer. The coercivity holds true if the function $f$ satisfies a suitable growth condition at infinity with respect to the last variable, so that any minimizing sequence turns out to be relatively compact in the weak topology of $W^{1,p}(\Omega)$. Then, extracting a weakly converging subsequence, its limit turns out to be a minimizer by the weak lower semicontinuity of $\mathcal{F}$, a property that it is well known to be equivalent to the quasiconvexity of the application $\xi \mapsto f(x, u, \xi)$.

S. Zagatti (✉): SISSA, Via Bonomea 265, 34136 Trieste, Italy.
e-mail: zagatti@sissa.it

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If we drop the quasiconvexity and lose the weak semicontinuity, the Direct Method fails and the functional may have no minimum points. However many efforts have been made in order to find classes of non semicontinuous functionals admitting minimizers, both in scalar ($m = 1$) and in vectorial case ($m > 1$). We quote the results contained in [1–4,6–10,13–19,21–29]. Due to the reasons just explained, these studies do not make use of weak lower semicontinuity and the procedure sketched above might work if we would be able to find minimizing sequences which converge strongly in $W^{1,p}(\Omega)$, since in such case the existence of a minimizer would follow by a simple application of Fatou lemma. Unfortunately there are no criteria ensuring the existence of strong converging minimizing sequences, and the approaches adopted in the literature are of another kind. The main way to treat non semicontinuous variational problems, as it is illustrated, for example, in papers [24,27,29], consists in looking for a minimizer of the functional $\mathcal{F}$ as a minimizer of the relaxed functional

$$\overline{\mathcal{F}}(u) = \int_{\Omega} \overline{f}(x, u, Du)dx,$$

solving almost everywhere the partial differential equation of Hamilton–Jacobi type

$$\overline{f}(x, u, Du) - f(x, u, Du) = 0,$$

where $\overline{f}$ is the lower quasiconvex envelope of $f$ with respect to the last variable. Hence the minimum problem turns into the solution of a fully nonlinear partial differential equation of the first order, in the restricted set of minimizers of the relaxed functional. In particular, in papers [24,27] and [29] we have introduced the integro-maximality method, with the aim of finding a general approach to this kind of problems, creating a link with viscosity theory for Hamilton–Jacobi equations and facing (in [29]) the higher technical difficulties of vectorial problems.

In the literature on Hamilton–Jacobi equations of the form $H(x, u, \nabla u) = 0$, the Hamiltonian $H$ is usually assumed to be continuous in the space variable $x$, while it is evident that in this setting, where $H = \overline{f} - f$, the natural condition is measurability. Moreover, applying the ideas used in paper [29], imposing continuity in the space variable corresponds to the too strong assumption of the existence of (piecewise) $C^1$ minimizers of the relaxed functional. For these reasons, in papers [30–32] and [33] we have investigated conditions on a Carathéodory Hamiltonian $H = H(x, u, \xi)$ for which a generalized solution does exist, with the relevant choice that, in contrast with the classical viscosity theory, we devoted our efforts to prove existence of at least one solution rather than uniqueness. The procedure adopted can be sketched in the simple case of the Dirichlet problem for eikonal equation:

$$\mathcal{P}(\Omega, \varphi, a) : \quad \begin{cases} \frac{1}{2}|\nabla u(x)|^2 - a(x) = 0 & \text{in } \Omega \\ u(x) = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

where the map $a \in L^\infty(\Omega)$ is discontinuous and the boundary datum $\varphi$ is a subsolution. Since the classical viscosity theory ensure that, whenever the map $a$ is continuous, there exists a unique viscosity solution, the idea is to consider a sequence $(a_n)$ of sufficiently regular functions, such that $a_n \to a$ almost everywhere in $\Omega$, and the corresponding sequence $(u_n)$ of viscosity solutions of the approximating problem $\mathcal{P}(\Omega, \varphi, a_n)$, with the aim of passing to the limit $n \to \infty$. Extracting if necessary a subsequence, $(u_n)$ converges weakly* in $W^{1,\infty}$ but, by the nonlinearity of the problem, its weak* limit is not necessarily a generalized solution of $\mathcal{P}(\Omega, \varphi, a)$. In order to achieve the claimed result, we need the convergence almost everywhere of the sequence of gradients $(\nabla u_n)$, or the strong convergence of $(u_n)$ in some Sobolev space $W^{1,p}(\Omega)$; hence we have been sent back to the starting problem of extracting strongly converging sequences out of weakly converging ones.

The elementary considerations summarized up to now lead us to infer that, when dealing with the different fields of non semicontinuous variational problems and fully nonlinear partial differential equations, the weak topologies in Sobolev spaces are inappropriate tools, since, in a certain sense, they "destroy the geometry" of the problem. This fact forces us to pursue the enunciated goal on strong precompactness, remarking that the existing precompactness criteria are of little help in our framework: the application of Rellich theorem to a sequence $(u_n)$ in $W^{1,p}(\Omega)$ consists in finding a uniform bound in $L^p(\Omega)$ on the sequence $\left(|D^2 u_n|\right)$, while other related tools (like concentrated compactness or semisubharmonicity) require similar conditions on the sequence $(\Delta u_n)$. Unfortunately, in our setting, there is no hope to obtain such kinds of estimates on second derivatives of minimizing sequences. Indeed, even in very simple one dimensional problems, the second derivatives of the elements of minimizing sequences are distributions with $\delta$-diverging terms. Hence we are interested in conditions ensuring strong compactness which do not involve second derivatives and, towards this goal, in this paper, we extend the ideas introduced in papers [31,32] and [33], which take inspiration from well known properties of semiconcave functions (see for example [5]). In such articles we have studied the Dirichlet problem for Hamilton–Jacobi equations, including the already mentioned eikonal case $\mathcal{P}(\Omega, \varphi, a)$ with a discontinuous term $a(\cdot)$, adopting the approximating procedure sketched above. In such framework we have proved and applied a pointwise convergence property in $W^{1,\infty}(\Omega)$ which preludes to the more general and finer results of this paper.

Now we sketch our main result in dimension equal to one and for $p = 1$ (see Theorem 1): take an interval $I$ of $\mathbb{R}$, a sequence $(v_n)$ in $W^{1,1}(I)$ converging in $L^1(I)$ to $v \in W^{1,1}(I)$, and assume that for every $n \in \mathbb{N}$, for almost every $x \in I$ and for $s \in \mathbb{R}$ sufficiently small, the following inequality holds true:

$$v_n(x+s) - v_n(x) - s v_n'(x) \le |s| G_n(x, s),$$

where $G_n$ is a Carathéodory function, even in $s$ and monotone nondecreasing for $s \ge 0$. Then assume that there exists a sequence $(h_n)$ in $\mathbb{R}^+$, with $h_n \to 0+$, such that

$$h_n^{-1} \|v_n - v\|_{L^1(I)} \to 0 \quad \text{and} \quad \int_I G_n(x, h_n)\, dx \to 0.$$

Under these conditions the sequence $(v_n)$ converges strongly to $v$ in $W^{1,1}(I)$. Roughly speaking, the functions $G_n$ test the frequency of the oscillations of the sequence $(v_n')$, while the sequence $(h_n)$ measures their amplitude. Hence our criterion says that if the decreasing of the amplitude of the oscillations is faster than the increasing of their frequency, then they coalesce on a null set and, consequently, the sequence $(v_n')$ converges strongly. We stress that this result is also necessary in the sense specified by Theorem 2. This argument is the basis for the result in several dimensions: take a polyinterval $I \doteq \times_{j=1}^d I_j \subseteq \mathbb{R}^d$, a sequence $u_n$ in $W^{1,1}(I)$ converging in $L^1(I)$ to $u \in W^{1,1}(I)$, and assume that the restrictions of $u_n$ and $u$ on almost every line segment parallel to coordinate axes satisfy the hypotheses imposed above on the functions of a single variable $v_n$ and $v$. Then, adding the uniform bound $|\nabla u_n| \leq f$ for some $f \in L^1(I)$, we have $u_n \to u$ in $W^{1,1}(I)$. The generalization to an arbitrary open subset $\Omega$ of $\mathbb{R}^d$ follows easily by localization.

For what concerns the applications of our criterion, we refer to the quoted papers [31,32] and [33], where, as already mentioned, we have used the pointwise $W^{1,\infty}$-version in the study of Hamilton–Jacobi equations. Further application will be considered in separated works since, in our humble opinion, the present results are worth to appear in a devoted paper.

The article is organized as follows. In Sect. 2 we list main notations and in Sect. 3 we prove our necessary and sufficient condition in dimension equal to one. In Sect. 4 we discuss the result in the general case of several dimensions, while Sect. 5 is devoted to the pointwise version of our criterion. Section 6 deals with the special case of the space $W^{1,\infty}$.

## 2. Notations

In this paper $\mathbb{R}^d$ is the euclidean $d$-dimensional space and by $|\cdot|$ and $\mathcal{E} \doteq \{e_1, \dots, e_d\}$ we denote the norm and the canonical basis in $\mathbb{R}^d$; a point $x \in \mathbb{R}^d$ is written as $x = (x_1, \dots, x_d)$ and the symbol $D_i$ denotes both classical and weak derivative with respect to the variable $x_i$. Given $E \subseteq \mathbb{R}^d$, $\partial E$ is the boundary, while $\mathrm{m}_k(E)$ is the $k$-dimensional Lebesgue measure. Given two measurable subset $A, B \subseteq \mathbb{R}^d$ with $A \subseteq B$, we say that $A$ is a subset of $B$ of full measure, or, simply, a full subset of $B$, if $\mathrm{m}_d(B \setminus A) = 0$. Given an open subset $U$ of $\mathbb{R}^d$, we use the spaces $C^k(U)$, $L^r(U)$, $W^{1,r}(U)$, for $k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $1 \leq r \leq \infty$, with their usual (strong and weak) topologies, adopting the standard notation for the conjugate index $r' \doteq r/(r-1)$. Dealing with a Sobolev function, we assume to use the precise representative and, when considering a function $f$ defined on $U$ and terms of the form $f(x \pm s)$, with $x \in U$, by saying that $s$ is sufficiently small we mean that $s$ is chosen in such way that $x \pm s \in U$.

**Notations 1.** Given $i \in \{1, \dots, d\}$ and $x \in \mathbb{R}^d$, write

$$x = (x_1, \dots, x_d) = (\hat{x}_i, x_i),$$

where

$$\hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathbb{R}^{d-1}.$$

Let $\{I_j \subseteq \mathbb{R}, j = 1, \dots d\}$, be a collection of $d$ open and bounded intervals of $\mathbb{R}$ and consider the polyinterval

$$I \doteq \times_{j=1}^d I_j \subseteq \mathbb{R}^d.$$

We write

$$\hat{I}_i \doteq \times_{j\neq i} I_j \subseteq \mathbb{R}^{d-1}, \quad I = \hat{I}_i \times I_i \quad i, j \in \{1, \dots, d\}.$$

Given a map $u : I \to \mathbb{R}$, an index $i \in \{1, \dots, d\}$ and a point $\hat{x}_i \in \hat{I}_i$ we set

$$u_{\hat{x}_i}(\cdot) \doteq u(\hat{x}_i, \cdot). \tag{2.1}$$

Analogously, for a sequence $(u_n)$ of functions $u_n : I \to \mathbb{R}$ we write

$$u_{n,\hat{x}_i}(\cdot) \doteq u_n(\hat{x}_i, \cdot). \tag{2.2}$$

Assuming that $u$ and $u_n$ are defined and integrable on $I$, we introduce the real values

$$r(x, u_n, u, I) \doteq \max\{\|u_n - u\|_{L^1(I)}, |u_n(x) - u(x)|\}, \text{ for } d = 1, \tag{2.3}$$

and, for $d \in \mathbb{N}$, $i \in \{1, \dots, d\}$,

$$r^i(x, u_n, u, I) \doteq \max\{\|u_{n,\hat{x}_i} - u_{\hat{x}_i}\|_{L^1(I_i)}, |u_n(x) - u(x)|\}. \tag{2.4}$$

## 3. The one dimensional case

We begin the discussion by giving our result in the one dimensional case: next Theorem 1 provides a sufficient condition for strong convergence on an open interval of $\mathbb{R}$ and is the basis of the proof in the case of several dimensions.

**Theorem 1.** *Let $I$ be an open interval of $\mathbb{R}$, $p \in [1, \infty[$, $(v_n)$ a sequence in $W^{1,p}(I)$ and $v \in W^{1,p}(I)$ such that*

$$v_n \overset{n\to\infty}{\longrightarrow} v \quad in \quad L^p(I).$$

*Assume that for every $n \in \mathbb{N}$ there exists a Carathéodory function $G_n : I \times \mathbb{R} \to [0, +\infty[$ satisfying the following properties:*

(i) $G_n \in L^p(I \times [0, 1])$ *and the map $\mathbb{R} \ni s \mapsto G_n(x, s)$ is even and monotone nondecreasing on $\mathbb{R}^+$ for almost every $x \in I$;*
(ii) *for almost every $x \in I$ and for every $s \in \mathbb{R}$ sufficiently small we have*

$$v_n(x + s) - v_n(x) - s v_n'(x) \leq |s| G_n(x, s); \tag{3.1}$$

(iii) *there exists a sequence $(h_n)$ in $]0, 1]$, with $h_n \to 0+$, such that*

$$\lim_{n\to\infty} \frac{1}{h_n} \|v_n - v\|_{L^p(I)} = 0 \tag{3.2}$$

*and*

$$\limsup_{n\to\infty} \frac{1}{h_n^{p+1}} \int_0^{h_n} \int_I s^p G_n(x, s)^p \, dx \, ds = 0. \tag{3.3}$$

*Then*

$$\limsup_{n\to\infty} \|v_n' - v'\|_{L^p(I)} = 0,$$

*that is to say that*

$$v_n \stackrel{n\to\infty}{\longrightarrow} v \textit{ strongly in } W^{1,p}(I).$$

*Proof.* In order to simplify the exposition, extend the maps $v$, $v'$, $v_n$, $v_n'$ on $\mathbb{R}\backslash I$ by setting $v'(x) = v_n'(x) = 0$ for almost every $x \in \mathbb{R}\backslash I$ and for every $n \in \mathbb{N}$.

Take a a point $x \in I$ such that (3.1) holds true and assume, in addition, that $x$ is a Lebesgue point of $v'$. Then, for $s \in \mathbb{R}\backslash\{0\}$, set

$$\tilde{\eta}(x, s) \doteq \frac{|v(x+s) - v(x) - v'(x)s|}{|s|}.$$

First of all we observe that

$$v(x+s) - v(x) = \int_0^s v'(x+r)\, dr$$

and then

$$\tilde{\eta}(x, s) \leq \frac{1}{s} \int_0^s |v'(x+r) - v'(x)|\, dr,$$

so that

$$\lim_{s\to 0} \tilde{\eta}(x, s) = 0.$$

Introduce the even function

$$\eta(x, s) \doteq \sup_{t\in[-s,s]} \tilde{\eta}(x, t),$$

and observe that we have

$$|v(x+s) - v(x) - u'(x)s| \leq |s|\eta(x, s) \quad \forall s. \tag{3.4}$$

In addition the map $\mathbb{R}^+ \ni s \mapsto \eta(x, s)$ is monotone nondecreasing, we have

$$\lim_{s\to 0} \eta(x, s) = 0 \tag{3.5}$$

and, for $\sigma > 0$,

$$\eta(x, s) \leq \eta(x, \sigma) \quad \forall |s| \leq \sigma, \tag{3.6}$$

so that, in particular, the map $I \ni x \mapsto \eta(x, s)$ is integrable for every $s \neq 0$. Consequently, by (3.5), (3.6) and dominated convergence, we have

$$\lim_{s\to 0} \int_I \eta(x, s)^p\, dx = 0. \tag{3.7}$$

Now write

$$\begin{aligned} s(v'(x) - v_n'(x)) = {} & sv'(x) - v(x+s) + v(x) \\ & + [v(x+s) - v_n(x+s)] + [v_n(x) - v(x)] \\ & + v_n(x+s) - v_n(x) - v_n'(x)s. \end{aligned} \tag{3.8}$$

From (3.8) we obtain

$$s(v'(x) - v_n'(x)) \leq |v(x+s) - v(x) - sv'(x)| + |v(x+s) - v_n(x+s)| + |v(x) - v_n(x)| + v_n(x+s) - v_n(x) - v_n'(x)s. \tag{3.9}$$

Using (3.1) and (3.4), formula (3.9) yields

$$s(v'(x) - v_n'(x)) \leq |v(x+s) - v_n(x+s)| + |v(x) - v_n(x)| + |s|\eta(x,s) + |s|G_n(x,s). \tag{3.10}$$

Take $h > 0$ and integrate both sides of (3.10) with respect to the variable $s$ from 0 to $h$. We obtain:

$$\frac{h^2}{2}(v'(x) - v_n'(x)) \leq \int_{-h}^{h} |v(x+s) - v_n(x+s)|\, ds + h|v(x) - v_n(x)| + \int_0^h sG_n(x,s)\, ds + \int_0^h s\,\eta(x,s)\, ds. \tag{3.11}$$

Consider again formula (3.8) for negative values of the variable $s$. We may write

$$|s|(v'(x) - v_n'(x)) = v(x+s) - v(x) - sv'(x) + [v_n(x+s) - v(x+s)] + [v(x) - v_n(x)] - [v_n(x+s) - v_n(x) - v_n'(x)s].$$

Then we have

$$|s|(v'(x) - v_n'(x)) \geq -|v(x+s) - v_n(x+s)| - |v_n(x) - v(x)| - |s|\eta(x,s) - |s|G_n(x,s). \tag{3.12}$$

Integrating both sides of (3.12) in the variable $s$ from $-h < 0$ to 0 and taking into account the evenness of the maps $s \mapsto \eta(x,s)$ and $s \mapsto G_n(x,s)$, we obtain

$$\frac{h^2}{2}(v'(x) - v_n'(x)) \geq -\int_{-h}^{h} |v(x+s) - v_n(x+s)|\, ds - h|v(x) - v_n(x)| - \int_0^h sG_n(x,s)\, ds - \int_0^h s\eta(x,s)\, ds. \tag{3.13}$$

Collecting (3.11) and (3.13) we obtain

$$\frac{h^2}{2}|v'(x) - v_n'(x)| \leq \int_{-h}^{h} |v(x+s) - v_n(x+s)|\, ds + h|v(x) - v_n(x)| + \int_0^h sG_n(x,s)\, ds + \int_0^h s\eta(x,s)\, ds. \tag{3.14}$$

Recalling that $h$ is positive, observe that, by Hölder inequality, we have

$$\int_{-h}^{h} |v_n(x+s) - v(x+s)|\, ds \leq (2h)^{\frac{1}{p'}} \left( \int_{-h}^{h} |v_n(x+s) - v(x+s)|^p\, ds \right)^{\frac{1}{p}}. \tag{3.15}$$

From (3.15) it follows that

$$\begin{aligned}\int_I \left(\int_{-h}^{h} |v_n(x+s) - v(x+s)|\,ds\right)^p dx\\ &\leq 2^p h^{p-1} \int_I \left(\int_{-h}^{h} |v_n(x+s) - v(x+s)|^p\right) dx\\ &\leq Ch^p \int_I |v_n(x) - v(x)|^p\,dx, \end{aligned} \tag{3.16}$$

where, from now on, by $C$ we mean a suitable positive constant possibly changing from line to line. By monotonicity and boundedness of $\mathbb{R} \ni s \mapsto \eta(x,s)$ we have

$$\left(\int_0^h s\eta(x,s)\,ds\right)^p \leq \frac{h^{2p}}{2^p}\eta(x,h)^p; \tag{3.17}$$

in addition, by Hölder inequality, we have

$$\left(\int_0^h sG_n(x,s)\,ds\right)^p \leq Ch^{p-1}\int_0^h s^p G_n(x,h)^p\,ds. \tag{3.18}$$

Then (3.14), (3.17) and (3.18) imply that

$$\begin{aligned}\frac{h^{2p}}{2^p}|v'(x) - v_n'(x)|^p &\leq C\left(\int_{-h}^{h} |v(x+s) - v_n(x+s)|\,ds\right)^p\\ &\quad + Ch^p|v_n(x) - v(x)|^p + Ch^{2p}\eta(x,h)^p\\ &\quad + Ch^{p-1}\int_0^h s^p G_n(x,s)^p\,ds. \end{aligned} \tag{3.19}$$

Multiplying both sides in (3.19) by $h^{-2p}$, integrating on $I$ and recalling (3.16), we have

$$\begin{aligned}\int_I |v'(x) - v_n'(x)|^p\,dx &\leq C\left[\frac{1}{h^p}\int_I |v_n(x) - v(x)|^p\,dx + \int_I \eta(x,h)^p\,dx\right]\\ &\quad + C\frac{1}{h^{p+1}}\int_0^h \int_I s^p G_n(x,s)^p\,dx\,ds. \end{aligned} \tag{3.20}$$

Now, in formula (3.20) replace the positive parameter $h$ by the element $h_n$ of the sequence given at point (iii) of the statement: we obtain

$$\begin{aligned}\int_I |v'(x) - v_n'(x)|^p\,dx &\leq \frac{C}{h_n^p}\int_I |v(x) - v_n(x)|^p\,dx\\ &\quad + C\int_I \eta(x,h_n)^p\,dx\\ &\quad + \frac{C}{h_n^{p+1}}\int_0^{h_n}\int_I s^p G_n(x,s)^p\,dx\,ds. \end{aligned}$$

Then, by virtue of (3.2), (3.3) and (3.7), we have

$$\begin{aligned}\limsup_{n\to\infty}\int_I |v'(x)-v_n'(x)|^p\,dx &\le C\limsup_{n\to\infty}\frac{1}{h_n^p}\|v-v_n\|_{L^p(I)}^p\\ &\quad + C\limsup_{n\to\infty}\int_I \eta(x,h_n)^p\,dx\\ &\quad + C\limsup_{n\to\infty}\frac{1}{h_n^{p+1}}\int_0^{h_n}\int_I s^p G_n(x,s)^p\,dx\,ds\\ &= 0.\end{aligned}$$

This ends the proof. □

*Remark 1.* In Theorem 1 we have imposed monotonicity and evenness on the function $s\mapsto G_n(x,s)$. It is evident that this assumption is not restrictive, since it can be obtained, if not satisfied, replacing $G_n$ by

$$\tilde{G}_n(x,s)\doteq\max_{t\in[-s,s]} G_n(x,t).$$

By monotonicity, it is immediate to verify that

$$\frac{1}{h_n^{p+1}}\int_0^{h_n}\int_I s^p G_n(x,s)^p\,dx\,ds \le C\int_I G_n(x,h_n)^p\,dx, \tag{3.21}$$

so that condition (3.3) can be replaced by the (simplest and) strongest one:

$$\limsup_{n\to\infty}\int_I G_n(x,h_n)^p\,dx = 0. \tag{3.22}$$

A simple example of a sequence $(G_n)$ satisfying the hypotheses of the Theorem 1 is given by the following formula:

$$G_n(x,s)=g_n(x)\omega(x,|s|),$$

where $(g_n)$ is a bounded sequence in $L^p(I)$ and $\omega$ is a nonnegative and bounded Carathéodory function such that $\omega(x,t)\to 0$ as $t\to 0+$ for almost every $x\in I$.

The sufficient condition given by Theorem 1 is also necessary, in the sense provided by the following

**Theorem 2.** *Let $I$ be an open interval of $\mathbb{R}$, $p\in[1,\infty[$, $(v_n)$ a sequence in $W^{1,p}(I)$ and $v\in W^{1,p}(I)$ such that*

$$v_n \overset{n\to\infty}{\longrightarrow} v \quad in \quad L^p(I).$$

*Assume that there exists a measurable subset $F\subseteq I$, with $m_1(F)>0$, and that for every $n\in\mathbb{N}$ there exists a Carathéodory function $H_n: F\times\mathbb{R}\to[0,+\infty[$ satisfying the following properties:*

(i) $H_n\in L^p(I\times[0,1])$ *and the map* $\mathbb{R}\ni s\mapsto H_n(x,s)$ *is even for almost every* $x\in F$*;*

(ii) *for almost every* $x \in F$ *and for every* $s \in \mathbb{R}$ *sufficiently small we have*

$$v_n(x+s) - v_n(x) - s v_n'(x) \geq |s| H_n(x,s); \tag{3.23}$$

(iii) *there exist a sequence* $(h_n)$ *in* $]0,1]$, *with* $h_n \to 0+$, *and a positive* $\delta$ *such that*

$$\lim_{n\to\infty} \frac{1}{h_n} \|v_n - v\|_{L^p(I)} = 0 \tag{3.24}$$

*and*

$$\liminf_{n\to\infty} \frac{1}{h_n^{p+1}} \int_0^{h_n} \int_I s^p H_n(x,s)^p \, dx \, ds \geq \delta. \tag{3.25}$$

*Then we have*

$$\liminf_{n\to\infty} \|v_n' - v'\|_{L^p(I)} > 0,$$

*so that the sequence* $(v_n')$ *does not converge to* $v'$ *in* $L^p(I)$ *and then the sequence* $(v_n)$ *does not converge to* $v$ *in* $W^{1,p}(I)$.

*Proof.* We adopt the notations and the arguments of Theorem 1. For $x \in F$, consider formula (3.8):

$$\begin{aligned} s(v'(x) - v_n'(x)) = {} & s v'(x) - v(x+s) + v(x) \\ & + [v(x+s) - v_n(x+s)] + [v_n(x) - v(x)] \\ & + v_n(x+s) - v_n(x) - v_n'(x) s. \end{aligned} \tag{3.26}$$

Assume $s > 0$ and observe that from (3.26) we have

$$\begin{aligned} s(v'(x) - v_n'(x)) \geq {} & -|v(x+s) - v_n(x+s)| - |v_n(x) - v(x)| \\ & - s\eta(x,s) + s H_n(x,s). \end{aligned} \tag{3.27}$$

Integrating both sides of (3.27) with respect to the variable $s$ from 0 to $h > 0$ we have

$$\begin{aligned} \frac{h^2}{2}(v'(x) - v_n'(x)) \geq {} & -\int_{-h}^{h} |v(x+s) - v_n(x+s)| \, ds - h|v_n(x) - v(x)| \\ & - \int_0^h s\eta(x,s) \, ds + \int_0^h s H_n(x,s) \, ds. \end{aligned} \tag{3.28}$$

Assume now $s < 0$ and observe that formula (3.26) may be written as

$$\begin{aligned} s(v_n'(x) - v'(x)) = {} & s v_n'(x) - v_n(x+s) + v_n(x) \\ & + [v_n(x+s) - v(x+s)] + [v(x) - v_n(x)] \\ & + v(x+s) - v(x) - v'(x) s. \end{aligned} \tag{3.29}$$

Multiplying both sides of (3.29) by $(-1)$ and recalling (3.4) and (3.23), we have

$$\begin{aligned} |s|(v_n'(x) - v'(x)) = {} & v_n(x+s) - v_n(x) - s v_n'(x) \\ & + [v(x+s) - v_n(x+s)] + [v_n(x) - v(x)] \\ & - v(x+s) + v(x) + v'(x) s. \\ \geq {} & -|v(x+s) - v_n(x+s)| - |v_n(x) - v(x)| \\ & - |s|\eta(x,s) + |s| H_n(x,s). \end{aligned} \tag{3.30}$$

Integrating (3.30) with respect to the variable $s$ from $-h < 0$ to 0, recalling the evenness in the second variable of $\eta$ and $H_n$, we obtain

$$\frac{h^2}{2}(v_n'(x) - v'(x)) \geq -\int_{-h}^{h} |v(x+s) - v_n(x+s)|\, ds - h|v_n(x) - v(x)| - \int_0^h s\eta(x,s)\, ds + \int_0^h sH_n(x,s)\, ds. \tag{3.31}$$

Collecting (3.28) and (3.31), we deduce that

$$\frac{h^2}{2}|v_n'(x) - v'(x)| \geq -\int_{-h}^{h} |v(x+s) - v_n(x+s)|\, ds - h|v_n(x) - v(x)| - \int_0^h s\eta(x,s)\, ds + \int_0^h sH_n(x,s)\, ds. \tag{3.32}$$

Reproducing the computations performed in the proof of theorem 1, and denoting by $\gamma$ a suitable positive constant possibly changing from line to line, (3.32) implies that

$$\frac{h^{2p}}{2^p}|v'(x) - v_n'(x)|^p \geq -\gamma \left( \int_{-h}^{h} |v(x+s) - v_n(x+s)|\, ds \right)^p - \gamma h^p |v_n(x) - v(x)|^p - \gamma h^{2p}\eta(x,h)^p + \gamma h^{p-1} \int_0^h s^p H_n(x,s)^p\, ds. \tag{3.33}$$

Recalling (3.16), multiplying both sides of (3.33) by $h^{-2p}$ and integrating on the set $F$ we obtain

$$\int_F |v_n'(x) - v'(x)|^p\, dx \geq -\gamma \frac{1}{h^p} \int_I |v_n(x) - v(x)|^p\, dx - \gamma \int_I \eta(x,h)^p\, dx + \frac{\gamma}{h^{p+1}} \int_0^h \int_F s^p H_n(x,s)^p\, dx\, ds. \tag{3.34}$$

Now, in (3.34), replace $h$ by the element $h_n$ of the sequence $(h_n)$ given in the statement: we have

$$\int_F |v_n'(x) - v'(x)|^p\, dx \geq -\frac{\gamma}{h_n^p} \|v_n - v\|^p_{L^p(I)} - \gamma \int_I \eta(x,h_n)^p\, dx + \frac{\gamma}{h_n^{p+1}} \int_0^{h_n} \int_F s^p H_n(x,s)^p\, dx\, ds. \tag{3.35}$$

Recalling (3.24), (3.25) and (3.7), we pass to the limit $n \to \infty$ in (3.35) and obtain

$$\liminf_{n\to\infty} \int_I |v_n'(x) - v'(x)|^p\, dx \geq \liminf_{n\to\infty} \int_F |v_n'(x) - v'(x)|^p\, dx \geq \gamma\delta,$$

as claimed. □

*Remark 2.* In order to give an example, we observe that a sequence $(H_n)$ satisfying the hypotheses of Theorem 2 is given by $H_n(x, s) = f_n(x)$ for almost every $x \in F$ and for every $s$, where the functions $f_n$ are nonnegative and there exists $\delta > 0$ such that $\int_F f_n(x)^p\,dx \geq \delta$ for every $n \in \mathbb{N}$.

*Remark 3.* We now compare our criterion with the classical Riesz-Kolmogorov condition involving mean continuity of $L^p$ spaces, which is well known to be equivalent to strong precompactness. For simplicity we limit ourselves to the case $p = 1$, since the general one is essentially identical. Following the notations and the exposition of paper [12], we consider an interval $I \subseteq \mathbb{R}$ and, for $J \subset\subset I$, $0 < s < \operatorname{dist}(J, I^c)$ and $f \in L^1(I)$, we set

$$\Gamma(f, J, s) \doteq \sup_{|t|\leq s}\left(\int_J |f(x+t) - f(x)|\,dx\right).$$

Given a family $\mathcal{F} \subseteq L^1(I)$, we know that $\mathcal{F}$ is relatively compact in $L^1(J)$ if and only if the following conditions hold true:

(1) there exists a constant $M > 0$ such that $\int_J |f|\,dx \leq M$ for every $f \in \mathcal{F}$;
(2) there exists a nondecreasing function $\omega : [0, +\infty[ \to [0, +\infty[$, with $\omega(t) \to 0+$ as $t \to 0+$ such that

$$\Gamma(f, J, s) \leq \omega(s) \quad \forall s \leq \operatorname{dist}(J, I^c) \quad \forall f \in \mathcal{F}. \tag{3.36}$$

In our setting we have $\mathcal{F} = \{v_n', n \in \mathbb{N}\}$, with $v_n \in W^{1,1}(I)$ for every $n \in \mathbb{N}$, $v_n \to v \in W^{1,1}(I)$ in $L^1(I)$ and we assume that $(v_n')$ is bounded in $L^1(I)$. Recalling inequality (3.1) in the statement of Theorem 1, we observe that for $x \in J$ and $0 < s < \operatorname{dist}(J, I^c)$ we may write

$$\begin{aligned} v_n(x+s) - v_n(x) - s v_n'(x) &= \int_0^s (v_n'(x+t) - v_n'(x))\,dt \\ &\leq \int_0^s |v_n'(x+t) - v_n'(x)|\,dt \\ &\leq s \cdot \sup_{|t|\leq s} |v_n'(x+t) - v_n'(x)|. \end{aligned} \tag{3.37}$$

Hence, if we set

$$V_n(x, s) \doteq \sup_{|t|\leq s} |v_n'(x+t) - v_n'(x)|,$$

we immediately obtain

$$v_n(x+s) - v_n(x) - s v_n'(x) \leq s V_n(x, s).$$

Assume that there exists a Carathéodory function $\tilde{V} : I \times [0, +\infty[ \to [0, +\infty[$ such that

$$V_n(x, s) \leq \tilde{V}(x, s) \quad \text{a.e } x \in J,\ \forall n \in \mathbb{N},\ \forall s < \operatorname{dist}(J, I^c) \tag{3.38}$$

and

$$\int_J \tilde{V}(x, h)\,dx \overset{h\to 0+}{\longrightarrow} 0.$$

Clearly our criterion turns out to be satisfied with $G_n(x, s) = V_n(x, s)$, since we can choose arbitrarily the sequence $(h_n)$.

On the other hand we notice that

$$\begin{aligned}\Gamma(v_n', J, s) &= \sup_{|t|\leq s} \int_J |v_n'(x+t) - v_n'(x)|\, dx \\ &\leq \int_J \sup_{|t|\leq s} |v_n'(x+t) - v_n'(x)|\, dx \\ &\leq \int_J V_n(x,s)\, dx \leq \int_J \tilde{V}(x,s)\, dx.\end{aligned}$$

Hence we may set

$$\omega(s) \doteq \int_J \tilde{V}(x,s)\, dx,$$

obtaining the precompacness result by Riesz-Kolmogorov condition.

These elementary computations help us to emphasize the elements of novelty of our result, as expressed in Theorem 1.

First of all we stress that, in contrast with the classical mean-continuity argument, we do not need the uniformity of the fundamental estimate with respect to $n$. Indeed, in formula (3.1), the bound $G_n$ at the right hand side depends on $n$ and we do not take the supremum for fixed $s$. In other words, the uniform bound (3.38) is unnecessary in our criterion, while it is needed in the classical method. In addition, in the fundamental estimate (3.1), we impose only an *upper bound* on the left hand side, while the corresponding argument of Riesz-Kolmogorov condition requires the uniform bound of the integral of the *absolute value* of the integrand in formula (3.36). To be more precise, in formula (3.37) we have introduced an unnecessary (for our criterion) estimate in term of the modulus of the integrand. We stress that the possibility of obtaining the compactness result by imposing only an upper bound, instead of a bound on the modulus, is due to the fact that our fundamental condition (3.1) is a pointwise inequality, while the mean-continuity condition (3.36) is an integral estimate, and to the fact that we deal with *derivatives* and not with *functions*.

## 4. The general case

We now turn our attention to the general case of functions of several variables belonging to the space $W^{1,p}(\Omega)$ and start by assuming that $\Omega$ is a bounded polyinterval of $\mathbb{R}^d$. Since our arguments are local, the generalization to an arbitrary open set $\Omega$ will follow straightforward. The proofs rely on the statements of previous section.

**Theorem 3.** *Let $I = \times_{j=1}^d I_j$ be a bounded open polyinterval of $\mathbb{R}^d$, $p \in [1, \infty[$, $(u_n)$ a sequence in $W^{1,p}(I)$, $u \in W^{1,p}(I)$ and $f$ a nonnegative function in $L^p(I)$ such that*

$$u_n \overset{n\to\infty}{\longrightarrow} u \text{ in } L^1(I) \tag{4.1}$$

*and*

$$|Du_n(x)| \leq f(x) \text{ for a.e. } x \in I,\ \forall n \in \mathbb{N}. \tag{4.2}$$

*Assume that for every* $n \in \mathbb{N}$ *and for every* $j \in \{1, \ldots, d\}$ *there exists a Carathéodory function* $G_n^j : I \times \mathbb{R} \to [0, +\infty[$ *satisfying the following properties:*

(i) $G_n^j \in L^p(I \times [0, 1])$ *and the map* $\mathbb{R} \ni s \mapsto G_n^j(x, s)$ *is even and monotone nondecreasing on* $\mathbb{R}^+$ *for almost every* $x \in I$*;*
(ii) *for almost every* $x \in I$ *and for every* $s \in \mathbb{R}$ *sufficiently small, we have*

$$u_n(x + se_j) - u_n(x) - sD_j u_n(x) \le |s| G_n^j(x, s);$$

(iii) *for almost every* $\hat{x}_j \in \hat{I}_j$ *there exists a sequence* $(h_n) = (h_n^{\hat{x}_j})$ *in* $]0, 1]$*, with* $h_n \to 0+$*, such that*

$$\lim_{n\to\infty} \frac{1}{h_n} \|u_{n,\hat{x}_j} - u_{\hat{x}_j}\|_{L^p(I_j)} = 0$$

*and*

$$\limsup_{n\to\infty} \frac{1}{h_n^{p+1}} \int_0^{h_n} \int_{I_j} s^p G_n^j(\hat{x}_j, x_j, s)^p \, dx_j \, ds = 0. \tag{4.3}$$

*Then*

$$\limsup_{n\to\infty} \|\nabla u_n - \nabla u\|_{L^p(I)} = 0,$$

*that is to say that*

$$u_n \overset{n\to\infty}{\longrightarrow} u \text{ strongly in } W^{1,p}(I).$$

*Proof.* Recall (2.1) and (2.2) from Notations 1, fix $i \in \{1, \ldots, d\}$ and observe that the convergence (4.1), Fubini Tonelli theorem and Fatou lemma imply the convergence

$$\lim_{n\to\infty} \|u_{n,\hat{x}_i} - u_{\hat{x}_i}\|_{L^p(I)} = 0 \text{ for a.e. } \hat{x}_i \in \hat{I}_i.$$

Indeed, assuming by contradiction that there exists $\hat{E}_i \subseteq \hat{I}_i$ of positive measure such that, for almost every $x \in \hat{E}_i$, we have

$$\liminf_{n\to\infty} \int_{I_i} |u_{n,\hat{x}_i}(x_i) - u_{\hat{x}_i}(x_i)|^p \, dx_i > 0,$$

we would have

$$\begin{aligned} 0 &< \int_{\hat{I}_i} \liminf_{n\to\infty} \int_{I_i} |u_{n,\hat{x}_i}(x_i) - u_{\hat{x}_i}(x_i)|^p \, dx_i \, d\hat{x}_i \\ &\le \liminf_{n\to\infty} \int_{\hat{I}_i} \int_{I_i} |u_{n,\hat{x}_i}(x_i) - u_{\hat{x}_i}(x_i)|^p \, dx_i \, d\hat{x}_i \\ &= \liminf_{n\to\infty} \int_I |u_n(x) - u(x)|^p \, dx = 0. \end{aligned}$$

Observe now that condition (3) is equivalent to assert that

$$u_{n,\hat{x}_i}(x_i + s) - u_{n,\hat{x}_i}(x_i) - su'_{n,\hat{x}_i}(x_i) \le |s| G_n^i(\hat{x}_i, x_i, s).$$

Then it is immediate to verify that for almost every $\hat{x}_i \in \hat{I}_i$ the sequence $(u_{n,\hat{x}_i})$ and the map $u_{\hat{x}_i}$ satisfy the hypotheses of Theorem 1 on the interval $I_i$. It follows that for almost every $\hat{x}_i \in \hat{I}_i$ we have

$$\lim_{n\to\infty} \int_{I_i} \left|D_i u_n(\hat{x}_i, x_i) - D_i u(\hat{x}_i, x_i)\right|^p \, dx_i = 0. \tag{4.4}$$

By virtue of the bound (4.2) and dominated convergence, (4.4) implies that

$$\begin{aligned}\lim_{n\to\infty} \int_I |D_i u_n(x) - D_i u(x)|^p \, dx \\ = \lim_{n\to\infty} \int_{\hat{I}_i} \int_{I_i} \left|D_i u_n(\hat{x}_i, x_i) - D_i u(\hat{x}_i, x_i)\right|^p \, d\hat{x}_i \, dx_i = 0.\end{aligned} \tag{4.5}$$

By the arbitrariness of the index $i$, this ends the proof. □

Reasoning as in the proof of Theorem 3 and invoking Theorem refnecess1 we easily obtain the following necessary condition.

**Theorem 4.** *Let $I = \times_{j=1}^d I_j$ be a bounded open polyinterval of $\mathbb{R}^d$, $p \in [1, \infty[$, $(u_n)$ a sequence in $W^{1,p}(I)$ and $u \in W^{1,p}(I)$ such that*

$$u_n \overset{n\to\infty}{\longrightarrow} u \text{ in } L^p(I).$$

*Assume that there exist $i \in \{1, \ldots, d\}$, a measurable subset $F = \hat{F}_i \times F_i \subseteq I$, with $m_d(F) > 0$, and, for every $n \in \mathbb{N}$, a Carathéodory function $H_n^i : I \times \mathbb{R} \to [0, +\infty[$ satisfying the following properties:*

(i) *$H_n^i \in L^p(I \times [0, 1])$ and the map $\mathbb{R} \ni s \mapsto H_n^i(x, s)$ is even for almost every $x \in F$;*
(ii) *for almost every $x \in F$ and for every $s \in \mathbb{R}$ sufficiently small we have*

$$u_n(x + se_i) - u_n(x) - sD_i u_n(x) \geq |s| H_n^i(x, s);$$

(iii) *for almost every $\hat{x}_i \in \hat{F}_i$ there exists a sequence $(h_n) = (h_n^{\hat{x}_i})$ in $]0, 1]$ and a positive $\delta = \delta^{\hat{x}_i}$ such that*

$$\lim_{n\to\infty} \frac{1}{h_n} \|u_{n,\hat{x}_i} - u_{\hat{x}_i}\|_{L^p(I_i)} = 0$$

*and*

$$\liminf_{n\to\infty} \frac{1}{h_n^{p+1}} \int_0^{h_n} \int_{F_i} s^p H_n(\hat{x}_i, x_i, s)^p \, dx_i \, ds \geq \delta^{\hat{x}_i}.$$

*Then*

$$\liminf_{n\to\infty} \int_I |D_i u_n(x) - D_i u(x)|^p \, dx > 0,$$

*and this implies that $(u_n)$ does not converge to $u$ in $W^{1,p}(I)$.*

*Proof.* Observe, first of all, that the form of the set $F = \hat{F}_i \times F_i \subseteq I$ is not restrictive, since an arbitrary measurable set $\tilde{F} \subseteq I$ of positive measure contains a subset of positive measure with the form of $F$.

Then we reason as in the proof of Theorem 3 and reduce ourselves to work on segments $I_i$ for almost every $\hat{x}_i \in \hat{F}_i$, so that the thesis is a consequence of Theorem 2 and of Fatou lemma. □

Consider now an arbitrary open subset $\Omega \subseteq \mathbb{R}^d$, a sequence in $W^{1,p}_{\mathrm{loc}}(\Omega)$, a map $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and suppose that

$$|Du_n(x)| \le f(x) \text{ for a.e. } x \in \Omega, \ \forall n \in \mathbb{N}, \tag{4.6}$$

for some $f \in L^p_{\mathrm{loc}}(\Omega)$. Take any bounded open polyinterval $I \subset\subset \Omega$ and assume that on $I$ the hypotheses of Theorems 3 and 4 are satisfied. By virtue of (4.6) it is evident that the conclusions of the quoted theorems remain valid and then the compactness property holds on the whole set $\Omega$. In other words we have immediately the following corollaries of Theorems 3 and 4.

**Corollary 1.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$, $p \in [1, \infty[$, $(u_n)$ a sequence in $W^{1,p}_{loc}(\Omega)$, $u \in W^{1,p}_{loc}(\Omega)$ and $f \in L^p_{loc}(\Omega)$. Assume that for every polyinterval $I$ compactly contained in $\Omega$ the sequence $(u_n|_I)$ and the function $u|_I$ satisfy the hypotheses of Theorem 3. Then $(u_n)$ converges strongly to $u$ in $W^{1,p}_{loc}(\Omega)$.*

**Corollary 2.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$, $p \in [1, \infty[$, $(u_n)$ a sequence in $W^{1,p}_{loc}(\Omega)$, $u \in W^{1,p}_{loc}(\Omega)$. Assume that there exists a polyinterval $I$ compactly contained in $\Omega$ such that the sequence $(u_n|_I)$ and the function $u|_I$ satisfy the hypotheses of Theorem 4. Then $(u_n)$ does not converge strongly to $u$ in $W^{1,p}_{loc}(\Omega)$.*

*Remark 4.* We stress that the bound (4.6) is required in order to pass to the limit in formula (4.5). Hence it is needed only in the case $d \ge 2$ and, in Theorem 3, it can be replaced by the weakest condition

$$\int_{I_i} \left|D_i u_n(\hat{x}_i, x_i)\right|^p dx_i \le F_i(\hat{x}_i), \text{ for a.e. } \hat{x}_i \in \hat{I}_i, \ \forall i \in \{1, \dots, d\}$$

for some $F_i \in L^1(\hat{I}_i)$, $i \in \{1, \dots, d\}$.

We remark, in addition, that the bound (4.6) turns out to be automatically satisfied in the special case in which $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}$.

The same conclusion of Remark 1 holds true, so that condition (4.3) can be replaced by the following one:

$$\limsup_{n\to\infty} \int_{I_j} G^j_n(x, h_n)^p \, dx = 0 \quad \forall j \in \{1, \dots, d\}.$$

## 5. A pointwise version

The convergence $u_n \to u$ in $W^{1,p}(\Omega)$ proved in previous section implies, in particular, the existence of a subsequence $(u_{n_k})$ such that $\nabla u_{n_k}(x) \to \nabla u(x)$ for almost every $x \in \Omega$. In this section we investigate conditions, closed to the ones used in Sect. 4, ensuring that the pointwise almost everywhere convergence holds true for the whole sequence $(\nabla u_n)$. In order to obtain such result we strengthen the assumptions on the pointwise behaviour of the sequence and, on the other hand, we are allowed to remove the boundedness assumption (4.2). Since we are interested in pointwise convergence, by virtue of the continuous embeddings $W^{1,p}_{\text{loc}} \hookrightarrow W^{1,1}_{\text{loc}}$ and $L^p_{\text{loc}} \hookrightarrow L^1_{\text{loc}}$, we may limit ourselves, without loss of generality, to the case $p = 1$. Moreover we assume that $\Omega$ is a bounded open polyinterval since it is clear that such assumption is not restrictive.

As in previous sections we start by proving the result on an interval $I \subseteq \mathbb{R}$, then we treat the general case of an open polyinterval $I$ of $\mathbb{R}^d$. We recall definition (2.3)–(2.4) from Notations 1 and start by the sufficient condition.

**Theorem 5.** *Let $I \subseteq \mathbb{R}$ be an open interval, $(v_n)$ a sequence in $W^{1,1}(I)$ and $v \in W^{1,1}(I)$ such that*

$$v_n \overset{n\to\infty}{\longrightarrow} v \quad \text{in} \quad L^1(I) \quad \text{and a.e. in} \quad I.$$

*Assume that for every $n \in \mathbb{N}$ and for almost every $x \in I$ there exists a continuous function*

$$G_n(x, \cdot) : \mathbb{R} \to [0, +\infty[,$$

*satisfying the following properties:*

(i) *the map $\mathbb{R} \ni s \mapsto G_n(x, s)$ is even and monotone nondecreasing on $\mathbb{R}^+$ for almost every $x \in I$;*
(ii) *for almost every $x \in E$ and for every $s \in \mathbb{R}$ sufficiently small we have*

$$v_n(x + s) - v_n(x) - v_n'(x)s \leq |s| G_n(x, s);$$

(iii) *for almost every every $x \in I$ for which $v_n(x) \to v(x)$ there exists a sequence $(h_n) = (h_n^x)$ in $\mathbb{R}^+$, with $h_n \to 0+$, such that*

$$\lim_{n\to\infty} \frac{r(x, v_n, v, I)}{h_n^2} = 0 \tag{5.1}$$

*and*

$$\limsup_{n\to\infty} G_n(x, h_n) = 0. \tag{5.2}$$

*Then, for almost every $x \in I$, we have*

$$v_n'(x) \overset{n\to\infty}{\longrightarrow} v'(x). \tag{5.3}$$

*Proof.* Call $E \subseteq I$ the subset of full measure of points $x \in I$ for which the function $v_n$ is differentiable at the point $x$ for every $n \in \mathbb{N}$. In addition call $L \subseteq I$ the full subset of Lebesgue points of $v'$ and $P$ the full subset of $I$ given by

$$P \doteq \{x \in I : v_n(x) \to v(x)\}.$$

Introduce the set

$$J \doteq E \cap P \cap L,$$

take $x \in J$ and recall notations and computations from the proof of Theorem 1, observing that, in the case $p = 1$, formula (3.19) takes the form

$$\begin{aligned}\frac{h^2}{2}|v'(x) - v_n'(x)| \leq \int_{-h}^{h} &|v(x+s) - v_n(x+s)|\, ds \\ &+ h|v_n(x) - v(x)| + \frac{h^2}{2}\eta(x,h) \\ &+ \frac{h^2}{2}\int_0^h sG_n(x,s)\, ds,\end{aligned}$$

so that we have:

$$\begin{aligned}\left|v'(x) - v_n'(x)\right| \leq\ & G_n(x,h) + \eta(x,h) \\ &+ \frac{2}{h^2}\|v - v_n\|_{L^1(I)} + \frac{2}{h}|v(x) - v_n(x)|. \end{aligned} \tag{5.4}$$

Consider the sequence $(h_n)$ given in the statement, recall that $x \in J \subseteq P$ is fixed and put $h = h_n$ in formula (5.4). We obtain

$$\begin{aligned}\left|v'(x) - v_n'(x)\right| \leq\ & G_n(x,h_n) + \eta(x,h_n) \\ &+ \left(\frac{2}{h_n^2} + \frac{2}{h_n}\right) r(x, v_n, v, I).\end{aligned}$$

Then, by (3.5), (5.1) and (5.2), we have

$$\limsup_{n\to\infty} \left|v'(x) - v_n'(x)\right| = 0,$$

that is to say (5.3). □

We skip the one dimensional necessary part and pass directly to the sufficient condition in several dimensions.

**Theorem 6.** *Let $I$ be an open bounded polyinterval of $\mathbb{R}^d$, $(u_n)$ a sequence in $W^{1,1}(I)$ and $u \in W^{1,1}(I)$ such that*

$$u_n \overset{n\to\infty}{\longrightarrow} u \quad in \quad L^1_{loc}(I) \quad and\ a.e.\ in \quad I.$$

*Assume that for almost every $x \in I$, for every $n \in \mathbb{N}$ and for every $i \in \{1, \ldots, d\}$, there exists a continuous function*

$$G_n^i(x,\cdot) : \mathbb{R} \to [0, +\infty[$$

*satisfying the following properties:*

(i) *the application* $\mathbb{R} \ni h \mapsto G_n^i(x,h)$ *is even and monotone nondecreasing on* $\mathbb{R}^+$;

(ii) *for almost every* $x \in I$ *for which* $u_n(x) \to u(x)$ *and for every* $s \in \mathbb{R}$ *sufficiently small we have*

$$u_n(x+se_i) - u_n(x) - D_i u_n(x)s \leq |s| G_n^i(x,s);$$

(iii) *for almost every* $x \in I$ *there exists a sequence* $(h_n^x) = (h_n)$ *in* $\mathbb{R}^+$, *with* $h_n \to 0+$, *such that*

$$\lim_{n\to\infty} \frac{r^i(x,u_n,u,I)}{h_n^2} = 0$$

*and*

$$\limsup_{n\to\infty} \; G_n^i(x,h_n) = 0.$$

*Then, for almost every* $x \in I$, *we have*

$$\nabla u_n(x) \stackrel{n\to\infty}{\longrightarrow} \nabla u(x).$$

*Proof.* Fix $i \in \{1,\ldots,d\}$, $\hat{x}_i \in \hat{I}_i$ and set

$$v = v_{\hat{x}_i} \doteq u(\hat{x}_i,\cdot), \quad w^i = w^i_{\hat{x}_i} \doteq D_i u(\hat{x}_i,\cdot),$$
$$v_n = v_{n,\hat{x}_i} \doteq u_n(\hat{x}_i,\cdot), \quad w^i_n = w^i_{n,\hat{x}_i} \doteq D_i u_n(\hat{x}_i,\cdot), \quad n \in \mathbb{N},$$

where all such functions are defined on the interval $I_i$ with values in $\mathbb{R}$. Then call $A_i \subseteq \hat{I}_i$ the set of points $\hat{x}_i \in \hat{I}_i$ such that

$$\begin{aligned} & v \in W^{1,1}(I_i), \quad w^i \in L^1(I_i), \\ & v_n \in W^{1,1}(I_i), \quad w^i_n \in L^1(I_i) \quad \forall n \in \mathbb{N}, \\ & v'(x_i) = w^i(x_i), \quad v'_n(x_i) = w^i_n(x_i), \quad \text{a.e. } x_i \in I_i, \ \forall n \in \mathbb{N}, \\ & v_n \stackrel{n\to\infty}{\longrightarrow} v \quad \text{in} \quad L^1(I_i) \quad \text{and a.e. in} \quad I_i. \end{aligned} \tag{5.5}$$

By classical results (see for example [11], ch. 4.9 pp. 162-164, ch. 1.4 pp. 22-24 and [20], lemma 6.5 p. 89) the set $A_i$ is a full measure subset of $\hat{I}_i$. Let now $\hat{x}_i \in A_i$ and introduce the sets

$$F_{\hat{x}_i} \doteq \left\{x_i \in I_i : (\hat{x}_i, x_i) \in F\right\}, L_{\hat{x}_i} \doteq \left\{x_i \in I_i : x_i \text{ is a Lebesgue point for } w^i\right\},$$

and

$$P_{\hat{x}_i} \doteq \{x_i \in I_i : v_n(x_i) \to v(x_i)\},$$

remarking that $F_{\hat{x}_i}$, $L_{\hat{x}_i}$ and $P_{\hat{x}_i}$ are subsets of $I = I_i$ of full measure (see again [20], Lemma 6.5, p. 89). Then define

$$H_i \doteq \left\{x = (\hat{x}_i, x_i) \in I : \hat{x}_i \in A_i, x_i \in F_{\hat{x}_i} \cap L_{\hat{x}_i} \cap P_{\hat{x}_i}\right\}.$$

Invoking well known properties of Lebesgue measure, it is immediate to see that $H_i$ is a full measure subset of $I$, hence, by the arbitrariness of $i \in \{1,\ldots,d\}$, the theorem is proved if we show that

$$D_i u_n(x) \stackrel{n\to\infty}{\longrightarrow} D_i u(x) \quad \forall x \in H_i.$$

We observe that, once fixed $i \in \{1, \dots, d\}$ and $\hat{x}_i \in A_i$, the set $H_i \cap F_{\hat{x}_i} \cap L_{\hat{x}_i} \cap P_{\hat{x}_i}$ is a full subset of $I_i$ and the family composed by the sets $I_i$ and $J \doteq H_i \cap F_{\hat{x}_i} \cap P_{\hat{x}_i} \cap L_{\hat{x}_i}$, the sequence $(v_n)$ and the map $v$ [defined in (5.5)], satisfies the conditions of Theorem 5. Hence, for every $x = (\hat{x}_i, x_i) \in H_i$, we have

$$D_i u_n(x) = D_i u_n(\hat{x}_i, x_i) = v_n'(x_i) \overset{n\to\infty}{\longrightarrow} v'(x_i) = D_i u(\hat{x}_i, x_i) = D_i u(x).$$

By the arbitrariness of $i \in \{1, \dots, d\}$ we have the thesis. □

We conclude this section with the necessary version of the pointwise condition, whose proof is a sort of interpolation of the arguments used up to now.

**Theorem 7.** *Let $I$ be an open bounded polyinterval of $\mathbb{R}^d$, $(u_n)$ a sequence in $W^{1,1}(I)$ and $u \in W^{1,1}(I)$ such that*

$$u_n \overset{n\to\infty}{\longrightarrow} u \quad \text{in} \quad L^1(I) \quad \text{and a.e. in} \quad I.$$

*Assume that there exist a measurable subset $F \subseteq I$, with $m_d(F) > 0$ and an index $i \in \{1, \dots, d\}$ such that for almost every $x \in F$ and for every $n \in \mathbb{N}$, there exists a continuous function*

$$H_n^i(x, \cdot) : \mathbb{R} \to [0, +\infty[$$

*satisfying the following properties:*

(i) *for almost every $x \in F$ and for every $s \in \mathbb{R}$ sufficiently small we have*

$$u_n(x + se_i) - u_n(x) - D_i u_n(x)s \geq |s| H_n^i(x, s);$$

(ii) *for almost every $x \in F$ for which $u_n(x) \to u(x)$ there exist a sequence $(h_n) = (h_n^x)$ in $\mathbb{R}^+$, with $h_n \to 0+$, and a positive $\delta^x$ such that*

$$\lim_{n\to\infty} \frac{r^i(x, u_n, u, I)}{h_n^2} = 0$$

*and*

$$\liminf_{n\to\infty} \frac{1}{h_n^2} \int_0^{h_n} s H_n^i(x, s)\, ds \geq \delta^x.$$

*Then, for almost every $x \in F$, we have*

$$\liminf_{n\to\infty} |D_i u_n(x) - D_i u(x)| \geq \delta^x,$$

*so that the sequence $(\nabla u_n)$ does not converge pointwise almost everywhere to $\nabla u$.*

*Proof.* It is clear that it is sufficient to prove the statement in dimension $d = 1$, so that we replace $u = u(\hat{x}_i, i)$ and $D_i u = D_i u(\hat{x}_i, x_i)$, defined for $x$ belonging to the polyinterval $I$, by $v = v(x)$ and $v' = v'(x)$ defined for $x$ belonging to the interval

$I \subseteq \mathbb{R}$. Hence we take a point $x \in F$ and observe that we may reproduce step by step the arguments of the proof of Theorem 2. In particular we take formula (3.32):

$$\frac{h^2}{2}|v_n'(x) - v'(x)| \geq -\int_{-h}^{h} |v(x+s) - v_n(x+s)|\, ds - h|v_n(x) - v(x)| \\ - \int_0^h s\eta(x,s)\, ds + \int_0^h sH_n(x,s)\, ds. \tag{5.6}$$

Considering the sequence $(h_n)$ of the statement, inequality (5.6) implies that

$$\begin{aligned} |v_n'(x) - v'(x)| &\geq -\frac{2}{h_n^2}\|v_n - v\|_{L^1(I)} - \frac{2}{h_n}|v_n(x) - v(x)| \\ &\quad -\eta(x, h_n) + \frac{2}{h_n^2}\int_0^{h_n} sH_n(x,s)\, ds \\ &= -\left(\frac{2}{h_n^2} + \frac{2}{h_n}\right) r(x, v_n, v, I) \\ &\quad -\eta(x, h_n) + \frac{2}{h_n^2}\int_0^{h_n} sH_n(x,s)\, ds. \end{aligned}$$

From this last inequality the thesis follows immediately.

The case $d > 1$ is treated by applying the scheme used in the proof of Theorem 5 to the argument developed above. □

## 6. The special case $W^{1,\infty}$

This section is devoted to the study of our pointwise convergence property in the space $W^{1,\infty}$. The result, in the sufficient part, has already been proved and applied in papers [31,32] and [33]; we reproduce it here for the sake of completeness, adding the necessary part.

**Theorem 8.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$, $(u_n)$ a sequence in $W^{1,\infty}(\Omega)$ and $u \in W^{1,\infty}(\Omega)$ such that*

$$u_n \overset{n\to\infty}{\longrightarrow} u \text{ uniformly on } \Omega.$$

*Assume that for almost every $x \in \Omega$, for every $n \in \mathbb{N}$ and for every $i \in \{1, \dots d\}$ there exists an even function $G_n^i(x, \cdot) : \mathbb{R} \to [0, +\infty[$ satisfying the following properties:*

(i) *for every $s \in \mathbb{R}$ sufficiently small we have*

$$u_n(x + se) - u_n(x) - D_i u(x)s \leq |s| G_n^i(x,s); \tag{6.1}$$

(ii) *for almost every $x \in \Omega$ there exists a sequence $(h_n) = (h_n^x)$ in $\mathbb{R}^+$, with $h_n \to 0+$, such that*

$$\lim_{n\to\infty} \frac{1}{h_n}\|u_n - u\|_\infty = 0 \tag{6.2}$$

*and*

$$\limsup_{n\to\infty} G_n^i(x, h_n) = 0. \tag{6.3}$$

*Then, for almost every* $x \in \Omega$, *we have*

$$\nabla u_n(x) \overset{n\to\infty}{\longrightarrow} \nabla u(x). \tag{6.4}$$

*Proof.* Fix an arbitrary $i \in \{1, \dots, d\}$, take $x \in \Omega$ be such that the hypothesis is satisfied and, in addition, $u$ and $u_n$ are differentiable at the point $x$ for every $n \in \mathbb{N}$. Set

$$z \doteq D_i u(x), \quad z_n \doteq D_i u_n(x),$$

so that the claim (6.4) of the theorem is

$$z_n \overset{n\to\infty}{\longrightarrow} z. \tag{6.5}$$

Let $\overline{h} \doteq \overline{h}_x > 0$ be such that the segment $[x - \overline{h}e_i, x + \overline{h}e_i]$ is contained in $\Omega$. By hypothesis (6.1), we have:

$$\frac{u_n(x + he_i) - u_n(x)}{h} - z_n \le G_n^i(x, h) \quad \forall h \in ]0, \overline{h}] \tag{6.6}$$

and

$$\frac{u_n(x + he_i) - u_n(x)}{h} - z_n \ge -G_n^i(x, |h|) \quad \forall h \in ]-\overline{h}, 0]. \tag{6.7}$$

We consider the sequence $(h_n)$ of the statement for $n$ large enough, so that $h_n \le \overline{h}$, and estimate the difference

$$\frac{u_n(x + h_n e_i) - u_n(x)}{h_n} - z.$$

We have

$$\begin{aligned}
&\left| \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} - z \right| \\
&\quad = \left| \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} - \frac{u(x + h_n e_i) - u(x)}{h_n} + \frac{u(x + h_n e_i) - u(x)}{h_n} - z \right| \\
&\quad \le \left| \frac{u_n(x + h_n e_i) - u(x)}{h_n} - \frac{u(x + he_i) - u(x)}{h_n} \right| + \left| \frac{u(x + h_n e_i) - u(x)}{h_n} - z \right| \\
&\quad \le \left| \frac{u_n(x + h_n e_i) - u(x + he_i)}{h_n} \right| + \left| \frac{u_n(x) - u(x)}{h_n} \right| + \left| \frac{u(x + h_n e_i) - u(x)}{h_n} - z \right| \\
&\quad \le 2\, \frac{\|u_n - u\|_{L^\infty(\Omega)}}{h_n} + \left| \frac{u(x + h_n e_i) - u(x)}{h_n} - z \right|.
\end{aligned}$$

By the differentiability of $u$ at the point $x$ and by (6.2), the r.h.s. of the last formula goes to zero as $n \to \infty$, hence we obtain

$$\lim_{n\to\infty} \left| \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} - z \right| = 0. \tag{6.8}$$

Clearly (6.8) still holds true if we replace $h_n$ by $k_n \doteq -h_n$. Now, recalling (6.6), observe that we have

$$z - z_n = z - \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} + \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} - z_n$$
$$\leq \left| z - \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} \right| + G_n^i(x, h_n). \tag{6.9}$$

By (6.3) and (6.8), it follows that

$$\limsup_{n\to\infty} (z - z_n) \leq 0. \tag{6.10}$$

Now set $k_n = -h_n \geq -\overline{h}$ for $n$ sufficiently large. As in (6.9), recalling (6.7), we have:

$$z - z_n = z - \frac{u_n(x + k_n e_i) - u_n(x)}{k_n} + \frac{u_n(x + k_n e_i) - u_n(x)}{k_n} - z_n$$
$$\geq - \left| z - \frac{u_n(x + k_n e_i) - u_n(x)}{k_n} \right| - G_n^i(x, h_n).$$

As above, by (6.3) and (6.8), it follows that

$$\liminf_{n\to\infty} (z - z_n) \geq - \limsup_{n\to\infty} G_n^i(x, h_n) \geq 0. \tag{6.11}$$

Collecting (6.10) and (6.11) we have achieved the proof of (6.5) and then, by the arbitrariness of the index $i$, of (6.4). □

We end the section with the necessary version.

**Theorem 9.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$, $(u_n)$ a sequence in $W^{1,\infty}(\Omega)$ and $u \in W^{1,\infty}(\Omega)$ such that*

$$u_n \overset{n\to\infty}{\longrightarrow} u \text{ uniformly on } \Omega.$$

*Assume that there exist a measurable subset $F \subseteq \Omega$, with $m_d(F) > 0$ and an index $i \in \{1, \ldots, d\}$ such that for almost every $x \in F$ and for every $n \in \mathbb{N}$ there exists an even function $H_n(x, \cdot) : \mathbb{R} \to [0, +\infty[$ such that the following properties hold true:*

(i) *for every $s \in \mathbb{R}$ sufficiently small we have*

$$u_n(x + se) - u_n(x) - D_i u(x)s \geq |s| H_n(x, s);$$

(ii) *for almost every $x \in F$ there exist a sequence $(h_n) = (h_n^x)$ in $\mathbb{R}^+$, with $h_n \to 0+$, and a positive $\delta^x$ such that*

$$\lim_{n\to\infty} \frac{1}{h_n} \|u_n - u\|_\infty = 0$$

*and*

$$\liminf_{n\to\infty} H_n(x, h_n) \geq \delta^x. \tag{6.12}$$

*Then, for almost every $x \in F$, we have*

$$\liminf_{n\to\infty} |D_i u_n(x) - D_i u(x)| \geq \delta^x,$$

*so that the sequence $(\nabla u)_n$ does not converge pointwise almost everywhere to $\nabla u$.*

*Proof.* Adopt the notations and the arguments of the proof of Theorem 9, take an arbitrary $x \in F$, and observe that in place of formula (6.6) we have

$$\frac{u_n(x + he_i) - u_n(x)}{h} - z_n \geq H_n^i(x, h) \quad \forall h \in ]0, \overline{h}]. \tag{6.13}$$

Once performed the same computations of previous proof, by (6.13) we obtain, in place of (6.9), the following inequality:

$$\begin{aligned} z - z_n &= z - \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} + \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} - z_n \\ &\geq -\left| z - \frac{u_n(x + h_n e_i) - u_n(x)}{h_n} \right| + H_n^i(x, h_n). \end{aligned} \tag{6.14}$$

By (6.8) and (6.12), (6.14) imply that

$$\liminf_{n\to\infty} |z - z_n| \geq \liminf_{n\to\infty} (z - z_n) \geq \delta^x,$$

and this ends the proof. □

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